A finite element method for time fractional partial differential equations

A FINITE ELEMENT METHOD FOR TIME FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

Fractional differential equations, particularly fractional partial differential equations (FPDEs) have many applications in areas such as diffusion processes, electromagnetics, electrochemistry, material science and turbulent flow. There are lots of work for the existence and uniqueness of the solutions for fractional partial differential equations. In recent years, people start to consider the numerical methods for solving fractional partial differential equation. The numerical methods include finite difference method, finite element method and the spectral method. In this dissertation, we mainly consider the finite element method, for the time fractional partial differential equation. We consider both time discretization and space discretization. We obtain the optimal error estimates both in time and space. The numerical examples demonstrate that the numerical results are consistent with the theoretical results.

Keywords:

- Fractional partial differential equations.
- Finite element method.
- Caputo fractional derivative.
- Riemann-Liouville fractional derivative.

This work is original and has not been previously submitted for any academic purpose.

Signed………………….

Date:……………………

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Table of Contents

List of Figures

Chapter 1

Introduction

Through the use of four sections this introductory chapter aims to highlight the research process in this dissertation. Section 1 and 2 of this chapter review and discuss the aims and objectives which have underpinned the research process. Finally, Section 3 outlines the organization of the dissertation undertaken.

1.1 Aim of the dissertation

The primary aim of the dissertation is to:

- Promote an engineer's ability to use fractional calculus as a modelling Instrument.

1.2 Objectives of the dissertation

This dissertation is a discussion of fractional partial differential equations and the numerical methods used to solve these equations. The evaluation of derivatives and integrals where the order of the derivative is not an integer is described as fractional calculus. Through a discussion of the different types of equations found in fractional calculus the author aims to provide an overview of the topic before moving on to the analysis of the finite element method for solving time fractional partial differential equations. From the discussion of the presented analysis the author will then identify potential areas of future work.

In this dissertation we shall:-

- Review the history of fractional calculus including some of the analytical methods used to solve the fractional differential equation
- Discuss the existing numerical methods for solving fractional differential equations.
- Introduce the standard finite element method for solving partial differential equations.
- Consider the finite element method for solving fractional PDEs.

1.3 Organization of this dissertation

In chapter two we describe the history of fractional calculus, and we present a brief survey of the possible uses of fractional differential equations. The chapter provides specific examples from recent applications enable us to understand the diversity of fields in which fractional calculus can be of use.

In chapter three we explain the tools and methods involved in solving fractional ordinary differential equations.

In chapter four we review two different algorithms for the numerical solution of fractional ordinary differential equation to provide the necessary background to enable a discussion of fractional Partial differential equations.

In chapter five we explore a background of using finite element method to solve parabolic partial differential equations.

In chapter six we consider the finite element methods for solving time fractional partial differential equations. We obtain the error estimates both in time and space. The numerical examples show that the numerical results are consistent with the theoretical results.

In chapter seven we summarize the dissertation, and give our conclusions.

1.4 Summary

A finite element method for time fractional partial differential equations is necessary to promote an engineer's ability in their work. This chapter explains the aim and objectives of the dissertation and then highlights the organization of the research by providing brief notes chapter-by-chapter.

Chapter 2

Fractional Calculus

Aim

We discuss the historical background for fractional calculus. The discussion will cover fractional integration and differentiation and provided two definitions of the fractional derivatives. The existence and uniqueness of the fractional differential equation will be demonstrated and the numerical methods for solving fractional differential equations are discussed.

2.1 A brief History of factional calculus

It is from a letter by G.F.A. de l'Hôpital in 1695 to G.W. Leibniz after created the notation $d^n y/dt^n$, that birth of fractional calculus is cited as it triggered a response which the development of this aspect of mathematics responded [18] to l'Hôpital's question of " what would be the result if $n = 1/2$?" Leibniz identified that there was an "*apparent paradox from which one day useful consequences will be drawn*" [18]. As it can be seen from various mathematical discussion writers such as Ross [19] and Gorenflo attribute this letter as the starting point of fractional calculus. It was later in the 18th century when L. Euler (1730) found that the evaluation of the derivative of the power function for non-integer order had a role to play in his Gamma function. Gorenflo and Mainardi assert that there were several important contributions from the 19th to the middle of the 20th century who further developed the knowledge and understanding of Fractional Calculus. Many mathematicians have experimented in Fractional Calculus including Fourier, Euler and Laplace [4] .These mathematicians used their own definitions and methods to support the concept of a non-integer order of an integral or derivative. The famous popularised definitions are from Reimann-Louville and Grünwald-Letnikov. However it has been the last century which has seen the most intriguing developments in engineering and scientific applications. In some cases mathematics has had to change in order to meet the requirements of physical reality. I. Podlubny [4] advises that there are numerous applications of fractional calculus to dynamical systems in control theory, such as fractance for electrical circuits, voltage divider generalisation, viscoelasticity, electrochemistry, fluid flow tracing, biological modelling of neurons and in electromagnetism the use of fractional-order multiples. It seems that Leibniz's response over 300 years ago appears to be at least half right as there are numerous applications and physical manifestations of fractional calculus.

While Leibniz's paradox is still present in $n = 1/2$, resulting in the physical meaning of equations that are difficult (arguably impossible) to grasp. The definitions and explorations are no more rigorous than those of their integer order counterparts. Laplace in 1812 defined a fractional derivative in terms of an integral while S.F. Lacroix [3] in 1819 provided the first defined expression of a fractional derivative. Lacroix [3] stated for $y(t)=t^m$, with m a positive integer, expressing its nth derivative in terms of the Legendre Symbol for Euler's Gamma function. Hence we have

$$\frac{d^n y(t)}{dt^n} = \frac{d^n t^m}{dt^n} = \frac{m!}{(m-n)!} t^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} t^{m-n}, \quad m \geq n.$$

Lacroix then set $m = 1$ and $n = 1/2$ thus producing the derivative of order half of the function t:

$$\frac{d^{1/2}t}{dt^{1/2}} = \frac{\Gamma(2)}{\Gamma(3/2)} t^{1/2} .$$

By expanding Γ(2) and Γ(3/2), the equation is then solely in terms of Γ(1/2). Using Euler's reflection theorem like Lacroix [3] we now have $\Gamma(t)\Gamma(1-t)=\pi/\sin t$ where $t < 1$, this equation determines Γ(1/2) and thus we have Lacroix's result:

$$\frac{d^{1/2}t}{dt^{1/2}} = \sqrt{\frac{4t}{\pi}} = \frac{2}{\sqrt{\pi}}\sqrt{t} .$$

J.B.J Fourier (1822) also commented on derivatives of arbitrary order [17]. But unfortunately none of these mathematicians gave any application of fractional calculus as it was not until N.H.Abel (1823) [11]. Abel [11] provided the first practical use of fractional calculus when he considered the tautochrone problem. The tautochrone problem is used to determine the shape of a fiction-less wire within a vertical plan and calculate how long it takes for a bead to slide along this wire to the lowest point of the wire that is independent of the start point [11]. In his consideration of the tautochrone problem Abel [11] deduced a fractional integral equation that he then converted into fractional differential equation. The fractional differential equation was then manipulated by Abel [11] so that the fractional differential operator was on a constant. By applying Lacroix's method to a known result of the fractional derivative of a constant Abel [11] adroitly calculated the curve of the tautochrone.

Joseph Liouville [13] was inspired by Abel's solution to start the first major study of fractional Calculus. From 1832 onwards Liouville [13] presented several papers and provided a definition of a fractional derivative based on an infinite series. However there is a drawback of this definition as the order of the fractional derivative can only have values for which the series converges. Liouville [13] addressed this issue by developing another definition by regarding and implementing a definite integral related to Euler's Gamma integral.

$$\int_o^{\infty} u^{a-1}\, e^{-tu} du = t^{-a} \int_0^{\infty} x^{a-1}\, e^{-x}\, dx = \frac{\Gamma(a)}{t^a}.$$

Taking the first and last expressions Liouville [13] formed the equation

$$t^{-a} = \frac{1}{\Gamma(a)} \int_0^{\infty} x^{a-1}\, e^{-x} dx\, .$$

From this equation Liouville now took the αth derivative of both sides assuming that $d^{\alpha}(e^{at})\, /d\, t^{\alpha} = a^{\alpha} e^{at}$, where $\alpha \in \mathbb{R}_{+}$, and derived his second definition:

$$\frac{d^{\alpha} t^{-a}}{dt^{\alpha}} = D^{\alpha}\, t^{-a} = \frac{(-1)^{\alpha}\Gamma(a+\alpha)}{\Gamma(a)}\, t^{-a-\alpha}.$$

The ordinary differential equation, $d^n y/dt^n = 0$, according to Liouville [14] has the complementary solution $y = c_0 + c_1 t + c_2\, t^2 + \cdots + c_{n-1} t^{n-1}$. It is then argued by Liouville that the arbitrary order equation $d^{\alpha} y/dt^{\alpha} = 0$ should have a corresponding series solution. Ignoring the trivial case $t = 0$ which gives a constant as the solution similarly the factional order derivative should also be zero providing a contradiction.

Later using the Lacroix method, W.Center (1850) demonstrated that the fractional derivative of a constant is not zero [3]. It was not until the end of the nineteenth century that this situation was resolved when consensus was reached by mathematicians for a robust definition of fractional derivative as part of the general theory of fractional operators [3].

As part of the general theory of fractional operators the new definition of fractional derivatives created a reconciliation of Lacroix classic-oriented approach with Liouville's methods. G.F.B. Riemann, as a student in 1847, developed an alternative theory of fractional operators utilising a Taylor series generalisation. However this expression for fractional integrals was not released until after Riemann's death in 1876. Davis [12] describes Riemann's expression for fractional integrals.

According to the efforts of these early Mathematicians of the late 19th and 20th centuries they were able to improve constructive definitions of both the fractional calculus and to find methods for solving various types of equations. Weilbeer [6] in his thesis of 2005 illustrated the development of fractional integral and derivative.

2.2 Fractional integration and differentiation

This section of the chapter is now divided into a number of sub-sections. Each sub-section will deal with one set of operations starting with the Rienmann-Liouville operators for fractional differentiation and integration,

2.2.1 The Riemann-Liouville fractional calculus

The Riemann-Liouville fractional integral

Definition 2.1 ([15] pp.33)

The Riemann-Liouville fractional integral of order $0 < \alpha < 1$, is denoted by the expression:

$$ {}_a^R D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau \,. \tag{2.1} $$

The Riemann-Liouville fractional derivative

Definition 2.2 ([15, pp.35])

Let $\alpha > 0$, the Riemann-Liouville fractional derivative is defined with $n-1 < \alpha \leq n$ by,

$$ {}_a^R D_t^{\alpha} f(t) = D^n \left[D_t^{\alpha-n} f(t) \right] = D^n \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \tag{2.2} $$

where $D^n = \frac{d^n}{dt^n}$ denotes the standard nth derivative.

Example 2.1 ([4, pp.56]). Let $f(t) = (t-a)^k$ for some $k > -1$ and $\alpha > 0$ then ,

$$ {}_a^R D_t^\alpha f(t) = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} (t-a)^{k-\alpha}. \tag{2.3}$$

The derivation moves in the direction to:

$$ {}_a^R D_t^\alpha ((t-a)^k) = \frac{1}{\Gamma(-\alpha)} \int_a^t (\tau - a)^k (t-\tau)^{-\alpha-1} d\tau. $$

Let $\tau = a + u(t-a)$ and using the definition of Beta function.

$$ {}_a^R D_t^\alpha ((t-a)^k) = \frac{1}{\Gamma(-\alpha)} (t-a)^{-\alpha+k} \int_0^1 u^k (1-u)^{-\alpha-1} du $$

$$ = \frac{1}{\Gamma(-\alpha)} \beta(-\alpha, k+1)(t-a)^{k-\alpha}. $$

Where β is a Beta function, and $\beta(-\alpha, k+1) = \frac{\Gamma(-\alpha)\Gamma(k+1)}{\Gamma(k+1-\alpha)}$.

Hence we get

$$ {}_a^R D_t^\alpha ((t-a)^k) = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} (t-a)^{k-\alpha}. $$

If we substitute $a = 0$, then the Riemann-Liouville fractional derivative of $f(t) = t^k$ is as follows:

$$ {}_a^R D_t^\alpha t^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} t^{k-\alpha} = \frac{k!}{\Gamma(k+1-\alpha)}. \tag{2.4}$$

2.2.2 The Caputo fractional derivative

M. Caputo [10] defines the Caputo fractional derivative ${}_a^C D_t^\alpha f(t) := D_t^{\alpha-n}[D^n f(t)]$ for $n - 1 < \alpha \leq n$ .We can define the Caputo fractional derivative of order α.

Definition 2.3

The Caputo fractional derivative of order $\alpha > 0$ is takes the form:

$$ {}_a^C D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} [D^n f(\tau)] d\tau, & \text{where } n-1 < \alpha < n, \\ \frac{d^n}{dt^n} f(t), & \text{where } \alpha = n. \end{cases} \tag{2.5}$$

The relationship between the Caputo derivative and the Riemann-Liouville derivative is the following, K. Diethelm [7],

Theorem 2.1 Let $\alpha > 0$, Assume that f is such that both ${}_a^R D_t^\alpha f$ and ${}_a^C D_t^\alpha f$ exist.

Then,

$${}_a^C D_t^\alpha f(t) = {}_a^R D_t^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1-\alpha)} (t-a)^{k-\alpha}. \tag{2.6}$$

Proof: Note that, Diethelm [7],

$${}_a^C D_t^\alpha f(t) = {}_a^R D_t^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} \, {}_a^R D_t^\alpha [(\cdot - a)^k](t).$$

By example 2.1, we have

$${}_a^C D_t^\alpha f(t) = {}_a^R D_t^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1-\alpha)} (t-a)^{k-\alpha}. \tag{2.7}$$

Theorem 2.2 Let $\alpha > 0, n-1 < \alpha < n$, then ,

$$\begin{aligned} {}_a^C D_t^\alpha f(t) &= {}_a^R D_t^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1-\alpha)} (t-a)^{k-\alpha} \\ &= {}_a^R D_t^\alpha [f(t) - T_{n-1}[f;a](t)]. \end{aligned}$$

Here $$T_{n-1}[f;a](t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k. \tag{2.8}$$

Denotes the Taylor polynomial of degree n-1 for the function f, centred at a; in the case $n = 0$ thus we define $T_{n-1}[f;a] := 0$.

Proof: from (2.7) we have

$${}_a^C D_t^\alpha f(t) = {}_a^R D_t^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1-\alpha)} (t-a)^{k-\alpha}.$$

By referring to equations (2.4) and (2.8), we get

$${}_a^C D_t^\alpha f(t) = {}_a^R D_t^\alpha [f(t) - T_{n-1}[f;a](t)] . \tag{2.9}$$

Note that if $\alpha = 0$, then

$$ {}_0^C D_t^\alpha f(t) = {}_0^R D_t^\alpha [f(t) - T_{n-1}[f;0](t)] \,, \tag{2.10}$$

where $$T_{n-1}[f;0](t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k, \qquad n-1 < \alpha < n. \tag{2.11}$$

When $0 < \alpha < 1$, $n = 1$, we have

$$ {}_0^C D_t^\alpha f(t) = {}_0^R D_t^\alpha [f(t) - f(0)]. \tag{2.12}$$

2.3 Fractional differential equations

The following subsection of this chapter will discuss the existence and uniqueness properties of fractional differential equations. The definition and proofs will be restricted to initial value problems and furthermore the assumptions without loss of generality will be that the fractional derivatives are developed at the point 0. As a consequence from now on ${}_0^R D_t^\alpha$ and ${}_0^C D_t^\alpha$ will be used as symbols for the Riemann-Liouville and the Caputo fractional derivatives developed at the point zero. The discussion will start with a formal definition of a fractional differential equation (FDE):

Let $\alpha > 0, \alpha \notin \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers.

Let $m = \lceil \alpha \rceil$, where $\lceil \alpha \rceil$ denotes the integer part of α, $f: A \subseteq \mathbb{R}^2 \to \mathbb{R}$, we consider the following fractional differential equation of Riemann-Liouville type

$$ {}_0^R D_t^\alpha y(t) = f\big(t, y(t)\big), \qquad t > 0. \tag{2.13}$$

The initial conditions for this type of (FDE) uses

$$ D^{\alpha-k} y(0) = b_k, \ (k = 1,2, \dots, m-1). \tag{2.14}$$

Similarly, we can consider the following fractional differential equation of Caputo type,

$$ {}_0^C D_t^\alpha y(t) = f(t, y(t)), \qquad t > 0. \tag{2.15}$$

In this case we use the initial condition

$$D^k y(0) = b_k, \quad (k = 0,1,2, \dots, m-1). \tag{2.16}$$

2.3.1 The existence and uniqueness of the solution

Theorem 2.3 [7] *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$ and $n = \lceil \alpha \rceil$. Moreover,*

let $K > 0$, $h^* > 0$, and $b_1, \dots, b_m \in \mathbb{R}$. Define

$$G := \{(t, y) \in \mathbb{R}^2 : 0 \le t \le h^*, y \in \mathbb{R} \, for \, t = 0 \, and$$

$$|t^{n-\alpha} y - \sum_{k=1}^{n} b_k \, t^{n-k} / \Gamma(\alpha - k + 1| < k \, else\}.$$

Assume that the function $f: G \to \mathbb{R}$ is continuous and bounded in G and that the equations fulfills a Lipschitz condition with respect to the second variable, i.e. there exists a constant $L > 0$ such that for all (t, y_1) and $(t, y_2) \in G$, we have

$$| f(t, y_1) - f(t, y_2| \le L|y_1 - y_2|.$$

Then the fractional differential equation of Riemann-Liouville type (2.13) *equipped with the initial conditions* (2.14) *has demonstrated a uniquely defined continuous solution $y \in C(0, h]$ where*

$h := min\{h^*, \tilde{h}, (\Gamma(\alpha + 1)K/M)^{1/n}\}$ *with $M := sup_{(t,z) \in G} |f(t, z)|$ and $\tilde{h}$ being an arbitrary positive number satisfying the constraint*

$$\tilde{h} < \frac{\Gamma(2\alpha - n + 1)}{(\Gamma(\alpha - n + 1)L)^{1/\alpha}}.$$

Theorem 2.4 [7] *Under the hypothesis of Theorem 1.3 the function $y \in C(0, h]$ is a solution to the fractional differential equation of Riemann-Liouville type* (2.13), *processed with the initial condition* (2.14), *if and only if it is a solution of the Volterra integral equation of the second kind*

$$y(t) = \sum_{k=1}^{n} \frac{b_k t^{\alpha-k}}{\Gamma(\alpha-k+1)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, y(\tau)) d\tau \tag{2.17}$$

For the fractional differential equation of Caputo type we can obtain a similar result in

Theorem 2.5 [7] Let $\alpha > 0$, $n \notin \mathbb{N}$ *and* $n = \lceil \alpha \rceil$. Let $M > 0$, $h^* > 0$, *and* $b_1, \ldots, b_{n-1} \in \mathbb{R}$. Define

$$G := [0, h^*] \times [b_0 - M, b_0 + M],$$

and let the function $f: G \to \mathbb{R}$ *be continuous. Then, there exists some* $h > 0$ *and a function* $y \in C[0, h]$ *solving the fractional differential equation of Caputo type* (2.15), *equipped with initial conditions* (2.16). *For the case* $\alpha \in (0,1)$ *the parameter h is given by*

$$h := min\{h^*, (K\Gamma(\alpha+1)/M)^{1/\alpha}\},\ with\ M := sup_{(t,z)}|f(t,z)|.$$

If furthermore f fulfils a Lipschitz condition with respect to the second variable, i.e.

$$| f(t, y_1) - f(t, y_2| \leq L|y_1 - y_2|,$$

with some constant $L > 0$ *independent of* t, y_1 *and* y_2, *then the function* $y \in C[0, h]$ *is unique.*

These results are very similar to their counterparts in the classical case of first-order equations. They are even proven in a similar way.

