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Error Analysis for Semilinear Stochastic Subdiffusion with Integrated Fractional Gaussian Noise

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Abstract: We analyze the error estimates of a fully discrete scheme for solving a semilinear stochastic subdiffusion problem driven by integrated fractional Gaussian noise with a Hurst parameter $H \in (0,1)$. The covariance operator Q of the stochastic fractional Wiener process satisfies $\|A^{-\rho}Q^{1/2}\|_{HS} < \infty$ for some $\rho \in [0,1)$, where $\|\cdot\|_{HS}$ denotes the Hilbert–Schmidt norm. The Caputo fractional derivative and Riemann–Liouville fractional integral are approximated using Lubich's convolution quadrature formulas, while the noise is discretized via the Euler method. For the spatial derivative, we use the spectral Galerkin method. The approximate solution of the fully discrete scheme is represented as a convolution between a piecewise constant function and the inverse Laplace transform of a resolvent-related function. By using this convolution-based representation and applying the Burkholder–Davis–Gundy inequality for fractional Gaussian noise, we derive the optimal convergence rates for the proposed fully discrete scheme. Numerical experiments confirm that the computed results are consistent with the theoretical findings.

Keywords: stochastic semilinear subdiffusion; fractional Gaussian noise; Caputo fractional derivative; spectral Galerkin method

MSC: 65M12; 65M06; 65M70; 35S10

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1. Introduction

Consider the following semilinear stochastic subdiffusion problem driven by integrated fractional Gaussian noise, with $0 < \alpha < 1$, $0 \le \gamma \le 1$,

$$ {}_0^C D_t^{\alpha} u(t) + Au(t) = F(u(t)) + {}_0^R D_t^{-\gamma} \frac{dW^H(t)}{dt}, \quad \text{for } 0 < t \le T, \quad \text{with } u(0) = 0, \quad (1) $$

where $A : \mathcal{D}(A) \to \mathbb{H} = L_2(\mathcal{D})$ with $\mathcal{D}(A) = H_0^1(\mathcal{D}) \cap H^2(\mathcal{D})$ is a linear elliptic operator and $\mathcal{D} \subset \mathbb{R}^d, d = 1,2,3$ is some regular domain with smooth boundary. Here, F is a smooth real-valued function specified in Section 2 and $\frac{dW^H(t)}{dt}$ a zero-mean, $\mathbb{H}$-valued Gaussian noise defined in a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{t\ge 0})$. Here ${}_0^C D_t^{\alpha} u(t)$ and ${}_0^R D_t^{-\gamma} u(t)$ denote the Caputo fractional derivative and Riemann–Liouville integral, respectively [1–3].

Let $\{\lambda_j, e_j\}_{j=1}^{\infty}$ be the eigenvalues and eigenfunctions of the elliptic operator A. We assume that $W^H(t)$ takes the following Fourier series form

$$ W^H(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} e_j \beta_j^H(t), $$

where $\beta_j^H(t)$ is one dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$ [2]. Let $Q : \mathbb{H} \to \mathbb{H}$ be the covariance operator of the Gaussian process $W^H(t)$ such that $Qe_j = \gamma_j e_j$ where Q is a self-adjoint, non-negative, linear bounded operator on $\mathbb{H}$.

We assume that A satisfies the following resolvent estimate, with $\theta \in (\pi/2, \pi)$, see Lubich et al. [4], Thomée [5],

$$\|(zI + A)^{-1}\| \le C|z|^{-1} \quad \text{for } z \in \Sigma_\theta = \{z \neq 0 : \ |\arg z| < \theta\}, \tag{2}$$

which implies that, with $0 < \alpha < 1$, see Yan et al. [6],

$$\|(z^\alpha I + A)^{-1}\| \le C|z|^{-\alpha} \quad \text{for } z \in \Sigma_\theta = \{z \neq 0 : \ |\arg z| < \theta\}. \tag{3}$$

Many application problems can be effectively described using (1) including thermal diffusion in media with fractional geometry [7], in highly heterogeneous aquifers [8], underground environmental challenges [9], and the study of random walks [10], among others. To provide further clarity, we can consider between phenomena with short-term memory characterized by H in the range $(0, \frac{1}{2})$ and those with long-term memory characterized by H in the range $(\frac{1}{2}, 1)$. From the viewpoint of mathematics, the Hurst index reflects the Hölder property of fractional Brownian motion's trajectory.

Let us consider the following physical problem which can be modeled by (1), [11,12]. Let the functions $u(t,x)$, $\tilde{E}(t,x)$, and $F(t,x)$ represent the body temperature, energy, and flux density, respectively. Then, for constants η and a with $\eta, a > 0$, we have

$$\begin{cases} \frac{\partial \tilde{E}}{\partial t}(t,x) = -\operatorname{div} F(t,x), \\ \tilde{E}(t,x) = \eta u(t,x), \\ F(t,x) = -a\nabla u(t,x). \end{cases}$$

The above equations lead to the classical heat equation:

$$\eta \frac{\partial u}{\partial t}(t,x) = a\Delta u.$$

However, in reality, the propagation speed is generally finite due to interruptions in heat flow caused by the material's response. If the material has thermal memory, we often use the following model to characterize it:

$$\tilde{E}(t,x) = \bar{\beta} u(t,x) + \int_0^t n(t-s)u(s,x)\, ds,$$

with the constant $\bar{\beta}$ and kernel $n(t)$. To account for external random effects, we express the energy term as

$$\tilde{E}(t,x) = \int_0^t k_1(t-s)u(s,x)\, ds + \int_0^t k_2(t-s)(b(u(s,x)) + \dot{W}^H(s,x))\, ds, \tag{4}$$

where $k_1(t) = \Gamma(1-\alpha)^{-1}t^{-\alpha}$, $k_2(t) = -\Gamma(\gamma+1)^{-1}t^{\gamma}$, and $\dot{W}^H(t,x) := \frac{dW^H(t)}{dt}$ is a centered Gaussian noise that is white in time and fractional in space. Differentiating (4), we obtain

$$\begin{aligned} -\operatorname{div} F &= \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} u(s,x)\, ds - \frac{1}{\Gamma(\gamma+1)} \frac{\partial}{\partial t} \int_0^t (t-s)^{\gamma} (b(u(s,x)) + \dot{W}^H(s,x))\, ds \\ &= {}_0^C D_t^\alpha u(t,x) - \frac{1}{\Gamma(\gamma)} \frac{\partial}{\partial t} \int_0^t \int_0^\tau (\tau - s)^\gamma (b(u(s,x)) + \dot{W}^H(s,x))\, ds\, d\tau \\ &= {}_0^C D_t^\alpha u(t,x) - {}_0^R D_t^{-\gamma} (b(u(t,x)) + \dot{W}^H(t,x)), \end{aligned}$$

which provides a physical explanation for the fractional derivative and fractional integral with respect to t in (1).

The investigation into the existence, uniqueness, and regularity of time-fractional Stochastic Partial Differential Equations (SPDEs) has been a subject of extensive research. Chen et al. [11] successfully demonstrated both the existence and uniqueness of a stochastic time-fractional PDE in both its divergence and non-divergence forms. Anh et al. [13] explored the weak-sense solution of a time-fractional SPDE that exhibits fractional spatial and temporal characteristics. Mijena and Nane [14] investigated the existence and uniqueness of a continuous random field solution to a space-time-fractional SPDE. In a subsequent study [15], they analyzed the weak intermittency in the solution and the propagation of intermittency fronts. Liu et al. [16] studied the existence and uniqueness of solutions to time-fractional SPDEs with a more general quasi-linear elliptic operator. Chen [17] conducted a comprehensive analysis of moments, Hölder continuity, and intermittency in the solution for one-dimensional nonlinear stochastic time-fractional diffusion, see also [18,19]. Shukla et al. [20] considered the approximate controllability of Hilfer fractional stochastic evolution inclusions of order $1 < q < 2$.

Recent advances have also been made in the field of numerical analysis for time-fractional SPDEs. Jin et al. [21] proposed numerical methods for stochastic linear time-fractional partial differential equations driven by integrated noise. Gunzburger, Li, and Wang [1,22] studied time discretization and finite element methods for approximating stochastic linear integrated-differential equations driven by space-time white noises. Wu et al. [23] considered the L1 scheme for approximating the linear stochastic subdiffusion problems driven by integrated space-time white noises. Cao et al. [24] considered the spatial semidiscretization of solving stochastic linear evolution equations driven by fractional noise, with the Hurst index $H \in (0,1)$. Deng et al. [25] investigated the existence, uniqueness, and spatial semidiscrete schemes for semilinear stochastic wave equations driven by fractional noise, with the Hurst index $H \in (\frac{1}{2}, 1)$. For numerical methods related to stochastic parabolic partial differential equations, refer to Wang et al. [26], Dai et al. [27], Liu [28], Yan [29], Kruse [30], Jentzen and Kloeden [31], Chen et al. [32], and the references therein. For recent numerical methods in deterministic time-fractional differential equations, see [33,34] and the references therein.

Nie and Deng [2] recently proposed a unified framework for the numerical analysis of stochastic semilinear fractional diffusion equations with $\alpha \in (0,1)$ and $H \in (0,1)$,

$$\partial_t u(t) + {}_0^R D_t^{1-\alpha} A u(t) = F(u(t)) + \frac{dW^H(t)}{dt}, \quad \text{for } 0 < t \leq T, \quad \text{with } u(0) = 0, \tag{5}$$

where they introduced a novel Burkholder–Davis–Gundy inequality for fractional Gaussian noise. They used the spectral Galerkin method and the convolution quadrature formula to discretize the Laplacian and Riemann–Liouville fractional derivatives, respectively, and provided error estimates for the proposed numerical methods.

In this paper, we adopt a similar approach to that in [2] to conduct a numerical analysis of (1). However, Equation (1) includes a more general nonlinear term $F(u)$ compared to that used in [2]. We approximate the Caputo time-fractional derivative and the Riemann–Liouville fractional integral using the first-order Lubich convolution quadrature formula, while discretizing the fractional noise via the Euler method. For spatial discretization, we use the spectral Galerkin method. Our fully discrete scheme is formulated by expressing the approximate solution as a convolution between a piecewise constant function and the inverse Laplace transform of a function associated with the resolvent. We obtain both the temporal and spatial regularity of the solutions. Optimal error estimates in the $L^p(\Omega, \mathbb{H})$, $p \geq 2$ norm are obtained via the Laplace transform method, showing precisely how the parameters $\alpha \in (0,1)$, $\gamma \in [0,1]$, and $H \in (0,1)$ influence the convergence orders. Numerical results are also provided to confirm that the computed outcomes are consistent with the theoretical findings.

The paper is organized as follows. In Section 2, we present some preliminaries and assumptions. In Section 3, we consider the spatial and temporal regularities of the solution in (1). In Section 4, we apply the spectral Galerkin method for spatial discretization. The time discretization of (1) is studied in Section 5. Finally, in Section 6, we provide some numerical simulations to validate the theoretically predicted convergence order discussed in Section 5. Throughout this paper, we use c, C to denote positive constants independent of the functions and parameters concerned, but not necessarily the same at different occurrences.

2. Preliminaries and Main Assumptions

Let $\dot{\mathbb{H}}^s, s \in \mathbb{R}$ be the Sobolev space defined by

$$\dot{\mathbb{H}}^s = D(A^{\frac{s}{2}}) = \{v \in \mathbb{H} : \|A^{\frac{s}{2}}v\|^2 = \sum_{j=1}^{\infty} \lambda_j^s (v, \varphi_j)^2 < \infty\},$$

with norm $|v|_s^2 = \|A^{\frac{s}{2}}\|^2 = \sum_{j=1}^{\infty} \lambda_j^s (v, \varphi_j)^2$.

Let $\mathcal{L}_2^0 = HS(Q^{\frac{1}{2}}(\mathbb{H}), \mathbb{H})$ be the Hilbert–Schmidt operators space from $Q^{\frac{1}{2}}(\mathbb{H})$ to $\mathbb{H}$ equipped with the following inner product and norm

$$< T_1, T_2 >= \sum_{j=1}^{\infty} (T_1 Q^{\frac{1}{2}} e_j, T_2 Q^{\frac{1}{2}} e_j), \quad \|T\|_{\mathcal{L}_2^0}^2 = \sum_{j=1}^{\infty} \|T Q^{\frac{1}{2}} e_j\|^2 < \infty.$$

We will provide some main assumptions on nonlinear term $F(u)$, fractional noise term $\frac{dW^H(t)}{dt}$, which will be used throughout this paper.

Assumption 1. *For the nonlinear term F, we assume that there exist $1 \le \nu < 2$, $1 \le \eta < 2$ such that*

$$\|F(u)\| \le C(1 + \|u\|), \quad u \in \mathbb{H}, \tag{6}$$

$$\|F'(u)v\| \le C\|v\|, \quad u, v \in \mathbb{H}, \tag{7}$$

$$\|F'(u)v\|_{\dot{\mathbb{H}}^{-\nu}} \le C(1 + \|u\|_{\dot{\mathbb{H}}^{\mu}})\|v\|_{\dot{\mathbb{H}}^{-\mu}}, \quad u \in \dot{\mathbb{H}}^{\mu}, v \in \dot{\mathbb{H}}^{-\mu}, 0 \le \mu < \nu < 2, \tag{8}$$

$$\|F''(u)(v_1, v_2)\|_{\dot{\mathbb{H}}^{-\eta}} \le C\|v_1\|\|v_2\|, \quad v_1, v_2 \in \mathbb{H}. \tag{9}$$

Assumption 2. *The space-time fractional Gaussian noise $\frac{dW^H(t)}{dt}$ takes the following Fourier series form*

$$\frac{dW^H(t)}{dt} = \sum_{j=1}^{\infty} \gamma_j^{1/2} e_j \frac{d\beta_j^H(t)}{dt}, \tag{10}$$

where $\beta_j^H(t)$ is one dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$.

Moreover, we assume

$$\|A^{-\rho}\|_{\mathcal{L}_2^0} < \infty, \quad \rho \in \Big[0, \min\{1, \frac{H}{\alpha} + 1 + \frac{\gamma - 1}{\alpha}\}\Big). \tag{11}$$

Lemma 1 ([24,35])**.** *For $H \in (0, 1/2)$ and $g_1(t), g_2(t) \in H_0^{\frac{1-2H}{2}}([0, T])$, we have*

$$\mathbb{E}\Big[\int_0^T g_1(r)\, d\beta^H(r) \int_0^T g_2(r)\, d\beta^H(r)\Big] = \frac{1}{2}H(1 - 2H) \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(g_1(r_1) - g_1(r_2))(g_2(r_1) - g_2(r_2))}{|r_1 - r_2|^{2-2H}}\, dr_1\, dr_2,$$

where $\beta^H(t)$ denotes the one-dimensional fractional Brownian motion with the Hurst parameter H.

Lemma 2 ([36,37])**.** *For $H \in (1/2, 1)$ and $g_1(t), g_2(t) \in L^2([0, T])$, there holds*

$$\mathbb{E}\Big[\int_0^T g_1(r)\, d\beta^H(r) \int_0^T g_2(r)\, d\beta^H(r)\Big] = H(2H - 1) \int_0^T \int_0^T g_1(r_1) g_2(r_2) |r_1 - r_2|^{2H-2}\, dr_1\, dr_2,$$

where $\beta^H(t)$ denotes the one-dimensional fractional Brownian motion with the Hurst parameter H.

Lemma 3 ([2])**.** *For $u \in \mathbb{H}^s(a,b)$ with $a,b \in \mathbb{R}, 0 \le s < \frac{1}{2}$, supp $v \subset (a,b)$, there exist positive c_1, c_2 such that*

$$c_1\| {}_a^R D_x^s \| \le |v|_{\mathbb{H}^s((a,b))} \le c_2 \| {}_a^R D_x^s \|.$$

Here $\mathbb{H}^s$ denote the fractional Sobolev space with $s \in (0,1)$.

