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Commun Nonlinear Sci Numer Simulat 148 (2025) 108860

Contents lists available at ScienceDirect

Communications in Nonlinear Science and Numerical Simulation

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Strong convergence for efficient full discretization of the stochastic Allen–Cahn equation with multiplicative noise

Xiao Qi [a], Lihua Wang [b], Yubin Yan [c,*]

[a] School of Artificial Intelligence, Jianghan University, Wuhan 430056, Hubei, China
[b] School of Medicine, Jianghan University, Wuhan 430056, Hubei, China
[c] School of Computer and Engineering Sciences, University of Chester, Parkgate Road, Chester CH1 4BJ, UK

ARTICLE INFO

MSC:
60H15
60H35
65C50
65M60

Keywords:
Stochastic Allen–Cahn equation
Multiplicative noise
Strong convergence
Semi-implicit tamed method
Finite element method

ABSTRACT

In this paper, we study the strong convergence of the full discretization based on a semi-implicit tamed approach in time and the finite element method with truncated noise in space for the stochastic Allen–Cahn equation driven by multiplicative noise. The proposed fully discrete scheme is efficient thanks to its low computational complexity and mean-square unconditional stability. The low regularity of the solution due to the multiplicative infinite-dimensional driving noise and the non-global Lipschitz difficulty introduced by the cubic nonlinear drift term make the strong convergence analysis of the fully discrete solution considerably complicated. By constructing an appropriate auxiliary procedure, the full discretization error can be cleverly decomposed, and the spatio-temporal strong convergence order is successfully derived under certain weak assumptions. Numerical experiments are finally reported to validate the theoretical result.

1. Introduction

The Allen–Cahn equation was originally proposed by Allen and Cahn in [1] as a mathematical model to describe the motion of antiphase boundaries in crystalline solids and has been widely used as a fundamental equation for many complex moving interface problems in materials science and fluid dynamics, see, e.g., [2–5] and references therein. However, due to the existence of material impurities, thermal fluctuations, or intrinsic instabilities in the evolution process, mathematicians generally believe that incorporating noise and/or uncertainty into the deterministic Allen–Cahn equation will be closer to reality. This has attracted widespread attention in the past few years to the study of the stochastic Allen–Cahn equation, especially the numerical approximation of this equation, see, e.g., [6–11] for strong convergence analysis, [12,13] for weak convergence analysis, and references therein, among which the semi-implicit tamed temporal discrete scheme is particular popular recently, see, e.g., [14–18]. Although great progress has been made in the numerical investigation of the stochastic Allen–Cahn equation, the numerical analysis and simulation of effective and stable fully discrete schemes for the stochastic Allen–Cahn equation are still far from satisfactory due to the essential difficulties caused by non–global Lipschitz nonlinearity, infinite–dimensional operators and driving noise. To the best of our knowledge, there has been no work on the fully discretized numerical analysis of the stochastic Allen–Cahn equation with multiplicative noise using semi-implicit tamed method and finite element method. The current paper attempts to fill this gap by providing a detailed strong convergence analysis of the "semi-implicit tamed method/ finite element method" full discretization to the stochastic Allen–Cahn equation perturbed by multiplicative noise.

* Corresponding author.
E-mail addresses: qixiao@jhun.edu.cn (X. Qi), lihuawang@jhun.edu.cn (L. Wang), y.yan@chester.ac.uk (Y. Yan).

https://doi.org/10.1016/j.cnsns.2025.108860
Received 14 October 2024; Received in revised form 7 April 2025; Accepted 9 April 2025
Available online 25 April 2025

The stochastic Allen–Cahn equations (SACEs) can be usually classified into two categories in terms of the different force noises, namely SACEs with additive noise or multiplicative noise. We just mention a few papers here on the numerical approximation of SACEs with additive noise. For instance, Qi et al. [19] showed an optimal spatio–temporal strong convergence rates for a backward Euler–Galerkin finite element fully discrete scheme of SACEs driven by additive noise. Bréhier et al. [20] proposed an explicit temporal splitting scheme for SACEs driven by additive noise, and derived its strong convergence rates. Wang et al. [17] analyzed the strong convergence order of a kind of nonlinearity–tamed accelerated exponential integrator scheme for SACEs with additive space–time white noise. Some other related works include Kovács et al. [21], Becker et al. [22], Cui et al. [23], Cai et al. [24], and Wang et al. [18], where different numerical schemes were constructed and investigated for different SACEs with additive noises. Compared to the case of additive noise, the numerical analysis of SACEs with multiplicative noise is commonly more subtle and challenging, and has received considerable attention in recent years. For example, Majee et al. [25] presented an optimal strong error estimate for a modified implicit Euler scheme for SACEs with multiplicative noise. Huang et al. [16] deduced the spatio–temporal strong convergence rate of a fully discrete scheme that is based on the spectral method in space and semi-implicit tamed scheme in time. Qi et al. [26] developed an extended Euler–Maruyama time discretization scheme that combines the semi-implicit scheme and scalar auxiliary variable method to numerically solve SACEs with multiplicative noise. In addition to the form of Allen–Cahn potential function, there has also been some works on the numerical solution of more general stochastic partial differential equations (SPDEs) with non-globally Lipschitz drift term. In this regard, we mention the work, for example, by Jentzen et al. [27] on a class of tamed space–time–noise discrete exponential Euler approximation for SPDEs with non-globally monotone nonlinearities, and the work by Liu et al. [28] on a general theory of optimal strong error analysis for SPDEs with monotone drift driven by a multiplicative infinite-dimensional noise.

The aim of this paper is to analyze the strong convergence order of a fully discrete scheme that is based on a semi-implicit tamed method in time and the finite element method in space for SACEs with multiplicative noise. The idea is to introduce an auxiliary approximation procedure and use an appropriate decomposition method to estimate the convergence error. The main contributions/novelties of this work are summarized as follows:

- An efficient time discretization scheme is proposed for solving the stochastic Allen-Cahn equation.
- The strong convergence rate of the proposed time scheme is rigorously proved.
- This method is computationally cheaper than the widely used backward Euler method [19,28].

The rest of paper is organized as follows. The well-posedness of the underlying problem is established under some standard assumptions in Section 2. The spatio-temporal full discretization are introduced in Section 3. In Section 4, the strong convergence analysis is performed using semigroup theory and stochastic calculus tools. Several numerical experiments are provided in Section 5 to validate the theoretical results and demonstrate the performance of the proposed method.

2. Problem and its well-posedness

Let $T > 0$, $D \in \mathbb{R}^d$, $d \in \{1, 2, 3\}$, be a bounded open spatial domain with smooth boundary. Specifically we consider $D := (0, 1)^d$ in this work. We are interested in the following stochastic Allen–Cahn equation written as:

$$\begin{aligned} \mathrm{d}u(t) &= \big(-Au(t) + F(u(t))\big)\mathrm{d}t + G(u(t))\mathrm{d}W(t),\ \ 0 < t < T,\\ u(x,t) &= 0, \quad 0 \le t \le T,\ \ x \in \partial D,\\ u(0) &= u_0, \end{aligned} \tag{2.1}$$

where $-A$ is the Laplace operator with the domain $\mathcal{D}(A) = H^2(D) \cap H_0^1(D)$, and $F(u)$ is the Nemytskij operator, given by $F(u)(x) := f(u(x))$, $x \in D$, with $f(v) := v - v^3$, $v \in \mathbb{R}$. $G(\cdot)$ is the nonlinear diffusion term and $W(t)$ is a smooth Q-Wiener process to be specified later.

In order to study the problem (2.1) theoretically and numerically, we review some notions and notations below. Let $L^{2p}(D)$, $p \ge 1$, be the classical Sobolev space and H ($H \subset L^2(D)$) be another separable Hilbert space. Let $(\cdot, \cdot)$ denote the $L^2(D)$-inner product and $\mathcal{L}(L^2(D))$ represent the space of bounded linear operators from $L^2(D)$ to $L^2(D)$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ be a filtered probability space with a normal filtration $\{\mathcal{F}_t\}_{t\ge0}$. Denote by $L^{2p}(\Omega, L^2(D))$ the space of $2p$-times integrable random variables, i.e., for $v \in L^{2p}(\Omega, L^2(D))$,

$$\|v\|_{L^{2p}(\Omega,L^2(D))} := \big(\mathbb{E}[\|v(\omega,\cdot)\|^{2p}_{L^2(D)}]\big)^{\frac{1}{2p}} < \infty, \quad \omega \in \Omega. \tag{2.2}$$

Here $\mathbb{E}[\cdot]$ stands for the expectation in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $W(t)$ be a $\mathcal{F}_t$-adapted H-valued Wiener process with covariance operator Q, where Q is a positive definite and symmetric trace class operator with orthonormal eigenfunctions $\{\phi_j(x) \in H : j \in \mathbb{N}\}$ and corresponding positive eigenvalues $\{q_j\}$, see, e.g., [29,30] for more details. Define Cameron–Martin space $Q^{\frac12}(H) := \{Q^{\frac12}v : v \in H\}$. Let $\mathcal{L}_0^2 := \{B | B : Q^{\frac12}(H) \to L^2(D)\}$ be the Hilbert–Schmidt operator space [26,31] equipped with norm

$$\|B\|_{\mathcal{L}_0^2} := \Big(\sum_{j=1}^{\infty} \|BQ^{\frac12}\phi_j\|^2_{L^2(D)}\Big)^{\frac12} < \infty, \quad \forall B \in \mathcal{L}_0^2.$$

Throughout the paper we use c, with or without subscripts, to mean generic positive constants, which may not be the same at different occurrences.

Denote by $\dot{H}^s$, $s \in \mathbb{R}$, the Hilbert space with norm $\|\cdot\|_{\dot{H}^s} := \|A^{\frac{s}{2}} \cdot\|_{L^2(D)}$. The mild solution related to the problem (2.1) is expressed as:

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau + \int_0^t S(t-\tau)G(u(\tau))dW(\tau), \tag{2.3}$$

where $S(t) := e^{-tA}$ is a semigroup generated by the operator A.

Some necessary assumptions on the nonlinear terms $F(\cdot)$ and $G(\cdot)$ are also needed for establishing the well-posedness of the problem (2.1), which are summarized as follows:

- Suppose $F(\cdot)$ is a mapping from $L^6(D)$ to $L^2(D)$ such that:

$$\big(v_1 - v_2, F(v_1) - F(v_2)\big) \le \|v_1 - v_2\|^2_{L^2(D)}, \quad \forall v_1, v_2 \in L^6(D). \tag{2.4}$$

- Suppose $G(\cdot)$ is a mapping from $L^2(D)$ to $\mathcal{L}_0^2$, such that:

$$\Big\|G(v_1) - G(v_2)\Big\|_{\mathcal{L}_0^2} \le c\|v_1 - v_2\|_{L^2(D)}, \quad \forall v_1, v_2 \in L^2(D). \tag{2.5}$$

- Suppose $A^{\frac{1}{2}}G(\cdot)$ maps from $\dot{H}^1$ to $\mathcal{L}_0^2$, such that:

$$\|A^{\frac{1}{2}}G(v)\|_{\mathcal{L}_0^2} \le c\big(1 + \|v\|_{\dot{H}^1}\big), \quad \forall v \in \dot{H}^1. \tag{2.6}$$

Remark 2.1. The assumption (2.4) is actually a one-sided Lipschitz condition, which is weaker than the global Lipschitz condition and is often used in the numerical analysis of SACEs, see, e.g., [19, Assumption 2.2]. The assumption (2.5) and assumption (2.6) have been considered by Liu et al. [28, Assumption 2.2], and some specific examples of $G(\cdot)$ satisfying (2.5) and (2.6) are also given in [28, Remark 2.2].

