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Research Paper

Long time stability and strong convergence of an efficient tamed scheme for stochastic Allen-Cahn equation driven by additive white noise

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ABSTRACT

Huang and Shen [*Math. Comput.* **92** (2023) 2685–2713] proposed a semi-implicit tamed scheme for the numerical approximation of stochastic Allen–Cahn equations driven by multiplicative trace-class noise. They showed that the scheme is unconditionally stable on finite time intervals and can be efficiently implemented. In this paper, we investigate the long-time stability of this tamed scheme for stochastic Allen–Cahn equations driven by additive white noise. We also address the strong convergence analysis of the associated fully discrete scheme within the Galerkin finite element framework. The main contributions of this work are as follows: (i) by constructing a suitable Lyapunov functional, we establish the unconditional long-time stability of the tamed method; (ii) we rigorously derive the strong convergence rates of the fully discrete scheme obtained by coupling the tamed approach with the finite element method. Numerical experiments are provided to validate the theoretical analysis and demonstrate the effectiveness of the proposed scheme.

1. Introduction

In the past decade, tamed methods have been widely applied for the numerical approximation of stochastic partial differential equations (SPDEs) with non-globally Lipschitz nonlinearities; see, e.g., [1–6] and the references therein. By incorporating “tamed factors” into these non-globally Lipschitz nonlinear terms, such methods effectively suppress numerical instabilities caused by superlinear growth. As a result, tamed methods exhibit remarkable potential in constructing stable and efficient numerical scheme for simulating complex stochastic systems. Among these applications, the numerical treatment of the stochastic Allen-Cahn equations (SACEs) using tamed schemes is especially appealing, see, e.g., [7–11] and references therein.

As a representative stochastic phase-field model with a non-globally Lipschitz drift, the SACE is widely used to simulate irregular phase transition phenomena. According to the type of driving noise, SACEs can be broadly classified into two categories: those driven by colored noise and those driven by white noise. In general, white noise is rougher than colored noise [12], which leads to significantly lower spatial regularity of the solution in the white-noise setting. Consequently, the numerical analysis of SACEs driven by white noise is considerably more challenging than that of their colored-noise counterparts [9]. Recent numerical studies on tamed methods for SACEs have primarily focused on the colored-noise setting, see, e.g., [5,7,10,11,13] and references therein, where different tamed methods have been employed to analyze the numerical behavior of the solution. In contrast, only a few works,

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see, e.g., [6,14] have investigated numerical performance of tamed schemes for SACEs under white-noise perturbations. Among the existing tamed approaches, the semi-implicit tamed scheme proposed by Huang and Shen [10] has received particular attention due to its unconditional stability and its computational efficiency. However, the numerical analysis of this scheme has so far been restricted mainly to colored-noise-driven SACEs [5,7], and existing stability results for this tamed scheme are limited to finite time intervals. To the best of our knowledge, no prior work has applied this tamed scheme to SACEs driven by white noise, nor has its long-time stability been examined in the white noise regime. Moreover, for SACEs perturbed by white noise, strong convergence analysis for the fully discrete scheme obtained by combining this tamed approach with the finite element method are also lacking. The present paper aims to fill these research gaps by providing a deeper and more systematic numerical analysis of this tamed scheme.

Consider the numerical approximation of the following stochastic Allen-Cahn equation driven by additive white noise:

$$\begin{aligned} &\mathrm{d}u(t) = \big(- Au(t) + F(u(t))\big)\mathrm{d}t + \mathrm{d}W(t),\ t > 0,\\ &u(0) = u_0, \end{aligned} \tag{1}$$

where $-A := \Delta$ denotes the Laplacian operator with homogeneous Dirichlet boundary conditions. (i.e., $u(t)$ satisfies homogeneous Dirichlet boundary conditions), the nonlinear drift term is defined by $F(u) := u - u^3$, and $W(t)$ represents cylindrical Wiener process [9] to be specified later.