Lemma 4 ([2])**.** *For $v \in L^2(\mathbb{R}), \mu \in (0,1/2)$ and supp $v \subset (c,d)$ with $c,d \in \mathbb{R}$, we have*

$$\int_c^d \int_c^d v(\eta)v(\zeta)|\zeta-\eta|^{2\mu-1}\, d\zeta\, d\eta \le C \int_c^d ({}_c^R D_s^{-\mu} v(s))^2 ds.$$

Lemma 5. *For $g \in L^2([0,T]), {}_0^R D_s^{\frac{1-2H}{2}} g \in L^2([0,T])$ and $H \in (0,1)$, we have*

$$\mathbb{E}\bigg(\int_0^T g(r) d\beta^H(r)\bigg)^2 \le C\| {}_0^R D_s^{\frac{1-2H}{2}} g\|^2_{L^2([0,T])}, \tag{12}$$

$$\mathbb{E}\bigg(\int_{t_1}^{t_2} g(t_2-r) d\beta^H(r)\bigg)^2 \le C\| {}_0^R D_s^{\frac{1-2H}{2}} g\|^2_{L^2([0,t_2-t_1])}. \tag{13}$$

Proof. We only prove (12) here. The proof of (13) is similar.

Case 1. When $H = \frac{1}{2}$, the Itô isometry may show

$$\mathbb{E}\bigg(\int_{t_1}^{t_2} g(t_2-r) d\beta^H(r)\bigg)^2 = \int_0^{t_2-t_1} |g(r)|^2\, dr = \| {}_0^R D_s^{\frac{1-2H}{2}} g\|^2_{L^2([0,t_2-t_1])}.$$

Case 2. When $H \in (0,1/2)$, let $\bar{g}(t_2-s)$ be the zero extension of $g(t_2-s)$ on $[t_1,t_2]$ such that supp $\bar{g}(t_2-s) \subset [t_1,t_2]$. By Lemma 1, the definition of the semi-norm in $\mathbb{H}^s$, and the fact $\mathbb{H}^s$ coincides with $\dot{\mathbb{H}}^s$, it holds

$$\begin{aligned}
\mathbb{E}\bigg(\int_{t_1}^{t_2} g(t_2-r) d\beta^H(r)\bigg)^2 &= \frac{1}{2}H(1-2H)\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(\bar{g}(t_2-r_1)-\bar{g}(t_2-r_2))^2}{|r_1-r_2|^{2-2H}}\, dr_1\, dr_2 \\
&= \frac{1}{2}H(1-2H)\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(\bar{g}(r_1)-\bar{g}(r_2))^2}{|r_1-r_2|^{2-2H}}\, dr_1\, dr_2 \\
&\le C|\bar{g}(r)|_{H^{\frac{1-2H}{2}}(\mathbb{R})} \le C|g(r)|_{H_0^{\frac{1-2H}{2}}([0,t_2-t_1])}.
\end{aligned}$$

Further by Lemma 3, we get

$$\mathbb{E}\bigg(\int_{t_1}^{t_2} g(t_2-r) d\beta^H(r)\bigg)^2 \le C\| {}_0^R D_s^{\frac{1-2H}{2}} g\|^2_{L^2([0,t_2-t_1])}.$$

Case 3. When $H \in (1/2,1)$, by Lemmas 2 and 4, we have

$$\begin{aligned}
\mathbb{E}\bigg(\int_{t_1}^{t_2} g(t_2-r) d\beta^H(r)\bigg)^2 &= H(2H-1)\int_{t_1}^{t_2}\int_{t_1}^{t_2} g(t_2-r_1)g(t_2-r_2)|r_1-r_2|^{2H-2} dr_1 dr_2 \\
&\le C\int_0^{t_2-t_1}\int_0^{t_2-t_1} g(r_1)g(r_2)|r_1-r_2|^{2H-2} dr_1 dr_2 \\
&\le C\int_0^{t_2-t_1} ({}_0^R D_s^{\frac{1-2H}{2}} g)^2 dt = C\| {}_0^R D_s^{\frac{1-2H}{2}} g\|^2_{L^2([0,t_2-t_1])}.
\end{aligned}$$

□

Lemma 6. *Let ${}_0^R D_s^{\frac{1-2H}{2}} \psi(s) \in \mathcal{L}_2^0$, then we have*

$$\mathbb{E}\bigg\|\int_0^t \psi(r)\, dW^H(r)\bigg\|^2 \le C\int_0^t \| {}_0^R D_r^{\frac{1-2H}{2}} \psi(r)\|^2_{\mathcal{L}_2^0}\, dr, \tag{14}$$

$$\mathbb{E}\left\|\int_s^t v(t-r)\, dW^H(r)\right\|^2 \leq C\int_0^{t-s} \|{}_0^R D_r^{\frac{1-2H}{2}} \psi(r)\|^2_{\mathcal{L}_2^0}\, dr. \tag{15}$$

Further we have, for $p \geq 2$,

$$\left\|\int_0^t \psi(r)dW^H(r)\right\|_{L^p(\Omega,\mathbb{H})} \leq C\left\|\left(\int_0^t \left\|{}_0^R D_r^{\frac{1-2H}{2}} \psi(r)\right\|^2_{\mathcal{L}_2^0} dr\right)^{\frac{1}{2}}\right\|_{L^p(\Omega,\mathbb{R})}, \tag{16}$$

$$\left\|\int_s^t \psi(t-r)dW^H(r)\right\|_{L^p(\Omega,\mathbb{H})} \leq C\left\|\left(\int_0^{t-s} \left\|{}_0^R D_r^{\frac{1-2H}{2}} \psi(r)\right\|^2_{\mathcal{L}_2^0} dr\right)^{\frac{1}{2}}\right\|_{L^p(\Omega,\mathbb{R})}. \tag{17}$$

Proof. See the Appendix A. □

3. Temporal and Spatial Regularities in the Solution of (1)

Denote $G(t) = F(u(t))$ and $f(t) = \frac{dW^H(t)}{dt}$, then (1) can be written as

$${}_0^C D_t^\alpha u(t) + Au(t) = G(t) + {}_0^R D_t^{-\gamma} f(t), \quad \text{for } 0 < t \leq T, \quad \text{with } u(0) = 0. \tag{18}$$

Taking the Laplace transform of (18), we have

$$z^\alpha \hat{u}(z) + A\hat{u}(z) = \hat{G}(z) + z^{-\gamma}\hat{f}(z),$$

which implies that

$$\hat{u}(z) = (z^\alpha + A)^{-1}\hat{G}(z) + z^{-\gamma}(z^\alpha + A)^{-1}\hat{f}(z).$$

By the inverse Laplace transform, we have

$$u(t) = \int_0^t \bar{E}(t-s)F(u(s))\, ds + \int_0^t \tilde{E}(t-s)\, dW^H(s), \tag{19}$$

where

$$\bar{E}(t) = \frac{1}{2\pi i}\int_\Gamma e^{zt}(z^\alpha + A)^{-1}\, dz, \quad \tilde{E}(t) = \frac{1}{2\pi i}\int_\Gamma e^{zt}(z^\alpha + A)^{-1}z^{-\gamma}\, dz,$$

and, with $\hat{v}(z)$ being the Laplace transform of $v(t)$,

$$\hat{\bar{E}}(z) = (z^\alpha + A)^{-1}, \quad \hat{\tilde{E}}(z) = (z^\alpha + A)^{-1}z^{-\gamma}.$$

Here $\Gamma = \{z : |\arg(z)| = \theta,\ \theta \in (\pi/2, \pi),\ \Im(z) \text{ increases from } -\infty \text{ to } \infty\}$.

According to the resolvent estimate (3) and interpolation theory, we have, for $r \in [0,1]$,

$$\|A^r \hat{\bar{E}}(z)\| = \|A^r(z^\alpha + A)^{-1}\| \leq C|z|^{\alpha(r-1)}, \tag{20}$$

$$\|A^r \hat{\tilde{E}}(z)\| = \|A^r(z^\alpha + A)^{-1}z^{-\gamma}\| \leq C|z|^{\alpha(r-1)-\gamma}. \tag{21}$$

Combining (20) and (21), we can easily get, with $0 \leq p-q \leq 2,\ 0 \leq p, q \leq 2$ and $l = 0, 1$,

$$|\partial_t^l \bar{E}(t)v|_p \leq Ct^{-\alpha\frac{p-q}{2}+\alpha-1-l}|v|_q, \tag{22}$$

$$|\partial_t^l \tilde{E}(t)v|_p \leq Ct^{-\alpha\frac{p-q}{2}+\alpha+\gamma-1-l}|v|_q. \tag{23}$$

Remark 1. *The solution operators $\bar{E}(t), \tilde{E}(t)$ also satisfy the following properties:*

$$\bar{E}(t) = t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha A), \quad \tilde{E}(t) = t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-t^\alpha A),$$

and, with $E(t) = E_{\alpha,1}(-t^\alpha A)$,

$$E'(t) = -A\bar{E}(t), \quad \tilde{E}'(t) = t^{\alpha+\gamma-2}E_{\alpha,\alpha+\gamma-1}(-t^\alpha A).$$

Here $E_{\alpha,\beta}(z)$, $z \in \mathbb{C}$ *with* $\alpha \in (0,1)$, $\beta > 0$ *denote the Mittag–Leffler functions.*

The existence and uniqueness of the mild solution of (1) in $C([0,T], L^p(\Omega,\mathbb{H}))$ with $p \geq 2$ can be considered by using the Banach contraction mapping theorem similar to [38]. To save space, we omit the proof here. Now we turn to the spatial and temporal regularities in the solution of (1).

Theorem 1. *Let* $u(t)$ *be the solution of* (1)*. Suppose that Assumptions 1 and 2 hold. Let* $\rho \in [0,1)$ *and* $\alpha \in (0,1), \gamma \in [0,1]$*. Then we have, with* $p \geq 2$,

$$\|u(t)\|_{L^p(\Omega,\dot{H}^\sigma)} \leq C,$$

where $0 \leq \sigma < \min\{2(1-\rho), \frac{2H}{\alpha} - 2\rho + 2 + \frac{2\gamma-2}{\alpha}\}$.

Proof. Simple calculations give

$$\|u(t)\|_{L^p(\Omega,\dot{H}^\sigma)} \leq \left\| \int_0^t \bar{E}(t-s)F(u(s))\,ds \right\|_{L^p(\Omega,\dot{H}^\sigma)} + \left\| \int_0^t \tilde{E}(t-s)dW^H(s) \right\|_{L^p(\Omega,\dot{H}^\sigma)} = I + II.$$

As for I, the estimate (22) with $l=0$, Assumption condition (6) lead to

$$I \leq C\int_0^t (t-s)^{\alpha(1-\frac{\sigma}{2})-1}(1+\|u(s)\|_{L^p(\Omega,\mathbb{H})})\,ds \leq C + C\int_0^t (t-s)^{\alpha(1-\frac{\sigma}{2})-1}\|u(s)\|_{L^p(\Omega,\mathbb{H})}ds,$$

where we require $\sigma < 2$.

When $\sigma = 2$, by (22) with $l=0$, the Assumption (7), and Remark 1, we have

$$\begin{aligned} I &\leq \left\| \int_0^t A\bar{E}(t-s)(F(u(s)) - F(u(t)))\,ds \right\|_{L^p(\Omega,\mathbb{H})} + \left\| \int_0^t A\bar{E}(t-s)F(u(t)\,ds \right\|_{L^p(\Omega,\mathbb{H})} \\ &\leq C\int_0^t (t-s)^{-1}\|u(s)-u(t)\|_{L^p(\Omega,\mathbb{H})}\,ds + \left\| \int_0^t E_s'(t-s)F(u(t)\,ds \right\|_{L^p(\Omega,\mathbb{H})} \\ &\leq C\int_0^t (t-s)^{-1}(t-s)^\varepsilon\,ds + \|E(0)-E(t)\|\|F(u(t)\|_{L^p(\Omega,\mathbb{H})}, \end{aligned} \tag{24}$$

where $\varepsilon > 0$ is determined by the temporal regularity in Theorem 2.

As for II, by (21), we have, noting that $0 \leq \sigma/2 + \rho \leq 1$, that is $\sigma \leq 2(1-\rho)$ which implies $\rho \leq 1$ when $\sigma \geq 0$,

$$\begin{aligned} II^2 &\leq C\left\| \left(\int_0^t \| {}_0^R D_s^{\frac{1-2H}{2}} A^{\sigma/2}\tilde{E}(s)\|^2_{\mathcal{L}_2^0}\,ds \right)^{\frac{1}{2}} \right\|^2_{L^P(\Omega,\mathbb{R})} \leq C_p\int_0^t \| {}_0^R D_s^{\frac{1-2H}{2}} A^{\sigma/2+\rho}\tilde{E}(s)\|^2_{\mathcal{L}(\mathbb{H})}\,ds\|A^{-\rho}\|^2_{\mathcal{L}_2^0} \\ &\leq C_p\int_0^t \left(\int_\Gamma |e^{zs}||z|^{\frac{1-2H}{2}+\alpha(\sigma/2+\rho-1)-\gamma}d|z| \right)^2 ds \leq C_p\int_0^t s^{2H-2\alpha(\sigma/2+\rho-1)+2\gamma-3}\,ds \leq C, \end{aligned} \tag{25}$$

where we need $2H - \alpha[\sigma + 2(\rho-1)] + 2\gamma - 2 > 0$ to preserve the boundedness of II, which leads to $\sigma < \frac{2H}{\alpha} - 2\rho + 2 + \frac{2\gamma-2}{\alpha}$ and $\rho < \frac{H}{\alpha} + 1 + \frac{\gamma-1}{\alpha}$.

For $0 < \sigma < 2$, we have

$$\|u(t)\|_{L^p(\Omega,\dot{H}^\sigma)} \leq C + C\int_0^t (t-s)^{\alpha(1-\frac{\sigma}{2})-1}\|u(s)\|_{L^p(\Omega,\dot{H}^\sigma)}ds,$$

which implies that, by Grönwall lemma,

$$\|u(t)\|_{L^p(\Omega,\dot{H}^\sigma)} \le C.$$

For $\sigma = 2$, we have, by (24) and (25),

$$\|u(t)\|_{L^p(\Omega,\dot{H}^\sigma)} \le C + C\|u(t)\|_{L^p(\Omega,H)} \le C.$$

Together these estimates complete the proof of Theorem 1. □

Theorem 2. *Let $u(t)$ be the mild solution of (1). Suppose that Assumptions 1 and 2 are fulfilled. Let $0 \le t_1 < t_2 \le T$, $\alpha \in (0,1)$, $\gamma \in [0,1]$, $\varepsilon \in (0,1)$, $\rho \in [0,1)$. Then we have, with $p \ge 2$,*

$$\|u(t_2) - u(t_1)\|_{L^p(\Omega,\mathbb{H})} \le C(t_2 - t_1)^\beta,$$

where $0 < \beta = \min\{\alpha, H - \alpha(\rho-1) + \gamma - 1 - \varepsilon\} < 1$.