Next, we focus on the well-posedness of the problem (2.1), that is, we aim to prove the existence, uniqueness and stability of the mild solution (2.3). To this end, we first define the space $\mathcal{H}_{2p}^T$ for $p \ge 1$, which is the Banach space of $\dot{H}^1$-valued predictable processes $\{v(\tau) : \tau \in [0,T]\}$, endowed with the norm

$$\|v\|_{\mathcal{H}_{2p}^T} := \sup_{\tau\in[0,T]} \|v(\tau)\|_{L^{2p}(\Omega,\dot{H}^1)} < \infty.$$

The well-posedness proof of the problem (2.1) is motivated by [28, Theorem 3.1], but adapted to the problem considered in this work.

Theorem 2.1. *Suppose that the initial data $u_0 \in L^{2p}(\Omega, \dot{H}^1)$, $p \ge 1$, is an $\mathcal{F}_0$-measurable random variable. Then, there exists a unique mild solution $u \in \mathcal{H}_{2p}^T$ to the problem (2.1). Furthermore, there exists a constant c such that the following moment estimate holds*

$$\|u\|_{\mathcal{H}_{2p}^T} \le c(1 + \|u_0\|_{L^{2p}(\Omega,\dot{H}^1)}). \tag{2.7}$$

Proof. Firstly, define the variational solution of the problem (2.1) as below:

$$u(t) = u_0 + \int_0^t \big(-Au(\tau) + F(u(\tau))\big)d\tau + \int_0^t G(u(\tau))dW(\tau), \quad t \in [0,T]. \tag{2.8}$$

According to the known result reported in [28, Lemma 2.3], one obtains that the variational solution (2.8) is also the mild solution of the problem (2.1) in the space $\mathcal{H}_{2p}^T$.

Next, we only need to prove the existence and uniqueness of the variational solution (2.8) in the space $\mathcal{H}_{2p}^T$, and prove that the inequality $\|u\|_{\mathcal{H}_{2p}^T} \le c(1 + \|u_0\|_{L^{2p}(\Omega,\dot{H}^1)})$ holds. In fact, this can be done by following the same lines as in [28, Theorem 3.1], where the imposed assumptions on $F(\cdot)$, $G(\cdot)$, and some basic knowledges such as dual argument, embedding inequality, and the known result shown in [32, Theorem 1.1] will be used to complete the proof (see [28, Theorem 3.1] for details). □

Note that based on the inequality (2.7), the Sobolev embedding results [28, (2.6)], and $F(u) = u - u^3$, it suffices to deduce that for $p \ge 1$,

$$\begin{aligned}\sup_{0\le t\le T} \|F(u(t))\|_{L^{2p}(\Omega,L^2(D))} &\le c\Big(\sup_{0\le t\le T} \|u(t)\|_{L^{2p}(\Omega,L^2(D))} + \sup_{0\le t\le T} \|u(t)\|^3_{L^{6p}(\Omega,L^6(D))}\Big) \\ &\le c\big(\sup_{0\le t\le T} \|u(t)\|_{L^{2p}(\Omega,\dot{H}^1)} + \sup_{0\le t\le T} \|u(t)\|^3_{L^{6p}(\Omega,\dot{H}^1)}\big) < \infty,\end{aligned} \tag{2.9}$$

which plays a key role in the subsequent error estimation.

3. Fully discrete scheme

This section is devoted to constructing a full discretization scheme for the problem (2.1). The proposed fully discrete scheme is based on the standard piecewise linear finite element method with truncated noise in space and semi-implicit tamed scheme in time.

Let $\Delta t := T/N$ be the uniform time step size for a positive integer N, and $\mathcal{T}_h$ be a regular triangulation. Denote the finite element space V_h by

$$V_h := \{v \in C^0(\bar{D}),\ v = 0 \text{ on } \partial D,\ v|_K \in \mathbb{P}_1(K) \text{ for all } K \in \mathcal{T}_h\},$$

where $\mathbb{P}_1(K)$ stands for the space of the polynomials of degree less than or equal to 1 defined in K. Let $\mathcal{P}_J^w$ be the projection from H to the finite-dimensional space $\text{span}\{\phi_1, \dots, \phi_J\}$ with J being a positive integer. Denote the orthogonal projection from $L^2(D)$ to V_h by $\mathcal{P}_h$. Let $A_h : V_h \to V_h$ be the finite-dimensional operator defined by

$$(A_h w, v) := (\nabla w, \nabla v), \quad \forall w, v \in V_h.$$

The fully discrete scheme, called hereafter "semi-implicit tamed/finite element method with truncated noise", reads: find $\mathcal{F}_{t_{n+1}}$-adapted V_h-valued random variable u_h^{n+1}, such that: for $n = 0, \dots, N-1$,

$$\begin{aligned}(I + \Delta t A_h)u_h^{n+1} &= u_h^n + \frac{\Delta t}{1 + \Delta t\|F(u_h^n)\|_{L^2(D)}^2}\mathcal{P}_h F(u_h^n) + \mathcal{P}_h G(u_h^n)\mathcal{P}_J^w \Delta W^n, \\ u_h^0 &:= \mathcal{P}_h u_0,\end{aligned} \tag{3.1}$$

where u_h^n is the fully discrete approximation to $u(t_n)$ with $t_n = n\Delta t$, and $\Delta W^n := W(t_{n+1}) - W(t_n)$.

Remark 3.1. Indeed, this semi-implicit tamed temporal discrete method has been applied to solve SPDEs with one-sided Lipschitz nonlinear drift terms, see, e.g., [16,18]. However, in [16], the spatial discretization method used is spectral method instead of finite element method, and in [18], the considered problem is SPDEs with additive noise (not multiplicative noise). It seems the current work is the first attempt to study the numerical approximation to the problem (2.1) using fully discrete scheme (3.1). One more thing is to note that the Q-Wiener process $W(t)$ is also a function of the spatial variable x, so we further consider the truncation of noise in the scheme (3.1) to make the space discretization more complete. This truncated noise has appeared in the literature [29,33].

One remarkable property of the above fully discrete scheme (3.1) is that it satisfies mean-square unconditional stability, as shown in the following proposition.

Proposition 3.1 (*Mean-Square Unconditional Stability*)**.** *The numerical solution of the discrete problem (3.1) is mean-square unconditionally stable in the sense that it satisfies:*

$$\|u_h^{n+1}\|_{L^2(\Omega,L^2(D))} \le c(1 + \|u_h^0\|_{L^2(\Omega,L^2(D))}), \quad n = 0, \dots, N-1. \tag{3.2}$$

Proof. One derives readily from (3.1) that

$$\begin{aligned}(u_h^{n+1} - u_h^n, v) + \Delta t(\nabla u_h^{n+1}, \nabla v) &= \frac{\Delta t}{1 + \Delta t\|F(u_h^n)\|_{L^2(D)}^2}(F(u_h^n), v) + (G(u_h^n)\mathcal{P}_J^w \Delta W^n, v),\ \forall v \in V_h, \\ u_h^0 &= \mathcal{P}_h u_0.\end{aligned} \tag{3.3}$$

Employing equality $a(a-b) = \frac{1}{2}(a^2 - b^2 + (a-b)^2)$ and choosing $v = u_h^{n+1}$ in (3.3) gives:

$$\begin{aligned}&\|u_h^{n+1}\|_{L^2(D)}^2 + \|u_h^{n+1} - u_h^n\|_{L^2(D)}^2 + 2\Delta t\|\nabla u_h^{n+1}\|_{L^2(D)}^2 \\ &= \|u_h^n\|_{L^2(D)}^2 + \frac{2\Delta t}{1 + \Delta t\|F(u_h^n)\|_{L^2(D)}^2}(F(u_h^n), u_h^{n+1}) + 2(G(u_h^n)\mathcal{P}_J^w \Delta W^n, u_h^{n+1}) \\ &\le \|u_h^n\|_{L^2(D)}^2 + \frac{2\Delta t}{1 + \Delta t\|F(u_h^n)\|_{L^2(D)}^2}(F(u_h^n), u_h^{n+1} - u_h^n) + 2\Delta t(F(u_h^n), u_h^n) + 2(G(u_h^n)\mathcal{P}_J^w \Delta W^n, u_h^{n+1}) \\ &\le \|u_h^n\|_{L^2(D)}^2 + 4\Delta t + \frac{1}{4}\|u_h^{n+1} - u_h^n\|_{L^2(D)}^2 - 2\Delta t\|u_h^n\|_{L^4(D)}^4 + 2\Delta t\|u_h^n\|_{L^2(D)}^2 + 2(G(u_h^n)\mathcal{P}_J^w \Delta W^n, u_h^n) \\ &\quad + 2(G(u_h^n)\mathcal{P}_J^w \Delta W^n, u_h^{n+1} - u_h^n) \\ &\le (1 + 2\Delta t)\|u_h^n\|_{L^2(D)}^2 + 4\Delta t + \frac{1}{2}\|u_h^{n+1} - u_h^n\|_{L^2(D)}^2 - 2\Delta t\|u_h^n\|_{L^4(D)}^4 + 4\|G(u_h^n)\mathcal{P}_J^w \Delta W^n\|_{L^2(D)}^2 \\ &\quad + 2(G(u_h^n)\mathcal{P}_J^w \Delta W^n, u_h^n).\end{aligned} \tag{3.4}$$

Therefore we obtain:

$$\|u_h^{n+1}\|_{L^2(D)}^2 \le (1 + 2\Delta t)\|u_h^n\|_{L^2(D)}^2 + 4\Delta t + 4\|G(u_h^n)\mathcal{P}_J^w \Delta W^n\|_{L^2(D)}^2 + 2(G(u_h^n)\mathcal{P}_J^w \Delta W^n, u_h^n). \tag{3.5}$$

Define $\delta := 1 + 2\Delta t$. A direct computation yields:

$$\lim_{N\to\infty} \delta^n = \lim_{N\to\infty} (1 + \frac{2T}{N})^n \le e^{2T}, \quad \forall n \le N. \tag{3.6}$$

Recursively applying (3.5) allows to get:

$$\begin{aligned} \|u_h^{n+1}\|_{L^2(D)}^2 \le{}& \delta^{n+1}\|u_h^0\|_{L^2(D)}^2 + 4\Delta t\sum_{k=0}^n \delta^k + 4\sum_{k=0}^n \delta^{n-k}\|G(u_h^k)\mathcal{P}_J^w \Delta W^k\|_{L^2(D)}^2 \\ &+ 2\sum_{k=0}^n \delta^{n-k}(G(u_h^k)\mathcal{P}_J^w \Delta W^k, u_h^k). \end{aligned} \tag{3.7}$$

Note that $\mathbb{E}[(G(u_h^k)\mathcal{P}_J^w \Delta W^k, u_h^k)] = 0$ since ΔW^k is independent of u_h^k. Thus taking expectation on both sides of (3.7) and using (3.6) gives:

$$\|u_h^{n+1}\|_{L^2(\Omega,L^2(D))}^2 \le c\|u_h^0\|_{L^2(\Omega,L^2(D))}^2 + cT + c\Delta t\sum_{k=0}^n \|G(u_h^k)\|_{L^2(\Omega,\mathcal{L}_0^2)}^2.$$

By assumption (2.6), we obtain:

$$\|u_h^{n+1}\|_{L^2(\Omega,L^2(D))}^2 \le c(1 + \|u_h^0\|_{L^2(\Omega,L^2(D))})^2 + c\sum_{k=0}^n \|u_h^k\|_{L^2(\Omega,L^2(D))}^2 \Delta t.$$

Making use of discrete Gronwall inequality deduces:

$$\|u_h^{n+1}\|_{L^2(\Omega,L^2(D))} \le c(1 + \|u_h^0\|_{L^2(\Omega,L^2(D))}), \quad n = 0, \dots, N-1.$$

It completes the proof. □

Another thing worth mentioning is that the full discretization scheme (3.1) treats the nonlinear drift term explicitly, which means that the overall computational cost of the scheme (3.1) is roughly equal to solving a second-order equation at each time step. Therefore this fully discrete scheme is efficient.