Although many researchers have conducted in-depth numerical analysis of problem (1) using a variety of numerical schemes, see, e.g., [15–20] and references therein, only a very limited number of works have investigated the long-time stability of numerical schemes. Liu [20,21] analyzed the long-time stability of the backward Euler scheme for SACEs driven by multiplicative noise, establishing the existence and uniqueness of invariant measure. Wang et al. [14] proposed a fully discrete "tamed accelerated exponential Euler/spectral Galerkin" scheme for SACEs with additive noise, proving time-uniform moment bounds over infinite time horizons and deriving weak error estimates on unbounded time intervals. Jiang et al. [22] developed a class of fully discrete "tamed exponential Euler/spectral Galerkin" schemes for SPDEs with non-globally Lipschitz coefficients, showing that these explicit methods are easy to implement, preserve the ergodicity of dissipative SPDEs, and admit uniform-in-time weak error bounds.

The main contributions/novelties of this work are summarized as follows:

- For problem (1) driven by additive white noise, we rigorously establish the long-time unconditional stability of the semi-implicit tamed method proposed in [10].
- We derive the strong convergence rates for a fully discrete scheme obtained by coupling the tamed scheme proposed in [10] with a finite element spatial discretization. The main theoretical result is the proof of the strong convergence order $\mathcal{O}\big(\Delta t^{\frac{\gamma}{2}} + h^{\gamma}\big)$, where h and Δt denote the spatial and temporal mesh sizes, respectively, and $\gamma \in [\frac{1}{3}, \frac{1}{2})$ is a parameter characterizing the regularity of the noise.

The remainder of the paper is organized as follows. Section 2 introduces the basic concepts and notations used throughout the paper. In Section 3, we present the tamed temporal discretization and establish the unconditional stability of the semi-discrete scheme. Section 4 is devoted to the strong convergence analysis of the fully discrete scheme based on the finite element method. Finally, Section 5 provides numerical experiments that demonstrate the efficiency of the proposed method.

2. Preliminaries

To facilitate the theoretical and numerical analysis of problem (1), we start by introducing some notations and notions. Let D be a bounded open physical domain. Denote by $C(D)$ a Banach space of continuous functions with usual norms. For $p \geq 1$, let $L^{2p}(D)$ be the standard Lebesgue space. Let $H := L^2(D)$ be a Hilbert space equipped with inner product $(\cdot,\cdot)$. For $s \in \mathbb{R}$, we define the Hilbert space $\dot{H}^s$ as the domain of the fractional power of the operator A, equipped with the norm

$$\|v\|_{\dot{H}^s} := \|A^{\frac{s}{2}} v\|_H, \quad \forall v \in \dot{H}^s, \tag{2}$$

where $\mathcal{D}(A) = H_0^1(D) \cap H^2(D)$. In particular, one has $\dot{H}^0 = H$, $\dot{H}^1 = H_0^1(D)$ and $\dot{H}^2 = \mathcal{D}(A)$ [23]. Let $\mathcal{L}(H)$ denote the space of bounded linear operators from H to H. Denote by $C_b(H)$ the space of bounded, continuous functions on H. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Denote by $L^{2p}(\Omega, H)$, $p \geq 1$, the space of H-valued random variables that are $2p$-integrable, endowed with the norm

$$\|v\|_{L^{2p}(\Omega,H)} := \big(\mathbb{E}[\|v\|_H^{2p}]\big)^{\frac{1}{2p}} < \infty, \quad \forall v \in L^{2p}(\Omega, H). \tag{3}$$

Here, $\mathbb{E}[\cdot]$ denotes the expectation with respect to the probability measure $\mathbb{P}$.

It is well known that Eq. (1) driven by additive white noise admits a mild solution only in the one-dimensional spatial setting [9]. Therefore, we restrict our analysis to a one-dimensional bounded spatial domain and, specifically, consider $D = (0,1)$ throughout this work. Denote by $\{\phi_j \in H : j \in \mathbb{N}_+\}$ the orthonormal eigenfunctions of A, and by $\{q_j, j \in \mathbb{N}_+\}$ the corresponding positive eigenvalues, that is $\{\phi_j = \sqrt{2}\sin(j\pi x), x \in D\}_{j\in\mathbb{N}_+}$ and $\{q_j = j^2\pi^2\}_{j\in\mathbb{N}_+}$. Let $W(t)$ be a $\mathcal{F}_t$-adapted cylindrical Wiener process represented by the Karhunen–Loève expansion

$$W(t) = \sum_{j=1}^{\infty} \phi_j \beta_j(t), \tag{4}$$

where $\{\beta_j(t)\}_{j\in\mathbb{N}_+}$ are independent and identically distributed (i.i.d.) $\mathcal{F}_t$-adapted Brownian motions. It is worth noting that this cylindrical Wiener process is a special case of a Q-Wiener process with the covariance operator $Q = I$, and it is well defined in the Hilbert space $\dot{H}^s$ for any $s < -\frac{1}{2}$ [12, Definition 10.14]. Throughout the paper, the symbol c, with or without subscripts, denotes a generic positive constant that may vary from line to line.