Proof. We first divide $\|u(t_2) - u(t_1)\|_{L^p(\Omega,\mathbb{H})}$ into four parts

$$\begin{aligned}
&\|u(t_2) - u(t_1)\|_{L^p(\Omega,\mathbb{H})} \\
\le& \left\| \int_0^{t_1} (\bar{E}(t_2-s) - \bar{E}(t_1-s))F(u(s))\,ds \right\|_{L^p(\Omega,\mathbb{H})} + \left\| \int_{t_1}^{t_2} \bar{E}(t_2-s)F(u(s))\,ds \right\|_{L^p(\Omega,\mathbb{H})} \\
&+ \left\| \int_0^{t_1} (\tilde{E}(t_2-s) - \tilde{E}(t_1-s))\,dW^H(s) \right\|_{L^p(\Omega,\mathbb{H})} + \left\| \int_{t_1}^{t_2} \tilde{E}(t_2-s)\,dW^H(s) \right\|_{L^p(\Omega,\mathbb{H})} \\
=& I + II + III + IV.
\end{aligned}$$

Using (22) with $l = 1$, and the Assumption (6) and Theorem 1, one yields

$$\begin{aligned}
I =& \left\| \int_0^{t_1} \int_{t_1}^{t_2} \bar{E}_t'(t-s)\,dt F(u(s))\,ds \right\|_{L^p(\Omega,\mathbb{H})} \le C \int_{t_1}^{t_2} \int_0^{t_1} (t-s)^{\alpha-2} \|F(u(s))\|_{L^p(\Omega,\mathbb{H})}\,ds\,dt \\
\le& C \int_{t_1}^{t_2} (t-t_1)^{\alpha-1}\,dt \le C(t_2-t_1)^\alpha.
\end{aligned} \tag{26}$$

Similarly we get

$$II \le C \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \|F(u(s))\|_{L^p(\Omega,\mathbb{H})}\,ds \le C(t_2-t_1)^\alpha. \tag{27}$$

As for III, applying the inequality (16), the resolvent estimate (21), and the fact $\|e^{z\tau} - 1\| \le C\tau^a|z|^a$, $a = H - \alpha(\rho-1) + \gamma - 1 - \epsilon \in (0,1)$ on Γ, we obtain

$$\begin{aligned}
III^2 \le& C \left\| \left(\int_0^{t_1} \| {}_0^R D_s^{\frac{1-2H}{2}} (\tilde{E}(t_2-s) - \tilde{E}(t_1-s)) \|_{\mathcal{L}_2^0}^2\,ds \right)^{\frac{1}{2}} \right\|_{L^P(\Omega,\mathbb{R})}^2 \\
\le& C_p \int_0^{t_1} \| {}_0^R D_s^{\frac{1-2H}{2}} A^\rho (\tilde{E}(t_2-s) - \tilde{E}(t_1-s)) \|_{\mathcal{L}(\mathbb{H})}^2\,ds\, \|A^{-\rho}\|_{\mathcal{L}_2^0}^2 \\
\le& C_p \int_0^{t_1} \left\| \int_\Gamma e^{z(t_1-s)} (e^{z(t_2-t_1)} - 1) A^\rho \hat{\tilde{E}}(z) z^{\frac{1-2H}{2}}\,dz \right\|_{\mathcal{L}(\mathbb{H})}^2 ds \\
\le& C_p (t_2-t_1)^{2a} \int_0^{t_1} \left(\int_\Gamma e^{-crs} r^{a + \frac{1-2H}{2} + \alpha(\rho-1) - \gamma}\,dr \right)^2 ds \\
\le& C_p (t_2-t_1)^{2a} \int_0^{t_1} s^{2(-a+H-\alpha(\rho-1)+\gamma-\frac{3}{2})}\,ds \\
\le& C_p (t_2-t_1)^{2a} \le C_p (t_2-t_1)^{2H-2\alpha(\rho-1)+2\gamma-2-2\epsilon}.
\end{aligned} \tag{28}$$

Here we choose $a = H - \alpha(\rho - 1) + \gamma - 1 - \epsilon$ to preserve $\int_0^{t_1} s^{2(-a+H-\alpha(\rho-1)+\gamma-\frac{3}{2})}\,ds < \infty$.

Similar to the estimate of III, by (17) and resolvent estimate (21), we get, noting that the Laplace transform of ${}_0^R D_s^{\frac{1-2H}{2}} v(t)$ satisfies $\widehat{{}_0^R D_s^{\frac{1-2H}{2}} v}(z) = z^{\frac{1-2H}{2}} \hat{v}(z)$ for sufficiently smooth $v(t)$,

$$\begin{aligned}
IV^2 &\le C \left\| \left(\int_0^{t_2-t_1} \| {}_0^R D_s^{\frac{1-2H}{2}} \tilde{E}(s) \|_{\mathcal{L}_2^0}^2 \, ds \right)^{\frac{1}{2}} \right\|_{L^p(\Omega,\mathbb{R})}^2 \le C_p \int_0^{t_2-t_1} \| {}_0^R D_s^{\frac{1-2H}{2}} A^\rho \tilde{E}(s) \|_{\mathcal{L}(\mathbb{H})}^2 \, ds \, \| A^{-\rho} \|_{\mathcal{L}_2^0}^2 \\
&\le C_p \int_0^{t_2-t_1} \left(\int_\Gamma |e^{zs}| |z|^{\frac{1-2H}{2}+\alpha(\rho-1)-\gamma} d|z| \right)^2 ds \le C_p \int_0^{t_2-t_1} s^{2H-2\alpha(\rho-1)+2\gamma-3}\, ds \\
&\le C_p (t_2 - t_1)^{2H-2\alpha(\rho-1)+2\gamma-2}.
\end{aligned} \tag{29}$$

By combining the above estimates (26)–(29), the proof is complete. □

Remark 2. *When $H = \frac{1}{2}, \alpha = 1, \rho = 0, \gamma = 0$, we get $\beta = \frac{1}{2}$ which is consistent with the well known results for the stochastic heat equation with trace class noise.*

4. Spatial Discretization

In this section, we shall use the spectral Galerkin method to discretize the spatial variable of Equation (1). Let $\mathbb{H}_N = \mathrm{span}\{e_1, e_2, \cdots, e_N\}$ be a finite dimensional subspace, and define the projection operator$P_N : \mathbb{H} \to \mathbb{H}_N$ by

$$P_N u = \sum_{j=1}^N (u, e_j) e_j, \quad \forall u \in \mathbb{H}.$$

Define $A_N : \mathbb{H}_N \to \mathbb{H}_N$ by $A_N = A P_N$ which generates a family of resolvent operators $\{\bar{E}_N(t)\}, \{\tilde{E}_N(t)\}$ in $\mathbb{H}_N$. Obviously, we have

$$\bar{E}_N(t) P_N = \bar{E}(t) P_N, \tilde{E}_N(t) P_N = \tilde{E}(t) P_N,$$

and

$$\| A^{-s}(I - P_N) \| = \sup_{k \ge N+1} \lambda_k^{-s} = \lambda_{N+1}^{-s}, \ s > 0. \tag{30}$$

The spectral Galerkin semeidiscrete scheme of (1) can be written as: find $u_N(t) \in \mathbb{H}_N$ such that

$${}_0^C D_t^\alpha u_N(t) + A_N u_N(t) = P_N F(u_N(t)) + P_N {}_0^R D_t^{-\gamma} \frac{dW^H(t)}{dt}, \quad \text{with } u_N(0) = 0. \tag{31}$$

Taking Laplace transform and inverse Laplace transform gives

$$u_N(t) = \int_0^t \bar{E}_N(t-s) P_N F(u_N(s))\, ds + \int_0^t \tilde{E}_N(t-s) P_N \, dW^H(s), \tag{32}$$

where $\bar{E}_N(t) = \frac{1}{2\pi i} \int_\Gamma e^{zt} (z^\alpha + A_N)^{-1} dz$, $\tilde{E}_N(t) = \frac{1}{2\pi i} \int_\Gamma e^{zt} (z^\alpha + A_N)^{-1} z^{-\gamma} dz$. These operators have the similar smoothing properties as $\bar{E}(t), \tilde{E}(t)$. Similar to the proofs of Theorems 1 and 2, one can get the same spatial and temporal regularities of u_N.

Now we turn to the error estimates of the spatial discretization.

Theorem 3. *Let $u(t)$, $u_N(t)$ be the solutions of (1) and (31), respectively. We assume that Assumptions 1 and 2 with $\rho \in [0, 1)$ are fulfilled. Let $\alpha \in (0, 1)$, $\gamma \in [0, 1], 0 \le \frac{\sigma}{2} < \min\{1 - \rho, \frac{H}{\alpha} - \rho + 1 + \frac{\gamma - 1}{\alpha}\}$, we have with $d = 1, 2, 3$ and $p \ge 2$*

$$\| u(t) - u_N(t) \|_{L^p(\Omega,\mathbb{H})} \le C_p (N+1)^{-\frac{\sigma}{d}}.$$

Proof. By (19) and (32), we have

$$\begin{aligned}
&\|u(t) - u_N(t)\|_{L^p(\Omega,\mathbb{H})} \\
\leq &\left\| \int_0^t \bar{E}(t-s)F(u(s))\,ds - \int_0^t \bar{E}(t-s)P_N F(u_N(s))\,ds \right\|_{L^p(\Omega,\mathbb{H})} \\
&+ \left\| \int_0^t \tilde{E}(t-s)\,dW^H(s) - \int_0^t \tilde{E}(t-s)P_N\,dW^H(s) \right\|_{L^p(\Omega,\mathbb{H})} \\
\leq &\left\| \int_0^t \bar{E}(t-s)[F(u(s)) - F(u_N(s))]\,ds \right\|_{L^p(\Omega,\mathbb{H})} + \left\| \int_0^t (\bar{E}(t-s)(I-P_N)F(u_N(s))\,ds \right\|_{L^p(\Omega,\mathbb{H})} \\
&+ \left\| \int_0^t \tilde{E}(t-s)(I-P_N)\,dW^H(s) \right\|_{L^p(\Omega,\mathbb{H})} = J_1 + J_2 + J_3.
\end{aligned} \tag{33}$$

The regularity of $\bar{E}(t)$ (22) with $l = 0$, and the Assumption (7) on F give

$$J_1 \leq \int_0^t \|\bar{E}(t-s)\|_{\mathcal{L}(\mathbb{H})} \|u(s) - u_N(s)\|_{L^p(\Omega,\mathbb{H})}\,ds \leq C\int_0^t (t-s)^{\alpha-1}\|u(s) - u_N(s)\|_{L^p(\Omega,\mathbb{H})}\,ds. \tag{34}$$

By the regularity of $\bar{E}(t)$ (22) with $l = 0$, the Assumption condition (6) on F, and (30), we have

$$\begin{aligned}
J_2 &\leq \int_0^t \|A^{\frac{\sigma}{2}}\bar{E}(t-s)\|_{\mathcal{L}(\mathbb{H})} \|A^{-\frac{\sigma}{2}}(I-P_N)\|_{\mathcal{L}(\mathbb{H})} \|F(u_N(s))\|_{L^p(\Omega,\mathbb{H})}\,ds \\
&\leq C\int_0^t (t-s)^{\alpha(1-\frac{\sigma}{2})-1}\,ds\,\lambda_{N+1}^{-\frac{\sigma}{2}}.
\end{aligned} \tag{35}$$

Applying the inequality (17) and (30), and the resolvent estimate (21), with $0 \leq \frac{\sigma}{2} < \min\{1-\rho, \frac{H}{\alpha} - \rho + 1 + \frac{\gamma-1}{\alpha}\}$, we obtain

$$\begin{aligned}
J_3^2 &\leq C_p \int_0^t \|{}_0^R D_s^{\frac{1-2H}{2}} A^{\rho}\tilde{E}(s)(I-P_N)\|^2\,ds\|A^{-\rho}\|^2_{\mathcal{L}_2^0} \\
&\leq C_p \int_0^t \|{}_0^R D_s^{\frac{1-2H}{2}} A^{\rho+\frac{\sigma}{2}}\tilde{E}(s)\|^2_{\mathcal{L}(\mathbb{H})}\,\|A^{-\frac{\sigma}{2}}(I-P_N)\|^2_{\mathcal{L}(\mathbb{H})}\,ds \\
&\leq C_p\lambda_{N+1}^{-\sigma}\int_0^t \left(\int_\Gamma e^{zs} z^{\frac{1-2H}{2}} z^{\alpha(\rho+\frac{\sigma}{2}-1)-\gamma}\,d|z|\right)^2 ds \\
&\leq C_p\lambda_{N+1}^{-\sigma}\int_0^t s^{2[\frac{2H-1}{2}-\alpha(\rho+\frac{\sigma}{2}-1)+\gamma-1]}\,ds \leq C_p\lambda_{N+1}^{-\sigma}.
\end{aligned} \tag{36}$$

From (34)–(36), we derive, noting $\lambda_j \geq j^{\frac{2}{d}}, j \geq 1$ [22],

$$\|u(t) - u_N(t)\|_{L^p(\Omega,\mathbb{H})} \leq C_p\lambda_{N+1}^{-\frac{\sigma}{2}} \leq C_p(N+1)^{-\frac{\sigma}{d}},$$

which complete the proof of Theorem 3. □

5. Time Discretization

Let $0 = t_0 < t_1 < t_2 < \cdots < t_M = T$ be a partition of $[0,T]$ and τ the time step size. At $t = t_n$, we consider the following approximations:

$${}_0^C D_t^{\alpha} u(t_n) = \tau^{-\alpha}\sum_{i=0}^{n} w_i^{(\alpha)} u(t_{n-i}) + O(\tau),$$

and

$${}_0^R D_t^{-\gamma} u(t_n) = \tau^{\gamma}\sum_{i=0}^{n} w_i^{(-\gamma)} u(t_{n-i}) + O(\tau),$$

where $w_i^{(\alpha)}, w_i^{(-\gamma)}, i = 1, 2, \ldots$ are generated by $(1-\xi)^\alpha, (1-\xi)^{-\gamma}$, respectively, that is,

$$(1-\xi)^\alpha = \sum_{i=0}^{\infty} w_i^{(\alpha)} \xi^i, \quad (1-\xi)^{-\gamma} = \sum_{i=0}^{\infty} w_i^{(-\gamma)} \xi^i,$$

see Jin et al. [21].

For the noise term $f(t) = \frac{dW^H(t)}{dt}$, we approximate it at $t = t_n$ using the Euler method as follows:

$$\frac{dW^H(t_n)}{dt} \approx \frac{W^H(t_n) - W^H(t_{n-1})}{\tau} = f^n, \ n = 1, 2, \ldots, M, \quad \text{with} \quad f^0 = 0.$$

Let u_N^n denote the fully approximate solution of $u(t_n)$, we define the following fully discretization scheme

$$\tau^{-\alpha} \sum_{i=0}^{n} w_{n-i}^{(\alpha)} u_N^i + A_N u_N^n = P_N F^n + \tau^\gamma \sum_{i=0}^{n} w_{n-i}^{(-\gamma)} P_N f^i. \tag{37}$$

Taking the discrete Laplace transform in both sides of (37), we have

$$\tau^{-\alpha} \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n} w_{n-i}^{(\alpha)} u_N^i \right) \xi^n + \sum_{n=1}^{\infty} (A_N u_N^n) \xi^n = \sum_{n=1}^{\infty} P_N F^n \xi^n + \tau^\gamma \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n} w_{n-i}^{(-\gamma)} P_N f^i \right) \xi^n.$$

Denote the discrete Laplace transform of $\{\omega_n^{(\alpha)}\}_{n=0}^\infty, \{\omega_n^{(-\gamma)}\}_{n=0}^\infty, \{u_N^n\}_{n=0}^\infty, \{F^n\}_{n=0}^\infty,$ $\{f^n\}_{n=0}^\infty$ by

$$\tilde{\omega}^{(\alpha)}(\xi) = \sum_{n=0}^{\infty} \omega_n^{(\alpha)} \xi^n, \quad \tilde{\omega}^{(-\gamma)}(\xi) = \sum_{n=0}^{\infty} \omega_n^{(-\gamma)} \xi^n,$$
$$\tilde{u}_N(\xi) = \sum_{n=0}^{\infty} u_N^n \xi^n, \quad \tilde{F}(\xi) = \sum_{n=0}^{\infty} F^n \xi^n, \quad \tilde{f}(\xi) = \sum_{n=0}^{\infty} f^n \xi^n,$$

respectively. We then have

$$(\tau^{-\alpha} \tilde{\omega}^{(\alpha)}(\xi) + A_N) \tilde{u}_N(\xi) = P_N \tilde{F}(\xi) + \tau^\gamma \tilde{\omega}^{(-\gamma)}(\xi) P_N \tilde{f}(\xi),$$

which implies

$$\tilde{u}_N(\xi) = (\tau^{-\alpha} \tilde{\omega}^{(\alpha)}(\xi) + A_N)^{-1} P_N \tilde{F}(\xi) + (\tau^{-\alpha} \tilde{\omega}^{(\alpha)}(\xi) + A_N)^{-1} \tau^\gamma \tilde{\omega}^{(-\gamma)}(\xi) P_N \tilde{f}(\xi).$$

By using the inverse discrete Laplace transform, we get, with $b \in (0,1), \xi = e^{-z\tau}$,

$$\begin{aligned}
u_N^n &= \frac{1}{2\pi i} \int_{|\xi|=b} \xi^{-(n+1)} \tilde{u}_N(\xi) \, d\xi \\
&= \frac{1}{2\pi i} \int_{|\xi|=b} \xi^{-(n+1)} \Big((\tau^{-\alpha} \tilde{\omega}^{(\alpha)}(\xi) + A_N)^{-1} P_N \tilde{F}(\xi) + (\tau^{-\alpha} \tilde{\omega}^{(\alpha)}(\xi) + A_N)^{-1} \tau^\gamma \tilde{\omega}^{(-\gamma)}(\xi) P_N \tilde{f}(\xi) \Big) \, d\xi \\
&= \frac{\tau}{2\pi i} \int_{\Gamma_\tau} e^{zt_n} [(\tau^{-\alpha} \tilde{\omega}^{(\alpha)}(e^{-z\tau}) + A_N)^{-1} P_N \tilde{F}(e^{-z\tau}) + (\tau^{-\alpha} \tilde{\omega}^{(\alpha)}(e^{-z\tau}) + A_N)^{-1} \tau^\gamma \tilde{\omega}^{(-\gamma)}(e^{-z\tau}) P_N \tilde{f}(e^{-z\tau})] \, dz,
\end{aligned}$$

where $\Gamma_\tau = \{z : z \in \Gamma, |\Im z| \leq \frac{\pi}{\tau}\}$.