4. Error estimate

This section aims to analyze the strong convergence of the fully discrete scheme (3.1) to the mild solution (2.3), where strong convergence is understood as convergence with respect to the norm $\|\cdot\|_{L^2(\Omega,L^2(D))}$.

Let $S_{h,\Delta t} := (I + \Delta t A_h)^{-1}$, the fully discrete scheme (3.1) can be rewritten as:

$$\begin{aligned} u_h^{n+1} &= S_{h,\Delta t}u_h^n + \frac{\Delta t}{1 + \Delta t\|F(u_h^n)\|_{L^2(D)}^2} S_{h,\Delta t}\mathcal{P}_h F(u_h^n) + S_{h,\Delta t}\mathcal{P}_h G(u_h^n)\mathcal{P}_J^w \Delta W^n, \\ u_h^0 &= \mathcal{P}_h u_0. \end{aligned} \tag{4.1}$$

Recurring (4.1) leads to the fully discrete approximation u_h^n satisfying:

$$u_h^n = S_{h,\Delta t}^n \mathcal{P}_h u_0 + \sum_{k=0}^{n-1} \frac{\Delta t}{1 + \Delta t\|F(u_h^k)\|_{L^2(D)}^2} S_{h,\Delta t}^{n-k}\mathcal{P}_h F(u_h^k) + \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} S_{h,\Delta t}^{n-k}\mathcal{P}_h G(u_h^k)\mathcal{P}_J^w dW(\tau). \tag{4.2}$$

Our goal in the following is to estimate the error $\|u(t_n)-u_h^n\|_{L^2(\Omega,L^2(D))}$. To this end, we first introduce an auxiliary process $\tilde{u}_h^n$, defined by: for $n = 0, \dots, N-1$,

$$\begin{aligned} (I + \Delta t A_h)\tilde{u}_h^{n+1} &:= \tilde{u}_h^n + \frac{\Delta t}{1 + \Delta t\|F(u_h^n)\|_{L^2(D)}^2}\mathcal{P}_h F(u(t_n)) + \mathcal{P}_h G(u(t_n))\Delta W^n, \\ \tilde{u}_h^0 &= \mathcal{P}_h u_0. \end{aligned} \tag{4.3}$$

Equivalently, the following recursion holds:

$$\begin{aligned} \tilde{u}_h^n &= S_{h,\Delta t}^n \mathcal{P}_h u_0 + \sum_{k=0}^{n-1} \frac{\Delta t}{1 + \Delta t\|F(u_h^k)\|_{L^2(D)}^2} S_{h,\Delta t}^{n-k}\mathcal{P}_h F(u(t_k)) + \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} S_{h,\Delta t}^{n-k}\mathcal{P}_h G(u(t_k))dW(\tau), \\ \tilde{u}_h^0 &= \mathcal{P}_h u_0. \end{aligned} \tag{4.4}$$

Hence the considered error $\|u(t_n) - u_h^n\|_{L^2(\Omega,L^2(D))}$ can be bounded by two parts below:

$$\|u(t_n) - u_h^n\|_{L^2(\Omega,L^2(D))} \le I_1 + I_2 \tag{4.5}$$

with

$$I_1 := \|u(t_n) - \tilde{u}_h^n\|_{L^2(\Omega,L^2(D))}, \quad I_2 := \|\tilde{u}_h^n - u_h^n\|_{L^2(\Omega,L^2(D))}. \tag{4.6}$$

We then estimate I_1 and I_2 individually, for which we need some preliminaries collected as follows:

• Suppose $u_0 \in L^2(\Omega, \mathcal{D}(A))$ is a $\mathcal{F}_0$-measurable random variable, and the eigenvalues of Q satisfy $q_j = \mathcal{O}(j^{-(2\gamma+1+\epsilon)})$[1] for some $\gamma \geq 1$ and $\epsilon > 0$. Then it holds:

$$\left\| G(u(t_n)) - G(u_h^n)\mathcal{P}_J^w \right\|_{L^2(\Omega,\mathcal{L}_0^2)} \leq c(\|u(t_n) - u_h^n\|_{L^2(\Omega,L^2(D))} + J^{-\gamma}). \tag{4.7}$$

Indeed, by the $\|\cdot\|_{\mathcal{L}_0^2}$ norm definition, (2.5), (2.6), and $q_j = \mathcal{O}(j^{-(2\gamma+1+\epsilon)})$, we obtain

$$\begin{aligned}
&\left\| G(u(t_n)) - G(u_h^n)\mathcal{P}_J^w \right\|_{L^2(\Omega,\mathcal{L}_0^2)} \leq \left\| G(u(t_n)) - G(u_h^n) \right\|_{L^2(\Omega,\mathcal{L}_{0}^2)} + \left\| G(u_h^n) - G(u_h^n)\mathcal{P}_J^w \right\|_{L^2(\Omega,\mathcal{L}_0^2)} \\
&\leq c\|u(t_n) - u_h^n\|_{L^2(\Omega,L^2(D))} + \mathbb{E}\Big[\sum_{j=1}^{\infty} \|G(u_h^n)(I - P_J^w)Q^{\frac{1}{2}}\phi_j\|_{L^2(D)}^2\Big]^{\frac{1}{2}} \\
&\leq c\|u(t_n) - u_h^n\|_{L^2(\Omega,L^2(D))} + c\mathbb{E}\Big[\|A^{\frac{1}{2}}G(u_h^n)\|_{\mathcal{L}(L^2(D))}^2 \sum_{j=1}^{\infty} \|(I - P_J^w)q_j^{\frac{1}{2}}\phi_j\|_{L^2(D)}^2\Big]^{\frac{1}{2}} \\
&\leq c\|u(t_n) - u_h^n\|_{L^2(\Omega,L^2(D))} + c\mathbb{E}\Big[(1 + \|u_h^n\|_{\dot{H}^1}^2) \sum_{j=J+1}^{\infty} q_j\Big]^{\frac{1}{2}} \\
&\leq c\|u(t_n) - u_h^n\|_{L^2(\Omega,L^2(D))} + c(\sum_{j=J+1}^{\infty} q_j)^{\frac{1}{2}} \leq c(\|u(t_n) - u_h^n\|_{L^2(\Omega,L^2(D))} + J^{-\gamma}).
\end{aligned}$$

This proves (4.7).

• If the initial value $u_0 \in L^{2p}(\Omega, \mathcal{D}(A))$, $p \geq 1$, is an $\mathcal{F}_0$-measurable random variable, then there exists an infinitesimal positive number ϵ_0 such that the mild solution (2.3) enjoys the following regularity:

$$\sup_{0\leq t\leq T} \|u(t)\|_{L^{2p}(\Omega,\dot{H}^{2-\epsilon_0})} < \infty. \tag{4.8}$$

Furthermore there exists a constant c depended on u_0 such that the mild solution u defined in (2.3) satisfies the following temporal Hölder regularity:

$$\|u(\tau_2) - u(\tau_1)\|_{L^{2p}(\Omega,L^2(D))} \leq c(\tau_2 - \tau_1)^{\frac{1}{2}}, \quad \forall\, 0 \leq \tau_1 \leq \tau_2 \leq T. \tag{4.9}$$

The proofs of (4.8) and (4.9) has been given by [28, Proposition 3.1] and [28, Theorem 3.2] respectively, which are omitted here. Basically, it makes use of the assumptions (2.4), (2.5), (2.6), and the stochastic calculus tools such as Burkholder–Davis–Gundy (BDG) inequality and semigroup inequality.

• Define $T_n := \left(e^{-t_n A} - S_{h,\Delta t}^n \mathcal{P}_h\right)$. Then for $0 \leq \nu \leq \mu \leq 2$, there exists a constant c such that:

$$\|T_n v\|_{L^2(D)} \leq c(h^\mu + \Delta t^{\frac{\mu}{2}})t_n^{-\frac{\mu-\nu}{2}} \|A^{\frac{\nu}{2}}v\|_{L^2(D)}, \quad \forall v \in \dot{H}^\nu, \quad n = 1, \ldots, N. \tag{4.10}$$

This is a classical result associated with the operator T_n, see, e.g., [19, Lemma 4.1].

• The following ultracontractive and smoothing properties of the semigroup $S(t)$ hold true [28,33]:

- For each $\alpha \geq 0$, there exists a constant c such that

$$\|A^\alpha S(t)\|_{\mathcal{L}(L^2(D))} \leq ct^{-\alpha}, \ \forall\, t > 0. \tag{4.11}$$

- For $0 \leq \alpha \leq 1$, there exists a constant c such that

$$\|A^{-\alpha}(I - S(t))\|_{\mathcal{L}(L^2(D))} \leq ct^\alpha, \ \forall\, t \geq 0. \tag{4.12}$$

• Suppose the initial value $u_0 \in L^{2p}(\Omega, \mathcal{D}(A))$, $p \geq 1$, is an $\mathcal{F}_0$-measurable random variable, then $\tilde{u}_h^n$ defined in (4.4) satisfies:

$$\|\tilde{u}_h^n\|_{L^{2p}(\Omega,\dot{H}^1)} \leq c(1 + \|u_0\|_{L^{2p}(\Omega,\dot{H}^1)}) < \infty, \quad n = 1, \ldots, N. \tag{4.13}$$

In fact, by virtue of $A^{\frac{1}{2}}F(u) = F'(u)A^{\frac{1}{2}}u = (1 - 3u^2)A^{\frac{1}{2}}u$, BDG inequality [34, Proposition 2.6], and $\sup_{n=1,\ldots,N} \|S_{h,\Delta t}^n \mathcal{P}_h v\|_{L^2(D)} \leq c\|v\|_{L^2(D)}$, $v \in L^2(D)$, (see, e.g., [28, (4.12)]), one derives readily that

[1] This $q_j = \mathcal{O}(j^{-(2\gamma+1+\epsilon)})$ has been used in the strong error estimation of truncated finite element method for SPDEs, see, e.g., [33, Lemma 10.33]. In particular, one checks easily that $\sum_{j=1}^{\infty} \|A^{\frac{1}{2}}\sqrt{q_j}\phi_j\|_{L^2(D)}^2 \leq c\sum_{j=1}^{\infty} j^{\frac{2}{d}-(2\gamma+1+\epsilon)} < \infty$ for $d = 1, 2, 3$, which means that even if the diffusion term $G(\cdot)$ is a constant, the assumption (2.6) still holds.

$$
\begin{aligned}
\|\tilde{u}_h^n\|_{L^{2p}(\Omega,\dot{H}^1)} &\le \|S_{h,\Delta t}^n \mathcal{P}_h A^{\frac{1}{2}} u_0\|_{L^{2p}(\Omega,L^2(D))} + \Delta t \sum_{k=0}^{n-1} \Big\| S_{h,\Delta t}^{n-k} \mathcal{P}_h A^{\frac{1}{2}} F(u(t_k)) \Big\|_{L^{2p}(\Omega,L^2(D))} \\
&\quad + \sum_{k=0}^{n-1} \Big\| \int_{t_k}^{t_{k+1}} S_{h,\Delta t}^{n-k} \mathcal{P}_h A^{\frac{1}{2}} G(u(t_k)) dW(\tau) \Big\|_{L^{2p}(\Omega,L^2(D))} \\
&\le c\|A^{\frac{1}{2}} u_0\|_{L^{2p}(\Omega,L^2(D))} + c\Delta t \sum_{k=0}^{n-1} \Big\| (1-3u(t_k)^2) A^{\frac{1}{2}} u(t_k) \Big\|_{L^{2p}(\Omega,L^2(D))} \\
&\quad + c \sum_{k=0}^{n-1} \Big(\int_{t_k}^{t_{k+1}} \Big\| A^{\frac{1}{2}} G(u(t_k)) \Big\|^2_{L^{2p}(\Omega,\mathcal{L}_0^2)} d\tau \Big)^{\frac{1}{2}}.
\end{aligned}
$$

It is known that $u(t_k) \in \dot{H}^{2-\epsilon_0}$ and $\dot{H}^{\delta_0} \hookrightarrow L^\infty(D)$ for $\delta_0 \in (\frac{3}{2}, 2)$, which means $\|u(t_k)\|_{L^\infty(D)} < \infty$. Together with (2.6) and (2.7), we get

$$
\|\tilde{u}_h^n\|_{L^{2p}(\Omega,\dot{H}^1)} \le c(1 + \|u_0\|_{L^{2p}(\Omega,\dot{H}^1)}) < \infty.
$$

This proves (4.13).