We are interested in the mild solution of the problem (1) given by

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau + \int_0^t S(t-\tau)dW(\tau), \tag{5}$$

where $S(t) := e^{-tA}$ denotes the analytic semigroup generated by the operator A. It is known from [12, Lemma 3.22] that the semigroup $S(t)$ satisfies the following smoothing and ultracontractive properties:

- For all $0 \le \alpha \le 1$, there exists a constant c such that

$$\|A^{-\alpha}(I - S(t))\|_{\mathcal{L}(H)} \le ct^{\alpha}, \ \forall t \ge 0. \tag{6}$$

- For any $\alpha \ge 0$, there exists a constant c such that

$$\|A^{\alpha}S(t)\|_{\mathcal{L}(H)} \le ct^{-\alpha}, \ \forall t > 0. \tag{7}$$

To investigate the strong convergence of the subsequent numerical scheme, we impose the following assumptions on the drift term $F(\cdot)$ and the operator A.

- Assume that $F(\cdot) : L^6(D) \to H$ satisfies the following one-sided Lipschitz condition

$$\big(v_1 - v_2, F(v_1) - F(v_2)\big) \le \|v_1 - v_2\|_H^2, \quad \forall v_1, v_2 \in L^6(D). \tag{8}$$

- We quantify the regularity of the noise by a parameter γ through the condition

$$\Big\|A^{\frac{\gamma-1}{2}}\Big\|_{\mathcal{L}_0^2} < \infty, \quad \frac{1}{3} \le \gamma < \frac{1}{2}, \tag{9}$$

where $\|\cdot\|_{\mathcal{L}_0^2}$ [2] denotes the norm of the Hilbert-Schmidt operator space [24].

Remark 1. The condition (8) is weaker than the global Lipschitz assumption and is commonly employed in the numerical analysis of the stochastic Allen-Cahn equation; see, e.g., [17, (2.7)]. In condition (9), the requirement $\gamma < 1/2$ ensures $\|A^{\frac{\gamma-1}{2}}\|_{\mathcal{L}_0^2} < \infty$. Indeed, for $\gamma < \frac{1}{2}$,

$$\|A^{\frac{\gamma-1}{2}}\|_{\mathcal{L}_0^2} = \Big(\sum_{j=1}^{\infty}\|A^{\frac{\gamma-1}{2}}\phi_j\|_H^2\Big)^{\frac{1}{2}} = \Big(\sum_{j=1}^{\infty}q_j^{\gamma-1}\Big)^{\frac{1}{2}} \le c\Big(\sum_{j=1}^{\infty}j^{2(\gamma-1)}\Big)^{\frac{1}{2}} < \infty.$$

Furthermore, we impose $\gamma \ge \frac{1}{3}$ in (9) to guarantee that the mild solution (5) possesses a certain spatial regularity, which is crucial for the use of the Sobolev embedding inequality in subsequent numerical analysis. The same assumption as (9) is also adopted by Qi et al. [17, (2.8)] in their numerical analysis of SACEs.

For a sufficiently large number $p_0 \in \mathbb{N}$, we assume that the initial value $u_0 \in L^{p_0}(\Omega, \dot{H}^1)$ is an $\mathcal{F}_0$-measurable random variable. Under these assumptions, for any $T > 0$, the SPDE (1) admits a unique mild solution $u(t) : [0,T]\times\Omega \to C(D)$ with continuous sample paths, given by (5). Moreover, for any $p \ge 1$, there exists constant $c > 0$ depending on T and p, such that

$$\sup_{0\le t\le T}\|u(t)\|_{L^{2p}(\Omega,C(D))} \le c(1 + \|u_0\|_{L^{2p}(\Omega,C(D))}), \tag{10}$$

$$\sup_{0\le t\le T}\|u(t)\|_{L^{2p}(\Omega,\dot{H}^{\gamma})} \le c(1 + \|u_0\|_{L^{2p}(\Omega,\dot{H}^{\gamma})}), \quad \gamma \in [\frac{1}{3}, \frac{1}{2}), \tag{11}$$

$$\|u(\tau_2) - u(\tau_1)\|_{L^{2p}(\Omega,H)} \le c(\tau_2 - \tau_1)^{\frac{\gamma}{2}}, \quad \forall\, 0 \le \tau_1 \le \tau_2 \le T,\ \gamma \in [\frac{1}{3}, \frac{1}{2}). \tag{12}$$

The well-posedness of problem (1), along with the regularity estimates (10)–(12), has already been established by Wang in [9, Theorem 2.6] for $t \in [0,T]$.