Denote $z_\tau = \frac{1-e^{-z\tau}}{\tau}$ where z_τ is a suitable approximation of $z \in \Gamma_\tau$, we then have

$$\tau^{-\alpha} \tilde{\omega}^{(\alpha)}(e^{-z\tau}) = z_\tau^\alpha, \quad \tau^\gamma \tilde{\omega}^{(-\gamma)}(e^{-z\tau}) = z_\tau^{-\gamma},$$

which implies

$$u_N^n = \frac{\tau}{2\pi i}\int_{\Gamma_\tau} e^{zt_n}[(z_\tau^\alpha + A_N)^{-1}P_N\tilde{F}(e^{-z\tau}) + (z_\tau^\alpha + A_N)^{-1}z_\tau^{-\gamma}P_N\tilde{f}(e^{-z\tau})]\,dz. \tag{38}$$

We will show that u_N^n can be expressed as the convolution of the piecewise constant function $\bar{F}(t)$, $\bar{f}(t)$. To obtain this, we first introduce the following piecewise constant function $\bar{f}(t), \bar{F}(t), t \geq 0$, defined by, with $\bar{f}(0) = \bar{F}(0) = 0$,

$$\bar{f}(t) = \begin{cases} f^j, \ t \in (t_{j-1}, t_j], \ j = 1, 2, \ldots, M, \\ 0, \quad t > T = t_M, \end{cases} \tag{39}$$

and

$$\bar{F}(t) = \begin{cases} F(u_N^{j-1}) = F^j, \ t \in (t_{j-1}, t_j], \ j = 1, 2, \ldots, M, \\ 0, \quad t > T = t_M. \end{cases} \tag{40}$$

Similar to the proof of Lemma 2.1 in Wu et al. [23], we may show that u_N^n in (38) can be written as

$$u_N^n = \int_0^{t_n} \bar{E}_\tau(t_n - s)P_N\bar{F}(s)\,ds + \int_0^{t_n} \tilde{E}_\tau(t_n - s)P_N\bar{f}(s)\,ds,$$

where

$$\bar{E}_\tau(t) = \frac{1}{2\pi i}\int_{\Gamma_\tau} e^{zt}(z_\tau^\alpha + A)^{-1}\frac{z\tau}{e^{z\tau} - 1}dz, \tag{41}$$

and

$$\tilde{E}_\tau(t) = \frac{1}{2\pi i}\int_{\Gamma_\tau} e^{zt}(z_\tau^\alpha + A)^{-1}z_\tau^{-\gamma}\frac{z\tau}{e^{z\tau} - 1}dz. \tag{42}$$

Denote

$$\partial_\tau W^H(t) = \begin{cases} 0, \quad t = 0, \\ f^j, \ t \in (t_{j-1}, t_j], \ j = 1, 2, \ldots, M, \\ 0, \quad t > T = t_M. \end{cases}$$

Then we have

$$u_N^n = \int_0^{t_n} \bar{E}_\tau(t_n - s)P_N\bar{F}(s)\,ds + \int_0^{t_n} \tilde{E}_\tau(t_n - s)P_N\partial_\tau W^H(s)\,ds. \tag{43}$$

Next, we introduce several lemmas that will be used in the error estimation of the fully discretized scheme.

Lemma 7 ([1]). *Let $z_\tau = \frac{1-e^{-z\tau}}{\tau}$ for $\forall z \in \Gamma_\tau$, then we have*

$$|z| \sim |z_\tau|, \quad |z - z_\tau| \leq C\tau|z|^2, \quad |z^\alpha - z_\tau^\alpha| \leq C\tau|z|^{\alpha+1},$$

where $|z| \sim |z_\tau|$ means that $\forall z \in \Gamma_\tau$, $|z|$ and $|z_\tau|$ are equivalent on Γ_τ.

Lemma 8. *Let $z_\tau = \frac{1-e^{-z\tau}}{\tau}$ for $z \in \Gamma_\tau$, we obtain*

$$\|(z^\alpha + A)^{-1} - (z_\tau^\alpha + A)^{-1}\frac{z\tau}{e^{z\tau} - 1}\| \leq C\tau|z|^{1-\alpha},$$

and, with $0 \leq \rho \leq 1$,

$$\|A^\rho[z^{-\gamma}(z^\alpha + A)^{-1} - z_\tau^{-\gamma}(z_\tau^\alpha + A)^{-1}\frac{z\tau}{e^{z\tau} - 1}]\| \leq C\tau|z|^{\alpha(\rho-1)-\gamma+1}.$$

Proof. Here, we only prove the latter. By using the resolvent estimate (2), the differential mean value theorem, and Lemma 7, we obtain:

$$\begin{aligned}
&\|A^{\rho}[z^{-\gamma}(z^{\alpha}+A)^{-1}-z_{\tau}^{-\gamma}(z_{\tau}^{\alpha}+A)^{-1}\frac{z\tau}{e^{z\tau}-1}]\| \\
\leq&\|A^{\rho}(z^{-\gamma}-z_{\tau}^{-\gamma})(z^{\alpha}+A)^{-1}\|+\|A^{\rho}z_{\tau}^{-\gamma}[(z^{\alpha}+A)^{-1}-(z_{\tau}^{\alpha}+A)^{-1}]\| \\
&+\|A^{\rho}z_{\tau}^{-\gamma}(z_{\tau}^{\alpha}+A)^{-1}(1-\frac{z\tau}{e^{z\tau}-1})\| \\
\leq&C|z|^{\alpha(\rho-1)}|z^{-\gamma}-z_{\tau}^{-\gamma}|+C\|z_{\tau}^{-\gamma}A^{\rho}(z^{\alpha}+A)^{-1}(z_{\tau}^{\alpha}+A)^{-1}(z^{\alpha}-z_{\tau}^{\alpha})\| \\
&+C|z|^{\alpha(\rho-1)}\|z_{\tau}^{-\gamma}(1-\frac{z\tau}{e^{z\tau}-1})\| \\
\leq&C|z|^{\alpha(\rho-1)}|z|^{-\gamma-1}|z-z_{\tau}|+C|z|^{\alpha(\rho-1)}|z|^{-\gamma}\|(z_{\tau}^{\alpha}+A)^{-1}(z^{\alpha}-z_{\tau}^{\alpha})\| \\
&+C|z|^{\alpha(\rho-1)}|z|^{-\gamma}|1-\frac{z\tau}{e^{z\tau}-1}| \\
\leq&C\tau|z|^{\alpha(\rho-1)-\gamma-1}|z|^{2}+C|z|^{\alpha(\rho-1)-\gamma}|z|^{-\alpha}|z|^{\alpha-1}|z-z_{\tau}| \\
&+C\tau|z|^{\alpha(\rho-1)-\gamma}|z|\leq C\tau|z|^{\alpha(\rho-1)-\gamma+1},
\end{aligned} \tag{44}$$

which completes the proof of the required inequality. □

We now turn to show the error estimate of fully discretization approximation.

Theorem 4. *Let $u(t), u_N^n$ be defined in (19) and (43), respectively. Let $0<\varepsilon<1$. Then we get, with $p\geq 2$,*

$$\|u(t_n)-u_N^n\|_{L^p(\Omega,\mathbb{H})}\leq C(N+1)^{-\frac{\sigma}{d}}+C\tau^{\beta},$$

where $0<\beta=\min\{\alpha, H+\alpha(1-\rho)+\gamma-1-\varepsilon\}$, $0\leq\frac{\sigma}{2}<\min\{1-\rho,\frac{H}{\alpha}-\rho+1+\frac{\gamma-1}{\alpha}\}$ with $\rho\in[0,1)$.

Proof. First, we have

$$\|u(t_n)-u_N^n\|_{L^p(\Omega,\mathbb{H})}\leq\|u(t_n)-u_N(t_n)\|_{L^p(\Omega,\mathbb{H})}+\|u_N(t_n)-u_N^n\|_{L^p(\Omega,\mathbb{H})}. \tag{45}$$

By subtracting (43) from (32), we have

$$\begin{aligned}
\|u_N(t_n)-u_N^n\|_{L^p(\Omega,\mathbb{H})}=&\|\int_0^{t_n}\bar{E}(t_n-s)P_NF(u_N(s))\,ds+\int_0^{t_n}\tilde{E}(t_n-s)P_N\,dW^H(s) \\
&-\int_0^{t_n}\bar{E}_{\tau}(t_n-s)P_N\bar{F}(s)\,ds-\int_0^{t_n}\tilde{E}_{\tau}(t_n-s)P_N\partial_{\tau}W^H(s)\,ds\|_{L^p(\Omega,\mathbb{H})} \\
\leq&\left\|\int_0^{t_n}(\bar{E}(t_n-s)-\bar{E}_{\tau}(t_n-s))P_NF(u_N(s))\,ds\right\|_{L^p(\Omega,\mathbb{H})} \\
&+\left\|\int_0^{t_n}\bar{E}_{\tau}(t_n-s)P_N[F(u_N(s))-\bar{F}(s)]\,ds\right\|_{L^p(\Omega,\mathbb{H})} \\
&+\left\|\int_0^{t_n}(\tilde{E}(t_n-s)-\tilde{E}_{\tau}(t_n-s))P_N\,dW^H(s)\right\|_{L^p(\Omega,\mathbb{H})} \\
&+\left\|\int_0^{t_n}\tilde{E}_{\tau}(t_n-s)P_N\,dW^H(s)-\int_0^{t_n}\tilde{E}_{\tau}(t_n-s)P_N\partial_{\tau}W^H(s)\,ds\right\|_{L^p(\Omega,\mathbb{H})} \\
=&I+II+III+IV.
\end{aligned}$$

For I, employing Assumption condition (6), the regularity of u_N, resolvent estimate (20) and the first inequality in Lemma 8, we have

$$I=\left\|\int_0^{t_n}\left(\frac{1}{2\pi i}\int_{\Gamma}e^{z(t_n-s)}(z^{\alpha}+A)^{-1}\,dz-\frac{1}{2\pi i}\int_{\Gamma_{\tau}}e^{z(t_n-s)}(z_{\tau}^{\alpha}+A)^{-1}\frac{z\tau}{e^{z\tau}-1}\,dz\right)P_NF(u_N(s))\,ds\right\|_{L^p(\Omega,\mathbb{H})}$$

$$
\begin{aligned}
\leq & C\int_0^{t_n}\left\|\int_{\Gamma_\tau} e^{z(t_n-s)}\left[(z^\alpha+A)^{-1}-(z_\tau^\alpha+A)^{-1}\frac{z\tau}{e^{z\tau}-1}\right]dz\right\|_{\mathcal{L}(\mathbb{H})}\cdot\|P_NF(u_N(s))\|_{L^p(\Omega,\mathbb{H})}\,ds\\
&+C\int_0^{t_n}\left\|\int_{\Gamma/\Gamma_\tau} e^{z(t_n-s)}(z^\alpha+A)^{-1}dz\right\|_{\mathcal{L}(\mathbb{H})}\cdot\|P_NF(u_N(s))\|_{L^p(\Omega,\mathbb{H})}\,ds\\
\leq & C\tau\int_0^{t_n}\int_{\Gamma_\tau}|e^{z(t_n-s)}||z|^{1-\alpha}\,d|z|\,ds+C\int_0^{t_n}\int_{\Gamma/\Gamma_\tau}|e^{z(t_n-s)}||z|^{-\alpha}\,|d|z|\,ds\\
\leq & C\tau\int_0^{t_n}\int_0^{\frac{1}{\tau}}e^{-cr(t_n-s)}r^{1-\alpha}\,dr\,ds+C\int_0^{t_n}\int_{\frac{1}{\tau}}^{\infty}e^{-cr(t_n-s)}r^{-\alpha+\epsilon}r^{-\epsilon}\,dr\,ds\\
\leq & C\tau\int_0^{\frac{1}{\tau}}r^{-\alpha}\,dr+C\tau^{\alpha-\varepsilon}\int_0^{t_n}\int_{\frac{1}{\tau}}^{\infty}e^{-crs}r^{-\varepsilon}\,dr\,ds\leq C\tau^\alpha.
\end{aligned}
$$

As for II, we first divide it into three parts by using the following Taylor expansion

$$F(u_N(s))=F(u_N(t_{i-1}))+F'(u_N(t_{i-1}))(u_N(s)-u_N(t_{i-1}))+R_{F,i-1}(s).$$

Thus we obtain

$$
\begin{aligned}
II=&\left\|\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\bar{E}_\tau(t_n-s)P_N[F(u_N(s))-F(u_N^{i-1})]\,ds\right\|_{L^p(\Omega,\mathbb{H})}\\
\leq & C\left\|\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\bar{E}_\tau(t_n-s)P_N[F(u_N(s))-F(u_N(t_{i-1}))]\,ds\right\|_{L^p(\Omega,\mathbb{H})}\\
&+C\left\|\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\bar{E}_\tau(t_n-s)P_N[F(u_N(t_{i-1}))-F(u_N^{i-1})]\,ds\right\|_{L^p(\Omega,\mathbb{H})}\\
\leq & C\left\|\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\bar{E}_\tau(t_n-s)P_NF'(u_N(t_{i-1}))(u_N(s)-u_N(t_{i-1}))\,ds\right\|_{L^p(\Omega,\mathbb{H})}\\
&+C\left\|\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\bar{E}_\tau(t_n-s)P_NR_{F,i-1}(s)ds\right\|_{L^p(\Omega,\mathbb{H})}\\
&+C\left\|\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\bar{E}_\tau(t_n-s)P_N[F(u_N(t_{i-1}))-F(u_N^{i-1})]\,ds\right\|_{L^p(\Omega,\mathbb{H})}\\
=&II_1+II_2+II_3,
\end{aligned}
$$

where

$$R_{F,i-1}(s)=\int_0^1F''(u_N(t_{i-1})+\lambda(u_N(s)-u_N(t_{i-1})))(u_N(s)-u_N(t_{i-1}),u_N(s)-u_N(t_{i-1}))(1-\lambda)d\lambda.$$