Theorem 2.6 [7] *Under the hypothesis of Theorem 2.5* $y \in C(0, h]$ *is a solution to the fractional differential equation of Caputo type* (2.15), *processed with the initial condition* (2.16), *if and only if it is a solution of the Volterra integral equation of the second kind*

$$y(t) = \sum_{k=0}^{n-1} \frac{b_k t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau \,. \quad (2.18)$$

The next section focuses on the uses of fractional differential equations (FDEs) in modelling physical phenomena and thus linking to practical applications of fractional calculus. The following example of electrical circuits with fractance will demonstrate the application of factional calculus to the electrode- electrolyte interface in circuitry.

2.4 Electrical Circuits

Integer-order models can be used to describe classical electrical circuits which consist of resistors and capacitors. But these circuits may have fractance which is an electrical element feature that can be described according to Le Mehaute and Crepy [1] with fractional-order impedance.

There are two kinds of fractances that can be considered in relation to electrical circuits: **(i) tree fractance** and **(ii) chain fractance.**

Tree fractance is considered by Nakagawa and Sorimachi [2], as a finite self-similar circuit with resistors that consists of resistance R and capacitors of capacitance C. The hindrance of the electron flow in the circuit as a result of the fractance is given by

$$Z(i\omega) = \sqrt{\frac{R}{C}}\,\omega^{-1/2} exp\left(-\frac{\pi i}{4}\right). \tag{2.19}$$

This tree fractance can be described by an associated fractional-order transfer function as demonstrated below in equation (2.20)

$$Z(s) = \sqrt{\frac{R}{C}}\,s^{-1/2}\ . \tag{2.20}$$

Oldham and Spanier [5] have demonstrated for a chain fractance where there are N pairs of resistor-capacitor connected in a chain that the transfer function is approximately

$$G(s) \approx \sqrt{\frac{R}{C}}\frac{1}{\sqrt{s}} \tag{2.21}$$

This chain fractance behaves as a fractional-order integrator of order 1/2 in the time domain as illustrated in $6RC \leq t < \left(\frac{1}{6}\right) N^2 RC$.

Electric batteries produce a limited amount of current due to microscopic electrochemical processes at the electrode-electrolyte interface. As the metalelectrolyte interfaces impedes function $Z(\omega)$ does not indicate the desired capacitive features for frequencies ω. Indeed, as $\omega \to 0$,

$$Z(\omega) \approx (i\omega)^{\eta}, \quad 0 < \eta < 1. \tag{2.22}$$

The Laplace transform space gives the equivalent impedance function

$$Z(s) \approx s^{-\eta}. \tag{2.23}$$

This function illustrates that the electrode-electrolyte interface is an applied example of a fractional-order process. The value of η is closely associated with the smoothness of the interface as the surface is infinitely smooth as $\eta \to 1$.

It was proposed by Kaplan et al. [21] that a physical model utilising the Cantor block to self-affine with N-stage electrical circuit of fractance type. Under suitable assumptions, Kaplan et al. [21] found the importance of the fractance circuit in the form as follows:

$$Z(\omega) = K(i\omega)^{-\eta}, \tag{2.24}$$

where $\eta = 2 - \log(N^2)/\log aK$ and a are constants, and $N^2 > a$ implies $0 < \eta < 1$. This model of Kaplan et al. also illustrates an example of the fractional-order electric circuit.

The inter conductor potential $\phi(x,t)$ or inter conductor current $i(x,t)$, in a resistive-capacitive transmission line model, satisfies the classical diffusion equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \ t > 0, \tag{2.25}$$

where the diffusion constant κ is replaced by $(RC)^{-1}$, R and C denotes the resistance and capacitance per unit length of the transmission line, while $u(x,t) = \phi(x,t)$ or $i(x,t)$.

Utilising the initial and boundary conditions

$$\phi(x,0) = 0 \ \ \forall\, x \in (0,\infty), \qquad \phi(x,t) \to 0 \ \ as\ x \to \infty, \tag{2.26}$$

the function produces an illustration of the current

$$i(0,t) = -\frac{1}{R}\frac{d}{dt}\,\phi(0,t) = \left(\frac{C}{R}\right)^{\frac{1}{2}} D_0^{\frac{1}{2}}\phi(0,t). \tag{2.27}$$

The current field in the transmission line of infinite length is expressed in terms of the fractional derivative of order 1/2 of the potential $\phi(0,t)$. The equation will be solved using numerical methods in this dissertation. The numerical method which is planned to be utilised is the finite element method.

Another example from electrical engineering is the involvement of fractional-order derivative in electrical transmission lines through generalised voltage dividers. It was observed by Westerlund [4] that both the tree fractance and chain fractance consist not only of resistors and capacitors properties, but additionally exhibit electrical properties with noninteger-order impedance. Westerlund [4] generalized that the classical voltage divider in which the fractional order impedances $F1$ and $F2$ represent impedances not only on Westerlund's capacitors, classical resistors, and induction coils, but also upon the impedance of tree fractance and chain fractance. The transfer function of Westerlund's voltage divider circuit is represented by the following function

$$H(s) = \frac{k}{s^{\alpha}+k}, \tag{2.28}$$

Where $-2 < \alpha < 2$ and k is a constant that depends on the elements of the voltage divider. The negative values of α correspond to a high pass filter; while the positive values of α correspond to a low pass filter. Some special cases of the transfer function (2.28) for voltage dividers were considered by Westerlund [4] when they consist of different combinations of resistors (R), capacitors (C), and induction coils(L). If $U_{in}(s)$ is the Laplace transform of the unit-step input signal $u_{in}(t)$, then the Laplace transform of the output signal $U_{out}(s)$ is represented by

$$U_{out}(s) = \frac{ks^{\alpha-1}}{s^{\alpha}+k}. \tag{2.29}$$

The inverse Laplace transformation of equation (2.29) is obtained from equation (9.8) in Westerlund [4] to obtain the output signal.

Where

$$U_{out}(t) = kt^{\alpha}E_{\alpha,\alpha+1}(-kt^{\alpha}) = k\mathcal{E}_0(t,-k;\alpha,\alpha+1). \tag{2.30}$$

Although the inverse Laplace transformation provides an exact solution for the output signal this function cannot describe physical properties of the signal. By evaluating the inverse Laplace transform in the complex s-plane numerous interesting physical properties of the output signal can be described for various values of α. For $1 < |\alpha| < 2$, the output signal exhibits oscillations.

These examples provide a demonstration for the application of factional derivative in electrical circuitr which is useful for electrical engineering. Fractance is presented in two forms: chain and tree fractance which are illustrated with their own individual equations.

This chapter has presented the historical background for fractional calculus. The discussion has covered fractional integration and differentiation and provided two definitions of the fractional derivatives. The existence and uniqueness of the solution of fractional differential equation has been demonstrated.

Chapter 3

The Tools and Analytical Methods of FODEs

Aim

In this chapter, we will introduce the tools and methods used to solve fractional differential equations.

3.1 Tools

By providing some relatively simple mathematical definitions the understanding of further definitions and their application in fractional calculus will become apparent. Some of the standard concepts are necessary that are provided below for the discussion of the application to fractional calculus: the Gamma function, the Beta function, the Laplace transform, the Mittag-Leffler function and Fourier transform.

3.1.1 Euler's Gamma Function

The Gamma function is a generalization of the factorial function n! where the factorial is only applicable to positive integer order n. The Gamma function can be used for any real number.

Definition 3.1 [4]

The Gamma function $\Gamma(x)$ is defined by the integral

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} \, dt$$

The Gamma function has one of the basic properties:

$$\Gamma(x+1) = x\Gamma(x) \tag{3.1}$$

Through the application of a partial integration, for the arbitrary $x > 0$, we can manipulate the integral in the definition of the Gamma function which yields:

$$\Gamma(x+1) = \int_0^\infty e^{-t}t^{x-1}\,dt = [-e^{-t}t^{x-1}]_0^\infty + x\int_0^\infty e^{-t}\,t^{x-1}\,dt = x\Gamma(x)$$

Obviously, $\Gamma(1) = 0! = 1$, and using (3.1) we obtain for $x = 1,2,3,\ldots$:

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1$$

$$\Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1 = 2!$$

$$\Gamma(4) = 3 \cdot \Gamma(3) = 3!$$

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$$\Gamma(n+1) = n\,\Gamma(n) = n \cdot (n-1)! = n!$$

3.1.2 Beta function

The Beta function is also known as the Euler integral of the first kind. The Beta function is an important relationship in fractional calculus as its solution is not only defined through the multiple use of Gamma functions, but additionally this function shares a form that is characteristically similar to the fractional integral/derivative of many functions. This fractional Integral/derivative is particularly found in the polynomials and the Mittag-Leffer function. The equation below demonstrates the Beta integral in terms of the Gamma function (see [4]).

$$\beta(p,q) = \int_0^1 (1-u)^{p-1}u^{q-1}\,du = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \qquad p,q \in \mathbb{R}_+. \tag{3.2}$$

In fact, we have

$$\Gamma(p)\Gamma(q) = \int_0^\infty e^{-t}\, t^{p-1} dt \int_0^\infty e^{-u} u^{q-1}\, du.$$

Set $t = x^2, u = y^2$, we obtain

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty e^{-x^2}\, x^{2p-1} dx \int_0^\infty e^{-y^2} y^{2q-1}\, dy,$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2p-1}\, y^{2q-1} dx dy.$$

Using $x = r\cos\theta, y = r\sin\theta$, $x^2 + y^2 = r^2$ we can then write

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty \int_0^{\pi/2} e^{-r^2} r^{2p+2q-2} (\cos\theta)^{2p-1} (\sin\theta)^{2q-1}\, r dr d\theta$$

$$= 2\int_0^\infty e^{-r^2}\, r^{2p+2q-1} dr \times 2\int_0^{\pi/2} (\cos\theta)^{2p-1} (\sin\theta)^{2q-1}\, d\theta.$$

With the re-substitution $r = \sqrt{t}$ and $\theta = \frac{\pi}{2} - \varphi$ we finally get

$$\Gamma(p)\Gamma(q) = \int_0^\infty e^{-t}\, t^{p+q-1} dt \times 2\int_0^{\pi/2} (\cos\varphi)^{2p-1} (\sin\varphi)^{2q-1}\, d\varphi.$$

From the last statement it follows that $\Gamma(p)\Gamma(q) = \Gamma(p+q)\beta(p,q)$.

The Beta function possesses the following properties:

1. For $Re(p), Re(q) > 0$, the definition (3.2) is equivalent to

$$\beta(p,q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}} dt \quad (3.3)$$

$$= 2\int_0^{\pi/2} (\sin t)^{2p-1} (\cos t)^{2q-1}\, dt. \quad (3.4)$$

2. $\beta(p+1, q+1)$ is the solution of Beta integral

$$\int_0^1 t^p(1-t)^q\,dt = \beta(p+1, q+1).$$

3. The following identities holds:

(a) $\beta(p,q) = \beta(q,p)$,

(b) $\beta(p,q) = \beta(p+1,q) + \beta(p,q+1)$,

(c) $\beta(p,q+1) = \frac{q}{p}\beta(p+1,q) = \frac{q}{p+q}\beta(p,q)$.

3.1.3 Laplace transform

A common method used in solving differential equations is the Laplace transform. With the Laplace transform it is possible to avoid working with equations of different differential orders by directly translating the problem into a domain where the solution arrives algebraically.

Definition 3.2 [4]

Let $f(t)$ be a function of $t > 0$ then we define the Laplace transform of $f(t)$ by

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty e^{-st} f(t)dt \tag{3.5}$$

Where s can be either real or complex number. The Laplace transform exists if the integral converges for some value of s. Correspondingly $f(t)$ is called the inverse Laplace transform of $F(s)$ which we denote as

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = \int_\Gamma e^{st}F(s)ds. \tag{3.6}$$

Here Γ is the straight line from $c - i\infty$ to $c + i\infty$, where $c = Re(s)$, $c > c_0$, c_0 lies in the right half plane of the absolute convergence of the Laplace integral [4].

For example, let $f(t) = e^t$. Then the absolute convergence of $\mathcal{L}(f)$ is

$$|\mathcal{L}(f)(s)| = \left|\int_0^\infty e^{-st} f(t)dt\right| = \left|\int_0^\infty e^{-st} e^{t} dt\right| = \left|\int_0^\infty e^{-(s-1)t}\, dt\right| < \infty, \qquad \text{if } Re(s) > 1$$

It is rather simple to calculate the Laplace transform of some elementary functions.

Example 3 .1

(a) For $f\ (t) \ = \ exp(at)$ with $a \in \mathbb{R}$ we have $\mathcal{L}(f)(s) = F(s) = \frac{1}{s-a}$,

whenever $s \ > \ a$.

(b) For $f\ (t) \ = \ t^q$ with $q \ > \ -1$ we find $\mathcal{L}(f)(s) = F(s) \ = \Gamma\ (q+1)/s^{q+1}$ whenever $s \ > \ 0$.

(c) For $f\ (t) = sin\omega t$ with ω >0 we have $\mathcal{L}(f)(s) = F(s) = \frac{\omega}{s^2+\omega^2}$, again for $s \ > 0$.

(d) For $f(t) = t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)$, then we have $\mathcal{L}(f)(s) = F(s) = \frac{1}{s^\alpha - 1}$.

The Laplace convolution is also commonly used and is presented in (3.7)

$$(f * g)(t) = \int_0^t f(u)g(t-u)du \tag{3.7}$$

The Laplace transform of equation (3.7) is presented in (3.8)

$$\mathcal{L}\{f * g\}(t) = \{\mathcal{L}(f)(s)\}.\{\mathcal{L}(g)(s)\} = F(s).G(s) \tag{3.8}$$

Proof: we can define $\mathcal{L}\{f * g\}(s)$ from (3.5) as following:

$$\mathcal{L}\{f * g\}(s) = \int_0^\infty e^{-st}(f * g)(s)\ dt$$

$$= \int_0^\infty e^{-st}\left[\int_0^t f(t-\tau)g(\tau)d\tau\right]dt$$

By changing the integrations we get

$$\mathcal{L}\{f * g\}(s) = \int_0^\infty \left[\int_\tau^\infty f(t-\tau)g(\tau)e^{-st}\, dt\right]d\tau = \int_0^\infty g(\tau)\left[\int_\tau^\infty f(t-\tau)e^{-st}dt\right] d\tau$$

By Substituting $t - \tau = y$,

$$\mathcal{L}\{f * g\}(s) = \int_0^{\infty} g(\tau)\left[\int_0^{\infty} f(y)e^{-s\tau-sy}\,dy\right]d\tau$$

$$\mathcal{L}\{f * g\}(s) = \left[\int_0^{\infty} g(\tau)e^{-s\tau}d\tau\right].\left[\int_0^{\infty} f(y)e^{-sy}dy\right] = F(s).G(s).$$

The n[th] order derivative of a function $f(t)$. can be expressed by

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{n-k-1}(0). \quad (3.9)$$

Lemma 3.1 [4]: (Laplace transform of the fractional integral)

$$\mathcal{L}\{D_0^{-\alpha} f(t)\}(s) = s^{-\alpha}F(s), \text{ where } \mathcal{L}\{f(t)\}(s) = F(s).$$

Proof:

$$\mathcal{L}\{D_0^{-\alpha} f(t)\}(s) = \frac{1}{\Gamma(\alpha)}\mathcal{L}\{t^{\alpha-1} * f(t)\} = \frac{1}{\Gamma(\alpha)}\cdot\mathcal{L}\{t^{\alpha-1}\}\cdot\mathcal{L}\{f(t)\},$$

$$\mathcal{L}\{D_0^{-\alpha} f(t)\}(s) = \frac{1}{\Gamma(\alpha)}\cdot\Gamma(\alpha)\cdot s^{-\alpha}\cdot F(s) = s^{-\alpha}F(s).$$

Podlubny has demonstrated in [4] that the Laplace transform of the Riemann-Liouville derivative is defined in the lemma below.

Lemma 3.2 [4]

$$\mathcal{L}\{{}_0^R D_t^{\alpha} f\}(s) = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^k \left[D_0^{(\alpha-k-1)} f(t)\right]_{t=0}, \text{ where } n-1 < \alpha < n$$

Proof: Let $g(t) = (D^{\alpha-n}f)(t)$, then

$$\mathcal{L}\{{}_0^R D_t^{\alpha} f\}(s) = \mathcal{L}\{\frac{d^n g(t)}{dt^n}\}(s) = s^n G(s) - \sum_{k=0}^{n-1} s^k g^{(n-k-1)}(0),$$

By using the definition of the Riemann-Liouville derivative, we have

$$\mathcal{L}\left\{\frac{d^n}{dt^n} g(t)\right\}(s) = \mathcal{L}\left\{\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau\right\}(s)$$

$$= \frac{1}{\Gamma(n-\alpha)}\cdot\mathcal{L}\{t^{n-\alpha-1} * f(t)\}(s)$$

$$\mathcal{L}\left\{\frac{d^n}{dt^n}g(t)\right\}(s) = \frac{1}{\Gamma(n-\alpha)}\cdot \mathcal{L}\{t^{n-\alpha-1}\}(s)\cdot \mathcal{L}\{f(t)\}(s).$$

By referring to (example 3.1 (b)), we have

$$\mathcal{L}\{t^{n-\alpha-1}\}(s) = \Gamma(n-\alpha)s^{-(n-\alpha)}.$$

Then

$$\mathcal{L}\left\{\frac{d^n}{dt^n}g(t)\right\}(s) = s^{-(n-\alpha)}F(s),$$

and

$$g^{(n-k-1)}(t) = \frac{d^{n-k-1}}{dt^{n-k-1}}g(t) = \frac{d^{n-k-1}}{dt^{n-k-1}}\, D_0^{-(n-\alpha)}f(t) = D_0^{\alpha-k-1}f(t),$$

Thus

$$\mathcal{L}\{{}_0^R D_t^\alpha F(s)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k\,[D_0^{(\alpha-k-1)}f(t)]_{t=0}.$$

Additionally the Laplace transform of the Caputo derivative is as follows.