Now, it is in a position to carry out strong error estimates for I_1 and I_2 respectively. We start with the error analysis of I_1.

Lemma 4.1 (*Error Estimate of I_1*)**.** *Suppose $u_0 \in L^{2p}(\Omega, \mathcal{D}(A))$, $p \ge 1$, is an $\mathcal{F}_0$-measurable random variable. Further assume that the time step size Δt and spatial mesh size h are coupled by $\Delta t = \mathcal{O}(h^2)$. Then there exists an arbitrarily small $\epsilon_0 > 0$ and a constant c independent of h and Δt, such that:*

$$
\|u(t_n) - \tilde{u}_h^n\|_{L^{2p}(\Omega,L^2(D))} \le c(\Delta t^{\frac{1}{2}} + h^{2-\epsilon_0}), \quad n = 1, \dots, N. \tag{4.14}
$$

Proof. Subtracting the first equation in (4.4) from (2.3) results in:

$$
\|u(t_n) - \tilde{u}_h^n\|_{L^{2p}(\Omega,L^2(D))} := e_1 + e_2 + e_3, \tag{4.15}
$$

where e_1, e_2, e_3 are given by

$$
\begin{aligned}
e_1 &:= \|S(t_n)u_0 - S_{h,\Delta t}^n \mathcal{P}_h u_0\|_{L^{2p}(\Omega,L^2(D))}, \\
e_2 &:= \Big\| \sum_{k=0}^{n-1} \Big(\int_{t_k}^{t_{k+1}} S(t_n-\tau)F(u(\tau))d\tau - \frac{\Delta t}{1+\Delta t\|F(u_h^k)\|^2_{L^2(D)}} S_{h,\Delta t}^{n-k}\mathcal{P}_h F(u(t_k)) \Big) \Big\|_{L^{2p}(\Omega,L^2(D))}, \\
e_3 &:= \Big\| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \big(S(t_n-\tau)G(u(\tau)) - S_{h,\Delta t}^{n-k}\mathcal{P}_h G(u(t_k)) \big) dW(\tau) \Big\|_{L^{2p}(\Omega,L^2(D))}.
\end{aligned}
$$

We are led to estimate e_1, e_2, and e_3 separately.

With the aid of (4.10), we get the boundedness of e_1 as follows:

$$
e_1 = \|T_n u_0\|_{L^{2p}(\Omega,L^2(D))} \le c(h^2 + \Delta t)\|A u_0\|_{L^{2p}(\Omega,L^2(D))} \le c(h^2 + \Delta t). \tag{4.16}
$$

Then we estimate e_2. Applying triangle inequality allows to get

$$
e_2 \le e_{2,1} + e_{2,2}, \tag{4.17}
$$

where $e_{2,1}$ and $e_{2,2}$ are represented below

$$
\begin{aligned}
e_{2,1} &:= \Big\| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (S(t_n-\tau) - S_{h,\Delta t}^{n-k}\mathcal{P}_h) \frac{F(u(t_k))}{1+\Delta t\|F(u_h^k)\|^2_{L^2(D)}} d\tau \Big\|_{L^{2p}(\Omega,L^2(D))}, \\
e_{2,2} &:= \Big\| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} S(t_n-\tau)\Big(F(u(\tau)) - \frac{F(u(t_k))}{1+\Delta t\|F(u_h^k)\|^2_{L^2(D)}} \Big) d\tau \Big\|_{L^{2p}(\Omega,L^2(D))}.
\end{aligned}
$$

Now we first estimate $e_{2,1}$. It is known from (4.10), $\|S(\cdot)\|_{\mathcal{L}(L^2(D))} \le c$, (4.11), and (4.12) that

$$
\begin{aligned}
&\|S(t_n-t_k) - S_{h,\Delta t}^{n-k}\mathcal{P}_h\|_{\mathcal{L}(L^2(D))} \le c\frac{\Delta t + h^2}{t_n - t_k}, \\
&\|S(t_n-\tau) - S(t_n-t_{n-1})\|_{\mathcal{L}(L^2(D))} \le c, \\
&\|S(t_n-\tau) - S(t_n-t_k)\|_{\mathcal{L}(L^2(D))} \le \|AS(t_n-\tau)\|_{\mathcal{L}(L^2(D))}\|A^{-1}(I - S(\tau-t_k))\|_{\mathcal{L}(L^2(D))} \\
&\qquad \le c\frac{\tau - t_k}{t_n - \tau} \le c\frac{\Delta t}{t_n - t_{k+1}}, \quad \tau \in (t_k, t_{k+1}), \quad k = 0, \dots, n-2.
\end{aligned} \tag{4.18}
$$

Through (4.18), (2.9), $n \le \frac{T}{\Delta t}$, and $\Delta t = \mathcal{O}(h^2)$, there exists an infinitesimal positive number ϵ_0 such that:

$$
\begin{aligned}
e_{2,1} &\le c\Big(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\Big\|(S(t_n-\tau)-S(t_n-t_k))\frac{F(u(t_k))}{1+\Delta t\|F(u_h^k)\|_{L^2(D)}^2}\Big\|_{L^{2p}(\Omega,L^2(D))}d\tau \\
&\quad+\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\Big\|(S(t_n-t_k)-S_{h,\Delta t^{n-k}}\mathcal{P}_h)\frac{F(u(t_k))}{1+\Delta t\|F(u_h^k)\|_{L^2(D)}^2}\Big\|_{L^{2p}(\Omega,L^2(D))}d\tau\Big) \\
&\le c\Big(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}\big[\big\|S(t_n-\tau)-S(t_n-t_k)\big\|_{\mathcal{L}(L^2(D))}^{2p}\big\|F(u(t_k))\big\|_{L^2(D)}^{2p}\big]^{\frac{1}{2p}}d\tau \\
&\quad+\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}\big[\big\|S(t_n-t_k)-S_{h,\Delta t^{n-k}}\mathcal{P}_h\big\|_{\mathcal{L}(L^2(D))}^{2p}\big\|F(u(t_k))\big\|_{L^2(D)}^{2p}\big]^{\frac{1}{2p}}d\tau\Big) \\
&\le c\sup_{0\le t\le T}\|F(u(t))\|_{L^{2p}(\Omega,L^2(D))}\big(\Delta t+\sum_{k=0}^{n-2}\int_{t_k}^{t_{k+1}}\frac{\Delta t}{t_n-t_{k+1}}d\tau+\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\frac{\Delta t+h^2}{t_n-t_k}d\tau\big) \\
&\le c\big(\Delta t+\Delta t\sum_{k=1}^{n}\frac{1}{k}+(\Delta t+h^2)\sum_{k=0}^{n-1}\frac{1}{n-k}\big)\le c\big(\Delta t+\Delta t\ln(\Delta t^{-1})+(\Delta t+h^2)\ln(\Delta t^{-1})\big) \\
&\le c(\Delta t^{1-\epsilon_0}+h^2\Delta t^{-\epsilon_0})\le c(\Delta t^{1-\epsilon_0}+h^{2-\epsilon_0}),
\end{aligned}
\tag{4.19}
$$

where we have used the fact that $\Delta t^{-\epsilon_0}$ dominates $\ln(\Delta t^{-1})$ for an arbitrarily small $\epsilon_0>0$.

To estimate $e_{2,2}$, we first note that

$$
e_{2,2} := \Big\|\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}S(t_n-\tau)\frac{F(u(\tau))-F(u(t_k))+\Delta t\|F(u_h^k)\|_{L^2(D)}^2F(u(\tau))}{1+\Delta t\|F(u_h^k)\|_{L^2(D)}^2}d\tau\Big\|_{L^{2p}(\Omega,L^2(D))}.
$$

Also note that for $\tau\in[t_k,t_{k+1})$,

$$
u(\tau)-u(t_k)=(S(\tau-t_k)-I)u(t_k)+\int_{t_k}^{\tau}S(\tau-r)F(u(r))dr+\int_{t_k}^{\tau}S(\tau-r)G(u(r))dW(r),
$$

and

$$
\begin{aligned}
F(u(t_k))-F(u(\tau)) &= F'(u(\tau))\big(u(t_k)-u(\tau)\big)+R_F(u(\tau),u(t_k)) = -F'(u(\tau))(S(\tau-t_k)-I)u(t_k) \\
&\quad-F'(u(\tau))\int_{t_k}^{\tau}S(\tau-r)F(u(r))dr-F'(u(\tau))\int_{t_k}^{\tau}S(\tau-r)G(u(r))dW(r)+R_F(u(\tau),u(t_k))
\end{aligned}
$$

with $R_F(u(\tau),u(t_k)) := \int_0^1 F''\big(u(\tau)+\xi(u(t_k)-u(\tau))\big)(u(t_k)-u(\tau))^2(1-\xi)d\xi$.

Therefore, by $\|F(u_h^k)\|_{L^2(D)}<\infty$ and triangle inequality, we obtain:

$$
\begin{aligned}
e_{2,2} &\le c\Big\|\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}S(t_n-\tau)(F(u(\tau))-F(u(t_k)))d\tau\Big\|_{L^{2p}(\Omega,L^2(D))} \\
&\quad+c\Delta t\Big\|\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}S(t_n-\tau)F(u(\tau))d\tau\Big\|_{L^{2p}(\Omega,L^2(D))}\le c(X_1+X_2+X_3+X_4+X_5),
\end{aligned}
$$

where

$$
\begin{aligned}
X_1 &:= \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\Big\|S(t_n-\tau)F'(u(\tau))(S(\tau-t_k)-I)u(t_k)\Big\|_{L^{2p}(\Omega,L^2(D))}d\tau, \\
X_2 &:= \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\Big\|S(t_n-\tau)F'(u(\tau))\int_{t_k}^{\tau}S(\tau-r)F(u(r))dr\Big\|_{L^{2p}(\Omega,L^2(D))}d\tau, \\
X_3 &:= \Big\|\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}S(t_n-\tau)F'(u(\tau))\int_{t_k}^{\tau}S(\tau-r)G(u(r))dW(r)d\tau\Big\|_{L^{2p}(\Omega,L^2(D))}, \\
X_4 &:= \Big\|\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}S(t_n-\tau)R_F(u(\tau),u(t_k))d\tau\Big\|_{L^{2p}(\Omega,L^2(D))}, \\
X_5 &:= \Delta t\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\Big\|S(t_n-\tau)F(u(\tau))\Big\|_{L^{2p}(\Omega,L^2(D))}d\tau.
\end{aligned}
$$

We next estimate X_1 to X_5 one by one.