As for II_1, noting that

$$u_N(s)=\int_0^s\bar{E}(s-\delta)P_NF(u_N(\delta))d\delta+\int_0^s\tilde{E}(s-\delta)P_N\,dW^H(\delta),$$

and

$$u_N(t_{i-1})=\int_0^{t_{i-1}}\bar{E}(t_{i-1}-\delta)P_NF(u_N(\delta))\,d\delta+\int_0^{t_{i-1}}\tilde{E}(t_{i-1}-\delta)P_NdW^H(\delta),$$

we get

$$
\begin{aligned}
II_1\leq & \Big\|\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\bar{E}_\tau(t_n-s)P_NF'(u_N(t_{i-1}))\Big[\int_0^s\bar{E}(s-\delta)P_NF(u_N(\delta))d\delta\\
&-\int_0^{t_{i-1}}\bar{E}(t_{i-1}-\delta)P_NF(u_N(\delta))\,d\delta\Big]\,ds\Big\|_{L^p(\Omega,\mathbb{H})}
\end{aligned}
$$

$$
\begin{aligned}
&+\Big\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\bar{E}_\tau(t_n-s)P_NF'(u_N(t_{i-1}))\Big[\int_0^s\tilde{E}(s-\delta)P_N\,dW^H(\delta)\\
&-\int_0^{t_{i-1}}\tilde{E}(t_{i-1}-\delta)P_NdW^H(\delta)\Big]ds\Big\|_{L^p(\Omega,\mathbb{H})}\\
\le&\Big\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\bar{E}_\tau(t_n-s)P_NF'(u_N(t_{i-1}))\int_0^{t_{i-1}}[\bar{E}(s-\delta)-\bar{E}(t_{i-1}-\delta)]P_NF(u_N(\delta))\,d\delta\,ds\Big\|_{L^p(\Omega,\mathbb{H})}\\
&+\Big\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\bar{E}_\tau(t_n-s)P_NF'(u_N(t_{i-1}))\int_{t_{i-1}}^{s}\bar{E}(s-\delta)P_NF(u_N(\delta))\,d\delta\,ds\Big\|_{L^p(\Omega,\mathbb{H})}\\
&+\Big\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\bar{E}_\tau(t_n-s)P_NF'(u_N(t_{i-1}))\int_0^{t_{i-1}}[\tilde{E}(s-\delta)-\tilde{E}(t_{i-1}-\delta)]P_N\,dW^H(\delta)\,ds\Big\|_{L^p(\Omega,\mathbb{H})}\\
&+\Big\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\bar{E}_\tau(t_n-s)P_NF'(u_N(t_{i-1}))\int_{t_{i-1}}^{s}\tilde{E}(s-\delta)P_N\,dW^H(\delta)\,ds\Big\|_{L^p(\Omega,\mathbb{H})}\\
=&II_{11}+II_{12}+II_{13}+II_{14}.
\end{aligned}
$$

Applying the assumptions (8), (6) on F, and Hölder inequality $\|uv\|_{L^p(\Omega)}\le\|u\|_{L^{2p}(\Omega)}\|v\|_{L^{2p}(\Omega)}$, and the spatial regularity of u_N, one can obtain

$$
\begin{aligned}
II_{11}\le&\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\|A^{\nu/2}\bar{E}_\tau(t_n-s)\|_{\mathcal{L}(\mathbb{H})}\|A^{-\nu/2}P_NF'(u_N(t_{i-1}))\int_0^{t_{i-1}}[\bar{E}(s-\delta)-\bar{E}(t_{i-1}-\delta)]P_NF(u_N(\delta))\,d\delta\|_{L^p(\Omega,\mathbb{H})}\,ds\\
\le&C\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_n-s)^{(1-\frac{\nu}{2})\alpha-1}(1+\|u_N(t_{i-1})\|_{L^{2p}(\Omega,\dot{H}^\mu)})\|\int_0^{t_{i-1}}\int_{t_{i-1}}^{s}\bar{E}'(\theta-\delta)P_NF(u_N(\delta))\,d\theta\,d\delta\|_{L^{2p}(\Omega,\dot{H}^{-\mu})}\,ds\\
\le&C\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_n-s)^{(1-\frac{\nu}{2})\alpha-1}\int_0^{t_{i-1}}\int_{t_{i-1}}^{s}(\theta-\delta)^{\alpha-2}d\theta\|F(u_N(\delta))\|_{L^{2p}(\Omega,\mathbb{H})}\,d\delta\,ds\\
\le&C\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_n-s)^{(1-\frac{\nu}{2})\alpha-1}(s-t_{i-1})^\alpha ds\le C\tau^\alpha.
\end{aligned}
$$

By simple calculation, we have

$$
\begin{aligned}
II_{12}\le&C\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_n-s)^{\alpha-1}\Big\|\int_{t_{i-1}}^{s}\bar{E}(s-\delta)P_NF(u_N(\delta))d\delta\Big\|_{L^p(\Omega,\mathbb{H})}\,ds\\
\le&C\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_n-s)^{\alpha-1}\int_{t_{i-1}}^{s}(s-\delta)^{\alpha-1}\,d\delta\,ds\le C\tau^\alpha.
\end{aligned}
$$

To estimate II_{13}, we proceed similarly to the estimate of II_{11}. By using the Hölder inequality, we obtain:

$$
\begin{aligned}
II_{13}\le&C\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_n-s)^{(1-\frac{\nu}{2})\alpha-1}\|P_NF'(u_N(t_{i-1}))\int_0^{t_{i-1}}(\tilde{E}(s-\delta)-\tilde{E}(t_{i-1}-\delta))P_N\,dW^H(\delta)\|_{L^p(\Omega,\dot{H}^{-\nu})}\,ds\\
\le&C\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_n-s)^{(1-\frac{\nu}{2})\alpha-1}\|\int_0^{t_{i-1}}A^{-\frac{\mu}{2}}(\tilde{E}(s-\delta)-\tilde{E}(t_{i-1}-\delta))P_N\,dW^H(\delta)\|_{L^{2p}(\Omega,\mathbb{H})}\,ds\\
\le&C_p\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_n-s)^{(1-\frac{\nu}{2})\alpha-1}\Big(\int_0^{t_{i-1}}\|{}_0^RD_\delta^{\frac{1-2H}{2}}A^{-\frac{\mu}{2}}(\tilde{E}(s-\delta)-\tilde{E}(t_{i-1}-\delta))P_N\|_{\mathcal{L}_2^0}^2d\delta\Big)^{\frac{1}{2}}\,ds\\
\le&C_p\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_n-s)^{(1-\frac{\nu}{2})\alpha-1}\Big(\int_0^{t_{i-1}}\|{}_0^RD_\delta^{\frac{1-2H}{2}}A^{\rho-\frac{\mu}{2}}(\tilde{E}(s-\delta)-\tilde{E}(t_{i-1}-\delta))\|_{\mathcal{L}_2^0}^2d\delta\Big)^{\frac{1}{2}}\,ds\|A^{-\rho}\|_{\mathcal{L}_2^0}.
\end{aligned}
$$

Case 1: For $1>\rho>\frac{\mu}{2}$, we have

$$
\int_0^{t_{i-1}}\|{}_0^RD_\delta^{\frac{1-2H}{2}}A^{\rho-\frac{\mu}{2}}(\tilde{E}(s-\delta)-\tilde{E}(t_{i-1}-\delta))\|^2\,d\delta
$$

$$\le C_p \int_0^{t_{i-1}} \| \int_\Gamma e^{z(t_{i-1}-\delta)}(e^{z(s-t_{i-1})}-1) z^{\frac{1-2H}{2}} A^{\rho-\frac{\mu}{2}} (z^\alpha + A)^{-1} z^{-\gamma}\, dz\|^2_{\mathcal{L}(\mathbb{H})}\, d\delta$$
$$\le C_p (s-t_{i-1})^{2a} \int_0^{t_{i-1}} \Big(\int_\Gamma e^{z\delta} |z|^{\alpha(\rho-\frac{\mu}{2}-1)-\gamma+a+\frac{1-2H}{2}}\, d|z|\Big)^2 d\delta$$
$$\le C_p (s-t_{i-1})^{2a} \int_0^{t_{i-1}} \delta^{2[\alpha(1-\rho+\frac{\mu}{2})+\gamma-a-1-\frac{1-2H}{2}]}\, d\delta \le C_p \tau^{2H+2\alpha(1-\rho+\frac{\mu}{2})+2\gamma-2-2\varepsilon}, \quad (46)$$

where we divide $\Gamma = \Gamma^\infty_{\delta^{-1}} \cup \gamma_{\theta,\delta^{-1}} = \{z : \arg(z) = \theta,\ \delta^{-1} \le |z| < \infty\} \cup \{z : |z| = \delta^{-1} e^{i\varphi}, -\theta \le \varphi \le \theta\}$ and choose $0 < a = H + \alpha(1-\rho+\frac{\mu}{2}) + \gamma - 1 - \varepsilon < 1$ due to $2[\alpha(1-\rho+\frac{\mu}{2})+\gamma-a-1-\frac{1-2H}{2}] > -1$, which implies $a < H + \alpha(1-\rho+\frac{\mu}{2})+\gamma-1$.

Case 2: For $\rho \le \frac{\mu}{2}$, we arrive, similarly to Case 1,

$$\int_0^{t_{i-1}} \| {}^R_0 D_\delta^{\frac{1-2H}{2}} A^{\rho-\frac{\mu}{2}} (\tilde{E}(s-\delta) - \tilde{E}(t_{i-1}-\delta)) \|^2\, d\delta$$
$$\le C_p \int_0^{t_{i-1}} \| \int_\Gamma e^{z(t_{i-1}-\delta)}(e^{z(s-t_{i-1})}-1) z^{\frac{1-2H}{2}} (z^\alpha + A)^{-1} z^{-\gamma}\, dz\|^2_{\mathcal{L}(\mathbb{H})}\, d\delta$$
$$\le C_p (s-t_{i-1})^{2a} \int_0^{t_{i-1}} \delta^{2[\alpha(1-\rho+\frac{\mu}{2})+\gamma-a-1-\frac{1-2H}{2}]}\, d\delta \le C_p \tau^{(2H+2\alpha+2\gamma-2-2\varepsilon)}. \quad (47)$$

As for II_{14}, similar to the estimate of II_{12}, note that ${}^R_0 D_\delta^{\frac{1-2H}{2}} A^\rho \tilde{E}(\delta) = \frac{1}{2\pi i} \int_\Gamma e^{z\delta} z^{\frac{1-2H}{2}} z^{\alpha(\rho-1)-\gamma}\, dz$, we obtain

$$II_{14} = \| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \bar{E}_\tau(t_n-s) P_N F'(u_N(t_{i-1})) \int_{t_{i-1}}^{t_i} \chi_{[t_{i-1},s]}(\delta) \tilde{E}(s-\delta) P_N\, dW^H(\delta)\, ds\|_{L^p(\Omega,\mathbb{H})}$$
$$\le C \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|\bar{E}_\tau(t_n-s) P_N F'(u_N(t_{i-1})) \int_{t_{i-1}}^{t_i} \chi_{[t_{i-1},s]}(\delta) \tilde{E}(s-\delta) P_N\, dW^H(\delta)\|_{L^p(\Omega,\mathbb{H})}\, ds$$
$$\le C \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_n-s)^{(1-\frac{\nu}{2})\alpha-1} \| \int_{t_{i-1}}^{t_i} A^{-\frac{\mu}{2}} \chi_{[t_{i-1},s]}(\delta) \tilde{E}(s-\delta) P_N\, dW^H(\delta)\|_{L^p(\Omega,\mathbb{H})}\, ds$$
$$\le C_p \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_n-s)^{(1-\frac{\nu}{2})\alpha-1} \Big(\int_{t_{i-1}}^{t_i} \| {}^R_0 D_\delta^{\frac{1-2H}{2}} A^{\rho-\frac{\mu}{2}} \chi_{[t_{i-1},s]} \tilde{E}(\delta)\|^2_{\mathcal{L}(\mathbb{H})} d\delta\Big)^{1/2} ds \cdot \|A^{-\rho}\|_{\mathcal{L}_2^0}$$
$$\le C_p \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_n-s)^{(1-\frac{\nu}{2})\alpha-1} \Big(\int_{s-t_i}^{s-t_{i-1}} \delta^{-2[\alpha(\rho-1-\frac{\mu}{2})-\gamma+\frac{1-2H}{2}+1]}\, d\delta\Big)^{1/2} ds$$
$$\le C_p \tau^{-\alpha(\rho-1-\frac{\mu}{2})+\gamma+H-1} \int_0^{t_n} (t_n-s)^{\alpha-1} \le C_p \tau^{\alpha(1+\frac{\mu}{2}-\rho)+\gamma+H-1}.$$

As for II_2, applying assumption (9) and Theorem 2 , we give

$$II_2 \le C \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|A^{\frac{\eta}{2}} \bar{E}_\tau(t_n-s) A^{-\frac{\eta}{2}} P_N R_{F,i-1}(s)\|_{L^p(\Omega,\mathbb{H})}\, ds$$
$$\le C \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_n-s)^{(1-\frac{\eta}{2})\alpha-1} \|R_{F,i-1}(s)\|_{L^p(\Omega,\dot{H}^{-\eta})}\, ds$$
$$\le C \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_n-s)^{(1-\frac{\eta}{2})\alpha-1} \|u_N(s) - u_N(t_{i-1})\|^2_{L^{2p}(\Omega,\mathbb{H})}\, ds$$
$$\le C\tau^{2\beta} \le C\tau^\beta. \quad (48)$$

Now we estimate II_3. Applying Assumption condition (7), we have

$$II_3 \le C \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|\bar{E}_\tau(t_n-s) P_N\|_{\mathcal{L}(\mathbb{H})} \|F(u_N(t_{i-1})) - F(u_N^{i-1})\|_{L^p(\Omega,\mathbb{H})}\, ds$$
$$\le C \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_n-s)^{\alpha-1} \|u_N(t_{i-1}) - u_N^{i-1}\|_{L^p(\Omega,\mathbb{H})}\, ds$$

$$\leq C\tau^{\alpha}\sum_{i=1}^{n}\|u_N(t_{i-1})-u_N^{i-1}\|_{L^p(\Omega,\mathbb{H})}. \tag{49}$$

For III, we can split it as

$$\begin{aligned}
III^2 &= \|\int_0^{t_n}(\tilde{E}(t_n-s)-\tilde{E}_\tau(t_n-s))P_N\,dW^H(s)\|^2_{L^p(\Omega,\mathbb{H})}\\
&\leq C_p\int_0^{t_n}\|{}_0^R D_s^{\frac{1-2H}{2}}(\tilde{E}(s)-\tilde{E}_\tau(s))P_N\|^2_{\mathcal{L}_2^0}ds\\
&\leq C_p\int_0^{t_n}\|{}_0^R D_s^{\frac{1-2H}{2}}A^\rho(\tilde{E}(s)-\tilde{E}_\tau(s))\|^2_{\mathcal{L}(\mathbb{H})}\,ds\,\|A^{-\rho}\|^2_{\mathcal{L}_2^0}\\
&\leq C_p\int_0^{t_n}\|\int_{\Gamma/\Gamma_\tau}e^{zs}z^{\frac{1-2H}{2}}A^\rho z^{-\gamma}(z^\alpha+A)^{-1}dz\|^2_{\mathcal{L}(\mathbb{H})}\,ds\\
&+C_p\int_0^{t_n}\|\int_{\Gamma_\tau}e^{zs}z^{\frac{1-2H}{2}}A^\rho\big(z^{-\gamma}(z^\alpha+A)^{-1}-z_\tau^{-\gamma}(z_\tau^\alpha+A)^{-1}\frac{z\tau}{e^{z\tau}-1}\big)dz\|^2_{\mathcal{L}(\mathbb{H})}\,ds\\
&= III_1+III_2.
\end{aligned} \tag{50}$$

For III_1, we have

$$\begin{aligned}
III_1 &\leq C_p\int_0^{t_n}(\int_{1/\tau}^\infty e^{-crt}r^{\frac{1-2H}{2}}r^{-\gamma}r^{\alpha(\rho-1)}r^{-b_2}r^{b_2}\,dr)^2\,dt\\
&\leq C_p\tau^{2b_2}\int_0^{t_n}(\int_0^\infty e^{-crt}r^{\frac{1-2H}{2}-\gamma+\alpha(\rho-1)+b_2}\,dr)^2\,dt\\
&\leq C_p\tau^{2b_2}\int_0^{t_n}t^{2H+2\gamma-2\alpha(\rho-1)-3-2b_2}\,dt\leq C_p\tau^{2b_2}=C_p\tau^{2H+2\alpha(1-\rho)+2\gamma-2-\epsilon}.
\end{aligned} \tag{51}$$

Here we choose $0<b_2=\frac{2H-1}{2}+\gamma+\alpha(1-\rho)-\frac{1}{2}-\frac{\epsilon}{2}$ with $\epsilon\in(0,1)$ such that $\int_0^\infty e^{-crt}r^{\frac{1-2H}{2}-\gamma+\alpha(\rho-1)+b_2}\,dr<\infty$ and $\int_0^{t_n}t^{2H-1+2\gamma-2\alpha(\rho-1)-2-b_2}<\infty$.