Lemma 3.3 [4]

$$\mathcal{L}\{{}_0^C D_t^\alpha f(t)\}(s) = s^\alpha F(s) - \textstyle\sum_{k=0}^{n-1} s^{\alpha-k-1}\, f^{(k)}(0) \tag{3.10}$$

Proof: Let $g(t) = D^n f(t)$, we have

$$\mathcal{L}\{{}_0^C D_t^\alpha f(t)\}(s) = \mathcal{L}\{D_0^{-(n-\alpha)}g(t)\}(s) = s^{-(n-\alpha)}G(s)\,,$$

$$\mathcal{L}\{{}_0^C D_t^\alpha f(t)\}(s) = s^{-(n-\alpha)}\left[s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1}\, f^{(k)}(0)\right],$$

thus

$$\mathcal{L}\{{}_0^CD_t^\alpha f(t)\}(s) = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0).$$

In particular, if we set $\alpha = \frac{1}{2}$, $n = 1$, we obtain

$$\mathcal{L}\{{}_0^CD_t^{1/2} f(t)\}(s) = s^{1/2} F(s) - s^{1/2} f(0),$$

and if we put $\alpha = \frac{3}{2}$, $n = 1$, we get

$$\mathcal{L}\{{}_0^CD_t^{3/2} f(t)\}(s) = s^{3/2} F(s) - [s^{1/2} f(0) + s^{-1/2} f^{(1)}(0)].$$

3.1.4 The Mittag-Leffler Function

The Mittag-Leffler function is an important function. This function has widespread use in the world of fractional calculus. In the same way as the Laplace transform the exponential naturally arises out of the solution to integer order differential equations. The Mittag-Leffler function plays an equivalent role in the solution of non-integer order differential equations. The standard definition of the Mittag-Leffler is given by [4]:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \qquad \alpha > 0. \tag{3.11}$$

It is also common practice to represent the Mittag-Leffler function in two arguments, α and β so that there is a similar equation as displayed in (3.11).

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad \alpha > 0, \beta > 0. \tag{3.12}$$

There are some relationships to other functions given by:

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z,$$

$$E_{1,2}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+2)}=\sum_{k=0}^{\infty}\frac{z^k}{(k+1)!}=\frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{k+1}}{(k+1)!}=\frac{e^z-1}{z},$$

$$E_{2,1}(z)=\cosh(\sqrt{z}),\qquad E_{2,2}(z)=\frac{\sinh(\sqrt{z})}{\sqrt{z}},$$

and

$$E_{\frac{1}{2},1}(\sqrt{z})=\frac{2}{\sqrt{\pi}}e^{-z}erfc(-\sqrt{z}).$$

Here the error function complement defined by ([4], pp.18).

$$erfc(z)=\frac{2}{\sqrt{\pi}}\int_z^{\infty}e^{-t^2}dt.$$

The following theorem utilizes some of the properties of the Mittag-Leffler function. These properties will be useful further on in the discussion in the analysis of ordinary and partial differential equations of fractional order.

Theorem 3.1 The Mittag-Leffler function possesses the following properties:

1. For $|z|<1$ the generalized Mittag-Leffler function satisfies

$$\int_0^{\infty}e^{-t}\,t^{\beta-1}E_{\alpha,\beta}(t^{\alpha}z)dt=\frac{1}{1-z}.$$

Proof: The Laplace transform of a function $t^k e^{qt}$ is deduced first using a series expansion of $\exp(z)$ and the definition of Euler's Gamma function. In fact, we have

$$\int_0^{\infty}e^{-t}e^{zt}\,dt=\sum_{k=0}^{\infty}\frac{z^k}{k!}\int_0^{\infty}e^{-t}t^k\,dt=\sum_{k=0}^{\infty}z^k=\frac{1}{1-z},$$

for $|z|<1$. Differentiating this statement k times with respect to z yields

$$\int_0^{\infty}e^{-t}t^ke^{zt}\,dt=\frac{k!}{(1-z)^{k+1}},\qquad |z|<1.$$

Substituting $z = 1 + q - p$ we then get the Laplace transform of the function $t^k e^{qt}$:

$$\int_0^\infty e^{-pt}\, t^k e^{qt} dt = \frac{k!}{(q-p)^{k+1}}, \quad Re(p) > |q|$$

If we now consider the Laplace transform of the Mittag-Leffler function, we can argue in the same manner as above:

$$\int_0^\infty e^{-t}\, t^{\beta-1} E_{\alpha.\beta}(zt^\alpha) dt = \int_0^\infty e^{-t} t^{\beta-1} \sum_{k=0}^{\infty} \frac{(zt^\alpha)^k}{\Gamma(\alpha k+\beta)} dt = \frac{1}{1-z}, \qquad (|z|<1).$$

(3.13)

We obtain from (3.13) a pair of the Laplace transforms of the function

$$t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm z t^\alpha), \quad \left(E_{\alpha,\beta}^{(k)}(y) \equiv \frac{d^k}{dy^k} E_{\alpha,\beta}(y)\right):$$

$$\int_0^\infty e^{-pt}\, t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm a t^\alpha) dt = \frac{k!\, p^{\alpha-\beta}}{(p^\alpha \pm a)^{k+1}}, \quad \left(Re(p) > |a|^{1/\alpha}\right).$$

(3.14)

The particular case of (3.14) for $\alpha = \beta = 1/2$,

$$\int_0^\infty e^{-pt}\, t^{\frac{k-1}{2}} E_{\frac{1}{2},\frac{1}{2}}^{(k)}(\pm a\sqrt{t}) dt = \frac{k!}{(\sqrt{p} \pm a)^{k+1}}, \qquad (Re(p) > a^2).$$

2. For $|z| < 1$, the Laplace transform of the Mittag-Leffler function $E_\alpha(z^\alpha)$, is given by,

$$\mathcal{L}\{E_\alpha(z^\alpha)\} = \int_0^\infty e^{-zt}\, E_\alpha(z^\alpha) dt = \frac{1}{z - z^{1-\alpha}}.$$

3.1.5 Fourier transform

Definition 3.4 [4] Let $f(t)$ is a continuous function of a real variable $t \in (-\infty, +\infty)$. Then the Fourier transform of $f(t)$ is defined by

$$\mathcal{F}\{f(t)\}(\omega) = \mathrm{F}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} f(t)dt,$$

Definition 3.5 Let $f(t)$ be a given function in a certain function space. Then the inverse Fourier transform is defined as

$$\mathcal{F}^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t} F(\omega)d\omega.$$

In a related approach to the Laplace transform we can specify the convolution property of two functions $f(t)$ and $g(t)$.

Definition 3.6 Let f, g be two functions. The Fourier convolution of f and g is denoted by $f * g$ and defined as

$$(f * g)(t) := \int_{-\infty}^{+\infty} f(t-u)g(u)du, \qquad t \in \mathbb{R}.$$

Taking the Fourier transform of the above yields

$$\mathcal{F}\{f * g\}(\omega) = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\} = F(\omega)G(\omega).$$

The Fourier transform of the Riemann-Liouville integral with a lower terminal of $-\infty$ is given by Podlubny [4, pp.111] as

$$\mathcal{F}\{{}_{-\infty}^{R}D_t^{-\alpha} f(t)\} = (-i\omega)^{-\alpha} F(\omega).$$

Here $F(\omega)$ is the Fourier transform of $f(t)$

For the Fourier transform of the Caputo fractional derivative with a lower terminal of $-\infty$, Podlubny [4, pp.111] gives it as

$$\mathcal{F}\{{}_{-\infty}^{C}D_t^{\alpha} f(t)\} = (-i\omega)^{\alpha} F(\omega).$$

Here $F(\omega)$ is the Fourier transform of $f(t)$.

This expression also gives the Fourier transform of the Riemann-Liouville and Grünwald-Letnikov derivatives.

3.2 The analytical solution of FODEs

3.2.1 Laplace Transforms Method

The Laplace transform is introduced by Podlubny [4] in his book. Using the expressions for the Riemann-Liouville derivative from (Section 3.1.3 of this chapter) it is possible to demonstrate the solving of fractional ordinary differential equations with constant coefficients.

Examples from [4, pp.140] of ordinary linear fractional differential equations which can be solved using the Laplace transform method are given below.

Example 3.2 Let $\alpha = 1/2$, solve the equation below with the Riemann-Liouville derivative and an initial value.

$$ {}_0^RD_t^{1/2}f(t) + af(t) = 0, \quad t > 0, \qquad {}_0^RD_t^{-\frac{1}{2}}f(t)\Big|_{t=0} = C \tag{3.15}$$

Solution:

Applying the Laplace transform we obtain

$$F(s) = \frac{C}{s^{1/2} + a}, \qquad C = {}_0^RD_t^{-1/2}f(t)\big|_{t=0}.$$

And the inverse transform with a help of (3.13) gives the solution of (3.15):

$$f(t) = Ct^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}\left(-a\sqrt{t}\right) = Ct^{-1/2}\sum_{k=0}^{\infty}\frac{\left(a\sqrt{t}\right)^k}{\Gamma\left(\frac{1}{2}k+\frac{1}{2}\right)},$$

If $a = 1$, we get

$$f(t) = Ct^{-1/2} \sum_{k=0}^{\infty} \frac{(t^{k/2})}{\Gamma\left(\frac{k+1}{2}\right)}. \tag{3.16}$$

Using series expansion (3.11) the solution (3.16) is identical to the solution

$$f(t) = C\left(\frac{1}{\sqrt{\pi t}} - e^t erfc(\sqrt{t})\right).$$

Example 3.3: Consider the following initial value problem for a non-homogeneous fractional differential equation:

$${}_0^R D_t^\alpha f(t) - \lambda f(t) = g(t), \quad where\ t > 0, m-1 < \alpha < m, \tag{3.17}$$

$$[{}_0^R D_t^{\alpha-k} f(t)]_{t=0} = b_k\,, \quad (k = 1,2, \dots, m) \tag{3.18}$$

Solution:

The solution can be provided by applying the Laplace transform to both sides of equation (3.17), and thus we obtain the following equation

$$s^\alpha F(s) - \lambda F(s) = G(s) + \sum_{k=1}^{m} s^{k-1} b_k,$$

from which

$$F(s) = \frac{G(s)}{s^\alpha - \lambda} + \sum_{k=1}^{m} b_k \frac{s^{k-1}}{s^\alpha - \lambda}$$

By applying the inverse Laplace transform of both sides of equation (3.17), we get

$$f(t) = \sum_{k=1}^{m} b_k t^{\alpha-k} E_{\alpha.\alpha-k+1}(\lambda t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\,(\lambda(t-\tau)^\alpha g(\tau) d\tau.$$

Here we use the following fact

$$(H * G)(s) = H(s) * g(s) \tag{3. 19}$$

If $H(s) = \frac{1}{s^{\alpha}-\lambda}$, then $\mathcal{L}^{-1}\{H(s)\} = h(t) = t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})$.

By utilising equation (3.5) on the inverse Laplace transform of (3.19) we arrive at the following equation.

$$\{h * g\}(t) = \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\,(\lambda(t-\tau)^{\alpha}g(\tau)d\tau.$$

Chapter 4

The Numerical Methods of Fractional Differential Equations

Aim

This chapter will discuss the numerical methods for solving fractional ordinary differential equations.

4.1 Diethelm's Backward Difference Method

This whole process was previously described in Diethelm's paper [8], in 1997, and for this reason the total method employed is called the Diethelm fractional backward difference method, see [39],[40].

In this section we consider the fractional ordinary differential equation in the Caputo type

$$ {}_{0}^{C}D_t^{\alpha}y(t) = \lambda y(t) + f(t),\ 0<\alpha<1,\ \lambda\le 0, \qquad 0\le t\le 1, \tag{4.1}$$

with the initial condition

$$y(0) = y_0. \tag{4.2}$$

This equation can be transformed into a fractional differential equation with the Riemann-Liouville derivative (see equations (2.10-2.12)),

$$ {}_{0}^{R}D_t^{\alpha}[y-y_0](t) = \lambda y(t) + f(t), \qquad 0\le t\le 1, \tag{4.3}$$

Initial condition: $y(0) = y_0$. (4.4)

Note that

$$ {}_{0}^{R}D_{t}^{\alpha}(y_0) = \frac{d}{dt}\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}(t-\tau)^{-\alpha}\, y_0 d\tau $$

$$ = \frac{y_0}{\Gamma(1-\alpha)}\frac{d}{dt}\left(\frac{1}{1-\alpha}t^{1-\alpha}\right) = \frac{y_0}{\Gamma(1-\alpha)}t^{-\alpha} $$

Diethlm [8] demonstrated a numerical algorithm which appears to use the Riemann-Liouville fractional derivative above in defining a backward difference formula generalization.

By interchanging differentiation and integration of the Riemann-Liouville fractional derivative (see definition 2.2) we get,

$$ {}_{0}^{R}D_{t}^{\alpha}y(t) = \frac{1}{\Gamma(-\alpha)}\int_{0}^{t}(t-\tau)^{-\alpha-1}\, y(\tau)d\tau, $$

where the integral is interpreted in a Hadamard finite-part [13].

Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of [0, 1].

Applying the approximation to the equispaced grid $t_j = j/n\,, j = 1,2,\dots,n,\ \Delta t = 1/n\,,$ is the time step. We obtain,

$$ {}_{0}^{R}D_{t}^{\alpha}y(t_j) = \frac{1}{\Gamma(-\alpha)}\int_{0}^{t_j}\frac{y(\tau)}{(t_j-\tau)^{\alpha+1}}d\tau, $$

setting $t_j - \tau = t_j\omega$, we get

$$ {}_{0}^{R}D_{t}^{\alpha}y(t_j) = \frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\int_{0}^{1}\frac{y(t_j-t_j\omega)-y(0)}{\omega^{\alpha+1}}d\omega \ = \frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\int_{0}^{1}g(\omega)\omega^{-\alpha-1}d\omega, $$

where

$$ g(\omega) = y(t_j - t_j\omega) - y(0). $$

We replace the integral by a compound quadrature formula [9], with equispaced nodes $0, \frac{1}{j}, \frac{2}{j}, \ldots, 1$ for each j, gives

$$ {}_0^R D_t^\alpha y(t_j) = \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \left[\sum_{k=0}^{j} \alpha_{kj} y(t_j - t_k) + R_j(g) \right], $$

thus the approximation can be represented by a quadrature formula of a product trapezoidal form

$$ Q_j[g] := \textstyle\sum_{k=0}^{j} \alpha_{kj} g(\frac{k}{j}) \approx \int_0^1 g(\omega)\omega^{-\alpha-1}\, d\omega, $$

where

$$ \int_0^1 g(\omega)\, \tau^{-\alpha-1} d\tau = Q_j[g] + R_j(g), $$

and the remainder term $R_j(g)$ satisfies

$$ \|R_j(g)\| \le C j^{\alpha-2} \sup_{0\le t\le T} \|y''(t_j - t_j\omega)\|. $$

Thus,

$$ \begin{aligned} {}_0^R D_t^\alpha y(t_j) &= \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{j} (-\alpha)(1-\alpha) j^{-\alpha}\, \alpha_{kj}\, y(t_j - t_k) + \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} R_j(g) \\ &= \Delta t^{-\alpha} \sum_{k=0}^{j} \frac{(-\alpha)(1-\alpha) j^{-\alpha}\, \alpha_{kj}}{\Gamma(2-\alpha)} y(t_j - t_k) + \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} R_j(g) \\ &= \Delta t^{-\alpha} \sum_{k=0}^{j} \omega_{kj}\, y(t_j - t_k) + \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} R_j(g). \end{aligned} $$

Here

$$ \Gamma(2-\alpha)\omega_{kj} = -\alpha(1-\alpha) j^{-\alpha} \alpha_{kj}\,, \tag{4.5} $$

where the weights ω_{kj} satisfies that [9]

$$\Gamma(2-\alpha)\omega_{kj} = \begin{cases} 1, & \text{for } k=0, \\ -2k^{1-\alpha}+(k-1)^{1-\alpha}+(k+1)^{1-\alpha}, & \text{for } k=1,2,\ldots,j-1, \\ -(\alpha-1)k^{-\alpha}+(k-1)^{1-\alpha}-k^{1-\alpha}, & \text{for k} = \text{j}, \end{cases} \quad (4.6)$$

and α_{kj} satisfies

$$\alpha(1-\alpha)j^{-\alpha}\alpha_{kj} = \begin{cases} -1, & \text{for } k=0, \\ 2k^{1-\alpha}-(k-1)^{1-\alpha}-(k+1)^{1-\alpha}, & \text{for } k=1,2,\ldots,j-1, \\ (\alpha-1)k^{-\alpha}-(k-1)^{1-\alpha}+k^{1-\alpha}, & \text{for k} = \text{j}. \end{cases} \quad (4.7)$$

Now we consider the finite difference method of

$${}_0^R D_t^\alpha[y-y_0](t) = \lambda y(t)+f(t), \quad \text{at} \quad t=t_j,$$

We get

$${}_0^R D_t^\alpha[y(t)-y(0)]|_{t=t_j} = \lambda y(t_j)+f(t_j),$$

$$\Delta t^{-\alpha}\textstyle\sum_{k=0}^{j}\omega_{kj}[\,y(t_j-t_k)-y(0)]+\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\;R_j(g) = \lambda y(t_j)+f(t_j),$$

Or

$${}_0^R D_t^\alpha[y(t)-y(0)]|_{t=t_j} = \Delta t^{-\alpha}\sum_{k=0}^{j}\omega_{kj}[\,y(t_j-t_k)-y(0)]+\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\;R_j(g)\,,$$

Denote $y_j \approx y(t_j)$ as the approximation of $y(t_j)$, we can define

$$\Delta t^{-\alpha}\textstyle\sum_{k=0}^{j}\omega_{kj}\big[\,y_{j-k}-y_0\big] = \lambda y_j+f_j,$$

Let k=0, we get

$$(\omega_{0j} - \Delta t^{\alpha}\lambda)y_j = \Delta t^{\alpha} f_j - \sum_{k=1}^{j} \omega_{kj}\, y_{j-k} + \sum_{k=0}^{j} \omega_{kj}\, y_0.$$

From (4.5) we can find

$$\sum_{k=0}^{j} \omega_{kj} = \frac{-\alpha(1-\alpha)j^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{j} \alpha_{kj}.$$

But

$$\sum_{k=0}^{j} \alpha_{kj} = \int_0^1 u^{-\alpha-1} du = -\frac{1}{\alpha}.$$

Thus

$$\sum_{k=0}^{j} \omega_{kj} = \frac{-\alpha(1-\alpha)j^{-\alpha}}{\Gamma(2-\alpha)}\left(-\frac{1}{\alpha}\right) = \frac{j^{-\alpha}}{\Gamma(1-\alpha)}.$$

The implicit formula below gives Diethelm's numerical algorithm for the equation (4.3) and (4.4):

$$y_j = (\omega_{0j} - \Delta t^{\alpha}\lambda)^{-1}[\Delta t^{\alpha} f_j - \textstyle\sum_{k=1}^{j} \omega_{kj}\, y_{j-k} + \frac{j^{-\alpha}}{\Gamma(1-\alpha)} y_0]. \tag{4.8}$$

Diethelm [8] provides that the error behaves as $O(h^{2-\alpha})$ when using functions that are sufficiently smooth. The method is analysed for $0 < \alpha < 1$. Diethelm provides that the extension to $1 < \alpha < 2$ should not present major difficulty.