For the estimation of X_1, applying (4.11) and the known result $\|A^{-\frac{\delta_0}{2}}v\|_{L^2(D)}\le c\|v\|_{L^1(D)}$, $\forall v\in L^1(D)$, $\delta_0\in(\frac{3}{2},2)$, (see, e.g., [19, (2.18)]), yields:

$$X_1 \le c\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \|A^{\frac{\delta_0}{2}}S(t_n-\tau)\|_{\mathcal{L}(L^2(D))}\Big\|A^{-\frac{\delta_0}{2}}F'(u(\tau))(S(\tau-t_k)-I)u(t_k)\Big\|_{L^{2p}(\Omega,L^2(D))}d\tau$$
$$\le c\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(t_n-\tau)^{-\frac{\delta_0}{2}}\Big\|F'(u(\tau))(S(\tau-t_k)-I)u(t_k)\Big\|_{L^{2p}(\Omega,L^1(D))}d\tau,\quad \forall\ \delta_0\in(\frac{3}{2},2).$$

Notice that $F(v)=v-v^3$, $v\in\mathbb{R}$. By (4.12), (4.8), and Hölder's inequality, there exists an arbitrarily small number $\varepsilon_0>0$ such that: for $\tau\in(t_k,t_{k+1})$,

$$\Big\|F'(u(\tau))(S(\tau-t_k)-I)u(t_k)\Big\|_{L^{2p}(\Omega,L^1(D))}\le c(1+\|u(\tau)\|^2_{L^{8p}(\Omega,L^4(D))})\|(S(\tau-t_k)-I)u(t_k)\|_{L^{4p}(\Omega,L^2(D))}$$
$$\le c(1+\|u(\tau)\|^2_{L^{8p}(\Omega,L^4(D))})\mathbb{E}\Big[\Big\|A^{-(1-\varepsilon_0)}(S(\tau-t_k)-I)\Big\|^{4p}_{\mathcal{L}(L^2(D))}\Big\|A^{(1-\varepsilon_0)}u(t_k)\Big\|^{4p}_{L^2(D)}\Big]^{\frac{1}{4p}}$$
$$\le c\Delta t^{1-\varepsilon_0}\big(1+\sup_{0\le t\le T}\|u(t)\|^2_{L^{8p}(\Omega,L^4(D))}\big)\sup_{0\le t\le T}\|u(t)\|_{L^{4p}(\Omega,\dot H^{2-\varepsilon_0})}$$
$$\le c\Delta t^{1-\varepsilon_0}.$$

Hence we deduce that for an infinitesimal positive number ε_0 and $\delta_0\in(\frac{3}{2},2)$,

$$X_1\le c\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(t_n-\tau)^{-\frac{\delta_0}{2}}\Big\|F'(u(\tau))(S(\tau-t_k)-I)u(t_k)\Big\|_{L^{2p}(\Omega,L^1(D))}d\tau$$
$$\le c\Delta t^{1-\varepsilon_0}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(t_n-\tau)^{-\frac{\delta_0}{2}}d\tau\le c\Delta t^{1-\varepsilon_0}\int_0^{t_n}(t_n-\tau)^{-\frac{\delta_0}{2}}d\tau\le c\Delta t^{1-\varepsilon_0}.$$

The estimation of X_2 follows from (4.11), (2.9), embedding result $\dot H^1\hookrightarrow L^4(D)$, (4.8), $\|S(\cdot)\|_{\mathcal{L}(L^2(D))}\le 1$, and $\|A^{-\frac{\delta_0}{2}}v\|_{L^2(D)}\le c\|v\|_{L^1(D)}$, $\delta_0\in(\frac{3}{2},2)$:

$$X_2\le c\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\int_{t_k}^{\tau}\|A^{\frac{\delta_0}{2}}S(t_n-\tau)\|_{\mathcal{L}(L^2(D))}\Big\|A^{-\frac{\delta_0}{2}}F'(u(\tau))S(\tau-r)F(u(r))\Big\|_{L^{2p}(\Omega,L^2(D))}drd\tau$$
$$\le c\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\int_{t_k}^{\tau}(t_n-\tau)^{-\frac{\delta_0}{2}}\Big\|F'(u(\tau))S(\tau-r)F(u(r))\Big\|_{L^{2p}(\Omega,L^1(D))}drd\tau$$
$$\le c\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\int_{t_k}^{\tau}(t_n-\tau)^{-\frac{\delta_0}{2}}(1+\|u(\tau)\|^2_{L^{8p}(\Omega,L^4(D))})\Big\|F(u(r))\Big\|_{L^{4p}(\Omega,L^2(D))}drd\tau$$
$$\le c\Delta t(1+\sup_{0\le t\le T}\|u(t)\|^2_{L^{8p}(\Omega,\dot H^1)})\sup_{0\le t\le T}\|F(u(t))\|_{L^{4p}(\Omega,L^2(D))}\int_0^{t_n}(t_n-\tau)^{-\frac{\delta_0}{2}}d\tau\le c\Delta t.$$

To estimate X_3, we denote the characteristic function of the set $[a,b)$ by $\chi_{[a,b)}(\cdot)$, $a,b\in\mathbb{R}$. Applying stochastic Fubini theorem (see, e.g., [35, Theorem 4.18]) and BDG inequality derives:

$$X_3=\Big\|\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}}\chi_{[t_k,\tau)}(r)S(t_n-\tau)F'(u(\tau))S(\tau-r)G(u(r))d\tau dW(r)\Big\|_{L^{2p}(\Omega,L^2(D))}$$
$$\le c\Big(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\Big\|\int_{t_k}^{t_{k+1}}S(t_n-\tau)F'(u(\tau))S(\tau-r)G(u(r))d\tau\Big\|^2_{L^{2p}(\Omega,\mathcal{L}_0^2)}dr\Big)^{\frac{1}{2}}$$
$$\le c\Big(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}}\Big\|S(t_n-\tau)F'(u(\tau))S(\tau-r)G(u(r))\Big\|^2_{L^{2p}(\Omega,\mathcal{L}_0^2)}d\tau dr\Big)^{\frac{1}{2}}.$$

Furthermore, by $\|S(\cdot)\|_{\mathcal{L}(L^2(D))}\le 1$, Hölder's inequality, embedding results $\dot H^1\hookrightarrow L^4(D)$, $\dot H^{\delta_0}\hookrightarrow L^\infty(D)$, $\delta_0\in(\frac{3}{2},2)$, (2.6) and (4.8), we obtain:

$$\Big\|S(t_n-\tau)F'(u(\tau))S(\tau-r)G(u(r))\Big\|_{L^{2p}(\Omega,\mathcal{L}_0^2)}$$
$$\le c\mathbb{E}\Big[\|F'(u(\tau))\|^{2p}_{L^4(D)}\Big(\sum_{j=1}^{\infty}\|S(\tau-r)G(u(r))Q^{\frac{1}{2}}\phi_j\|^2_{L^4(D)}\Big)^p\Big]^{\frac{1}{2p}}$$
$$\le c\mathbb{E}\Big[\|F'(u(\tau))\|^{2p}_{L^4(D)}\Big(\sum_{j=1}^{\infty}\|A^{\frac{1}{2}}S(\tau-r)G(u(r))Q^{\frac{1}{2}}\phi_j\|^2_{L^2(D)}\Big)^p\Big]^{\frac{1}{2p}}$$
$$\le c\mathbb{E}\Big[\|F'(u(\tau))\|^{2p}_{L^4(D)}\|A^{\frac{1}{2}}G(u(r))\|^{2p}_{\mathcal{L}_0^2}\Big]^{\frac{1}{2p}}\le c\mathbb{E}\Big[\|F'(u(\tau))\|^{2p}_{L^4(D)}(1+\|u(r)\|^{2p}_{\dot H^1})\Big]^{\frac{1}{2p}}$$
$$\le c\mathbb{E}\Big[\|F'(u(\tau))\|^{4p}_{L^4(D)}\Big]^{\frac{1}{4p}}\mathbb{E}\Big[(1+\|u(r)\|^{4p}_{\dot H^1})\Big]^{\frac{1}{4p}}\le c\mathbb{E}\Big[(1+\|u(\tau)\|^8_{L^\infty(D)})^p\Big]^{\frac{1}{4p}}\mathbb{E}\Big[(1+\|u(r)\|^{4p}_{\dot H^1})\Big]^{\frac{1}{4p}}$$
$$\le c(1+\|u(\tau)\|^2_{L^{8p}(\Omega,\dot H^{\delta_0})})(1+\|u(r)\|_{L^{4p}(\Omega,\dot H^1)})\le c,\quad \forall\delta_0\in(\frac{3}{2},2).$$

Thus it gives:

$$X_3 \le c\Delta t^{\frac{1}{2}}.$$

Next we focus on the boundedness of the part X_4. For $\delta_0 \in (\frac{3}{2}, 2)$, using (4.12) with $\alpha = \frac{\delta_0}{2}$, $\|A^{-\frac{\delta_0}{2}} v\|_{L^2(D)} \le c\|v\|_{L^1(D)}$, embedding result $\dot{H}^{\delta_0} \hookrightarrow L^\infty(D)$, and (4.9) yields:

$$\begin{aligned}
X_4 &\le c\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} (t_n-\tau)^{-\frac{\delta_0}{2}} \Big\|A^{-\frac{\delta_0}{2}} R_F(u(\tau),u(t_k))\Big\|_{L^{2p}(\Omega,L^2(D))} d\tau \\
&\le c\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} (t_n-\tau)^{-\frac{\delta_0}{2}} \Big\|R_F(u(\tau),u(t_k))\Big\|_{L^{2p}(\Omega,L^1(D))} d\tau \\
&\le c\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} (t_n-\tau)^{-\frac{\delta_0}{2}} \int_0^1 \Big(\Big\|\big((1-\xi)\|u(\tau)\|_{L^\infty(D)} + \xi\|u(t_k)\|_{L^\infty(D)}\big)\|u(\tau)-u(t_k)\|^2_{L^2(D)}\Big\|_{L^{2p}(\Omega)}\Big) d\xi d\tau \\
&\le c \sup_{0\le t\le T}\|u(t)\|_{L^{4p}(\Omega,\dot{H}^{\delta_0})} \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} (t_n-\tau)^{-\frac{\delta_0}{2}} \|u(\tau)-u(t_k)\|^2_{L^{8p}(\Omega,L^2(D))} d\tau \\
&\le c\Delta t\int_0^{t_n}(t_n-\tau)^{-\frac{\delta_0}{2}} d\tau \le c\Delta t.
\end{aligned}$$

The estimate of X_5 can be deduced by employing $\|S(\cdot)\|_{\mathcal{L}(L^2(D))} \le c$ and (2.9),

$$X_5 = \Delta t\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \Big\|S(t_n-\tau)F(u(\tau))\Big\|_{L^{2p}(\Omega,L^2(D))} d\tau \le c \sup_{0\le t\le T}\|F(u(t))\|_{L^{2p}(\Omega,L^2(D))}\Delta t \le c\Delta t.$$

Combining all above estimates from X_1 to X_5 allows to get:

$$e_{2,2} \le c\Delta t^{\frac{1}{2}}. \tag{4.20}$$

By (4.17), (4.19) and (4.20), we have

$$e_2 \le c(\Delta t^{\frac{1}{2}} + h^{2-\varepsilon_0}) \tag{4.21}$$

with ε_0 being an arbitrarily small number.