For III_2, by Cauchy–Schwarz inequality and Lemma 8, there holds

$$\begin{aligned}
III_2 &\leq C_p\int_0^{t_n}\big(\int_{\Gamma_\tau}1d|z|\big)\int_{\Gamma_\tau}\|e^{zt}z^{\frac{1-2H}{2}}A^\rho[z^{-\gamma}(z^\alpha+A)^{-1}-z_\tau^{-\gamma}(z_\tau^\alpha+A)^{-1}\frac{z\tau}{e^{z\tau}-1}]\|^2_{\mathcal{L}(\mathbb{H})}\,d|z|\,dt\\
&\leq C\tau^2\int_0^{t_n}(\int_0^{1/\tau}1\,dr)(\int_0^{1/\tau}e^{-crt}r^{1-2H}r^{2-2\alpha(1-\rho)-2\gamma}\,dr)\,dt\\
&\leq C\tau\int_0^{t_n}\big(\int_0^{1/\tau}e^{-crt}r^{3-2H-2\alpha(1-\rho)-2\gamma}\,dr\big)\,dt.
\end{aligned} \tag{52}$$

On one hand, we have

$$III_2\leq C_p\tau\int_0^{1/\tau}r^{2-2H-2\alpha(1-\rho)-2\gamma}dr\leq C\tau^{2H-2+2\alpha(1-\rho)+2\gamma},$$

where we require $2H-2+2\alpha(1-\rho)+2\gamma<1$ to preserve the boundedness of III_2.

On the other hand, we have

$$III_2\leq C_p\tau\int_0^{t_n}t^{2H-4+2\alpha(1-\rho)+2\gamma}\,dt\leq C\tau.$$

Meanwhile, we need $2H-2+2\alpha(1-\rho)+2\gamma>1$ to preserve the boundedness of III_2.

Finally, we turn to estimate IV. Denote $G_\tau(r)=\frac{1}{\tau}\int_{t_i}^{t_{i+1}}\tilde{E}_\tau(\bar{r})\,d\bar{r}$. By the inequality (17) and variable substitution, we have

$$\begin{aligned}
IV^2 &= \|\int_0^{t_n}\tilde{E}_\tau(t_n-s)P_N\,dW^H(s)-\int_0^{t_n}\tilde{E}_\tau(t_n-s)\partial_\tau W^H(s)P_N\,ds\|^2_{L^p(\Omega,\mathbb{H})}\\
&= \|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\tilde{E}_\tau(t_n-s)P_N\,dW^H(s)-\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\tilde{E}_\tau(t_n-\bar{s})P_N(\frac{1}{\tau}\int_{t_i}^{t_{i+1}}dW^H(s))d\bar{s}\|^2_{L^p(\Omega,\mathbb{H})}
\end{aligned}$$

$$
\begin{aligned}
&= \| \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (\tilde{E}_\tau(t_n - s) - \frac{1}{\tau} \int_{t_i}^{t_{i+1}} \tilde{E}_\tau(t_n - \bar{s}) d\bar{s}) dW^H(s) \|^2_{L^p(\Omega, \mathbb{H})} \\
&\le C_p \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \| {}_0^R D_r^{\frac{1-2H}{2}} A^\rho [\tilde{E}_\tau(r) - \frac{1}{\tau} \int_{t_i}^{t_{i+1}} \tilde{E}_\tau(\bar{r})\, d\bar{r}] \|^2_{\mathcal{L}(\mathbb{H})} \, dr \| A^{-\rho} \|^2_{L_2^0} \\
&\le C_p \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \| {}_0^R D_r^{\frac{1-2H}{2}} A^\rho [\tilde{E}_\tau(r) - G_\tau(r)] \|^2_{\mathcal{L}(\mathbb{H})} \, dr.
\end{aligned}
$$

Due to Lemmas A1 and A2 in Appendix A, one can split IV^2 into three parts.

$$
\begin{aligned}
IV^2 \le & C_p \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \| {}_0^R D_r^{\frac{1-2H}{2}} A^\rho [\tilde{E}_\tau(r) - G_\tau(r)] \|^2_{\mathcal{L}(\mathbb{H})} \, dr \\
\le & C_p \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \| \frac{1}{2\pi i} \int_{\Gamma/\Gamma_\tau} e^{zr} A^\rho \hat{G}_\tau(z) z^{\frac{1-2H}{2}} dz \|^2_{\mathcal{L}(\mathbb{H})} \, dr \\
& + C_p \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \| \frac{1}{2\pi i} \int_{\Gamma_\tau} e^{zr} A^\rho [\hat{\tilde{E}}_\tau(z) - \hat{G}_\tau(z)] z^{\frac{1-2H}{2}} \, dz \|^2_{\mathcal{L}(\mathbb{H})} \, dr \\
& + C \int_0^{t_1} \| {}_0^R D_r^{\frac{1-2H}{2}} A^\rho [\tilde{E}_\tau(r) - G_\tau(r)] \|^2_{\mathcal{L}(\mathbb{H})} \, dr \\
= & IV_1 + IV_2 + IV_3.
\end{aligned}
$$

Here we split $[0, t_n]$ into $[0, t_1]$ and $[t_1, t_n]$ to guarantee $r\cos\theta + (1 - \alpha - \gamma + \alpha\rho)\tau < 0$.

Note that $0 < \rho < \min\{1, \frac{H}{\alpha} + 1 + \frac{\gamma - 1}{\alpha}\}$, then one has $t\cos\theta + (1 - \alpha - \gamma + \alpha\rho)\tau < t\cos\theta + \tau < 0$ for $t > t_1$. Thus, by Lemmas A2 and A3 in Appendix A, we have the following cases:

If $1 - \alpha(1-\rho) - \gamma \ge 0$,

$$
\begin{aligned}
IV_1 &\le C_p \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \| \frac{1}{2\pi i} \int_{\Gamma/\Gamma_\tau} e^{zt} A^\rho \hat{G}_\tau(z) z^{\frac{1-2H}{2}} dz \|^2_{L(H)} \, dt \\
&\le C_p \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \Big(\int_{\Gamma/\Gamma_\tau} |e^{zt}| |z|^{\alpha(\rho-1)-\gamma} e^{(1-\alpha-\gamma+\alpha\rho)|z|\tau} |z|^{\frac{1-2H}{2}} \, d|z| \Big)^2 \, dt \\
&\le C_p \int_{t_1}^{t_n} \Big(\int_{\frac{1}{\tau}}^{\infty} e^{(t\cos\theta + (1-\alpha-\gamma+\alpha\rho)\tau)|z|} r^{\frac{1-2H}{2} + \alpha(\rho-1) - \gamma} \, dr \Big)^2 \, dt \\
&\le C_p \int_{t_1}^{t_n} \Big(\int_{\frac{1}{\tau}}^{\infty} e^{-crt} r^{\frac{1-2H}{2} + \alpha(\rho-1) - \gamma} \, dr \Big)^2 \, dt \\
&\le C_p \int_{t_1}^{t_n} \Big(\int_{\frac{1}{\tau}}^{\infty} e^{-crt} r^{\frac{1-2H}{2} + \alpha(\rho-1) - \gamma - \frac{\epsilon}{2}} r^{\frac{\epsilon}{2}} \, dr \Big)^2 \, dt \\
&\le C_p \tau^{2H + 2\alpha(1-\rho) + 2\gamma - 1 + \epsilon} \int_{t_1}^{t_n} \Big(\int_0^{\infty} e^{-crt} r^{\frac{\epsilon}{2}} \, dr \Big)^2 \, dt \\
&\le C_p \tau^{2H + 2\alpha(1-\rho) + 2\gamma - 1 + \epsilon} \int_{t_1}^{t_n} t^{-\epsilon - 2} \, dt \\
&\le C_p \tau^{2H + 2\alpha(1-\rho) + 2\gamma - 2}.
\end{aligned} \tag{53}
$$

If $1 - \alpha(1-\rho) - \gamma < 0$, then

$$
\begin{aligned}
IV_1 &\le C_p \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \| \frac{1}{2\pi i} \int_{\Gamma/\Gamma_\tau} e^{zt} A^\rho \hat{G}_\tau(z) z^{\frac{1-2H}{2}} dz \|^2_{\mathcal{L}(\mathbb{H})} \, dt \\
&\le C_p \tau^{2\alpha(1-\rho) + 2\gamma - 2 + 2H - 2\varepsilon} \int_{t_1}^{t_n} \Big(\int_{\frac{1}{\tau}}^{\infty} |e^{zt}| |z|^{-\frac{1}{2} - \varepsilon} \, d|z| \Big)^2 \, dt
\end{aligned}
$$

$$\leq C_p \tau^{2\alpha(1-\rho)+2\gamma-2+2H-2\varepsilon} \int_{t_1}^{t_n} t^{2\varepsilon-1}\, dt \leq C_p \tau^{2\alpha(1-\rho)+2\gamma-2+2H-2\varepsilon}. \tag{54}$$

As for IV_2, by Lemma 8, the Assumption condition 2, and $0 < \rho < \min\{1, \frac{H}{\alpha} + 1 + \frac{\gamma-1}{\alpha}\}$, there holds

$$\begin{aligned}
IV_2 &\leq \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \| \int_{\Gamma_\tau} e^{zt} A^\rho \big[z_\tau^{-\gamma} (z_\tau^\alpha + A)^{-1} \frac{z\tau}{e^{z\tau}-1} - z_\tau^{1-\gamma} z^{-1} (z_\tau^\alpha + A)^{-1} \big] z^{\frac{1-2H}{2}} dz \|_{\mathcal{L}(\mathbb{H})}^2 dt \\
&\leq C_p \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \{ \int_0^{\frac{1}{\tau}} d|z| \cdot \int_0^{\frac{1}{\tau}} |e^{zt}|^2 \| A^\rho [z_\tau^{-\gamma} (z_\tau^\alpha + A)^{-1} \frac{z\tau}{e^{z\tau}-1} - z_\tau^{-\gamma} (z_\tau^\alpha + A)^{-1}] \|_{\mathcal{L}(\mathbb{H})}^2 |z|^{1-2H} d|z| \} dt \\
&\leq C_p \tau \int_{t_1}^{t_n} \int_0^{\frac{1}{\tau}} e^{-crt} r^{-2\alpha(1-\rho)+2(1-\gamma)+1-2H} dr dt \\
&\leq C_p \tau \int_{t_1}^{t_n} t^{2\alpha(1-\rho)+2\gamma+2H-4}\, dt.
\end{aligned} \tag{55}$$

If $-1 < 2\alpha(1-\rho) + 2\gamma + 2H - 3 < 0$, then

$$IV_2 \leq C_p \tau t_1^{2\alpha(1-\rho)+2\gamma+2H-3} \leq C_p \tau^{2H+2\alpha(1-\rho)+2\gamma-2}.$$

If $0 \leq 2\alpha(1-\rho) + 2\gamma + 2H - 3 < 1$, we have

$$IV_2 \leq C_p \tau t_n^{2\alpha(1-\rho)+2\gamma+2H-3} \leq C_p \tau.$$

For IV_3, we first have the following estimate, with $t \in [0, t_1]$,

$$\begin{aligned}
&\| A^\rho [\tilde{E}_\tau(t) - G_\tau(t)] \|_{\mathcal{L}(\mathbb{H})} = \| A^\rho [\tilde{E}_\tau(t) - \frac{1}{\tau} \int_0^{t_1} \tilde{E}_\tau(\bar{t}) d\bar{t}] \|_{\mathcal{L}(\mathbb{H})} \\
&= \| A^\rho [\frac{1}{\tau} \int_0^{t_1} (\tilde{E}_\tau(t) - \tilde{E}_\tau(\bar{t})) d\bar{t}] \|_{\mathcal{L}(\mathbb{H})} \\
&\leq C\tau^{-1} \| \int_0^{t_1} \int_{\Gamma_\tau} e^{zt} (1 - e^{z(\bar{t}-t)}) A^\rho (z^\alpha + A)^{-1} z^{-\gamma}\, dz\, d\bar{t} \|_{\mathcal{L}(\mathbb{H})} \\
&\leq C\tau^a \int_{\Gamma_\tau} |e^{zt}| |z|^{\alpha(\rho-1)-\gamma+a}\, d|z| \leq C\tau^a t^{\alpha(1-\rho)+\gamma-a-1},
\end{aligned} \tag{56}$$

where we have used $|1 - e^{z(\bar{t}-t)}| \leq C\tau^a |z|^a$, $a \in (0,1)$.