4.2 Adams-Bashforth-Moulton method [7]

In this section we will present the algorithm for the fractional differential equation in the Caputo type:

$${}_0^C D_t^\alpha y(t) = f\big(t, y(t)\big),$$

with the initial condition

$$D^k y(0) = b_k \ , \quad k = 0,1,2,\dots,n-1 \ ,$$

The algorithm to solve the fractional differential equation of Caputo type is based on the fractional formulation of the classical Adams-Bashforth-Moulton method. In particular by using the formulation of the problem in Abel-Volterra integral form, i.e

$$y(t) = \sum_{k=0}^{n-1} \frac{D^k y(0) t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f\big(\tau, y(\tau)\big) d\tau \tag{4.9}$$

In order to discuss differential equations of fractional order it is necessary to review the classical differential equations and the methods used to numerically solve these equations. From the classical algorithms it is possible to extend the resulting formulas to the fractional differential equation so it is important to understand this common background for numerical methods. It must be noted that many classical numerical schemes can be extended in more than one way which can led to issues within literature as different equations could be conveyed in a similar manner creating a potential source of confusion. For example, the fractional Adams–Moulton rules of Galeone and Garrappa [16] do not coincide with the methods of the same name as it will be demonstrated later in this section below.

4.2.1 Classical Formulation

Diethelm [7] identifies that the classical Adams–Bashforth–Moulton algorithm for first-order equations should be reviewed to enable a starting point by using the familiar initial-value problem for the first-order differential equation

$$Dy(t) = f(t, Y(t)), \tag{4.10}$$

$$y(0) = y_0. \tag{4.11}$$

It is assumed that the function f will be a unique solution that exists on some interval $[0, T]$. Following Hairer & Wanner [20], Diethelm [7] advises to use the predictor-corrector technique of Adams where it is assumed that for simplicity that mathematician is working on a uniform grid $\{t_i = ih : i = 0,1, \dots, N\}$ with some integer N and $h = T/N$. While in some applications it would be more efficient to utilise a non-uniform grid and this will be identified to the reader and thus a generalised sense of numerical approximation formulas will be utilised. When reviewing the properties of the scheme the author will restrict themselves to the isometric case.

Basically it assumed that the approximations have already been calculated as $y_{i\approx}y(t_i)$, $(i = 1,2, \dots, n)$. While trying to obtain the approximation y_{n+1} by means of the equation

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(z, y(z))dz. \tag{4.12}$$

Following the integration of equation (4.10) on the interval $[t_{n,}t_{n+1}]$ without knowing either of the expressions on the right-hand side of equation (4.12) exactly. Yet there is an approximation for $y(t_n)$, namely y_n that can exploit instead. The integral is then replaced by the two-point trapezoidal quadrature formula

$$\int_a^b g(z)dz \approx \frac{b-a}{2}(g(a) + g(b)), \tag{4.13}$$

Thus giving an equation for the unknown approximation y_{k+1}, it being

$$y_{n+1} = y_n + \frac{t_{n+1}-t_n}{2}\Big(f\big(t_n, y(t_n)\big) + f\big(t_{n+1}, y(t_{n+1})\big)\Big). \tag{4.14}$$

Again $y(t_n)$ and $y(t_{n+1})$ are replaced by their approximations y_n and y_{n+1} respectively and this produces the equation for the implicit one-step Adams–Moulton method, which is

$$y_{n+1} = y_n + \frac{t_{n+1}-t_n}{2}\big(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\big).$$

Diethelm [7] advises that the so-called predictor or preliminary approximation y_{n+1}^p is similarly obtained by only replacing the trapezoidal quadrature formula in the rectangle rule giving the explicit forward Euler method to produce the following formula:

$$y_{n+1}^p = y_n + hf(t_n, y_n). \tag{4.15}$$

And

$$y_{n+1} = y_n + \frac{h}{2}\Big(f(t_n, y_n) + f\big(y_{n+1}, y_{n+1}^p\big)\Big), \tag{4.16}$$

This approach is known as the one-step Adams-Bashforth-Moulton method,

The convergence order of (4.16) is 2, i.e,

$$\max_{i=1,2,\ldots,N}|y(t_i) - y_i| = O(h^2). \tag{4.17}$$

Where $y(t_i)$ is an exact solution and y_i is an approximate solution.

4.2.2 Fractional Formulation

From the classical algorithms it is possible to transfer the essential concepts over to the fractional-order problems of courses with some necessary adaptions. The key to addressing this application to fractional-order problems is to develop an equation which is similar to (4.16) according Diethelm [7] but the equation will be different due to the range of integration which now starts at 0 instead of t_k.

By using the product trapezoidal quadrature formula to replace the integral, for example the nodes $t_i\{ i = 0,1,2, \dots , k + 1\}$ and thus interpret the function $(t_{k+1} - \cdot)$ as a weight function for the integral. By apply the approximation

$$\int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} g(t)dt \approx \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1}\, g_{k+1}^*(t)dt,$$

where g_{k+1}^* is the piecewise linear interpolant for g with nodes and knots chosen at the $t_i, i = 0,1,2,\dots, k + 1$. From this construction it demonstrates that the weighted trapezoidal quadrature formula can be represented as a weighted sum of function values of the integrand g , taken at the points t_i . Explicitly, the integral on the right-hand side of (4.18) can be expressed as

$$\int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1}\, g_{k+1}^*(t)dt = \sum_{i=0}^{k+1} a_{i,k+1}\, g(t_i). \tag{4.18}$$

Where

$$a_{i,k+1} = \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1}\, \phi_{i,k+1}(t)dt, \tag{4.19}$$

and

$$\phi_{i,k+1}(t) = \begin{cases} \frac{t-t_{i-1}}{t_i-t_{i-1}}, & \text{if } t_{i-1} < t \le t_i, \\ \frac{t_{i+1}-t}{t_{i+1}-t_i}, & \text{if } t_i < t < t_{i+1}, \\ 0, & \text{else}. \end{cases} \tag{4.20}$$

This is clear because the functions $\phi_{i,k+1}$ satisfy

$$\phi_{i,k+1}(t_\mu) = \begin{Bmatrix} 1 & \text{if} & i \neq \mu \\ 0 & \text{if} & i = \mu \end{Bmatrix}$$

And that they are continuous and piecewise linear with breakpoints at the nodes t_μ , and thus must integrated exactly by the developed formula.

An easy explicit calculation produces that, for an arbitrary choice of the t_i, (4.19) and (4.20) result in (4.21)

$$a_{0,k+1} = \frac{(t_{k+1} - t_1)^{\alpha+1} + t_{k+1}^{\alpha}[t_1 + t_1 - t_{k+1}]}{t_1\alpha(\alpha+1)},$$

$$a_{i,k+1} = \frac{(t_{k+1} - t_{i-1})^{\alpha+1} + (t_{k+1} - t_i)^{\alpha}[\alpha(t_{i-1} - t_i) + t_{i-1} - t_{k+1}]}{(t_i - t_{i-1})\alpha(\alpha+1)} +$$

$$\frac{(t_{k+1} - t_{i+1})^{\alpha+1} - (t_{k+1} - t_i)^{\alpha}[\alpha(t_i - t_{i+1}) - t_{i+1} + t_{k+1})}{(t_{i+1} - t_i)\alpha(\alpha+1)}, \quad 1 \le i \le k,$$

$$a_{k+1,k+1} = \frac{(t_{k+1} - t_k)^{\alpha}}{\alpha(\alpha+1)}.$$

(4.21)

The isometric nodes ($t_i = ih$ with some fixed h) are reduced to the following equations

This then provides a factional variant of the one –step Adams–Moulton method by providing the correct formula which is

$$a_{i,k+1} = \begin{cases} \frac{h^{\alpha}}{\alpha(\alpha+1)}(k^{\alpha+1} - (k-\alpha)(k+1)^{\alpha}) & \text{if } i = 0 \\ \frac{h^{\alpha}}{\alpha(\alpha+1)}((k-i+2)^{\alpha+1} + (k-i)^{\alpha+1} \\ -2(k-i+1)^{\alpha+1}) & \text{if } 1 \le i \le k \\ \frac{h^{\alpha}}{\alpha(\alpha+1)} & \text{if } i = k+1 \end{cases} \quad (4.22)$$

This then provides a factional variant of the one –step *Adams–Moulton method* by providing the correct formula which is

$$y_{k+1} = \sum_{i=0}^{m-1} \frac{t_{k+1}^{i}}{i!} y_0^{(i)} + \frac{1}{\Gamma(n)}\left(\sum_{i=0}^{k} a_{i,k+1} f(t_i, y_i) + a_{k+1,k+1} f(t_{k+1}, y_{k+1}^{p})\right). \quad (4.23)$$

What remains are the resolution of the predictor formula and thus the required calculation of the value y_{k+1}^{p}. The same concept that was utilized to generalize the Adams–Moulton technique is applied to the one-step Adams–Bashforth method by replacing the integral with the product of rectangle rule

$$\int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} g(t)dt \approx \sum_{i=0}^{k} b_{i,k+1} g(t_i), \tag{4.24}$$

where

$$b_{i,k+1} = \int_{t_i}^{t_{i+1}} (t_{k+1} - t)^{\alpha-1} dt = \frac{(t_{k+1}-t_i)^{\alpha} - (t_{k+1}-t_{i+1})^{\alpha}}{\alpha} . \tag{4.25}$$

Similarly to the method utilized in the equations (4.22)-(4.24) the weight can be consequential calculated. Yet there is a requirement to utilize a piecewise constant approximation and not a piecewise linear one, and hence there is a requirement to

$$b_{i,k+1} = \frac{h^{\alpha}}{\alpha} ((k + 1 - i)^{\alpha} - (k - i)^{\alpha}). \tag{4.26}$$

Thus, the predictor y_{k+1}^{p} is determined by the fractional Adams–Bashforth method

$$y_{k+1}^{p} = \sum_{i=0}^{m-1} \frac{t_{k+1}^{i}}{i!} y_0^{(i)} + \frac{1}{\Gamma(n)} \sum_{i=0}^{k} b_{i,k+1} f(t_i , y_i) , \tag{4.27}$$

The fractional Adams–Bashforth–Moulton method, is therefore completed and described by the formula expressions (4.27) and (4.23) with the weights $a_{i,k+1}$ and $b_{i,k+1}$ as defined according to (4.21) and (4.26), respectively.

This chapter has presented the historical background of analytical and numerical scheme for the solution of the fractional ordinary differential equations.

In the next two chapters we will discuss fractional partial differential equations; we will introduce the finite element method to solve these equations numerically.

Chapter 5

Introduction of Finite Element Method for Solving Parabolic PDEs

Aim

In this chapter we will give a background of using finite element method to solve parabolic partial differential equations.

A linear homogeneous partial differential equation of order two in two variables t, x in general is:

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial t^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = 0\,,$$

which is parabolic equation if $b^2 - ac = 0$. The standard diffusion or heat equation is a parabolic partial differential equation.

5.1 Finite element methods

Partial differential equations are solved by numerous methods. The popular methods amongst them are **finite element methods** and **finite difference methods.** We will discuss finite element methods in this dissertation.

Let us consider finite element method to solve the following heat equation [38],

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), \quad 0 \le x \le 1, \qquad 0 < t \le T, \tag{5.1}$$

$$\text{I.C} \quad u(0,x) = g(x), \qquad 0 \le x \le 1\,, \tag{5.2}$$

$$\text{B.C} \quad u(t,0) = u(t,1) = 0, \quad 0 < t \le T\,. \tag{5.3}$$

Denote that $H_0^1(0,1) = H_0^1 = \{v(x) \mid v(x)\ and\ v'(x)$ are square integrable on (0,1),

i.e, $\{v \in L^2(\Omega), v' \in L^2(\Omega)\ and\ v(0) = v(1) = 0\}$.

The inner product in $L^2(0,1)$ is defined by

$$(f,g) = \int_0^1 f(x)g(x)dx, \quad \forall f,g \in L^2(0,1). \tag{5.4}$$

Step 1: Find the weak solution of (5.1)-(5.3). Suppose that u is the solution of (5.1)-(5.3) then u satisfies

$$\left(\frac{\partial u}{\partial t}, v\right) + \left(-\frac{\partial^2 u}{\partial x^2}, v\right) = (f,v), \qquad \forall v \in H_0^1.$$

By integration by parts, we have

$$\left(-\frac{\partial^2 u}{\partial x^2}, v\right) = \int_0^1 \left(\frac{\partial^2 u}{\partial x^2}\right) v(x)dx = \int_0^1 \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} dx = (u', v')$$

In other words, the solution u of (5.1)-(5.3) satisfies

$$\left(\frac{\partial u}{\partial t}, v\right) + (u', v') = (f,v), \qquad \forall v \in H_0^1 \tag{5.5}$$

$$u(0) = g(x) \tag{5.6}$$

In (5.5)-(5.6), we only require u has first derivative with respect to x. Note that we require u has second derivative with respect to x in (5.1)-(5.3). Therefore (5.5)-(5.6) is the weak form of (5.1)-(5.3). It is easy to find the solution of (5.5)-(5.6) mathematically. The solution of (5.5)-(5.6) is also called the variational solution of (5.1)-(5.3),

i.e. Find $u(t) \in H_0^1$, $u(0) = g(x)$, such that

$$\left(\frac{\partial u}{\partial t}, v\right) + (u', v') = (f,v), \qquad \forall v \in H_0^1. \tag{5.7}$$

Step 2: We will find the finite element approximation of (5.5)-(5.6). Let us introduce a linear finite element space S_h.

Let $x_0 = 0 < x_1 < x_2 < \cdots < x_m = 1$ be a partition of space interval [0, 1].

Denote $\phi_j(x)$ be the basis function at $x = x_j$

$$\phi_j(x_m) = \begin{cases} 1, & \text{if } x_m = x_j, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{Where } j, m = 0,1,2, \dots, M \tag{5.8}$$

Move precisely, we have

$$\phi_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}}, & \text{if } x_{j-1} < x < x_j, \\ \frac{x - x_{j+1}}{x_j - x_{j+1}}, & \text{if } x_j < x < x_{j+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.9}$$

Here $\phi_j(x)$ is a piecewise-linear hat function on the mesh $x_0 < x_1 < x_2 < \cdots < x_m$. It has two desirable properties: (i) $\phi_j(x)$ is unity at node j and vanishes at all other nodes and (ii) ϕ_j is only nonzero on those elements containing node j. The first property simplifies the determination of solutions at nodes while the second simplifies the solution of the algebraic system that finite element discretization.

Denote $\{S_h\}$ is the finite dimensional space, with grid parameter h generated by the basis function ϕ_j, i.e

$S_h = \{\alpha_0\phi_0 + \alpha_1\phi_1 + \cdots + \alpha_m\phi_m \mid \alpha_j \in \mathbb{R}\},$

$= \{$all piecewise linear continuous function on $[0, 1]\}$.

The finite element method of (5.1)-(5.3) is to find $u_h(t) \in S_h$ for fixed $t \in [0, T]$ such that

$$\left(\frac{\partial u_h(t)}{\partial t}, \chi\right) + (u_h'(t), \chi') = (f, \chi), \quad \forall \chi \in S_h. \tag{5.10}$$

Here u_h' denotes the derivative with respect to the space variable x. χ' denotes the derivative with respect to the space variable x.

Step 3: We consider the time discretization. Let $0 = t_0 < t_1 < \cdots < t_n = T$ be a time partition.

We use backward Euler method, i.e.

$$\left.\frac{\partial u_h(t)}{\partial t}\right|_{t=t_n} \approx \frac{u_h(t_n)-u_h(t_{n-1})}{k}. \qquad (5.11)$$

Here k is the time step, $k = \Delta t = t_n - t_{n-1}$.

Denote $U_n(x) \approx u_h(t_n)$ be the approximation of $u_h(t_n)$.

The backward Euler method of (5.10) is to find $U_n(x) \in S_h$, such that

$$\left(\frac{U_n(x)-U_{n-1}(x)}{k}, \chi\right) + (U_n'(x), \chi') = (f(t_n), \chi), \quad \forall \chi \in S_h. \qquad (5.12)$$

We substitute $k = \Delta t$ into equation (5.12) to obtain,

$$(U_n(x), \chi) + \Delta t(U_n'(x), \chi') = \Delta t(f(t_n), \chi) + (U_{n-1}(x), \chi), \quad \forall \chi \in S_h \qquad (5.13)$$

Let $U_n(x) = \sum_{j=1}^{m-1} \alpha_j \phi_j$, here ϕ_0, ϕ_m are homogeneous boundary conditions and choose

$\chi = \phi_i$, $i = 1,2, \dots, m-1$, we obtain:

$$\left(\sum_{j=1}^{m-1} \alpha_j \phi_j, \phi_i\right) + \Delta t \sum_{j=1}^{m-1} \alpha_j (\phi_j', \phi_i') = \Delta t(f(t_{n-1}), \phi_i) + (U_{n-1}(t), \phi_i) \qquad (5.14)$$

Let $n = 1$, we know $U_0(x) = g(x)$, the initial value.

Step 4: Let us find $U_1(x)$, that is

$$(U_1(x), \chi) + \Delta t(U_1'(x), \chi') = \Delta t(f(x_0), \chi) + (U_0(t), \chi). \qquad (5.15)$$

Let $\phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ be a basis functions defined by, for example

$$\phi_1(t) = \begin{cases} \frac{x-x_0}{x_1-x_0} & x_0 < x < x_1, \\ \frac{x-x_2}{x_1-x_2} & x_1 < x < x_2. \\ 0 & otherwise, \end{cases} \qquad (5.16)$$

$$\phi_2(t) = \begin{cases} \frac{x-x_1}{x_2-x_1}, & x_1 < x < x_2, \\ \frac{x-x_3}{x_3-x_2}, & x_2 < x < x_3. \\ 0, & otherwise. \end{cases} \qquad (5.17)$$

Let $U_1(x) = \alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3 + \alpha_4\phi_4$,

here we do not need to consider ϕ_0, ϕ_5, because we consider the homogeneous Dirichlet boundary conditions.

Choose $\chi = \phi_i, i = 1,2,3,4$.

$\sum_{j=1}^{4} \alpha_j(\phi_j, \phi_i) + \Delta t \sum_{j=1}^{4} \alpha_j (\phi_j', \phi_i') = \Delta t(f(t_0), \phi_i) + (U_0(x), \phi_i)$. (5.18)

Denote

$$\text{Mass} = \begin{bmatrix} (\phi_1,\phi_1) & (\phi_1,\phi_2) & (\phi_1,\phi_3) & (\phi_1,\phi_4) \\ (\phi_2,\phi_1) & (\phi_2,\phi_2) & (\phi_2,\phi_3) & (\phi_2,\phi_4) \\ (\phi_3,\phi_1) & (\phi_3,\phi_2) & (\phi_3,\phi_3) & (\phi_3,\phi_4) \\ (\phi_4,\phi_1) & (\phi_4,\phi_2) & (\phi_4,\phi_3) & (\phi_4,\phi_4) \end{bmatrix},$$

$$\text{Stiffness} = \begin{bmatrix} (\phi_1',\phi_1') & (\phi_1',\phi_2') & (\phi_1',\phi_3') & (\phi_1',\phi_4') \\ (\phi_2',\phi_1') & (\phi_2',\phi_2') & (\phi_2',\phi_3') & (\phi_2',\phi_4') \\ (\phi_3',\phi_1') & (\phi_3',\phi_2') & (\phi_3',\phi_3') & (\phi_3',\phi_4') \\ (\phi_4',\phi_1') & (\phi_4',\phi_2') & (\phi_4',\phi_3') & (\phi_4',\phi_4') \end{bmatrix},$$

and

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}, \qquad F = \begin{bmatrix} (f(t_0),\phi_1) \\ (f(t_0),\phi_2) \\ (f(t_0),\phi_3) \\ (f(t_0),\phi_4) \end{bmatrix}, \qquad R = \begin{bmatrix} (U_0(t),\phi_1) \\ (U_0(t),\phi_2) \\ (U_0(t),\phi_3) \\ (U_0(t),\phi_4) \end{bmatrix}.$$

Substituting by Mass, stiffness, α, F and R expression into equation (5.18). Then we get

$$\text{Mass} * \alpha + \Delta t * \text{Stiffness} * \alpha = \Delta t\, F + R\,.$$

Let

$$A\alpha = \text{Mass} * \alpha + \Delta t * \text{Stiffness} * \alpha.$$

Then we have

$$\alpha = (\text{Mass} + \Delta t * \text{Stiffness})^{-1}(\Delta t\, F + R) = A^{-1} * (\Delta t\, F + R).$$

Here

$$\alpha = \begin{bmatrix} U_1(x_1) \\ U_1(x_2) \\ U_1(x_3) \\ U_1(x_4) \end{bmatrix}, \text{ denote the values of } U_1(x) \text{ at grid points } x_1, x_2, x_3, x_4.$$

Step5: We will construct the Mass matrix.