All that remains is to estimate e_3. Making use of triangle inequality gives $e_3 \le \sum_{i=1}^3 e_{3,i}$ with

$$\begin{aligned}
e_{3,1} &:= \Big\|\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \big(S(t_n-\tau)-S(t_n-t_k)\big)G(u(\tau))dW(\tau)\Big\|_{L^{2p}(\Omega,L^2(D))}, \\
e_{3,2} &:= \Big\|\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \big(S(t_n-t_k)-S_{h,\Delta t}^{n-k}\mathcal{P}_h\big)G(u(\tau))dW(\tau)\Big\|_{L^{2p}(\Omega,L^2(D))}, \\
e_{3,3} &:= \Big\|\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} S_{h,\Delta t}^{n-k}\mathcal{P}_h\big(G(u(\tau))-G(u(t_k))\big)dW(\tau)\Big\|_{L^{2p}(\Omega,L^2(D))}.
\end{aligned}$$

Then we analyze each term individually. For the part $e_{3,1}$, applying BDG inequality gives:

$$\begin{aligned}
e_{3,1} &\le c\Big(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \Big\|\big(S(t_n-\tau)-S(t_n-t_k)\big)G(u(\tau))\Big\|^2_{L^{2p}(\Omega,\mathcal{L}_0^2)} d\tau\Big)^{\frac{1}{2}} \\
&\le c\Big(\sum_{k=0}^{n-2}\int_{t_k}^{t_{k+1}} \mathbb{E}\Big[\Big\|A^{-\frac{1}{2}}(S(t_n-\tau)-S(t_n-t_k))\Big\|^{2p}_{\mathcal{L}(L^2(D))}\Big\|A^{\frac{1}{2}}G(u(\tau))\Big\|^{2p}_{\mathcal{L}_0^2}\Big]^{\frac{1}{p}} d\tau\Big)^{\frac{1}{2}} \\
&\quad + c\Big(\int_{t_{n-1}}^{t_n} \mathbb{E}\Big[\Big\|S(t_n-\tau)-S(t_n-t_{n-1})\Big\|^{2p}_{\mathcal{L}(L^2(D))}\Big\|G(u(\tau))\Big\|^{2p}_{\mathcal{L}_0^2}\Big]^{\frac{1}{p}} d\tau\Big)^{\frac{1}{2}}.
\end{aligned}$$

On one hand, from $\|S(t_n-\tau)-S(t_n-t_{n-1})\|_{\mathcal{L}(L^2(D))} \le c$, embedding inequality, (2.6) and (2.7), we obtain:

$$\begin{aligned}
&\Big(\int_{t_{n-1}}^{t_n} \mathbb{E}\Big[\Big\|S(t_n-\tau)-S(t_n-t_{n-1})\Big\|^{2p}_{\mathcal{L}(L^2(D))}\Big\|G(u(\tau))\Big\|^{2p}_{\mathcal{L}_0^2}\Big]^{\frac{1}{p}} d\tau\Big)^{\frac{1}{2}} \\
&\le c\Big(\int_{t_{n-1}}^{t_n} \mathbb{E}\Big[\Big\|A^{\frac{1}{2}}G(u(\tau))\Big\|^{2p}_{\mathcal{L}_0^2}\Big]^{\frac{1}{p}} d\tau\Big)^{\frac{1}{2}} \le c\Big(\int_{t_{n-1}}^{t_n} \big(1+\|u(\tau)\|^2_{L^{2p}(\Omega,\dot{H}^1)}\big)d\tau\Big)^{\frac{1}{2}} \\
&\le c\Big(\int_{t_{n-1}}^{t_n} (1+\|u_0\|^2_{L^{2p}(\Omega,\dot{H}^1)})d\tau\Big)^{\frac{1}{2}} \le c\Delta t^{\frac{1}{2}}.
\end{aligned}$$

On the other hand, by (4.11) and (4.12), we have

$$\left\|A^{-\frac{1}{2}}(S(t_n-\tau)-S(t_n-t_k))\right\|_{\mathcal{L}(L^2(D))} \le \left\|A^{\frac{1}{2}}S(t_n-\tau)\|_{\mathcal{L}(L^2(D))}\|A^{-1}\left(I-S(\tau-t_k)\right)\right\|_{\mathcal{L}(L^2(D))}$$
$$\le c\frac{\tau-t_k}{(t_n-\tau)^{\frac{1}{2}}} \le c\frac{\Delta t}{(t_n-t_{k+1})^{\frac{1}{2}}}, \quad \tau\in(t_k,t_{k+1}).$$

Using (2.6), (2.7), and $\ln(\Delta t^{-1}) \le c\Delta t^{-\epsilon_0}$ for an infinitesimal $\epsilon_0>0$, yields:

$$\Big(\sum_{k=0}^{n-2}\int_{t_k}^{t_{k+1}}\mathbb{E}\Big[\left\|A^{-\frac{1}{2}}(S(t_n-\tau)-S(t_n-t_k))\right\|^{2p}_{\mathcal{L}(L^2(D))}\left\|A^{\frac{1}{2}}G(u(\tau))\right\|^{2p}_{\mathcal{L}_0^2}\Big]^{\frac{1}{p}}d\tau\Big)^{\frac{1}{2}}$$
$$\le c\Big(\sum_{k=0}^{n-2}\int_{t_k}^{t_{k+1}}\frac{\Delta t^2}{t_n-t_{k+1}}d\tau\Big)^{\frac{1}{2}} \le c\Delta t(\ln(\Delta t^{-1}))^{\frac{1}{2}} \le c\Delta t^{1-\epsilon_0}.$$

Therefore we get by triangle inequality that

$$e_{3,1} \le c\Delta t^{\frac{1}{2}}.$$

The estimate of $e_{3,2}$ follows from BDG inequality, (4.10) with $\mu=2$ and $\nu=0$, embedding inequality, (2.6), (2.7), and $\Delta t=\mathcal{O}(h^2)$:

$$e_{3,2} \le c\Big(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\left\|\left(S(t_n-t_k)-S_{h,\Delta t}^{n-k}\mathcal{P}_h\right)G(u(\tau))\right\|^2_{L^{2p}(\Omega,\mathcal{L}_0^2)}d\tau\Big)^{\frac{1}{2}}$$
$$\le c\Big(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}\Big[\left\|T_{n-k}\right\|^{2p}_{\mathcal{L}(L^2(D))}\left\|G(u(\tau))\right\|^{2p}_{\mathcal{L}_0^2}\Big]^{\frac{1}{p}}d\tau\Big)^{\frac{1}{2}}$$
$$\le c\Big(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}\Big[\Big(\frac{\Delta t+h^2}{t_{n-k}}\Big)^{2p}\left\|A^{\frac{1}{2}}G(u(\tau))\right\|^{2p}_{\mathcal{L}_0^2}\Big]^{\frac{1}{p}}d\tau\Big)^{\frac{1}{2}}$$
$$\le c\Big(\frac{(\Delta t+h^2)^2}{\Delta t}\sum_{k=0}^{n-1}\frac{1}{(n-k)^2}\Big)^{\frac{1}{2}} \le c\Big(\frac{(\Delta t+h^2)^2}{\Delta t}\Big)^{\frac{1}{2}} \le c\Delta t^{\frac{1}{2}}.$$

For the last part $e_{3,3}$, using $\|S_{h,\Delta t}^{n-k}\|_{\mathcal{L}(L^2(D))} \le 1$, BDG inequality, (2.5) and (4.9) deduces:

$$e_{3,3} \le c\Big(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\left\|S_{h,\Delta t}^{n-k}\mathcal{P}_h(G(u(\tau))-G(u(t_k)))\right\|^2_{L^{2p}(\Omega,\mathcal{L}_0^2)}d\tau\Big)^{\frac{1}{2}}$$
$$\le c\Big(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\left\|G(u(\tau))-G(u(t_k))\right\|^2_{L^{2p}(\Omega,\mathcal{L}_0^2)}d\tau\Big)^{\frac{1}{2}}$$
$$\le c\Big(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\left\|u(\tau)-u(t_k)\right\|^2_{L^{2p}(\Omega,L^2(D))}d\tau\Big)^{\frac{1}{2}} \le c\Delta t^{\frac{1}{2}}.$$

According to all estimates from $e_{3,1}$ to $e_{3,3}$, we have

$$e_3 \le c\Delta t^{\frac{1}{2}}. \tag{4.22}$$

Finally, gathering (4.15), (4.16), (4.21), (4.22), and triangle inequality to conclude. It completes the proof. □

Remark 4.1. The condition $\Delta t=\mathcal{O}(h^2)$ shown in Lemma 4.1 is actually the Courant–Friedrichs–Lewy (CFL) condition, which is often a necessary stability condition. It is also used to balance the errors from spatial and temporal discretization in the numerical analysis of SPDEs; see, e.g., [31,33,34].

It is followed by the estimation of I_2, whose analysis involves the application of the martingale property of stochastic integral.

Lemma 4.2 (*Error Estimate of* I_2). *Let assumptions in Lemma 4.1 be fulfilled. Further assume the eigenvalues of Q satisfy $q_j=\mathcal{O}(j^{-(2\gamma+1+\epsilon)})$ for some $\gamma\ge 1$ and $\epsilon>0$, and $J^{-\gamma}=\mathcal{O}(h^2)$. Then there exists an arbitrarily small $\epsilon_0>0$ and a constant c independent of h and Δt, such that*

$$\|\tilde{u}_h^n-u_h^n\|^2_{L^2(\Omega,L^2(D))} \le c\Big[(\Delta t^{\frac{1}{2}}+h^{2-\epsilon_0})^2+\sum_{k=0}^{n-1}\|u(t_k)-u_h^k\|^2_{L^2(\Omega,L^2(D))}\Delta t\Big], \quad n=1,\dots,N.$$

Proof. Let $e_h^n:=\tilde{u}_h^n-u_h^n$, then $e_h^0=0$ and $e_h^n\in V_h$. Through (3.1) and (4.3), we obtain

$$u_h^{n+1}=u_h^n-\Delta tA_hu_h^{n+1}+\frac{\Delta t}{1+\Delta t\|F(u_h^n)\|^2_{L^2(D)}}\mathcal{P}_hF(u_h^n)+\mathcal{P}_hG(u_h^n)\mathcal{P}_J^w\Delta W^n,$$
$$\tilde{u}_h^{n+1}=\tilde{u}_h^n-\Delta tA_h\tilde{u}_h^{n+1}+\frac{\Delta t}{1+\Delta t\|F(u_h^n)\|^2_{L^2(D)}}\mathcal{P}_hF(u(t_n))+\mathcal{P}_hG(u(t_n))\Delta W^n.$$

Subtracting the above two equations allows to get:

$$e_h^{n+1} - e_h^n = -\Delta t A_h e_h^{n+1} + \frac{\Delta t}{1+\Delta t\|F(u_h^n)\|^2_{L^2(D)}}\mathcal{P}_h\big(F(u(t_n)) - F(u_h^n)\big) + \mathcal{P}_h(G(u(t_n)) - G(u_h^n)\mathcal{P}_J^w)\Delta W^n.$$

Taking inner product of both sides of the above equation with e_h^{n+1} yields:

$$\begin{aligned}(e_h^{n+1} - e_h^n, e_h^{n+1}) + \Delta t\|\nabla e_h^{n+1}\|^2_{L^2(D)} &= \frac{\Delta t}{1+\Delta t\|F(u_h^n)\|^2_{L^2(D)}}\Big[\big(F(u(t_n)) - F(\tilde{u}_h^n), e_h^{n+1}\big)\\ &+ \big(F(\tilde{u}_h^n) - F(u_h^n), e_h^n\big) + \big(F(\tilde{u}_h^n) - F(u_h^n), e_h^{n+1} - e_h^n\big)\Big] + \Big(e_h^{n+1}, \int_{t_n}^{t_{n+1}}(G(u(t_n)) - G(u_h^n)\mathcal{P}_J^w)dW(\tau)\Big).\end{aligned} \tag{4.23}$$

By $F(v) = v - v^3$, $v \in \mathbb{R}$, and the dual estimation (see, e.g., [28, (4.16)]), it holds:

$$\|A^{-\frac{1}{2}}(F(v_1) - F(v_2))\|_{L^2(D)} \le c(1 + \|v_1\|^2_{\dot{H}^1} + \|v_2\|^2_{\dot{H}^1})\|v_1 - v_2\|_{L^2(D)}, \quad \forall v_1, v_2 \in \dot{H}^1. \tag{4.24}$$