If $H = \frac{1}{2}$, note that $0 < H + \alpha(1-\rho) + \gamma - 1 < 1$, by choosing $a = 1$, we have

$$\begin{aligned}
IV_3 &\leq C_p \int_0^{t_1} \| A^\rho [\tilde{E}_\tau(t) - G_\tau(t)] \|_{\mathcal{L}(\mathbb{H})}^2\, dt \leq C\tau^2 \int_0^{t_1} t^{2\alpha(1-\rho)+2\gamma-4}\, dt \\
&\leq C\tau^{2\alpha(1-\rho)+2\gamma-1} = C\tau^{2\alpha(1-\rho)+2\gamma+2H-2}.
\end{aligned} \tag{57}$$

If $H \in (\frac{1}{2}, 1)$, by choosing $0 < a = H + \alpha(1-\rho) + \gamma - 1 - \epsilon < 1$, we have

$$\begin{aligned}
IV_3 &\leq C_p \int_0^{t_1} \| {}_0^R D_t^{\frac{1-2H}{2}} A^\rho [\tilde{E}_\tau(t) - G_\tau(t)] \|_{\mathcal{L}(\mathbb{H})}^2\, dt \\
&\leq C_p \int_0^{t_1} \Big(\int_0^t (t-s)^{\frac{2H-1}{2}-1} \| A^\rho (\tilde{E}_\tau(s) - G_\tau(\bar{s})) \|_{\mathcal{L}(\mathbb{H})}\, ds \Big)^2 dt \\
&\leq C_p \tau^{2a} \int_0^{t_1} \Big(\int_0^t (t-s)^{\frac{2H-1}{2}-1} s^{\alpha(1-\rho)+\gamma-a-1}\, ds \Big)^2 dt \\
&\leq C_p \tau^{2a} \int_0^{t_1} t^{2[\frac{2H-1}{2}+\alpha(1-\rho)+\gamma-a-1]}\, dt \\
&\leq C_p \tau^{2H+2\alpha(1-\rho)+2\gamma-2}.
\end{aligned} \tag{58}$$

If $H \in (0, \frac{1}{2})$, by choosing $0 < a = H + \alpha(1-\rho) + \gamma - 1 - \epsilon < 1$, we have

$$\begin{aligned}
IV_3 &\leq C_p \int_0^{t_1} \| {}_0^R D_t^{\frac{1-2H}{2}} A^\rho [\tilde{E}_\tau(t) - G_\tau(t)] \|^2_{\mathcal{L}(\mathbb{H})} \, dt \\
&\leq C_p \int_0^{t_1} \Big(\frac{d}{dt} \int_0^t (t-s)^{\frac{2H-1}{2}} \| A^\rho (\tilde{E}_\tau(s) - G_\tau(\bar{s})) \|_{\mathcal{L}(\mathbb{H})} \, d\bar{s} \, ds \Big)^2 dt \\
&\leq C_p \tau^{2a} \int_0^{t_1} \Big(\frac{d}{dt} \int_0^t (t-s)^{\frac{2H-1}{2}} s^{\alpha(1-\rho)+\gamma-a-1} \, ds \Big)^2 dr \\
&\leq C_p \tau^{2a} \int_0^{t_1} [(t^{\frac{2H-1}{2} + \alpha(1-\rho) + \gamma - a})']^2 \, dt \\
&\leq C_p \tau^{2a} \int_0^{t_1} t^{2[\frac{2H-1}{2} + \alpha(1-\rho) + \gamma - a - 1]} \, dt \\
&\leq C_p \tau^{2H + 2\alpha(1-\rho) + 2\gamma - 2}.
\end{aligned} \tag{59}$$

Finally combining the above estimates, and employing Theorem 3 we have

$$\begin{aligned}
&\| u(t_n) - u_N^n \|_{L^p(\Omega, \mathbb{H})} \\
&\leq C_p (N+1)^{-\frac{\sigma}{d}} + C_p \tau^\alpha + C_p \tau^{\alpha - \epsilon} + C_p \tau^{H + \alpha(1-\rho) + \gamma - 1 - \epsilon} + C_p \tau^{H + \alpha(1-\rho) + \gamma - 1} + C\tau^\beta \\
&\quad + C\tau^\alpha \sum_{i=1}^n \| u(t_{j-1}) - u_N^{j-1} \|_{L^p(\Omega, \mathbb{H})} \, ds,
\end{aligned}$$

where $0 < \beta = \min\{\alpha, H + \alpha(1-\rho) + \gamma - 1 - \epsilon\}$. By discrete Grönwall inequality, one has

$$\| u(t_n) - u_N^n \|_{L^p(\Omega, \mathbb{H})} \leq C(N+1)^{-\frac{\sigma}{d}} + C\tau^\beta,$$

where $0 < \beta = \min\{\alpha, H + \alpha(1-\rho) + \gamma - 1 - \epsilon\} < 1$.

The proof of Theorem 4 is complete. □

6. Numerical Experiments

In this section, we provide numerical results for the following time-fractional semilinear stochastic partial differential Equation (SPDE), where $\alpha \in (0, 1)$ and $\gamma \in [0, 1]$.

$${}_0^C D_t^\alpha u(t) + Au(t) = \sin(u(t)) + {}_0^R D_t^{-\gamma} \frac{dW^H(t)}{dt}, \quad \text{for } 0 < t \leq T, \quad \text{with } u(0) = 0, \tag{60}$$

Assume that the covariance operator Q of the $\mathbb{H}$-valued fractional Wiener process $W^H(t)$ has eigenvalues given by $\gamma_k = k^m$ for $k = 1, 2, \ldots$, where $m \leq 0$. Two cases are of particular interest in applications:

Case 1. When $m = 0$, $W^H(t)$ is referred to as white noise since $\mathrm{tr}(Q) = \sum_{k=1}^\infty \gamma_k = \sum_{k=1}^\infty k^0 = \infty$.

Case 2. When $m < -1$, $W^H(t)$ is known as trace-class noise, as $\mathrm{tr}(Q) = \sum_{k=1}^\infty \gamma_k = \sum_{k=1}^\infty k^m < \infty$.

Under the assumption that $\|A^{-\rho}\|_{\mathcal{L}_2^0} = \|A^{-\rho} Q^{1/2}\| HS < \infty$, it follows that ρ is approximately $\frac{(m+1)d}{4}$, where $d = 1, 2, 3$ denotes the dimension of the space variable [2]. This result is derived from the following observation, noting $\lambda_k \approx k^{2/d}$,

$$\|A^{-\rho} Q^{1/2}\|^2_{HS} = \sum_{k=1}^\infty \lambda_k^{2\rho} \gamma_k = \sum_{k=1}^\infty (k^{2/d})^{-2\rho} k^m = \sum_{k=1}^\infty k^{-\frac{4\rho}{d} + m} < \infty,$$

if $\rho > \frac{(m+1)d}{4}$. Based on this, we observe that in the trace-class case, when $m = -1$, we have $\rho \approx 0$. In contrast, in the white noise case, when $m = 0$, we have $\rho \approx d/4$.

By Theorem 4, the convergence rate in time is given by:

$$\| u(t_n) - u_N^n \|_{L^p(\Omega, \mathbb{H})} \leq C\tau^\beta,$$

where $0 < \beta = \min\{\alpha, H + \alpha(1-\rho) + \gamma - 1 - \epsilon\}$.

In the numerical simulations presented below, we experimentally determine the convergence rates for the following two cases:

Case 1. When $m = 0$, $W^H(t)$ corresponds to white noise. In this case, we have $\rho \approx 1/4$ in the one-dimensional case. Therefore, the theoretical convergence rate is $O(\tau^{\min(\alpha, H+\frac{3}{4}\alpha+\gamma-1)})$.

Case 2. When $m < -1$, $W^H(t)$ corresponds to trace-class noise. Here, we have $\rho \approx 0$, leading to a theoretical convergence rate of $O(\tau^{\min(\alpha, H+\alpha+\gamma-1)})$.

Let Δt represent the time step size for the partition $0 = t_1 < t_2 < \cdots < t_M = T$, where T is the final time. We demonstrate the numerical simulations using a one-dimensional example on the unit interval $D = (0,1)$. In our computations, we set $N = 4, 8, 16, 32, 64$ and the time step size $\Delta t = T/N$. All expected values are computed using $M = 60$ trajectories. We focus solely on examining the temporal convergence rates.

The final time is set to $T = 0.1$. The reference solution is computed using a much finer temporal mesh with $N = 512$. The numerical results for various combinations of the Hurst parameter H, the fractional orders α and γ, as well as for both trace-class noise ($m = -1$) and white noise ($m = 0$), are provided in Tables 1–4. All numerical simulations were conducted using MATLAB R2018a (version 9.4).

In Tables 1–4, the numbers in parentheses in the "rate" column indicate the theoretical rates predicted by Theorem 4. The experimentally determined convergence rates in time are in good agreement with the theoretical predictions, fully confirming the error analysis, despite the relatively small number of trajectories used to compute the expectations. The convergence improves consistently as the fractional orders α and γ and H increase, reflecting enhanced temporal regularity of the solution. The running times are provided in the final columns in Tables 1–4.

In Table 1, we set $\gamma = 0$ and $H = 0.5$. For $m = 0$ (the white noise case), the theoretical convergence order is $O(\tau^{\min(\alpha, H+\frac{3}{4}\alpha+\gamma-1)}) = O(\tau^{\min(\alpha, \frac{3}{4}\alpha-0.5)})$. For $m = -1$ (the trace-class noise case), the theoretical convergence rate becomes $O(\tau^{\min(\alpha, H+\alpha+\gamma-1)}) = O(\tau^{\alpha-0.5})$. In both cases, we observe that the experimentally determined convergence orders exceed the theoretical expectations.

In Table 2, we set $\gamma = 0$ and $H = 0.8$. For $m = 0$ (the white noise case), the theoretical convergence order is $O(\tau^{\min(\alpha, H+\frac{3}{4}\alpha+\gamma-1)}) = O(\tau^{\min(\alpha, \frac{3}{4}\alpha-0.2)})$. For $m = -1$ (the trace-class noise case), the theoretical convergence rate is $O(\tau^{\min(\alpha, H+\alpha+\gamma-1)}) = O(\tau^{\alpha-0.2})$. Again, the observed experimental orders are higher than the theoretical values.

In Table 3, we set $\gamma = 1$ and $H = 0.5$. For $m = 0$ (the white noise case), the theoretical convergence order is $O(\tau^{\min(\alpha, H+\frac{3}{4}\alpha+\gamma-1)}) = O(\tau^{\min(\alpha, \frac{3}{4}\alpha+0.5)})$. For $m = -1$ (the trace-class noise case), the theoretical convergence rate is $O(\tau^{\min(\alpha, H+\alpha+\gamma-1)}) = O(\tau^{\alpha})$. The experimentally determined orders, once more, surpass the theoretical predictions.

In Table 4, we set $\gamma = 1$ and $H = 0.8$. For $m = 0$ (the white noise case), the theoretical convergence order is $O(\tau^{\min(\alpha, H+\frac{3}{4}\alpha+\gamma-1)}) = O(\tau^{\min(\alpha, \frac{3}{4}\alpha+0.8)})$. For $m = -1$ (the trace-class noise case), the theoretical convergence rate is $O(\tau^{\min(\alpha, H+\alpha+\gamma-1)}) = O(\tau^{\alpha})$. As observed in the previous cases, the experimental convergence rates exceed the theoretical predictions.

Table 1. The $L^2(\Omega; H)$-error at $t = 0.1$ with $\gamma = 0$ and $H = 0.5$.

m	$\alpha \backslash N$	4	8	16	32	64	Rate	CPU Time
0	0.3	4.45×10^{-0}	4.36×10^{-0}	4.23×10^{-0}	4.22×10^{-0}	3.52×10^{-0}	0.08 (−0.27)	165 s
	0.5	2.04×10^{-0}	1.93×10^{-0}	1.81×10^{-0}	1.75×10^{-0}	1.37×10^{-0}	0.11 (−0.12)	165 s
	0.7	8.36×10^{-1}	7.41×10^{-1}	6.45×10^{-1}	5.73×10^{-1}	4.23×10^{-1}	0.15 (0.02)	165 s
	0.9	3.35×10^{-1}	2.81×10^{-1}	2.26×10^{-1}	1.87×10^{-1}	1.34×10^{-1}	0.20 (0.17)	165 s
−1	0.3	4.25×10^{-0}	4.15×10^{-0}	4.05×10^{-0}	4.04×10^{-0}	3.35×10^{-0}	0.08 (−0.20)	165 s
	0.5	1.79×10^{-0}	1.65×10^{-0}	1.57×10^{-0}	1.53×10^{-0}	1.16×10^{-0}	0.11 (0.00)	165 s
	0.7	6.13×10^{-1}	4.89×10^{-1}	4.03×10^{-1}	3.68×10^{-1}	2.48×10^{-1}	0.30 (0.20)	165 s
	0.9	1.92×10^{-1}	1.29×10^{-1}	7.82×10^{-2}	5.98×10^{-2}	3.56×10^{-2}	0.50 (0.40)	165 s

Table 2. The $L^2(\Omega; H)$-error at $t = 0.1$ with $\gamma = 0$ and $H = 0.8$.

m	$\alpha \backslash N$	4	8	16	32	64	Rate	CPU Time
0	0.3	2.75×10^{-1}	2.57×10^{-1}	2.46×10^{-1}	2.25×10^{-1}	1.67×10^{-1}	0.10 (0.02)	169 s
	0.5	1.59×10^{-1}	1.35×10^{-1}	1.18×10^{-1}	9.91×10^{-2}	6.73×10^{-2}	0.25 (0.17)	169 s
	0.7	8.63×10^{-2}	6.47×10^{-2}	4.71×10^{-2}	3.40×10^{-2}	2.11×10^{-2}	0.45 (0.33)	169 s
	0.9	4.49×10^{-2}	3.03×10^{-2}	1.88×10^{-2}	1.22×10^{-2}	7.16×10^{-3}	0.60 (0.48)	169 s
−1	0.3	2.63×10^{-1}	2.45×10^{-1}	2.36×10^{-1}	2.17×10^{-1}	1.60×10^{-1}	0.17 (0.10)	169 s
	0.5	1.44×10^{-1}	1.18×10^{-1}	1.04×10^{-1}	8.85×10^{-2}	5.84×10^{-2}	0.32 (0.30)	169 s
	0.7	7.24×10^{-2}	4.81×10^{-2}	3.17×10^{-2}	2.25×10^{-2}	1.29×10^{-2}	0.62 (0.50)	169 s
	0.9	3.33×10^{-2}	1.91×10^{-2}	8.98×10^{-3}	4.84×10^{-3}	2.42×10^{-3}	0.90 (0.70)	169 s

Table 3. The $L^2(\Omega; H)$-error at $t = 0.1$ with $\gamma = 1$ and $H = 0.5$.

m	$\alpha \backslash N$	4	8	16	32	64	Rate	CPU Time
0	0.3	2.68×10^{-3}	1.80×10^{-3}	1.26×10^{-3}	1.16×10^{-3}	7.20×10^{-4}	0.40 (0.30)	165 s
	0.5	5.73×10^{-3}	23.62×10^{-3}	2.20×10^{-3}	1.89×10^{-3}	1.06×10^{-3}	0.60 (0.50)	165 s
	0.7	8.15×10^{-3}	4.87×10^{-3}	2.36×10^{-3}	1.72×10^{-3}	8.99×10^{-4}	0.74 (0.70)	165 s
	0.9	8.74×10^{-3}	5.08×10^{-3}	2.19×10^{-3}	1.34×10^{-3}	7.14×10^{-4}	0.92 (0.90)	165 s
−1	0.3	2.67×10^{-3}	1.79×10^{-3}	1.25×10^{-3}	1.16×10^{-3}	7.18×10^{-4}	0.47 (0.30)	165 s
	0.5	5.69×10^{-3}	3.56×10^{-3}	2.15×10^{-3}	1.86×10^{-3}	1.04×10^{-3}	0.61 (0.50)	165 s
	0.7	8.03×10^{-3}	4.71×10^{-3}	2.18×10^{-3}	1.62×10^{-3}	8.45×10^{-4}	0.80 (0.60)	165 s
	0.9	8.51×10^{-3}	4.84×10^{-3}	1.98×10^{-3}	1.22×10^{-3}	6.51×10^{-4}	0.92 (0.80)	165 s

Table 4. The $L^2(\Omega; H)$-error at $t = 0.1$ with $\gamma = 1$ and $H = 0.8$.

m	$\alpha \backslash N$	4	8	16	32	64	Rate	CPU Time
0	0.3	4.12×10^{-4}	2.34×10^{-4}	1.19×10^{-4}	7.66×10^{-5}	4.10×10^{-5}	0.80 (0.30)	169 s
	0.5	9.27×10^{-4}	5.23×10^{-4}	2.47×10^{-4}	1.40×10^{-4}	6.91×10^{-5}	0.93 (0.50)	169 s
	0.7	1.48×10^{-3}	8.19×10^{-4}	3.55×10^{-4}	1.79×10^{-4}	8.57×10^{-5}	1.02 (0.70)	169 s
	0.9	1.85×10^{-3}	1.00×10^{-3}	4.35×10^{-4}	2.11×10^{-4}	9.91×10^{-5}	1.05 (0.90)	169 s
−1	0.3	4.11×10^{-4}	2.33×10^{-4}	1.19×10^{-4}	7.64×10^{-5}	4.09×10^{-5}	0.83 (0.30)	169 s
	0.5	9.24×10^{-4}	5.19×10^{-4}	2.44×10^{-4}	1.38×10^{-4}	6.83×10^{-5}	0.93 (0.50)	169 s
	0.7	1.47×10^{-3}	8.08×10^{-4}	3.46×10^{-4}	1.74×10^{-4}	8.33×10^{-5}	1.03 (0.70)	169 s
	0.9	1.83×10^{-3}	9.82×10^{-4}	4.23×10^{-4}	2.05×10^{-4}	9.65×10^{-5}	1.06 (0.90)	169 s

7. Conclusions

In this work, we developed a fully discrete scheme to approximate the stochastic time-fractional diffusion problem driven by integrated fractional noise with a Hurst parameter $H \in (0, 1)$. The Caputo time-fractional derivative and fractional integral were approximated using a first-order convolution quadrature formula, while the fractional noise was approximated using the Euler method. For spatial discretization, we used the spectral

Galerkin method. By applying the convolution-based expression of the approximate solution, we obtained the error estimates for the proposed fully discrete scheme. In future work, we aim to extend these techniques to develop numerical approximations for nonlinear stochastic subdiffusion problems driven by multiplicative integrated fractional noise.