From the definition of variational form (5.4), we have

$$(\phi_1, \phi_1) = \int_0^1 \phi_1(x)\, \phi_1(x) dx = \int_{x_0}^{x_1} \phi_1(x)\, \phi_1(x) dx + \int_{x_1}^{x_2} \phi_1(x)\, \phi_1(x) dx \, .$$

From (5.16) we obtain

$$(\phi_1, \phi_1) \; = \int_{x_0}^{x_1} \left(\frac{x - x_0}{x_1 - x_0}\right)^2 dx + \int_{x_1}^{x_2} \left(\frac{x - x_2}{x_2 - x_1}\right)^2 dx$$

And set $x - x_0 = t, x - x_2 = t$, we get

$$(\phi_1, \phi_1) \; = \int_0^h \frac{t^2}{h^2}\, dt + \int_{-h}^0 \frac{t^2}{h^2}\, dt = \frac{1}{3}\, h + \frac{1}{3}\, h = \frac{2}{3}\, h.$$

$$(\phi_1, \phi_2) = \int_0^1 \phi_1(x)\, \phi_2(x) dx = \int_{x_1}^{x_2} \phi_1(x)\, \phi_2(x) dx = \int_{x_1}^{x_2} (\frac{x - x_2}{x_1 - x_2})(\frac{x - x_1}{x_2 - x_1})\, dx,$$

Where

$$\phi_2(x) = \begin{cases} \frac{x-x_1}{x_2-x_1} & x_1 < x < x_2, \\ \frac{x-x_3}{x_2-x_3} & x_2 < x < x_3. \\ 0 & \text{otherwise} \end{cases}$$

And set $x - x_1 = t$, we get

$$(\phi_1, \phi_2) = \int_0^h \frac{t-h}{-h} \cdot \frac{t}{h}\, dt = \frac{1}{6}h.$$

And $(\phi_1, \phi_3) = 0$, $(\phi_1, \phi_4) = 0$.

Similarly $(\phi_2, \phi_1) = (\phi_1, \phi_2) = \frac{1}{6}h$, $(\phi_2, \phi_2) = \frac{2}{3}h$, $(\phi_2, \phi_3) = \frac{1}{6}h$,

$$(\phi_2, \phi_4) = 0, \quad (\phi_3, \phi_2) = \frac{1}{6}h, \quad (\phi_3, \phi_3) = \frac{2}{3}h, \quad (\phi_3, \phi_4) = \frac{1}{6}h,$$

$$(\phi_4, \phi_1) = 0, \quad (\phi_4, \phi_2) = 0, \quad (\phi_4, \phi_3) = \frac{1}{6}h, \quad (\phi_4, \phi_4) = \frac{2}{3}h.$$

The Mass Matrix as follows:

$$\text{Mass} = h\begin{bmatrix} 2/3 & 1/6 & 0 & 0 \\ 1/6 & 2/3 & 1/6 & 0 \\ 0 & 1/6 & 2/3 & 1/6 \\ 0 & 0 & 1/6 & 2/3 \end{bmatrix}$$

Step 6: Now construct the Stiffness Matrix

$$(\phi_1', \phi_1') = \int_0^1 \phi_1'(x)\, \phi_1'(x) dx = \int_{x_0}^{x_1} \phi_1'(x)\, \phi_1'(x) + \int_{x_1}^{x_2} \phi_1'(x)\, \phi_1'(x) dx,$$

where

$$\phi_1(x) = \begin{cases} \frac{x-x_0}{x_1-x_0} & x_0 < x < x_1, \\ \frac{x-x_2}{x_1-x_2} & x_1 < x < x_2. \\ 0 & otherwise. \end{cases}$$

Then we get

$$(\phi_1', \phi_1') = \int_{x_0}^{x_1} \left(\frac{1}{x_1 - x_0}\right)^2 dx + \int_{x_1}^{x_2} \left(\frac{1}{x_1 - x_2}\right)^2 dx = \frac{1}{h} + \frac{1}{h} = \frac{2}{h}$$

$$(\phi_1', \phi_2') = \int_0^1 \phi_1'(x)(\phi_2'(x)dx = \int_{x_1}^{x_2} \phi_1'(x)\, \phi_2'(x)dx\,,$$

where

$$\phi_2(x) = \begin{cases} \frac{x-x_1}{x_2-x_1}, & x_1 < x < x_2, \\ \frac{x-x_3}{x_2-x_3}, & x_2 < x < x_3, \\ 0, & otherwise. \end{cases}$$

Therefore

$$(\phi_1', \phi_2') = \int_{x_1}^{x_2} \frac{1}{x_1 - x_2} \cdot \frac{1}{x_2 - x_1} dx = -\frac{1}{h}\cdot$$

Note that

$(\phi_1', \phi_3') = 0, (\phi_1', \phi_4') = 0, (\phi_2', \phi_1') = -\frac{1}{h}, (\phi_2', \phi_2') = \frac{2}{h}, (\phi_2', \phi_3') = -\frac{1}{h}, (\phi_2', \phi_4') = 0,$

$(\phi_3', \phi_1') = 0, (\phi_3', \phi_2') = -\frac{1}{h}, (\phi_3', \phi_3') = \frac{2}{h}, (\phi_3', \phi_4') = -\frac{1}{h}, (\phi_4', \phi_1') = 0, (\phi_4', \phi_2') = 0,$

$(\phi_4', \phi_3') = -\frac{1}{h}, (\phi_4', \phi_4') = \frac{2}{h}.$

We can write Stiffness matrix as follows:

$$\text{Stiffness} = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Step 7: Construct the right vector

$$\text{F}=\begin{bmatrix} (f(t_0), \phi_1) \\ (f(t_0), \phi_2) \\ (f(t_0), \phi_3) \\ (f(t_0), \phi_4) \end{bmatrix}$$

Where

$$(f(t_0), \phi_1) = \int_0^1 f(t_0) \cdot \phi_1 dx = \int_{x_0}^{x_1} f(t_0)\, \phi_1 dx + \int_{x_1}^{x_2} f(t_0)\, \phi_1 dx$$

$$(f(t_0), \phi_1) \approx f(t_0) \cdot \frac{1}{2}(x_1 - x_0) + f(t_0) \cdot \frac{1}{2}(x_2 - x_1)$$

$$= \frac{1}{2} h\, f(t_0) + \frac{1}{2} h\, f(t_0) = h\, f(t_0),$$

$$(f(t_0), \phi_2) = \int_0^1 f(t_0) \cdot \phi_2 dx = \int_{x_1}^{x_2} f(t_0)\, \phi_2 dx + \int_{x_2}^{x_3} f(t_0)\, \phi_2 dx$$

$$\approx f(t_0) \cdot \frac{1}{2}(x_2 - x_1) + f(t_0) \cdot \frac{1}{2}(x_3 - x_2)$$

$$= \frac{1}{2} h\, f(t_0) + \frac{1}{2} h\, f(t_0) = h\, f(t_0),$$

$$(f(t_0), \phi_3) = hf(t_0), \quad (f(t_0), \phi_4) = hf(t_0)\,.$$

We can calculate the algorithm of the finite element method using MATLAB software.

Chapter 6

A finite Element Method for Solving Time Fractional Partial Differential Equations

Aim

In this chapter we will discuss fractional partial differential Equations (FPDE) to provide the finite element method which is solving these equations numerically.

6.1 Introduction

Time fractional partial differential equations have been used in various areas such as , diffusion processes material science, turbulent flow, electromagnetics, electrochemistry, etc.[22], [23], [24], [25], [26], [27], [37],. Analytical solutions of time fractional partial differential equations have been focused on using Green's functions or Fourier-Laplace transforms [4],[28],[29],[30]. Numerical methods for fractional partial differential equations were considered by some authors. Liu et al. [31] used the finite difference method in both space and time and analysed the stability condition. Sun and Wu [32] advised a finite difference method for the fractional diffusion-wave equation.

Ervin and Roop [33], [34] employed finite element method to get the variational solution of the fractional advection dispersion equation, where the fractional derivative based on the space, related to the nonlocal operator. Li et al. [35] studied a time fractional partial differential equation by using the finite element method and obtained error estimates in both semi-discrete and fully discrete cases.

Jiang et al. [36] considered a high-order finite element method for the time fractional partial differential equations and proved the optimal order error estimates.

In this Chapter, we will consider finite element method to solve the time fractional partial differential equation

$$ {}_{0}^{C}D_t^{\alpha}u(t,x) - \Delta u(t,x) = f(t,x), \ x \in \Omega, \ 0 < t < T, \tag{6.1} $$

Initial conditions: $u(0,x) = 0, \ x \in \Omega,$ (6.2)

Boundary conditions: $u(t,0) = 0, \quad x \in \Omega, \qquad 0< t < T.$ (6.3)

Where $0 < \alpha < 1$ and $\Omega = [0,1]$, here $\Delta = \frac{\partial^2}{\partial x^2}$ denotes the Laplacian operator with respect to the x variable and $u(t,x)$ depends the space variable $x \in [0,1]$, and time variable $t \in [0,T]$. $u(0,x) = g(t)$ is the initial condition. $u(t,0) = u(t,1) = 0$ is the Dirichlet boundary condition, ${}_{0}^{C}D_t^{\alpha}u(t,x)$ denotes the Caputo fractional derivative with respect to the time variable t defined by

$$ {}_{0}^{C}D_t^{\alpha}u(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u(\tau,x)}{(t-\tau)^{\alpha}} d\tau, \quad 0 < \alpha < 1. \tag{6.4} $$

In Chapter 6 we demonstrated finite element method to solve the equation (5.1)-(5.3) and we recalled the idea of finite element method for solving the parabolic equation.

In this Chapter we use finite element method for solving fractional partial differential equations.

6.2 Finite element method for solving FPDEs

In this section, we will consider how to solve the one dimension time fractional partial differential equation by using finite element method.

Consider the time fractional partial differential equation with the Caputo type:

$$ {}_0^C D_t^{\alpha} u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), \quad 0 \le x \le 1, \quad 0 < t \le T, \tag{6.5}$$

Initial condition: $u(0,x) = u_0, \qquad 0 \le x \le 1, \qquad (6.6)$

Boundary condition: $u(t,0) = u(t,1) = 0, \qquad 0 < t < T,\ 0 < \alpha < 1. \qquad (6.7)$

We know that

$$ {}_0^C D_t^{\alpha} u(x,t) = {}_0^R D_t^{\alpha}[u(x,t) - u_0] $$

Hence the equations (6.5)-(6.7) reduces to

$$ {}_0^R D_t^{\alpha}[u(t,x) - u_0] - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), \quad 0 \le x \le 1, \quad 0 < t < T, \tag{6.8}$$

$$ u(t,0) = u(t,1) = 0, \qquad 0 \le t \le T. \tag{6.9}$$

Here ${}_0^R D_t^{\alpha} u(t,x)$, denotes the Riemann-Liouville fractional derivative with respect to the time variable t defined by

$$ {}_0^R D_t^{\alpha} u(t,x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(\tau,x)}{(t-\tau)^{\alpha}} d\tau, \quad 0 < \alpha < 1, \tag{6.10}$$

where Γ denotes the Gamma function.

The variational form (see chapter 5) is to find $u(t) \in H_0^1(0,1)$ such that

$$ ({}_0^R D_t^{\alpha}[u(t,x) - u_0], v(x)) + (\frac{\partial u_h}{\partial x}, \frac{\partial v}{\partial x}) = (f,v), \quad \forall\, v \in H_0^1 , \tag{6.11}$$

The finite element method is to find a solution $u_h(t) \in S_h$. Such that

$$ {}_0^R D_t^{\alpha}[u(t,x) - u_0], \chi) + \left(\frac{\partial u_h}{\partial x}, \frac{\partial v}{\partial x}\right) = (f,\chi), \quad \forall\, \chi \in S_h \tag{6.12}$$

Let $0 = t_0 < t_1 < \cdots < t_n = T$ be a time discretization

Note that, (see section 4.1 in chapter 4)

$$ {}_0^R D_t^\alpha y(t)|_{t=t_j} = \Delta t^{-\alpha} \sum_{k=0}^{j} \omega_{kj}[\, y(t_j - t_k) - y(0)] + \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \; R_j(g)\,, \tag{6.13}$$

where

$$\Gamma(2-\alpha)\omega_{kj} = \begin{cases} 1, & \text{for } k = 0, \\ -2k^{1-\alpha} + (k-1)^{1-\alpha} + (k+1)^{1-\alpha}, & \text{for } k = 1,2,\dots,j-1, \\ -(\alpha-1)k^{-\alpha} + (k-1)^{1-\alpha} - k^{1-\alpha}, & \text{for k = j}, \end{cases}$$

And the remainder term $R_j(g)$ satisfies

$$\|R_j(g)\| \le Cj^{\alpha-2} \sup_{0\le t\le T} \|y''(t_j - t_j\omega)\|, \qquad 0 < \omega < 1.$$

Denote $U^j \approx u_h(t_j)$ as the approximation of $u_h(t)$ at $t = t_j$.

Then we can define the following time discretization, with $f^j = f(t_j)$,

$$\Delta t^{-\alpha} \sum_{k=0}^{j} \omega_{kj} \left(U^{j-k} - u_0\right),\chi) + \left(\frac{\partial U^j}{\partial x}, \frac{\partial \chi}{\partial x}\right) = (f^j,\chi), \quad j = 0,1,2,\dots \quad \forall\chi \in S_h \tag{6.14}$$

Or

$$\Delta t^{-\alpha}\omega_{0j}(U^j,\chi) + \left(\frac{\partial U^j}{\partial x}, \frac{\partial \chi}{\partial x}\right) = (f^j,\chi) - (\Delta t^{-\alpha} \sum_{k=1}^{j} \omega_{kj} \left(U^{j-k} - u_0\right),\chi) \tag{6.15}$$

$+\Delta t^{-\alpha}\omega_{0j}(u_0,\chi), \quad \forall\chi \in S_h,$ for $j = 0,1,2,\dots,n$

Now find U^j, for $j = 0,1,2,\dots,n$.

Step 1: if we set j=0, then we will get $U^0 = u_0$

Step 2: we put j=1, then we have

$$\Delta t^{-\alpha}\omega_{01}(U^1,\chi) + \left(\frac{\partial U^1}{\partial x}, \frac{\partial \chi}{\partial x}\right) = (f^1,\chi) - (\Delta t^{-\alpha} \sum_{k=1}^{j} \omega_{k1} \left(U^0 - u_0\right),\chi)$$

$+\Delta t^{-\alpha}\omega_{01}(u_0,\chi), \quad \forall\chi \in S_h.$ (6.16)

And we know that $U^1 = \sum_{\ell=1}^{M-1} \alpha_\ell \phi_\ell(x)$, where $\phi_1(x), \phi_2(x), \dots, \phi_{M-1}(x)$, are the basis functions of the finite element space S_h, and then we have

$$\Delta t^{-\alpha}\omega_{01}(\sum_{\ell=1}^{M-1} \alpha_\ell(\phi_\ell(x), \chi) + \sum_{\ell=1}^{M-1} \alpha_\ell \left(\frac{\partial \phi_\ell}{\partial x}, \frac{\partial \chi}{\partial x}\right) = (f^1, \chi) - (\Delta t^{-\alpha}\omega_{11}(U^0 - u_0), \chi)$$

$$+\Delta t^{-\alpha}\omega_{01}(u_0, \chi), \quad \forall \chi \in S_h,$$

Choose $\chi = \phi_m(x)$, for $m = 1,2, \dots, M-1$, and we substitute into equation (6.16)

$$\Delta t^{-\alpha}\omega_{01}\left(\sum_{\ell=1}^{M-1} \alpha_\ell(\phi_\ell(x), \phi_m(x)\right) + \sum_{\ell=1}^{M-1} \alpha_\ell \left(\frac{\partial \phi_\ell(x)}{\partial x}, \frac{\partial \phi_m(x)}{\partial x}\right) = \left(f^1, \phi_m(x)\right) -$$

$$(\Delta t^{-\alpha}\omega_{11}(U^0 - u_0), \phi_m(x)) + \Delta t^{-\alpha}\omega_{01}\left(u_0, \phi_m(x)\right), \tag{6.17}$$

Then we get

$$\Delta t^{-\alpha}\omega_{01}(\text{Mass} * \text{V}^1) + \text{stiff} * \text{V}^1 = \text{F}^1 - \Delta t^{-\alpha}\omega_{11}\text{V}^0 + \Delta t^{-\alpha}\sum_{k=0}^{1} \omega_{k1} u^0 \tag{6.18}$$

Denote

$$\text{Mass}=\begin{bmatrix} (\phi_1, \phi_1) & (\phi_2, \phi_1) & \cdots & (\phi_{M-1}, \phi_1) \\ (\phi_1, \phi_2) & (\phi_2, \phi_2) & \cdots & (\phi_{M-1}, \phi_2) \\ \vdots & \vdots & & \vdots \\ (\phi_1, \phi_{M-1}) & (\phi_2, \phi_{M-1}) & \cdots & (\phi_{M-1}, \phi_{M-1}) \end{bmatrix}$$

$$\text{Stiff}=\begin{bmatrix} \left(\frac{\partial \phi_1}{\partial x}, \frac{\partial \phi_1}{\partial x}\right) & \left(\frac{\partial \phi_2}{\partial x}, \frac{\partial \phi_1}{\partial x}\right) & \cdots & \left(\frac{\partial \phi_{M-1}}{\partial x}, \frac{\partial \phi_1}{\partial x}\right) \\ \left(\frac{\partial \phi_1}{\partial x}, \frac{\partial \phi_2}{\partial x}\right) & \left(\frac{\partial \phi_2}{\partial x}, \frac{\partial \phi_2}{\partial x}\right) & \cdots & \left(\frac{\partial \phi_{M-1}}{\partial x}, \frac{\partial \phi_2}{\partial x}\right) \\ \vdots & \vdots & & \vdots \\ \left(\frac{\partial \phi_1}{\partial x}, \frac{\partial \phi_{M-1}}{\partial x}\right) & \left(\frac{\partial \phi_2}{\partial x}, \frac{\partial \phi_{M-1}}{\partial x}\right) & \cdots & \left(\frac{\partial \phi_{M-1}}{\partial x}, \frac{\partial \phi_{M-1}}{\partial x}\right) \end{bmatrix}$$

$$\text{F}^1 = \begin{pmatrix} (f^1, \phi_1) \\ (f^1, \phi_2) \\ \vdots \\ (f^1, \phi_{M-1}) \end{pmatrix}, \quad \text{V}^1 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{M-1} \end{pmatrix}, \quad \text{V}^0 = \begin{pmatrix} (U^0, \phi_1) \\ (U^0, \phi_2) \\ \vdots \\ (U^0, \phi_{M-1}) \end{pmatrix}, \quad u^0 = \begin{pmatrix} (u_0, \phi_1) \\ (u_0, \phi_2) \\ \vdots \\ (u_0, \phi_{M-1}) \end{pmatrix}$$

Step 3: Let us compute U^2.