Using (4.24), (2.4), $\|F(v)\|^2_{L^2(D)} \le c(\|v\|^2_{L^2(D)} + \|v\|^6_{L^6(D)}), \forall\, v \in L^6(D)$, and Young inequality gives the boundedness of the right hand side of (4.23):

$$\begin{aligned}&\frac{\Delta t}{1+\Delta t\|F(u_h^n)\|^2_{L^2(D)}}\Big[\big(F(u(t_n)) - F(\tilde{u}_h^n), e_h^{n+1}\big) + \big(F(\tilde{u}_h^n) - F(u_h^n), e_h^n\big)\\ &\quad + \big(F(\tilde{u}_h^n) - F(u_h^n), e_h^{n+1} - e_h^n\big)\Big] + \Big(e_h^{n+1}, \int_{t_n}^{t_{n+1}}(G(u(t_n)) - G(u_h^n)\mathcal{P}_J^w)dW(\tau)\Big)\\ &\le \Delta t\|A^{-\frac{1}{2}}(F(u(t_n)) - F(\tilde{u}_h^n))\|_{L^2(D)}\|A^{\frac{1}{2}}e_h^{n+1}\|_{L^2(D)} + \Delta t\|e_h^n\|^2_{L^2(D)} + \Delta t^2\|F(\tilde{u}_h^n) - F(u_h^n)\|^2_{L^2(D)}\\ &\quad + \frac{1}{4}\|e_h^{n+1} - e_h^n\|^2_{L^2(D)} + \int_{t_n}^{t_{n+1}}\big(e_h^{n+1}, (G(u(t_n)) - G(u_h^n)\mathcal{P}_J^w)dW(\tau)\big)\\ &\le c\Delta t(1 + \|u(t_n)\|^2_{\dot{H}^1} + \|\tilde{u}_h^n\|^2_{\dot{H}^1})^2\|u(t_n) - \tilde{u}_h^n\|^2_{L^2(D)} + \Delta t\|\nabla e_h^{n+1}\|^2_{L^2(D)} + \Delta t\|e_h^n\|^2_{L^2(D)}\\ &\quad + c\Delta t^2(\|\tilde{u}_h^n\|^2_{L^2(D)} + \|\tilde{u}_h^n\|^6_{L^6(D)} + \|u_h^n\|^2_{L^2(D)} + \|u_h^n\|^6_{L^6(D)}) + \frac{1}{4}\|e_h^{n+1} - e_h^n\|^2_{L^2(D)}\\ &\quad + \int_{t_n}^{t_{n+1}}\big(e_h^{n+1}, (G(u(t_n)) - G(u_h^n)\mathcal{P}_J^w)dW(\tau)\big).\end{aligned} \tag{4.25}$$

We next focus on the stochastic integral term in the last line of (4.25). According to the martingale property of the stochastic integral [28, (4.26)], Young inequality, Itô isometry and (4.7), we obtain that the expectation of this stochastic term satisfies:

$$\begin{aligned}\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\big(e_h^{n+1}, (G(u(t_n)) - G(u_h^n)\mathcal{P}_J^w)dW(\tau)\big)\Big] &= \mathbb{E}\Big[\big(e_h^{n+1} - e_h^n, (G(u(t_n)) - G(u_h^n)\mathcal{P}_J^w)\Delta W(\tau)\big)\Big]\\ &\le \frac{1}{4}\mathbb{E}\big[\|e_h^{n+1} - e_h^n\|^2_{L^2(D)}\big] + \mathbb{E}\big[\|(G(u(t_n)) - G(u_h^n)\mathcal{P}_J^w)\Delta W^n\|^2_{L^2(D)}\big]\\ &= \frac{1}{4}\mathbb{E}\big[\|e_h^{n+1} - e_h^n\|^2_{L^2(D)}\big] + \Delta t\mathbb{E}\big[\|G(u(t_n)) - G(u_h^n)\mathcal{P}_J^w\|^2_{\mathcal{L}_0^2}\big]\\ &\le \frac{1}{4}\mathbb{E}\big[\|e_h^{n+1} - e_h^n\|^2_{L^2(D)}\big] + c\Delta t(\|u(t_n) - u_h^n\|^2_{L^2(\Omega, L^2(D))} + J^{-2\gamma}).\end{aligned} \tag{4.26}$$

Therefore taking expectation on both sides of (4.23), and employing (4.25), (4.26) and the equality $a(a-b) = \frac{1}{2}(a^2 - b^2) + \frac{1}{2}(a-b)^2$, $a, b \in \mathbb{R}$, allows to get:

$$\begin{aligned}\frac{1}{2}\mathbb{E}\big[\|e_h^{n+1}\|^2_{L^2(D)} - \|e_h^n\|^2_{L^2(D)}\big] &\le c\Delta t\big(1 + \mathbb{E}[\|u(t_n)\|^8_{\dot{H}^1}] + \mathbb{E}[\|\tilde{u}_h^n\|^8_{\dot{H}^1}]\big)^{\frac{1}{2}}\mathbb{E}[\|u(t_n) - \tilde{u}_h^n\|^4_{L^2(D)}]^{\frac{1}{2}} + \Delta t\mathbb{E}[\|e_h^n\|^2_{L^2(D)}]\\ &\quad + c\Delta t^2\mathbb{E}\big[\|\tilde{u}_h^n\|^2_{L^2(D)} + \|\tilde{u}_h^n\|^6_{L^6(D)} + \|u_h^n\|^2_{L^2(D)} + \|u_h^n\|^6_{L^6(D)}\big] + c\Delta t(\|u(t_n) - u_h^n\|^2_{L^2(\Omega, L^2(D))} + J^{-2\gamma}).\end{aligned}$$

By virtue of (4.13), $\sup_{n=1,\dots,N}\|u_h^n\|_{L^{2p}(\Omega, \dot{H}^1)} < \infty$, $p \ge 1$, $\dot{H}^1 \hookrightarrow L^6(D)$ [28, (2.6)], (4.14), $J^{-\gamma} = \mathcal{O}(h^2)$, there exists an infinitesimal number $\epsilon_0 > 0$ such that:

$$\frac{1}{2}\|e_h^{n+1}\|^2_{L^2(\Omega, L^2(D))} \le (\frac{1}{2} + \Delta t)\|e_h^n\|^2_{L^2(\Omega, L^2(D))} + c\Delta t(\Delta t^{\frac{1}{2}} + h^{2-\epsilon_0})^2 + c\Delta t\|u(t_n) - u_h^n\|^2_{L^2(\Omega, L^2(D))}.$$

Denote $\rho(\Delta t) := 1 + 2\Delta t$. It holds:

$$\|e_h^{n+1}\|^2_{L^2(\Omega, L^2(D))} \le \rho(\Delta t)\|e_h^n\|^2_{L^2(\Omega, L^2(D))} + c\Delta t(\Delta t^{\frac{1}{2}} + h^{2-\epsilon_0})^2 + c\Delta t\|u(t_n) - u_h^n\|^2_{L^2(\Omega, L^2(D))}.$$

Recursive deduction yields:

$$\begin{aligned}\|e_h^n\|^2_{L^2(\Omega, L^2(D))} &\le \rho(\Delta t)^n\|e_h^0\|^2_{L^2(\Omega, L^2(D))} + c\sum_{k=0}^{n-1}\rho(\Delta t)^k\|u(t_{n-k-1}) - u_h^{n-k-1}\|^2_{L^2(\Omega, L^2(D))}\Delta t\\ &\quad + c\Delta t(\Delta t^{\frac{1}{2}} + h^{2-\epsilon_0})^2\sum_{k=0}^{n-1}\rho(\Delta t)^k.\end{aligned}$$

A simple calculation gives $\lim_{N\to\infty}(1+\frac{2T}{N})^n = \lim_{N\to\infty}\rho(\Delta t)^n \le e^{2T}$ for $n \le N$, which means $\sum_{k=0}^{n-1}\rho(\Delta t)^k \le c$. Further using $e_h^0 = 0$ derives

$$\|e_h^n\|^2_{L^2(\Omega,L^2(D))} \le c(\Delta t^{\frac{1}{2}} + h^{2-\epsilon_0})^2 + c\sum_{k=0}^{n-1}\|u(t_k)-u_h^k\|^2_{L^2(\Omega,L^2(D))}\Delta t.$$

This completes the proof. □

Thanks to the results elaborated in Lemmas 4.1–4.2, it is now sufficient to derive the full discretization error bound, which is stated in the following theorem.

Theorem 4.1. *Let u be the mild solution defined in (2.3), and u_h^n be the numerical solution of (3.1). Then under the assumptions shown in Lemmas 4.1–4.2, there exists an infinitesimal positive number ϵ_0 and a constant c independent of Δt and h, such that*

$$\|u(t_n)-u_h^n\|_{L^2(\Omega,L^2(D))} \le c\left(\Delta t^{\frac{1}{2}} + h^{2-\epsilon_0}\right), \quad n = 1,\dots,N.$$

Proof. On the basis of (4.5), Lemmas 4.1–4.2, and the triangle inequality, we obtain:

$$\|u(t_n)-u_h^n\|^2_{L^2(\Omega,L^2(D))} \le c\left[\left(\Delta t^{\frac{1}{2}} + h^{2-\epsilon_0}\right)^2 + \sum_{k=0}^{n-1}\|u(t_k)-u_h^k\|^2_{L^2(\Omega,L^2(D))}\Delta t\right], \; n = 1,\dots,N,$$

where ϵ_0 is an arbitrarily small positive number.

Then using discrete Gronwall inequality yields:

$$\|u(t_n)-u_h^n\|^2_{L^2(\Omega,L^2(D))} \le c\left(\Delta t^{\frac{1}{2}} + h^{2-\epsilon_0}\right)^2, \quad n = 1,\dots,N.$$

This ends the proof. □

5. Numerical experiments

In this section, we take one-dimensional SACE with multiplicative noise as an example to validate the theoretical analysis and demonstrate the performance of the proposed numerical scheme (3.1). The test model is represented as:

$$\begin{aligned} &du(x,t) = \left(\epsilon\partial_{xx}u + (u-u^3)\right)dt + \sigma(1-|u|)dW(x,t), \; 0<t<T, \; x\in(0,1),\\ &u(0,t) = u(1,t) = 0, \quad 0\le t\le T,\\ &u(x,0) = u_0(x) = \sin(4\pi x), \; x\in[0,1], \end{aligned} \tag{5.1}$$

where σ and ϵ denote respectively the noise intensity and diffusion coefficient, and $W(x,t)$ is defined by

$$W(x,t) := \sum_{j=1}^{\infty}\sqrt{q_j}\phi_j(x)\beta_j(t) = \sum_{j=1}^{\infty}\sqrt{2q_j}\sin(j\pi x)\beta_j(t),$$

with $\beta_j(t)$ being the standard Brownian motion and $q_j := \mathcal{O}(j^{-(2\gamma+1+\epsilon)})$ for an arbitrary small $\epsilon > 0$ and $\gamma \ge 1$. Note here that $G(u) = \sigma(1-|u|)$, $\phi_j(x) = \sqrt{2}\sin(j\pi x)$. Thus for $v_1, v_2 \in L^2(D)$, one checks easily that

$$\|G(v_1)-G(v_2)\|^2_{\mathcal{L}_0^2} \le c\sum_{j=1}^{\infty}q_j\|(|v_1|-|v_2|)\phi_j\|^2_{L^2(D)} \le c(\sum_{j=1}^{\infty}q_j)\|v_1-v_2\|^2_{L^2(D)} \le c\|v_1-v_2\|^2_{L^2(D)},$$

and

$$\|A^{\frac{1}{2}}G(v)\|^2_{\mathcal{L}_0^2} \le c\sum_{j=1}^{\infty}q_j\|A^{\frac{1}{2}}(1-|v|)\phi_j\|^2_{L^2(D)} \le c(\sum_{j=1}^{\infty}q_j)\left\|1-|v|\right\|^2_{\dot{H}^1} \le c(1+\|v\|_{\dot{H}^1})^2, \; \forall v\in\dot{H}^1.$$

It means $G(u) = \sigma(1-|u|)$ satisfies assumptions (2.5) and (2.6).