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Appendix A

In this Appendix, we provide the proof of Lemma 6 and introduce several lemmas that are essential for proving the main theorems presented in this paper.

Proof of Lemma 6. We will prove the first equality; the others can be demonstrated in a similar manner. Note that

$$\begin{aligned}\left\|\int_0^t g(s)\,dW^H(s)\right\|^2 &= \|\sum_{j=1}^{\infty}\int_0^t g(s)\gamma_j^{1/2}e_j\,d\beta_j^H(s)\|^2 = \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}\int_0^t g(s)\gamma_j^{1/2}e_j\,d\beta_j^H(s),\phi_k\right)^2\\
&= \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}\int_0^t (g(s)\gamma_j^{1/2}e_j,\phi_k)\,d\beta_j^H(s)\right)^2\\
&= \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}\int_0^t (g(s)\gamma_j^{1/2}e_j,\phi_k)\,d\beta_j^H(s),\sum_{l=1}^{\infty}\int_0^t (g(s)\gamma_l^{1/2}e_l,\phi_k)\,d\beta_l^H(s)\right).\end{aligned}$$

Thus we have, using the orthogonality of basis functions,

$$\begin{aligned}\mathbb{E}\left\|\int_0^t g(s)\,dW^H(s)\right\|^2 &= \mathbb{E}\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}\int_0^t (g(s)\gamma_j^{1/2}e_j,\phi_k)\,d\beta_j^H(s),\sum_{l=1}^{\infty}\int_0^t (g(s)\gamma_l^{1/2}e_l,\phi_k)\,d\beta_l^H(s)\right)\\
&= \mathbb{E}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\left(\int_0^t (g(s)\gamma_j^{1/2}e_j,\phi_k)\,d\beta_j^H(s)\cdot\int_0^t (g(s)\gamma_l^{1/2}e_l,\phi_k)\,d\beta_l^H(s)\right).\end{aligned}$$

Since the fractional Brownian motions $\beta_j^H(t)$ and $\beta_l^H(t)$ are independent and the cross terms for $j \neq l$ have mean zero and vanish, we arrive at

$$\begin{aligned}\mathbb{E}\left\|\int_0^t g(s)\,dW^H(s)\right\|^2 &= \mathbb{E}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left(\int_0^t (g(s)\gamma_j^{1/2}e_j,\phi_k)\,d\beta_j^H(s)\right)^2\\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\mathbb{E}\left(\int_0^t (g(s)\gamma_j^{1/2}e_j,\phi_k)\,d\beta_j^H(s)\right)^2.\end{aligned}$$

By Lemma 2.5, we get

$$\mathbb{E}\left\|\int_0^t g(s)\,dW^H(s)\right\|^2 \le C\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\|\,{}_0^R D_s^{\frac{1-2H}{2}}(g(s)\gamma_j^{1/2}e_j,\phi_k)\|^2_{L^2([0,t])}$$

$$
\begin{aligned}
&= C\sum_{j=1}^{\infty}\int_0^t \sum_{k=1}^{\infty}\left|{}_0^R D_s^{\frac{1-2H}{2}}(g(s)\gamma_j^{1/2}e_j,\phi_k)\right|^2 ds\\
&= C\sum_{j=1}^{\infty}\int_0^t \|{}_0^R D_s^{\frac{1-2H}{2}} g(s)\gamma_j^{1/2}e_j\|^2 ds\\
&= C\int_0^t \|{}_0^R D_s^{\frac{1-2H}{2}} g(s)\|^2_{\mathcal{L}_2^0} ds,
\end{aligned}
$$

which completes the proof of the first equality. □

Lemma A1. *Let $\bar{E}_\tau(t)$ and $\tilde{E}_\tau(t)$ be defined in (41) and (42), respectively, then we have*

$$
{}_0^R D_t^{\frac{1-2H}{2}} \bar{E}_\tau(t) = \frac{1}{2\pi i}\int_{\Gamma_\tau} e^{zt} z^{\frac{1-2H}{2}} (z_\tau^\alpha + A)^{-1}\frac{z\tau}{e^{z\tau}-1}\,dz,
$$

$$
{}_0^R D_t^{\frac{1-2H}{2}} \tilde{E}_\tau(t) = \frac{1}{2\pi i}\int_{\Gamma_\tau} e^{zt} z^{\frac{1-2H}{2}} z_\tau^{-\gamma}(z_\tau^\alpha + A)^{-1}\frac{z\tau}{e^{z\tau}-1}\,dz.
$$

Proof. We only prove ${}_0^R D_t^{\frac{1-2H}{2}} \tilde{E}_\tau(t)$. Note that

$$
\begin{aligned}
\hat{\tilde{E}}_\tau(\xi) &= \int_0^\infty e^{-\xi t}\tilde{E}_\tau(t)dt\\
&= \int_0^\infty e^{-\xi t}[\frac{1}{2\pi i}\int_{\Gamma_\tau} e^{zt} z_\tau^{-\gamma}(z_\tau^\alpha + A)^{-1}\frac{z\tau}{e^{z\tau}-1}dz]dt\\
&= \frac{1}{2\pi i}\int_{\Gamma_\tau}[\int_0^\infty e^{-(\xi-z)t}dt] z_\tau^{-\gamma}(z_\tau^\alpha + A)^{-1}\frac{z\tau}{e^{z\tau}-1}dz\\
&= \frac{1}{2\pi i}\int_{\Gamma_\tau}\frac{1}{\xi - z} z_\tau^{-\gamma}(z_\tau^\alpha + A)^{-1}\frac{z\tau}{e^{z\tau}-1}dz.
\end{aligned}
$$

Then by Cauchy integral formula, one obtain

$$
\begin{aligned}
{}_0^R D_t^{\frac{1-2H}{2}} \tilde{E}_\tau(t) &= \frac{1}{2\pi i}\int_{\Gamma} e^{-\xi t}\xi^{\frac{1-2H}{2}}[\frac{1}{2\pi i}\int_{\Gamma_\tau}\frac{1}{\xi - z} z_\tau^{-\gamma}(z_\tau^\alpha + A)^{-1}\frac{z\tau}{e^{z\tau}-1}dz]d\xi\\
&= \frac{1}{2\pi i}\int_{\Gamma_\tau}[\frac{1}{2\pi i}\int_{\Gamma}\frac{e^{\xi t}\xi^{\frac{1-2H}{2}}}{\xi - z}d\xi] z_\tau^{-\gamma}(z_\tau^\alpha + A)^{-1}\frac{z\tau}{e^{z\tau}-1}dz\\
&= \frac{1}{2\pi i}\int_{\Gamma_\tau} e^{zt} z^{\frac{1-2H}{2}} z_\tau^{-\gamma}(z_\tau^\alpha + A)^{-1}\frac{z\tau}{e^{z\tau}-1}dz.
\end{aligned}
$$

The proof of the first equality is complete. □

Lemma A2. *Let*

$$
G_\tau(t) = \begin{cases} 0, \quad t = 0,\\ \frac{1}{\tau}\int_{t_i}^{t_{i+1}} \tilde{E}_\tau(\bar{r})d\bar{r},\ t \in (t_i, t_{i+1}],\ j = 0, 1, \ldots, n-1. \end{cases}
$$

Denote $z_\tau = \frac{1-e^{-z\tau}}{\tau}$. Then we get

$$
\widehat{G_\tau}(z) = z^{-1} z_\tau^{1-\gamma}(z_\tau^\alpha + A)^{-1}.
$$

Proof. Extending the definition of $G_\tau(t)$ to any $t > 0$ (still denote $G_\tau(t)$) by

$$
\bar{G}_\tau(t) = \begin{cases} 0, \quad t = 0,\\ \frac{1}{\tau}\int_{t_i}^{t_{i+1}} \tilde{E}_\tau(\bar{r})d\bar{r},\ t \in (t_i, t_{i+1}],\ j = 0, 1, \ldots, n-1,\\ \cdots. \end{cases}
$$

which is possible since $\tilde{E}_\tau(t)$ is defined for any $t > 0$.

By Laplace transform, we may have, with $t > 0$,

$$ {}_0^R D_t^{\frac{1-2H}{2}} G_\tau(t) = \frac{1}{2\pi i} \int_\Gamma e^{zt} z^{\frac{1-2H}{2}} \hat{G}_\tau(z)\, dz, $$

and

$$ \begin{aligned} \hat{G}_\tau(z) &= \int_0^\infty e^{-zt} G_\tau(t)\, dt = \sum_{i=0}^{\infty} \int_{t_i}^{t_{i+1}} e^{-zt} \Big(\frac{1}{\tau} \int_{t_i}^{t_{i+1}} \tilde{E}_\tau(\bar{r})\, d\bar{r}\Big)\, dt \\ &= \left(\sum_{i=0}^{\infty} [\frac{1}{\tau} \int_{t_i}^{t_{i+1}} \tilde{E}_\tau(\bar{r})\, d\bar{r}] (e^{-z\tau})^i \right) \frac{1 - e^{-z\tau}}{z}. \end{aligned} \tag{A1} $$

Denote $G_i = \frac{1}{\tau} \int_{t_i}^{t_{i+1}} \tilde{E}_\tau(\bar{r}) d\bar{r}$, we have, by definition of $\tilde{E}_\tau(\bar{r})$,

$$ \begin{aligned} G_i &= \frac{1}{\tau} \int_{t_i}^{t_{i+1}} \tilde{E}_\tau(\bar{r}) d\bar{r} = \frac{1}{\tau} \int_{t_i}^{t_{i+1}} [\frac{1}{2\pi i} \int_{\Gamma_\tau} e^{z\bar{r}} z_\tau^{-\gamma} (z_\tau^\alpha + A)^{-1} \frac{z\tau}{e^{z\tau} - 1} dz]\, d\bar{r} \\ &= \frac{1}{2\pi i} \int_{\Gamma_\tau} [\frac{1}{\tau} \int_{t_i}^{t_{i+1}} e^{z\bar{r}} d\bar{r}] z_\tau^{-\gamma} (z_\tau^\alpha + A)^{-1} \frac{z\tau}{e^{z\tau} - 1}\, dz = \frac{1}{2\pi i} \int_{\Gamma_\tau} \frac{e^{zt_{i+1}} - e^{zt_i}}{z\tau} z_\tau^{-\gamma} (z_\tau^\alpha + A)^{-1} \frac{z\tau}{e^{z\tau} - 1}\, dz \\ &= \frac{1}{2\pi i} \int_{\Gamma_\tau} e^{zt_i} z_\tau^{-\gamma} (z_\tau^\alpha + A)^{-1}\, dz. \end{aligned} \tag{A2} $$

Further, we have, with $0 < b < 1$,

$$ G_i = \frac{1}{2\pi i} \int_{|\xi| = b} \xi^{-(i+1)} [\sum_{i=0}^{\infty} G_i \xi^i]\, d\xi = \frac{\tau}{2\pi i} \int_{\Gamma_\tau} e^{zt_i} [\sum_{i=0}^{\infty} G_i (e^{-z\tau})^i]\, dz = \frac{1}{2\pi i} \int_{\Gamma_\tau} e^{zt_i} [\sum_{i=0}^{\infty} \tau G_i (e^{-zt_i})]\, dz. $$

Thus, by applying the above analysis, we obtain

$$ \sum_{i=0}^{\infty} G_i (e^{-zt_i}) = \frac{1}{\tau} z_\tau^{-\gamma} (z_\tau^\alpha + A)^{-1}. $$

Hence,

$$ \hat{G}_\tau(z) = (\sum_{i=0}^{\infty} G_i e^{-zt_i}) \frac{1 - e^{-z\tau}}{z} = \frac{1}{\tau} z_\tau^{-\gamma} (z_\tau^\alpha + A)^{-1} \frac{1 - e^{-z\tau}}{z}. $$

Note that $z_\tau = \frac{1-\xi}{\tau} = \frac{1-e^{-z\tau}}{\tau}$, we obtain

$$ \hat{G}_\tau(z) = \frac{1}{z} z_\tau^{1-\gamma} (z_\tau^\alpha + A)^{-1}. $$

□

Lemma A3. *Let* $0 \le s \le 1$, *we have*

$$ \|A^s \hat{G}_\tau(z)\| = \begin{cases} C|z|^{-\alpha(1-s)-\gamma} e^{(1-\alpha-\gamma+\alpha s)|z|\tau}, & z \in \Gamma/\Gamma_\tau,\ 1 - \alpha(1-s) - \gamma \ge 0, \\ C\tau^{(1-s)\alpha+\gamma-1} |z|^{-1}, & z \in \Gamma/\Gamma_\tau,\ 1 - \alpha(1-s) - \gamma < 0, \\ C|z|^{\alpha(s-1)-\gamma}, & z \in \Gamma_\tau. \end{cases} \tag{A3} $$

Proof. When $z \in \Gamma_\tau$, the desired estimate can be obtained by Lemma 7, resolvent estimate and Lemma A2. That is

$$ \|A^s \hat{G}_\tau(z)\| = \|z^{-1} z_\tau^{1-\gamma} A^s (z_\tau^\alpha + A)^{-1}\| \le C|z|^{\alpha(s-1)-\gamma}. $$

As for $z \in \Gamma/\Gamma_\tau$, simple calculation gives

$$|z_\tau| = |\frac{1-e^{-z\tau}}{\tau}| = \frac{|1-e^{-|z|\tau\cos\theta - i|z|\tau\sin\theta}|}{\tau} \geq \frac{e^{-|z|\tau\cos\theta}-1}{\tau}.$$

Since $|z| \geq \frac{\pi}{\tau\sin\theta}$, one can choose a suitable $\theta \in (\frac{\pi}{2}, \pi)$, such that $|z_\tau| \geq \frac{e^{-\pi\cot\theta}-1}{\tau} \geq \frac{e-1}{\tau}$. Let τ be small enough to satisfy $(\frac{e-1}{\tau})^\alpha > \lambda_k$. Obviously

$$\Re(z_\tau) = \frac{|1-e^{-|z|\tau\cos\theta - i\cos(|z|\tau\sin\theta)}|}{\tau} \geq 0,$$

which shows $(z_\tau^\alpha + A)^{-1}$ exists, for $z \in \Sigma_\theta$. Thus

$$|\lambda_k^s \hat{G}_\tau(z)| \leq |z|^{-1}|z_\tau|^{1-\gamma}|z_\tau|^{-\alpha}|z_\tau|^{\alpha s} \leq C|z|^{-1}|z_\tau|^{1-\alpha-\gamma+\alpha s}.$$

According to the fact $z_\tau = \frac{1-e^{-z\tau}}{\tau} \leq |z|\sum_{k=1}^{\infty}\frac{(z\tau)^{k-1}}{k!} \leq |z|e^{|z\tau|}$, the desired result is reached. More precisely, if $1-\alpha(1-s)-\gamma \geq 0$, we obtain

$$|\lambda_k^s \hat{G}_\tau(z)| \leq |z|^{-\alpha(1-s)-\gamma}e^{(-\alpha(1-s)-\gamma+1)|z|\tau},\ z \in \Gamma/\Gamma_\tau.$$

If $1-\alpha(1-s)-\gamma < 0$, we obtain

$$|\lambda_k^s \hat{G}_\tau(z)| \leq C\tau^{\alpha(1-s)+\gamma-1}|z|^{-1},\ z \in \Gamma/\Gamma_\tau.$$

The proof is complete. □

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