To compute U^2 we set $j = 2$ into equation (6.11), then we have

$$\Delta t^{-\alpha}\omega_{02}(U^2,\chi) + \left(\frac{\partial U^2}{\partial x},\frac{\partial \chi}{\partial x}\right) = (f^2,\chi) - (\Delta t^{-\alpha}\textstyle\sum_{k=1}^{2}\omega_{k2}\,(U^{2-k} - u_0),\chi) \tag{6.19}$$

$$+\Delta t^{-\alpha}\textstyle\sum_{k=1}^{2}\omega_{k2}\,(u_0,\chi), \qquad \forall\chi \in S_h.$$

Let $U^2 = \sum_{\ell=1}^{M-1}\alpha_\ell\,\phi_\ell(x)$ as we have done before. Then the equation (6.19) is equivalent to

$$\Delta t^{-\alpha}\omega_{02}(\textstyle\sum_{\ell=1}^{M-1}\alpha_\ell(\phi_\ell(x),\phi_m(x))) + \sum_{\ell=1}^{M-1}\alpha_\ell\left(\frac{\partial\phi_\ell}{\partial x},\frac{\partial\phi_m(x)}{\partial x}\right) =$$

$$(f^2,\phi_m(x)) - ((\Delta t)^{-\alpha}\textstyle\sum_{k=1}^{2}\omega_{k2}\,(U^{2-k} - u_0),\phi_m(x)) + (\Delta t)^{-\alpha}\sum_{k=1}^{2}\omega_{k2}\,(u_0,\phi_m(x)), \tag{6.20}$$

and finally we get

$$(\Delta t)^{-\alpha}\omega_{02}(\text{Mass} * \text{V}^2) + \text{stiff} * \text{V}^2 = \text{F}^2 - (\Delta t)^{-\alpha}\textstyle\sum_{k=1}^{2}\omega_{k2}\,\text{V}^{2-\text{k}} + (\Delta t)^{-\alpha}\sum_{k=0}^{2}\omega_{k2}u^0$$

Denote

$$\text{F}^2 = \begin{pmatrix}(f^2,\phi_1)\\(f^2,\phi_2)\\\vdots\\(f^2,\phi_{M-1})\end{pmatrix},\quad \text{V}^2 = \begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_{M-1}\end{pmatrix},\quad \text{V}^0 = \begin{pmatrix}(U^0,\phi_1)\\(U^0,\phi_2)\\\vdots\\(U^0,\phi_{M-1})\end{pmatrix},\quad u^0 = \begin{pmatrix}(u_0,\phi_1)\\(u_0,\phi_2)\\\vdots\\(u_0,\phi_{M-1})\end{pmatrix},$$

and note that Mass and stiffness expressions are the same in Chapter 5

Step 4: We continue this process to obtain $\boldsymbol{U^n \approx u_h(t_n)}$, the approximation solution of $\boldsymbol{u_h(t_n)}$ at time $t = t_n$ for $n = 0,1,2, \ldots$.

$$\Delta t^{-\alpha}\omega_{0n}(U^n,\chi) + \left(\frac{\partial U^n}{\partial x},\frac{\partial \chi}{\partial x}\right) = (f^n,\chi) - (\Delta t^{-\alpha}\textstyle\sum_{k=1}^{M-1}\omega_{kn}\,(U^{n-k} - u_0),\chi)$$

$$+\Delta t^{-\alpha}\textstyle\sum_{k=1}^{M-1}\omega_{kn}\,(u_0,\chi), \qquad \forall\chi \in S_h. \tag{6.21}$$

To calculate U^n we have to follow the same steps as in step 2 and 3. Based on the idea above, we can design the algorithm of the finite element method and solve the system by using MATLAB software.

6.3 The Error Estimates

In this section, we prove the error estimates in a time discretization and a space discretization scheme.

- **Local errors:**

The local errors define as the difference between the exact solution and the approximate solution of the method, if there is no error in earlier steps.

- **Global errors:**

The errors below together referred to as global errors.

- **Truncation errors:**

They are achieved when an incidence method is terminated and the approximate solution changes from the exact solution.

- **Round-off errors:**

They happen due to lack of the accuracy in rounding of unit quantities by computer.

- **Discretization errors:**

They occur when the solution of the discrete problem does not correspond with the solution of the continuous problem.

6.3.1 Time discretization

We will consider the error estimate of the finite element approximation and the stability result of the following fractional partial differential equation with the Riemann-Liouville type

$$ {}_0^R D_t^{\alpha}[u(t,x) - u_0] - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), \quad 0 \le x \le 1, \quad t > 0, \tag{6.22}$$

$$ u(t,0) = u(t,1) = 0, \qquad 0 \le t \le 1,\ 0 < \alpha < 1. \tag{6.23}$$

Define $A = \frac{\partial^2}{\partial x^2}, D(A) = H_0^1 \cap H^2 = \{u \mid u', u'' \in L_2(0,1), u(0) = u(1) = 0\}$,

where $L_2(0,1) = \left\{f : \int_0^1 f^2 dx < \infty\right\}$, then the system (6.22)-(6.23) can be written in the abstract form

$$\textbf{FODE:} \quad {}_0^R D_t^{\alpha}[u(t) - u_0] + Au(t) = f(t), \ 0 \le x \le 1, \quad t > 0. \tag{6.24}$$

First let us consider the error estimates for the time discretization of the abstract problem (6.24).

Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be the time partition of [0, 1]. Then, for fixed t_j,

$j = 1,2,\dots,n,$ we have (see Section 4.2)

$$ {}_0^R D_t^{\alpha}[u(t) - u_0]|_{t=t_j} = \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \sum_{k=0}^{j} \alpha_{kj}[u(t_j - t_k) - u_0] + R_j(g),$$

where

$$\alpha(1-\alpha) j^{-\alpha} \alpha_{kj} = \begin{cases} -1, & \text{for } k = 0, \\ 2k^{1-\alpha} - (k-1)^{1-\alpha} - (k+1)^{1-\alpha}, & \text{for } k = 1,2,\dots,j-1, \\ (\alpha-1)k^{-\alpha} - (k-1)^{1-\alpha} + k^{1-\alpha}, & \text{for k = j}, \end{cases}$$

and

$$\|R_j(g)\| \le C j^{\alpha-2} \sup_{0\le t\le 1} \|u_{tt}''(t_j - t_j t)\|,$$

where

$$\sup_{0\le t\le 1} \|u_{tt}''(t_j - t_j t)\| = \|u_{tt}''(t_j - t_j t)\|_{L_\infty}.$$

Thus we get

$$\frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \left[\sum_{k=0}^{j} \alpha_{kj}[u(t_j - t_k) - u_0] + R_j(g)\right] + Au(t_j) = f(t_j), \tag{6.25}$$

Rewriting equation (6.25) when $k = 0$ we obtain

$$[\alpha_{0j}+t_j^\alpha\Gamma(-\alpha)A]u(t_j) = t_j^\alpha\Gamma(-\alpha)f_j - \sum_{k=1}^{j}\alpha_{kj}u(t_j - t_k) + \sum_{k=0}^{j}\alpha_{kj}u_0 - R_j(g) \quad (6.26)$$

Denote $U^j \approx u(t_j)$ as the approximation of $u(t_j)$. We can define the following time stepping method

$$[\alpha_{0j}+t_j^\alpha\Gamma(-\alpha)A]U^j = t_j^\alpha\Gamma(-\alpha)f_j - \sum_{k=0}^{j}\alpha_{kj}U^{j-k} + \sum_{k=0}^{j}\alpha_{kj}U^0 \quad (6.27)$$

Let $\varepsilon^j = U^j - u(t_j)$ denotes the error. Then we have the following error estimate:

Theorem 6.1 Let U^j and $u(t_j)$ be the solution of (6.22)-(6.23), then we have

$$\varepsilon^j \le C\Delta t^{2-\alpha} + \|u(t_0) - U^0\|, \qquad \text{where } \varepsilon_0 = \|u(t_0) - U^0\|$$

Proof:

Subtracting (6.27) from (6.25), we get the error equation

$$(\alpha_{0j}+t_j^\alpha\Gamma(-\alpha)A)\varepsilon_j = -\sum_{k=0}^{j}\alpha_{kj}\,\varepsilon^{j-k} - R_j, \quad (6.28)$$

Rewriting (6.28), then we have

$$\varepsilon^j = (-\alpha_{0j} - t_j^\alpha\Gamma(-\alpha)A)^{-1}\left(\sum_{k=0}^{j}\alpha_{kj}\,\varepsilon^{j-k} + R_j\right), \quad (6.29)$$

Where

$$\|R_j\| \le \sup_{0\le t\le 1}\left\|u_{tt}''(t_j - t_j t)\right\|,$$

Taking the L_2 norm for (6.29), we get

$$\|\varepsilon^j\| \le \|(-\alpha_{0j} - t_j^\alpha\Gamma(-\alpha)A)^{-1}\|\left[\sum_{k=0}^{j}\alpha_{kj}\,\|\varepsilon_{j-k}\| + \|R_j\|\right] \quad (6.30)$$

Note that A is a positive definite elliptic operator. The eigenvalues of A are $\lambda_j = j^2\pi^2$, $j = 1,2,3,\ldots$. For any function $g(x)$ we have, by spectal method,

$$\|g(A)\| = \sup_{\lambda>0}|g(\lambda)|$$

From (6.30) and (4.7), we have

$$\|(-\alpha_{0j} - t_j^\alpha\Gamma(-\alpha)A)^{-1}\| = \left\|\left(\frac{1}{\alpha(1-\alpha)j^{-\alpha}} - t_j^\alpha\Gamma(-\alpha)A\right)^{-1}\right\|$$

$$= \left\|\alpha(1-\alpha)j^{-\alpha}\left(1-\alpha(1-\alpha)j^{-\alpha}t_j^{\alpha}\Gamma(-\alpha)A\right)^{-1}\right\|$$

$$\|(-\alpha_{0j} - t_j^{\alpha}\Gamma(-\alpha)A)^{-1}\| = \alpha(1-\alpha)j^{-\alpha}\sup_{\lambda>0}\left(1-\alpha(1-\alpha)j^{-\alpha}t_j^{\alpha}\Gamma(-\alpha)\lambda\right)^{-1}$$

Since $\Gamma(-\alpha) > 0$, we find that

$$\sup_{\lambda>0}\left(1-\alpha(1-\alpha)j^{-\alpha}t_j^{\alpha}\Gamma(-\alpha)\lambda\right)^{-1} \leq 1.$$

Hence

$$\|(-\alpha_{0j} - t_j^{\alpha}\Gamma(-\alpha)A)^{-1}\| \leq \alpha(1-\alpha)j^{-\alpha}.$$

Thus (6.30) implies that

$$\|\varepsilon^j\| \leq \alpha(1-\alpha)j^{-\alpha}\left[\sum_{k=0}^{j}\alpha_{kj}\,\|\varepsilon_{j-k}\| + Cj^{\alpha}n^{-2}\sup_{0\leq t\leq 1}\|u''_{tt}(t_j-t_jt)\|\right], \qquad (6.31)$$

Where we use the fact, noting that $t_n = n \cdot \Delta t = 1$,

$$u''_{tt}(t_j - t_jt) = u''(t_j - t_jt)\cdot t_j^2 = j^2\,\Delta t^2 u''(t_j - t_jt)$$

$$= j^2\,n^{-2}u''(t_j - t_jt),$$

Further (6.31) can be written into the form

$$\|\varepsilon^j\| \leq \alpha(1-\alpha)j^{-\alpha}Cn^{-2}\|u''\|_{L_\infty} + \alpha(1-\alpha)j^{-\alpha}\sum_{k=0}^{j}\alpha_{kj}\,\|\varepsilon_{j-k}\|.$$

Denote $a = \alpha(1-\alpha)j^{-\alpha}Cn^{-2}\|u''\|_{L_\infty}$.

Choose: $j = 1$.

Then we have

$$\|\varepsilon^1\| \leq a + \alpha(1-\alpha)1^{-\alpha}\alpha_{11}\|\varepsilon_0\|$$

$$= a \cdot d_1 + r_1\|\varepsilon_0\|,$$

Here $d_1 = 1, r_1 = \alpha(1-\alpha)1^{-\alpha}\alpha_{11}$.

Choose: $j = 2$, we get

$$\|\varepsilon^2\| \le a + \alpha(1-\alpha)2^{-\alpha}\left[\sum_{k=1}^{2-1} \alpha_{k2}\,\|\varepsilon_{2-k}\| + \alpha_{22}\|\varepsilon_0\|\right]$$

$$\|\varepsilon^2\| \;\le\; a + \alpha(1-\alpha)2^{-\alpha}\left[\sum_{k=1}^{2-1} \alpha_{k2}\,(ad_{2-k} + r_{2-k}\|\varepsilon_0\|) + \alpha_{22}\|\varepsilon_0\|\right]$$

$$= a\left[1 + \alpha(1-\alpha)2^{-\alpha}\sum_{k=1}^{2-1} \alpha_{k2}\, d_{2-k}\right] + \alpha(1-\alpha)2^{-\alpha}\left[\sum_{k=1}^{2-1} \alpha_{k2}\, r_{2-k}\right]\|\varepsilon_0\|$$

$$= ad_2 + r_2\|\varepsilon_0\|.$$

Here

$$d_2 = 1 + \alpha(1-\alpha)2^{-\alpha}\sum_{k=1}^{2-1} \alpha_{k2}\, d_{2-k},$$

$$r_2 = \alpha(1-\alpha)2^{-\alpha}\sum_{k=1}^{2-1} \alpha_{k2}\, r_{2-k}, r_0 = 1,$$

In general, we obtain

$$\left\|\varepsilon^j\right\| \le ad_j + r_j\|\varepsilon_0\|, \qquad j = 1,2,3,\dots, \tag{6.32}$$

Next we will find d_j and r_j, where

$$d_j = 1 + \alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j-1} \alpha_{kj}\, d_{j-k}, \quad j = 2,3,\dots,$$

$$r_j = \alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j} \alpha_{kj}\, r_{j-k}, \; j = 1,2,3,\dots, \;\; r_0 = 1.$$

Lemma 6.1 [9] for $0 < \alpha < 1$, let the sequence $\{d_j\}$, $j = 1,2,\dots$ be given by $d_1 = 1$ and

$$d_j = 1 + \alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j-1} \alpha_{kj}\, d_{j-k}, \quad j = 2,3,\dots,$$

then,

$$1 \le d_j \le \frac{\sin \pi\alpha}{\pi\alpha(1-\alpha)} j^{\alpha}, \quad j = 1,2, \ldots.$$

Lemma 6.2 Assume that if $r_0 = 1,\ r_j = \alpha(1-\alpha)j^{-\alpha} \sum_{k=1}^{j} \alpha_{kj}\, r_{j-k},\quad j = 1,2,3, \ldots,$

then

$$r_j \le 1.$$

Proof:

Step 1: If we have $r_0 = 1$. Then

$$r_1 = \alpha(1-\alpha)1^{-\alpha}\alpha_{11}r_0 = \alpha(1-\alpha)\alpha_{11} = (\alpha-1)1^{-\alpha} + 1^{1-\alpha} = \alpha < 1.$$

Step 2: Assume that $r_j < \alpha < 1$, thet

$$r_{j+1} = \alpha(1-\alpha)j^{-\alpha} \sum_{k=1}^{j} \alpha_{kj}\, r_{j-k} \le \alpha(1-\alpha)j^{-\alpha} \sum_{k=1}^{j} \alpha_{kj} \cdot 1$$

$$= \alpha(1-\alpha)j^{-\alpha} \left[\sum_{k=0}^{j} \alpha_{kj} - \alpha_{0j} \right] = \alpha(1-\alpha)j^{-\alpha} \left[-\frac{1}{\alpha} + \frac{1}{\alpha(1-\alpha)j^{-\alpha}} \right]$$

$$= \alpha(1-\alpha)j^{-\alpha} \left[-\frac{1}{\alpha} + \frac{1}{\alpha(1-\alpha)j^{-\alpha}} \right] \quad < \alpha(1-\alpha)j^{-\alpha} \frac{1-(1-\alpha)j^{-\alpha}}{\alpha(1-\alpha)j^{-\alpha}}.$$

Hence, we get

$$r_{j+1} < \alpha(1-\alpha)j^{-\alpha} \frac{1}{\alpha(1-\alpha)j^{-\alpha}} = 1.$$

The proof of the Lemma 6.2 is complete.

By using Lemma 6.1 and Lemma 6.2, we obtain from (6.32) the follows

$$\|\varepsilon^j\| \le\ ad_j + r_j\|\varepsilon_0\| \le \alpha(1-\alpha)Cn^{-2}\|u''\|_{L_\infty} \cdot d_j + r_j\|\varepsilon_0\|$$

$$\le \alpha(1-\alpha)Cn^{-2}\|u''\|_{L_\infty} \frac{\sin \pi\alpha}{\pi\alpha(1-\alpha)} j^{\alpha} + \|\varepsilon_0\| \le 1 \le C\Delta t^{2-\alpha} + \|\varepsilon_0\|.$$

The proof of the Theorem 6.1 is complete.

Second: we will consider a stability result of the time discretization of the FPDEs (6.22) and (6.23).