We first compare the proposed scheme (3.1) with the drift-implicit Euler–Galerkin finite element scheme introduced in [28, Section 4.1], focusing on computational efficiency. Numerical experiments are conducted on a computer equipped with 32 GB of memory and a "Core Ultra 7 155H" CPU. Letting $T = 0.1$, $\epsilon = 0.01$, $h = 1/128$, and $\sigma = 0.5$, we evaluate the CPU time required by both methods under various time step sizes. The corresponding numerical results are presented in Table 1, which clearly shows that the semi-implicit tamed scheme (3.1) significantly reduces computational time compared to the classical drift-implicit scheme. Hence, the proposed method is more computationally efficient.

The strong convergence rate in space and time is measured in terms of mean-square approximation errors at the endpoint $T = 0.1$. Notably the exact solution of the problem (5.1) is unknown, and we will use the reference solution computed in the fine space–time mesh size as the exact solution. Specifically, we take the reference solution calculated by $h = 1/128$, $\Delta t = 10^{-6}$ to test time accuracy, and use the computed solution with $h = 1/512$, $\Delta t = 10^{-5}$ for spatial accuracy test. Moreover, we employ the mean of 200 samples

Table 1
CPU time consumption at different time steps.

Time step Δt	Semi-implicit scheme (3.1) CPU time (s)	Drift-implicit scheme CPU time (s)
1/100	0.127635	0.820542
1/200	0.152599	1.294839
1/500	0.210357	1.932968
1/1000	0.287350	2.303738

Table 2
Temporal (left table) and spatial (right table) convergence rates.

Time step Δt	Fully discrete (3.1)		Mesh size h	Fully discrete (3.1)	
	u_{error}	Order		u_{error}	Order
1.00E−2	2.5189E−3	–	1/16	4.2388E−2	–
5.00E−3	1.8461E−3	0.45	1/32	1.1340E−2	1.90
2.50E−3	1.3257E−3	0.48	1/64	3.0124E−3	1.91
1.25E−3	9.8854E−4	0.42	1/128	7.9262E−4	1.93
6.25E−4	6.7973E−4	0.54	1/256	1.9585E−4	2.02

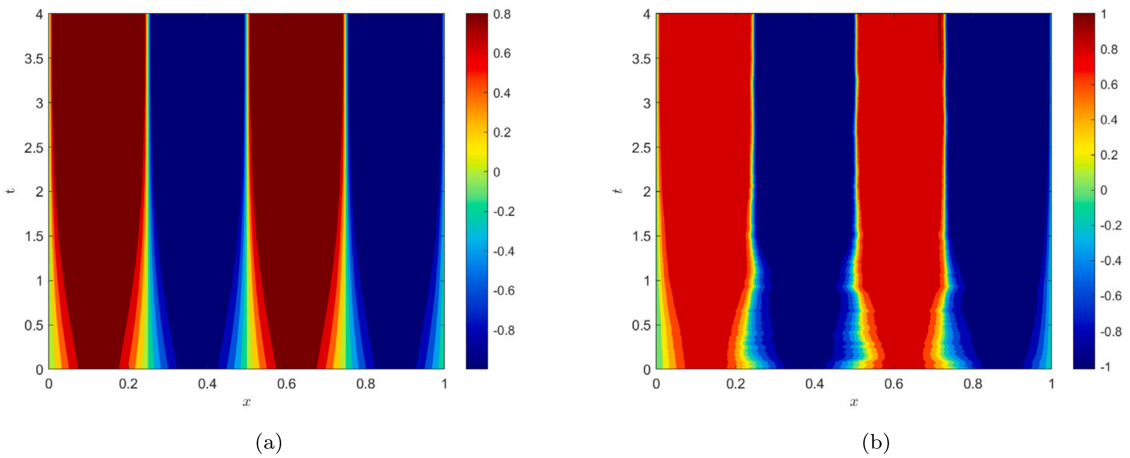

Fig. 1. Time evolution of the numerical solution of one-dimensional deterministic and stochastic Allen–Cahn equation. (a): deterministic. (b): stochastic version.

to approximate the expectation of the numerical error, i.e., the mean-square error $\mathbb{E}[\|u(T)-u_h^N\|_{L^2(D)}^2]^{\frac{1}{2}}$ is computed approximately by

$$\mathbb{E}[\|u(T)-u_h^N\|_{L^2(D)}^2]^{\frac{1}{2}} \approx \left(\frac{1}{200}\sum_{j=1}^{200}\|u_j^{\text{ref}}-u_{j,h}^N\|_{L^2(D)}^2\right)^{\frac{1}{2}} =: u_{\text{error}},$$

where u_j^{ref} and $u_{j,h}^N$ denote separately the reference solution and the fully discrete numerical solution of the jth sample. By taking $\epsilon = 10^{-3}$, $\gamma = 1$, $\sigma = \frac{1}{2}$, the numerical error u_{error} under different time steps and mesh sizes is calculated in Table 2, from which we see that the strong convergence order is as predicted by the theory.

Additionally, the time evolution of the numerical solution of the problem (5.1) is compared to that of the deterministic Allen–Cahn equation (i.e., $\sigma = 0$) to demonstrate the influence of random perturbations. By setting the initial input to be $T = 4$, $\epsilon = 10^{-5}$, $\Delta t = 10^{-4}$, $h = 1/128$, $\gamma = 1$, we calculate the mean value of the numerical solutions of 10 samples and show its contour figure in Fig. 1, from which we observe that, after the incorporation of noise, some small-scale structures are generated, and the static kinks corresponding to the deterministic model are changed. These kinks can interact with each other or even annihilate each other, and some new kinks are produced.

Next, we use another numerical example of the evolution of the mean curvature surface of the two-dimensional Allen–Cahn equation to further demonstrate the perturbation effect of random factors on the numerical solution. Given the initial condition $u(x,0) = u_0(x)$ and the homogeneous Dirichlet boundary condition, we consider the deterministic and stochastic Allen–Cahn equations:

$$\begin{aligned} &u_t(x,t) = -Au + \frac{u-u^3}{\epsilon^2}, \\ &du(x,t) = -Audt + \frac{u-u^3}{\epsilon^2}dt + 5(1-|u|)dW(x,t), \end{aligned} \qquad t \in (0,T),\ x \in (0,1)^2, \tag{5.2}$$

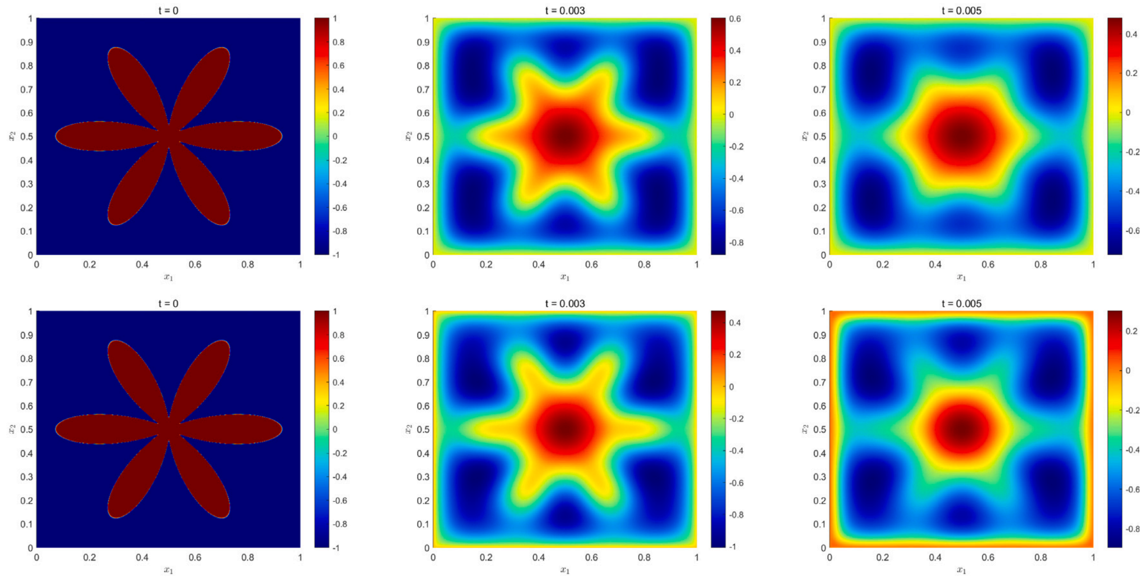

Fig. 2. Evolution of a star-shaped interface for $t = 0, 0.003, 0.005$. The first row: deterministic case. The second row: random case with $G(u) = 5(1 - |u|)$.

where ϵ stands for the interface scale parameter. Let $x = (x_1, x_2)^T$, $x_1, x_2 \in (0,1)$. Define $W(x,t)$ as

$$W(x,t) := W(x_1, x_2, t) = \sum_{j_1,j_2=1}^{\infty} \sqrt{q_{j_1,j_2}} \sin(j_1 \pi x_1) \sin(j_2 \pi x_2) \beta_{j_1,j_2}(t),$$

where $q_{j_1,j_2} = \exp(-\frac{j_1^2+j_2^2}{200})$ and $\beta_{j_1,j_2}(t)$ are the i.i.d Brownian motions. Similar to the one-dimensional case, it is readily to verify that $G(u) = 5(1-|u|)$ also satisfies (2.5) and (2.6) using $\sum_{j_1,j_2=1}^{\infty} q_{j_1,j_2} < \infty$.

It is known for the deterministic Allen–Cahn equation that when $\epsilon \to 0$, the zero level set of u denoted by $\Gamma_t^\epsilon := \{x \in D : u(x,t) = 0\}$ approaches a surface Γ_t whose evolution follows the geometric law:

$$V = -\frac{1}{R} = -\kappa, \tag{5.3}$$

where V is the normal velocity of the surface Γ_t at each point, κ is its mean curvature, and R is the principal radii of curvature at the point of the surface [36,37]. If we denote the radius at time t by $R(t)$ and set the initial radii to be R_0, then $R(t) = \sqrt{R_0^2 - 2t}$.

Let computational domain $D = (0,1) \times (0,1)$ be divided into a mesh of 512×512, $\epsilon = 7.5 \times 10^{-4}$, $\Delta t = 5 \times 10^{-5}$. Supplement the initial configuration to be

$$\begin{cases} u(x,0) = u(x_1,x_2,0) = \tanh \frac{1.5+1.2\cos(6\theta)-2\pi r}{\sqrt{2}\epsilon}, \\ \theta = \arctan \frac{x_2-0.5}{x_1-0.5}, \\ r = \sqrt{(x_1-0.5)^2 + (x_2-0.5)^2}. \end{cases}$$

We calculate the 5 sample mean of the numerical solutions and plot the contour figures at different times given above each subfigure for both deterministic case and random case, as shown in Fig. 2, where the first row shows the evolution of a star-shaped interface in a curvature-driven flow of the deterministic Allen–Cahn model and we observe the well known result that the tips of the star move inward, while the gaps between the tips move outward, and the whole shape shows a trend of shrinking towards the center. The second row of Fig. 2 presents the evolution of the star interface for the case with $G(u) = 5(1-|u|)$. It can be seen that noise plays a significant role, changing the properties of the solution. Moreover, similar to the case in one-dimensional. The kinks interact, even cancel each other out, and new kinks may appear.

6. Conclusions

This paper studied a stochastic Allen–Cahn equation driven by Q-Wiener multiplicative force noise. The well-posedness of the underlying equation was elaborated. An efficient fully discrete scheme, based on semi-implicit tamed method and finite element method with truncated noise, was proposed. The strong convergence rates of the spatio-temporal full discretization scheme was successfully derived using an appropriate decomposition method. Numerical experiments were finally provided to verify the theoretical results and show the effectiveness of the proposed method. In the future, we will focus on the numerical analysis and computation for the stochastic fractional Allen–Cahn equation.

CRediT authorship contribution statement

Xiao Qi: Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Data curation. **Lihua Wang:** Resources, Methodology, Investigation, Conceptualization. **Yubin Yan:** Writing – review & editing, Supervision, Resources, Project administration, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Xiao Qi reports financial support was provided by the Research Fund of Jianghan University under Grant No. 2024JCYJ04. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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