Theorem 6.2 Let U^j be the approximate solution of (6.27), then we have

$$\|U^j\| \leq 2\|U^0\| + \frac{\sin \pi\alpha}{\pi}|\Gamma(-\alpha)|t_j^\alpha\|f\|_{L_\infty}$$

Before proving this Theorem we have the following steps:

Step 1: Substituting by the expression $\sum_{k=0}^{j}\alpha_{kj} = -\frac{1}{\alpha}$, into (6.27), we get

$$(\alpha_{0j}+t_j^\alpha\Gamma(-\alpha)A)U^j = t_j^\alpha\Gamma(-\alpha)f_j - \sum_{k=1}^{j}\alpha_{kj}U^{j-k} - \frac{1}{\alpha}U^0, \quad \text{for } j = 1,2,3,\ldots.$$

Or

$$(-\alpha_{0j} - t_j^\alpha\Gamma(-\alpha)A)U^j = \sum_{k=1}^{j}\alpha_{kj}U^{j-k} + \frac{1}{\alpha}U^0 - t_j^\alpha\Gamma(-\alpha)f_j, \tag{6.33}$$

Multiplying on both sides of (6.33) by $\alpha(1-\alpha)j^{-\alpha}$, and use the fact,

$$\alpha(1-\alpha)j^{-\alpha}(-\alpha_{0j}) = 1,$$

then we obtain the follows

$$[1 + \alpha(1-\alpha)j^{-\alpha}(-t_j^\alpha)\Gamma(-\alpha)A]U^j = \alpha(1-\alpha)j^{-\alpha}\left(\sum_{k=1}^{j}\alpha_{kj}U^{j-k} + \frac{1}{\alpha}U^0 - t_j^\alpha\Gamma(-\alpha)f_j\right).$$

Step 2: Assume that $u_j = \alpha(1-\alpha)j^{-\alpha}(-t_j^\alpha)\Gamma(-\alpha)$, then we get

$$U^j = (1 + u_jA)^{-1}\alpha(1-\alpha)j^{-\alpha}\left(\sum_{k=1}^{j}\alpha_{kj}U^{j-k} + \frac{1}{\alpha}U^0 - t_j^\alpha\Gamma(-\alpha)f_j\right).$$

We denote that the norm $\|(1 + u_jA)^{-1}\| = \sup_{\lambda>0}|(1 + u_j\lambda)^{-1}| < 1$, then

$$\|U^j\| \leq \left\|(1 + u_jA)^{-1}\right\| \alpha(1-\alpha)j^{-\alpha}\left(\sum_{k=1}^{j}\alpha_{kj}\|U^{j-k}\| + \frac{1}{\alpha}\|U^0\| + |t_j^\alpha\Gamma(-\alpha)|\|f\|_{L_\infty}\right)$$

$$\leq \alpha(1-\alpha)j^{-\alpha}\left(\sum_{k=1}^{j}\alpha_{kj}\left\|U^{j-k}\right\|\right)+(1-\alpha)j^{-\alpha}\|U^0\|+\alpha(1-\alpha)\Delta t^{\alpha}|\Gamma(-\alpha)|\|f\|_{L_\infty}$$

$$\leq \alpha(1-\alpha)j^{-\alpha}\textstyle\sum_{k=1}^{j}\alpha_{kj}\left\|U^{j-k}\right\|+(1-\alpha)j^{-\alpha}\|U^0\|+a, \tag{6.34}$$

Here

$a=\alpha(1-\alpha)\Delta t^{\alpha}|\Gamma(-\alpha)|\|f\|_{L_\infty}$.

Denote that when $j=1$,

$$\|U^1\|\leq a+\alpha(1-\alpha)1^{-\alpha}\alpha_{11}\|U^0\|+(1-\alpha)1^{-\alpha}\|U^0\|.$$

Suppose that $d_1=1,\ b_1=(1-\alpha)1^{-\alpha},\ r_1=\alpha(1-\alpha)1^{-\alpha}$,

then we have

$$\|U^1\|\leq ad_1+b_1\|U^0\|+r_1\|U^0\|.$$

In general, we can write that

$$\left\|U^j\right\|\leq ad_j+b_j\|U^0\|+r_j\|U^0\|,\ j=1,2,3,.... \tag{6.35}$$

Here

$$\begin{cases} d_1=1, \\ d_j=1+\alpha(1-\alpha)j^{-\alpha}\displaystyle\sum_{k=1}^{j-1}\alpha_{kj}d_{j-k},\ j=2,3,4,\dots, \end{cases}$$

$$\begin{cases} b_1=(1-\alpha)1^{-\alpha}, \\ b_j=(1-\alpha)j^{-\alpha}+\alpha(1-\alpha)j^{-\alpha}\displaystyle\sum_{k=1}^{j-1}\alpha_{kj}b_{j-k},\ j=2,3,4,\dots, \end{cases}$$

and

$$\begin{cases} r_0 = 1, \\ r_j = \alpha(1-\alpha)j^{-\alpha} \sum_{k=1}^{j} \alpha_{kj} r_{j-k}, \quad j = 1,2,3, \dots . \end{cases}$$

Step 3: Suppose that, for some fixed numbers $j = 1,2,3, \dots,$

$\|U^j\| \leq ad_j + b_j\|U^0\| + r_j\|U^0\|$.

Then by (6.34), we have

$$\|U^{j+1}\| \leq \alpha(1-\alpha)j^{-\alpha} \sum_{k=1}^{j} \alpha_{kj}\|U^{j-k}\| + (1-\alpha)j^{-\alpha}\|U^0\| + a$$

$$\leq \alpha(1-\alpha)j^{-\alpha} \sum_{k=1}^{j} \alpha_{kj}\left[ad_{j-k} + b_{j-k}\|U^0\| + r_{j-k}\|U^0\|\right]$$

$$+(1-\alpha)j^{-\alpha}\|U^0\| + a$$

$$= \left[\alpha(1-\alpha)j^{-\alpha} \sum_{k=1}^{j} \alpha_{kj} d_{j-k}\right]$$

$$+ \left[(1-\alpha)j^{-\alpha} + \alpha(1-\alpha)j^{-\alpha} \sum_{k=1}^{j} \alpha_{kj} b_{j-k}\right] \|U^0\|$$

$$+ \left[\alpha(1-\alpha)j^{-\alpha} \sum_{k=1}^{j} \alpha_{kj} r_{j-k}\right] \|U^0\|$$

$$= ad_{j+1} + b_{j+1}\|U^0\| + r_{j+1}\|U^0\|.$$

Which shows that (6.35) holds.

Lemma 6.3: Assume that, for $0 < \alpha < 1$,

Choose: $j = 1$, then $b_1 = (1-\alpha)j^{-\alpha}$,

$$b_j = (1-\alpha)j^{-\alpha} + \alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j}\alpha_{kj}b_{j-k}, \quad \text{for } j = 2,3,4,\ldots,$$

Then we have

$$b_j \leq 1.$$

Proof: we know that

$$b_1 = (1-\alpha)j^{-\alpha} < 1.$$

By mathematics induction principle, suppose that

$b_j < 1$, then we have

$$b_{j+1} = (1-\alpha)(j+1)^{-\alpha} + \alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j-1}\alpha_{kj}\, b_{j-k}$$

$$\leq (1-\alpha)(j+1)^{-\alpha} + \alpha(1-\alpha)j^{-\alpha}\left(\sum_{k=0}^{j-1}\alpha_{kj} - \alpha_{0j}\right)$$

$$\leq (1-\alpha)(j+1)^{-\alpha} + \alpha(1-\alpha)j^{-\alpha}\left(-\frac{1}{\alpha} + \frac{1}{\alpha(1-\alpha)j^{-\alpha}}\right)$$

$$\leq (1-\alpha)(j+1)^{-\alpha} + \alpha(1-\alpha)j^{-\alpha}\frac{1-(1-\alpha)j^{-\alpha}}{\alpha(1-\alpha)j^{-\alpha}}$$

$$\leq (1-\alpha)(j+1)^{-\alpha} + 1 - (1-\alpha)j^{-\alpha} < 1.$$

The proof of the Lemma 6.3 is complete.

Proof of the Theorem 6.2: by the expression (6.35), we have

$$\|U^j\| \leq ad_j + b_j\|U^0\| + r_j\|U^0\|, \quad j = 1,2,3,\ldots,$$

here d_j, b_j and r_j are given before.

Using Lemma 6.1-6.3, we obtain

$$\|U^j\| \le ad_j + b_j\|U^0\| + r_j\|U^0\| \le a\frac{\sin \pi\alpha}{\pi\alpha(1-\alpha)}j^{-\alpha} + \|U^0\| + \|U^0\|$$

$$= \alpha(1-\alpha)\Delta t^{\alpha}|\Gamma(-\alpha)|\|f\|_{L_\infty}\frac{\sin \pi\alpha}{\pi\alpha(1-\alpha)}j^{-\alpha} + 2\|U^0\|$$

$$\le 2\|U^0\| + \frac{\sin \pi\alpha}{\pi}|\Gamma(-\alpha)|t_j^{\alpha}\|f\|_{L_\infty}.$$

The proof of the Theorem 6.2 is complete.

6.3.2 Space discretization

Let us consider the finite element approximation (the space discretization) of the equations (6.22)-(6.23).

Let S_h denote the piecewise linear continuous finite element space. More precisely, let $0 = x_0 < x_1 < \cdots < x_m = 1$ be space partition of [0, 1]. Denote

$$S_h = \{v_h(x) \mid v_h(x) \text{ is piecewise linear continuous function on } [0,1]\}$$

The variational form of (7.1) is to find the solution $u(t) \in H_0^1(0,1)$ such that

$$({}_0^RD_t^{\alpha}[u(t,x) - u_0], v(x)) + (\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}) = (f(t), v(x)), \quad \forall\, v \in H_0^1. \tag{6.36}$$

The finite element method is to find $u_h(t) \in S_h$, such that

$$({}_0^RD_t^{\alpha}[u_h(t,x) - u_0], \chi) + (\frac{\partial u_h}{\partial x}, \frac{\partial \chi}{\partial x}) = (f(t), \chi), \quad \forall \chi \in S_h. \tag{6.37}$$

Denote $A_h = -\Delta_h$: $S_h \to S_h$, which satisfies

$$(A_h u_h, \chi) = ((\frac{\partial u_h}{\partial x}, \frac{\chi}{\partial x}), \quad \forall \chi \in S_h.$$

Let P_h: $H \to S_h$ be the L_2 projection operator is defined by

$$(P_h v, \chi) = (v, \chi), \quad \forall \chi \in S_h, \quad v \in L_2.$$

We can write (6.33) into the abstract form

$$({}_{0}^{R}D_{t}^{\alpha}[u_h(x,t) - u_0] + A_h u_h = P_h f, \quad t > 0. \tag{6.38}$$

Let $0 = t_0 < t_1 < \cdots < t_j < \cdots < t_n = 1$ be the time partition, Δt be the time step.

Denote $U^j \approx u(t_j)$ as the approximation of $u(t_j)$. We can define the following time stepping method as (6.23),

$$\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\left[\sum_{k=0}^{j} \alpha_{kj}\left(U^{j-k} - U^0\right)\right] + A_h U^j = f_j, \quad j = 1,2,3,\dots. \tag{6.39}$$

We have the following theorem.

Theorem 6.3: Let $u(t_j)$ and U^j be the solutions of (6.24) and (6.39). Then we have

$$\left\|U^j - u(t_j)\right\| \le 2\|U^0 - R_h u_0\| + O(\Delta t^{2-\alpha} + h^2),$$

here

$h = \Delta x$: Space step size

Let $R_h\colon H_0^1 \to S_h$ be the Ritz projection or the elliptic projection defined by

$(\nabla R_h v, \nabla \chi) = (\nabla v, \nabla \chi), \quad \forall \chi \in S_h$.

We write

$$\varepsilon^j = U^j - u(t_j) = U^j - R_h u(t_j) + R_h u(t_j) - u(t_j)$$

$$= \theta^j + \rho^j, \quad j = 1,2,3,\dots,$$

where

$$\theta^j = U^j - R_h u(t_j),$$

$$\rho^j = R_h u(t_j) - u(t_j).$$

Step 1: Estimate θ^j, we have the error equation obtained from (6.39),

$$\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\sum_{k=0}^{j}\alpha_{kj}\left(\theta^{j-k}-\theta^0\right)+A_h\theta^j=\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\sum_{k=0}^{j}\alpha_{kj}\left(U^{j-k}-U^0\right)+A_hU^j$$

$$-\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\sum_{k=0}^{j}\alpha_{kj}\,R_h\big(u(t_{j-k})-u_0\big)+A_hR_h\mathrm{u}(t_j)$$

$$=P_hf_j-R_h\left[\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\sum_{k=0}^{j}\alpha_{kj}\left(u(t_{j-k})-u_0\right)\right]+P_hA_hu(t_j)$$

$$=P_h\big({}_0^RD_t^\alpha[u(t_j)-u_0]\big)-P_h\left[\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\sum_{k=0}^{j}\alpha_{kj}\left(u(t_{j-k})-u_0\right)\right]$$

$$=-P_h\omega^j,$$

Here,

$$\omega^j=-{}_0^RD_t^\alpha[u(t_j)-u_0]+R_h\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\sum_{k=0}^{j}\alpha_{kj}\left(u(t_{j-k})-u_0\right)$$

$$\omega^j=(R_h-I)\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\sum_{k=0}^{j}\alpha_{kj}\left(u(t_{j-k})-u_0\right)$$

$$+\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\sum_{k=0}^{j}\alpha_{kj}\left(u(t_{j-k})-u_0\right)\;-\;{}_0^RD_t^\alpha[u(t_j)-u_0]$$

$$=\sigma^j+\tau^j,$$

where

$$\sigma^j=(R_h-I)\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\sum_{k=0}^{j}\alpha_{kj}\left(u(t_{j-k})-u_0\right),\tag{6.40}$$

and

$$\tau^j = \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \sum_{k=0}^{j} \alpha_{kj}\left(u(t_{j-k}) - u_0\right) - {}_0^R D_t^\alpha\left[u(t_j) - u_0\right]. \tag{6.41}$$

Thus we get

$$\frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \sum_{k=0}^{j} \alpha_{kj}\left(\theta^{j-k} - \theta^0\right) + A_h \theta^j = P_h\left(\sigma^j + \tau^j\right).$$

Using the stability results (Theorem 6.2), we obtain

$$\left\|\theta^j\right\| \le 2\|\theta^0\| + \frac{\sin \pi\alpha}{\pi} |\Gamma(-\alpha)| t_j^{-\alpha} \left\|\rho_h\left(\sigma^j + \tau^j\right)\right\|.$$

Here

$$\left\|\tau^j\right\| \le \left\| \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \sum_{k=0}^{j} \alpha_{kj}\left(u(t_{j-k}) - u_0\right) - {}_0^R D_t^\alpha\left[u(t_j) - u_0\right] \right\|$$

$$\le \left\| \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} R_j \right\| \le \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} C j^{\alpha-2} \left\| \frac{\partial^2 (u(t_j - t_j t) - u_0}{\partial t^2} \right\|$$

$$\le C t_j^{-\alpha} j^{\alpha-2} t_j^2 \left\| u''^{(t_j - t_j t)} \right\| = C \Delta t^{2-\alpha} \left\| u''^{(t_j - t_j t)} \right\|,$$

and

$$t_j^\alpha \left\|\sigma^j\right\| = t_j^\alpha \left\| (R_h - I) \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \sum_{k=0}^{j} \alpha_{kj} \left(u(t_{j-k}) - u_0\right) \right\|,$$

$$t_j^\alpha \left\|\sigma^j\right\| \le C h^2 \left\| \sum_{k=0}^{j} \alpha_{kj} \left(u(t_{j-k}) - u_0\right) \right\|_{H^2}$$

$$\le C h^2 \left\| \sum_{k=0}^{j} \alpha_{kj} u(t_{j-k}) \right\|_{H^2} + \left\| \sum_{k=0}^{j} \alpha_{kj} u_0 \right\|_{H^2},$$

Where $\|.\|_{H^2}$ denote the sobolev norm.

Note that, with $g(t) = u(t_j - t_j t)$,

$$\sum_{k=0}^{j} \alpha_{kj} u(t_{j-k}) = \sum_{k=0}^{j} \alpha_{kj} g(\frac{k}{j}) = \int_0^1 g(t) t^{-1-\alpha} dt + R_j,$$

where

$\int_0^1 \grave{g}(t) t^{-1-\alpha} dt,$ denotes the finite Hadamard integral [13] , and

$$|R_j| \leq j^{\alpha-2} \|g''_{tt}\|_{H^2} = \left\|j^{\alpha-2} t_j^2 u''_{tt}(t_j - t_j t)\right\|_{H^2}$$

$$\leq \Delta t^2 j^{\alpha} \|u''\|_{H^2} \leq \Delta t^{2-\alpha} t_j^{\alpha} \|u''_{tt}\|_{H^2}$$

Further, we have

$$\int_0^1 g(t) t^{-1-\alpha}\, dt = \int_0^1 u(t_j - t_j\, t) t^{-1-\alpha} dt,$$

setting $t_j - t_j t = \tau$, we obtain

$$\int_0^1 g(t) t^{-1-\alpha}\, dt = \int_0^{t_j} u(\tau) \left(\frac{t_j - \tau}{t_j}\right)^{-1-\alpha} \frac{1}{t_j} d\tau$$

$$= t_j^{\alpha} \int_0^{t_j} (t_j - \tau)^{-1-\alpha}\, u(\tau) d\tau$$

$$= t_j^{\alpha}\ {}_0^R D_t^{\alpha} u(t_j) \Gamma(-\alpha).$$

Thus

$$\left\|\sum_{k=0}^{j} \alpha_{kj} u(t_{j-k})\right\|_{H^2} \leq t_j^{\alpha} |\Gamma(-\alpha)| \left\|\ {}_0^R D_t^{\alpha} u(t_j)\right\|_{H^2} + \Delta t^{2-\alpha} t_j^{\alpha} \|u''_{tt}\|_{H^2},$$

which implies that

$$t_j^{\alpha} \|\sigma^j\| \leq C h^2 t_j^{\alpha} \left[|\Gamma(-\alpha)| \left\|\ {}_0^R D_t^{\alpha} u(t_j)\right\|_{H^2} + \Delta t^{2-\alpha} \|u''_{tt}\|_{H^2}\right].$$

Hence, we obtain

$$\|\theta^j\| \le 2\|\theta^0\| + \frac{\sin\pi\alpha}{\pi}|\Gamma(-\alpha)|t_j^\alpha\|\rho_h(\sigma^j+\tau^j)\| \tag{6.42}$$

$$\le 2\|\theta^0\| + Ct_j^\alpha\Delta t^{2-\alpha}\|u''_{tt}\|_{H^2}$$

$$+Ch^2t_j^\alpha\left(\left\| {}_0^RD_t^\alpha u(t_j)\right\|_{H^2} + \Delta t^{2-\alpha}\|u''_{tt}\|_{H^2}\right)$$

$$\le 2\|\theta^0\| + O(\Delta t^{2-\alpha}+h^2).$$

Thus

$$\|\varepsilon^j\| \le \|\theta^j\| + \|\rho^j\| \le 2\|\theta^0\| + O(\Delta t^{2-\alpha}+h^2) + \|\rho^j\|.$$

By the error estimates of the Ritz projection, we have [37], [38]

$$\|\rho^j\| = \|R_hu(t_j) - u(t_j)\| = Ch^2\|u(t_j)\|_{H^2} \tag{6.43}$$

Together with these estimates we get,

$$\|\varepsilon^j\| \le 2\|\theta^0\| + O(\Delta t^{2-\alpha}+h^2).$$

The proof of the Theorem 6.3 is complete.

6.4 Numerical simulation

In this section, we will consider two numerical examples.

Example 6.1. Consider the time fractional partial differential equation, with $0 < \alpha < 1$,

$$\frac{\partial^\alpha}{\partial t^\alpha}u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x) = f(t,x) \tag{6.44}$$

$$\textbf{I.c: } u(0,x) = u_0, \qquad 0 \le x \le 1, \tag{6.45}$$

$$\textbf{B.c: } u(t,0) = u(t,1) = 0, \qquad t > 0, \quad 0 < \alpha < 1 \tag{6.46}$$

The exact solution is

$$u(t,x) = \sin(\pi t)\sin(\pi x).$$

The write hand side of the function

$$f(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \pi(t-s)^{-\alpha} \cos(\pi s) \sin(\pi x)\, ds - \pi^2 \sin(\pi t) \sin(\pi x)$$

We choose $\alpha = 0.2, \quad \Delta x = h = 0.01, \quad T = 1, \ \Delta t = k = 1/32, \ N = T/\Delta t$.

Let U^n denote the approximate solution and $u(t_n)$ denote the exact solution at $t = t_n$.

Let $\varepsilon^n = U^n - u(t_n)$ denote their error at $t = t_n$. We plot the exact solution, approximate solution at $t_N = 1$, in Figure 1. We plot the error at $t_N = 1$ in Figure 2.

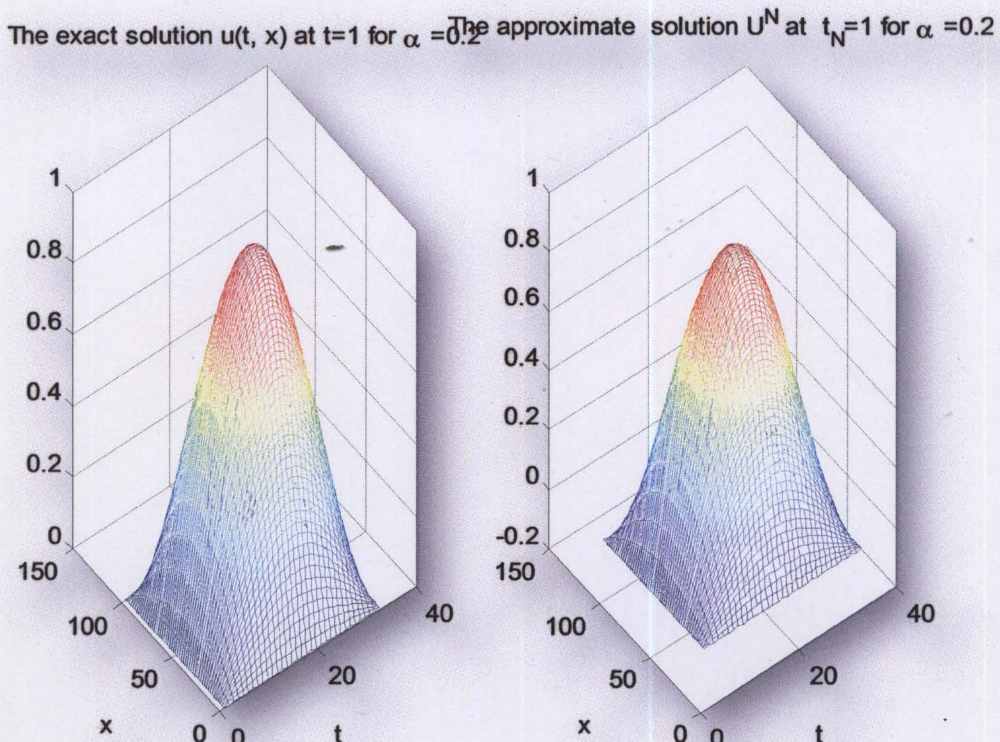


Figure 1. The approximate and exact solutions at $t_N = 1$

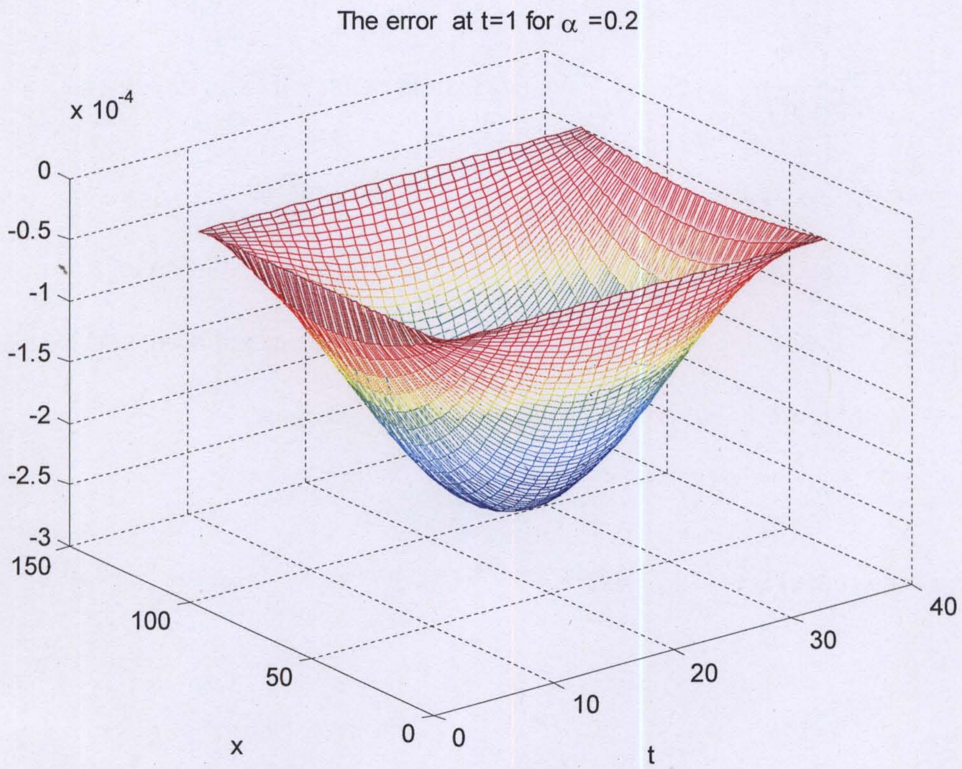


Figure 2: The error at $t_N = 1$

Example 6.2. Consider, with $0 < \alpha < 1$

$$\frac{\partial^\alpha}{\partial t^\alpha} u(t,x) - \frac{\partial^2}{\partial x^2} u(t,x) = f(t,x), \qquad t > 0, \quad 0 < x < 1 \tag{6.47}$$

I. c: $u(0,x) = u_0,$ (6.48)

B. c: $u(t,0) = u(t,1) = 0,$ (6.49)

The exact solution is

$$u(t,x) = \sin(\pi t)\sin(\pi x).$$

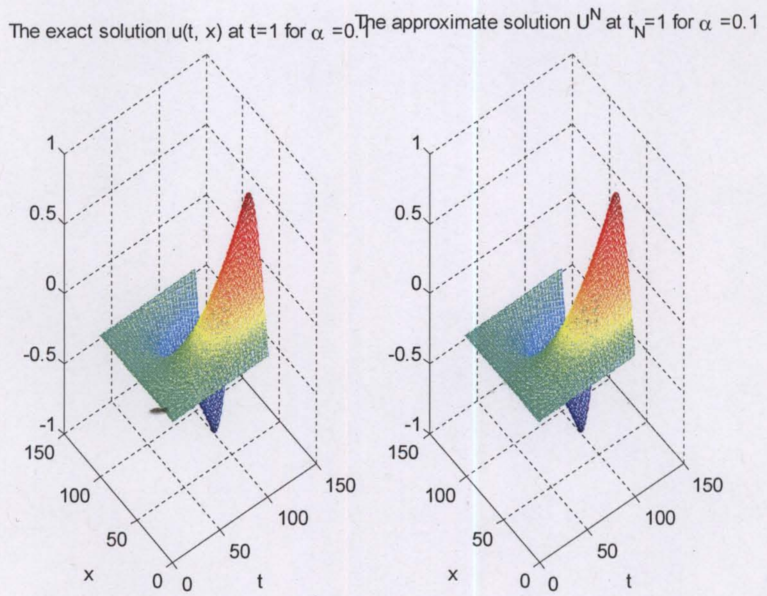
The right hand side function

$$f(t,x) = 2t^{2-\alpha}\sin(2\pi x)/\Gamma(3-\alpha) - 4\pi^2\sin(2\pi x)t^2.$$

We choose $\alpha = 0.2,\ \ \Delta x = 0.01,\ \ T = 1,\ \ \Delta t = 0.01,\ \ N = T/\Delta t.$

Let U^n denote the approximate solution and $u(t_n)$ denote the exact solution at $t = t_n$.

Let $\varepsilon^n = U^n - u(t_n)$ denote the error at $t = t_n$. We plot the exact solution and the approximate solution at $t_N = 1$ in Figure 3, and we plot the error at $t_N = 1$ in Figure 4.



Figur 3. The approximate and exact solutions at $t_N = 1$

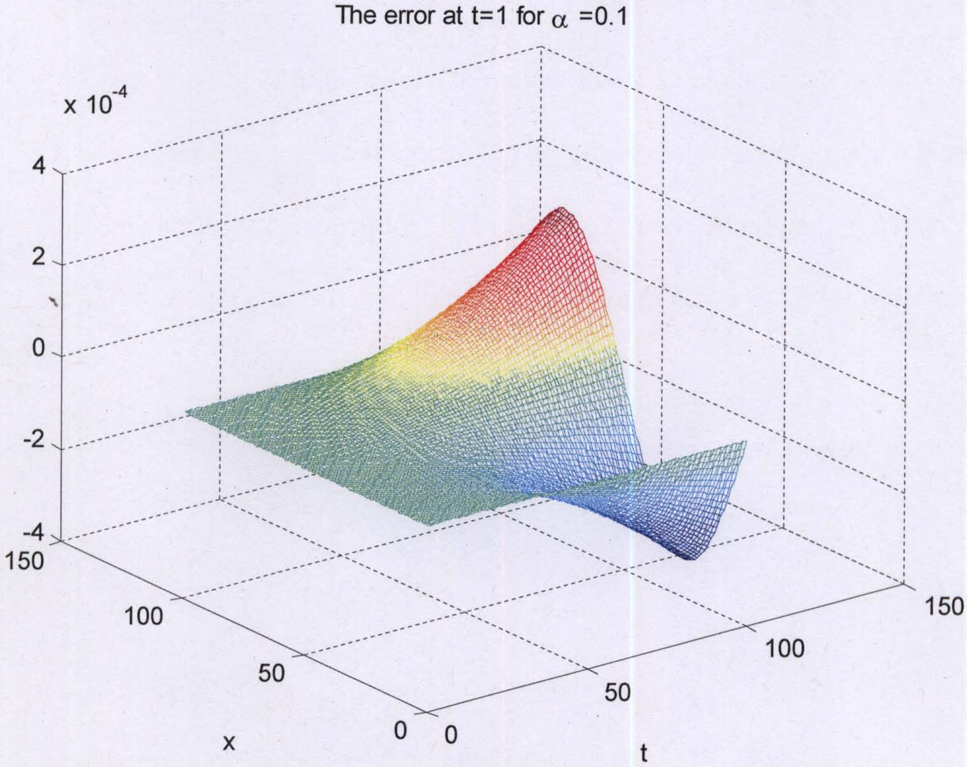


Figure 4. The error at $t_N = 1$

Chapter 7

Conclusion and Future Research

7.1 Conclusion

In this dissertation we discuss the finite element method for the time fractional partial differential equations. We first introduce the finite element method for solving parabolic partial differential equation. Then we extend the method to the time fractional partial differential equation.

We obtain the error estimates in the L_2-norm between the exact solution and the approximate solution in fully discrete case. The numerical examples show that the numerical results are consistent with the theoretical results.

7.2 Future Research

The main objective of this chapter is to highlight areas where further research might be pursued in order to contribute to the understanding and advancement of finite element method for solving partial differential equations in fractional order. We only demonstrate the finite element method for solving 1D linear fractional partial differential equations in this dissertation.

In the future research we will extend the present approach to the 2D fractional partial differential equations. We will also study finite element methods for the non-linear fractional partial differential equations, and consider the error estimates. It is also very interesting to consider the finite element method for fractional partial differential equations where both time and space derivative are of fractional order.

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Appendix

MATLAB program for example 6.1

```
function [ dA,db,localmass ] = elementcontributions( t,x,nodes,e1,w0,al )
%ELEMENTCONTRIBUTION Summary of this function goes here
%   Detailed explanation goes here
n1=nodes(e1,1);
n2=nodes(e1,2);
x1=x(n1);
x2=x(n2);
length=x2-x1;
f=[right(t,x1,al);right(t,x2,al)];
%f=0;
localmass=[1/3*length 1/6*length; 1/6*length 1/3*length];
localstiffness=w0*[1/length -1/length; -1/length 1/length];
dA=localmass+localstiffness;
db=w0*localmass*f;
%db=0;
end
(w).....
function y=w(k,j,q)
if k==0;
    y=1/gamma(2-q);
else if j==1 && k==j
        y=-q/gamma(2-q);
    else if  k==1 && j>=2;
          y=(2^(1-q)-2)/gamma(2-q);
    else if k>=2 && k<=j-1;
           y=((k-1)^(1-q)+(k+1)^(1-q)-2*k^(1-q))/gamma(2-q);
        else  k==j && j>=2;
            y=((k-1)^(1-q)-(q-1)*k^(-q)-k^(1-q))/gamma(2-q);
        end
    end
  end
end
end


function [y ] = right( t,x,al )

ee=1e-5;
y=1/gamma(1-al)*quad(@(ta)funexp1(ta,t,x,al),0,t-ee)-fun(t,x);

end



function y=funexp1(ta,t,x,al)
y=pi*(t-ta).^(-al).*cos(pi*ta).*sin(pi*x);
end
```

```
function y=fun(t,x)
y=-pi^2.*sin(pi*t).*sin(pi*x);
end


%To solve the time fractional partial differential equation
%
%
%D^{alpha}_{t} u (x, t) - D^{2}_{x}  u (t, x) = f(t,x), 0 < alpha <1,
%u(0,x)=0
%u(t,0)=u(t,1)=0;
%
% The exact solution is
%
% u(t,x)=sin(pi t)*sin(pi x);
%
% Here f(x, t) is caculated by using quadrature formula "quad"



%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear
al=0.2;   % fractional order
h=1/100;  % space stepsize
x=[0:h:1];
n=size(x,2);
nodes1=1:n-1;
nodes2=2:n;

for i=1:n-1;
    nodes(i,1)=nodes1(i);
    nodes(i,2)=nodes2(i);
end

T=1;
k=1/32;   % time stepsize
NT=T/k;

U00=ones(size(x'))*0;   %initial value


t=[1:1:NT]'*k;
exact=sin(pi*t)*sin(pi*x);  % exact solution
exact=exact';
exact=[U00 exact];


UU=U00;


tic

for j=1:NT
```

```
    w0=1/w(0,j,al)*k^(al);

    R0=w0*(t(j))^(-al)/gamma(1-al)*U00;


    A=zeros(n,n);
    b=zeros(n,1);
    mass=zeros(n,n);
    U=zeros(n,1);
    ssum=zeros(n,1);
    F=zeros(n,1);


    for e1=1:n-1
    [dA,db,localmass]=elementcontributions(t(j),x,nodes,e1,w0,al);
    nn=nodes(e1,:);
    A(nn,nn)=A(nn,nn)+dA;
    b(nn)=b(nn)+db;
    mass(nn,nn)=mass(nn,nn)+localmass;
    end


    for g=2:j+1
        ssum=ssum+w(g-1,j,al)*UU(:,j-g+2);
        %w denotes the coefficients of the fractional time derivative
approximation
    end
    R=w0/k^(al)*ssum;

    b=b-mass*R+mass*R0;


    innodes=2:n-1;
    A1=A(innodes,innodes);
    b1=b(innodes);
    U1=A1\b1;
    U(innodes)=U1;

    UU=[UU U];
end

error=UU-exact;


figure(1)
subplot(1,2,1)
mesh(exact)
xlabel('t');ylabel('x');
title('The exact solution u(t, x) at t=1 for \alpha =0.2')
subplot(1,2,2)
mesh(UU)
xlabel('t');ylabel('x');
title('The approximate  solution U^{N} at  t_{N}=1 for \alpha =0.2')
```

```
figure(2)
mesh(error)
xlabel('t');ylabel('x');
title('The error  at t=1 for \alpha =0.2')

toc
```

MATLAB program for example 6.2

```
%To solve the time fractional partial differential equation
%
%
%D^{alpha}_{t} u (x, t) - D^{2}_{x}  u (t, x) = f(t,x), 0 < alpha <1,
%u(0,x)=0
%u(t,0)=u(t,1)=0;
%
% The exact solution is
%
% u(t,x)= t^2 *sin(2*pi x);
%
% Here
% f(x, t) = 2 * t^(2- alpha) * sin (2 *pi*x)/Gamma(3-alpha)
%             - 4 * pi^2 *sin(2 * pi*x) *t^2


clear
al=0.1;      % fractional order
h=1/100;      % space stepsize
x=[0:h:1];
n=size(x,2);
nodes1=1:n-1;
nodes2=2:n;

for i=1:n-1;
    nodes(i,1)=nodes1(i);
    nodes(i,2)=nodes2(i);
end

T=1;
k=0.01;    % time stepsize
NT=T/k;

U00=ones(size(x'))*0; %initial value
t=[1:1:NT]'*k;
exact=(t.^2)*sin(2*pi*x);  % exact solution
exact=exact';
exact=[U00 exact];

%U0=U00;
UU=U00;
```

```
tic

for j=1:NT

    w0=1/w(0,j,al)*k^(al);

    R0=w0*(t(j))^(-al)/gamma(1-al)*U00;



    A=zeros(n,n);
    b=zeros(n,1);
    mass=zeros(n,n);
    U=zeros(n,1);
    ssum=zeros(n,1);
    F=zeros(n,1);


    for e1=1:n-1
    [dA,db,localmass]=elementcontributions(t(j),x,nodes,e1,w0,al);
    nn=nodes(e1,:);
    A(nn,nn)=A(nn,nn)+dA;
    b(nn)=b(nn)+db;
    mass(nn,nn)=mass(nn,nn)+localmass;
    end


    for g=2:j+1
        ssum=ssum+w(g-1,j,al)*UU(:,j-g+2);
   %w denotes the coefficients of the fractional time derivative
approximation
    end
    R=w0/k^(al)*ssum;


    b=b-mass*R+mass*R0;


    innodes=2:n-1;
    A1=A(innodes,innodes);
    b1=b(innodes);
    U1=A1\b1;
    U(innodes)=U1;

    UU=[UU U];
end

figure(1)
subplot(1,2,1)
mesh(exact)
xlabel('t');ylabel('x');
title('The exact solution u(t, x) at t=1 for \alpha =0.1')
subplot(1,2,2)
```

```
mesh(UU)
xlabel('t');ylabel('x');
title('The approximate solution U^{N} at t_{N}=1 for \alpha =0.1')


error=UU-exact;


figure(2)
mesh(error)
xlabel('t');ylabel('x');
title('The error at t=1 for \alpha =0.1')

toc

function [ dA,db,localmass ] = elementcontributions( t,x,nodes,e1,w0,al )
%ELEMENTCONTRIBUTION Summary of this function goes here
%   Detailed explanation goes here
n1=nodes(e1,1);
n2=nodes(e1,2);
x1=x(n1);
x2=x(n2);
length=x2-x1;
f=[right(t,x1,al);right(t,x2,al)];
%f=0;
localmass=[1/3*length 1/6*length; 1/6*length 1/3*length];
localstiffness=w0*[1/length -1/length; -1/length 1/length];
dA=localmass+localstiffness;
db=w0*localmass*f;
%db=0;
end

function y=w(k,j,q)
if k==0;
    y=1/gamma(2-q);
else if j==1 && k==j
        y=-q/gamma(2-q);
    else if  k==1 && j>=2;
           y=(2^(1-q)-2)/gamma(2-q);
    else if k>=2 && k<=j-1;
            y=((k-1)^(1-q)+(k+1)^(1-q)-2*k^(1-q))/gamma(2-q);
        else  k==j && j>=2;
             y=((k-1)^(1-q)-(q-1)*k^(-q)-k^(1-q))/gamma(2-q);
        end
    end
  end
end
end
```