Based on the regularity estimate (11), the explicit form of the drift term $F(u) = u - u^3$, and the Sobolev embedding $\dot{H}^{\frac{1}{3}} \hookrightarrow L^6(D)$ in one-dimensional case [17, (2.17)], it follows that, for any $p \ge 1$ and $T > 0$,

$$\begin{aligned}\sup_{0\le t\le T}\|F(u(t))\|_{L^{2p}(\Omega,H)} &\le c\big(\sup_{0\le t\le T}\|u(t)\|_{L^{2p}(\Omega,H)} + \sup_{0\le t\le T}\|u(t)\|^3_{L^{6p}(\Omega,L^6(D))}\big)\\ &\le c(\sup_{0\le t\le T}\|u(t)\|_{L^{2p}(\Omega,\dot{H}^{\gamma})} + \sup_{0\le t\le T}\|u(t)\|^3_{L^{6p}(\Omega,\dot{H}^{\gamma})}) < \infty.\end{aligned} \tag{13}$$

This bound plays a crucial role in the subsequent error analysis.

[2] For $\psi \in \mathcal{L}_0^2$, $\|\psi\|_{\mathcal{L}_0^2} := \big(\sum_{j=1}^{\infty}\|\psi\phi_j\|_H^2\big)^{\frac{1}{2}}$.

3. Tamed scheme and its long-time stability

This section is devoted to the long-time stability analysis of a tamed time-discrete scheme for problem (1). Let $\Delta t \in (0,1)$ denote a uniform time step and set $t_n := n\Delta t$ for $n \in \mathbb{N}_+$. Motivated by the work of Shen et al. [10], we define the tamed factor

$$r(\cdot) := \frac{1}{1 + \Delta t \|F(\cdot)\|_H^2}. \tag{14}$$

This factor depends on the H-norm of the nonlinear drift term $F(\cdot)$. The temporal discretization of (1), referred to hereafter as the "*drift-tamed Euler-Maruyama*" scheme, is formulated as follows: find an $\mathcal{F}_{t_n}$-adapted Markov chain $\{u^n\}_{n\in\mathbb{N}_+}$, such that

$$\begin{aligned} (I + \Delta t A)u^{n+1} = \quad & u^n + r(u^n)F(u^n)\Delta t + \Delta W^n, \quad n \in \mathbb{N}_+, \\ u^0 := \quad & u_0, \end{aligned} \tag{15}$$

where $\Delta W^n := W(t_{n+1}) - W(t_n)$. This temporal discretization has previously been applied to the numerical approximation of SACEs with trace-class noise; see, e.g., [5,7], where it was shown to be unconditionally stable on arbitrary finite time intervals. However, to the best of our knowledge, no existing work has investigated the long-time stability of this scheme for SACEs driven by white noise. The present study appears to be the first to analyze its long-time stability. The scheme (15) is drift-explicit and its computational cost is essentially that of solving a second-order linear elliptic problem at each time step, which makes it more efficient than drift-implicit alternatives [17,21].

In what follows, we show that the scheme (15) is long-time unconditionally stable.

Proposition 1 (Long-time unconditional stability). *Let $u_0 \in L^{2p}(\Omega, \dot H^1)$, $p \ge 1$. Then the numerical solution $\{u^n\}_{n\in\mathbb{N}_+}$ of the semi-discrete problem (15) is unconditionally stable in the mean-square sense. More precisely, there exists a constant c independent of Δt such that*

$$\|u^n\|^2_{L^2(\Omega,H)} \le c e^{-\frac{\pi^2-2}{1+2\pi^2} t_n} \|u_0\|^2_{L^2(\Omega,\dot H^1)} + c, \quad n \in \mathbb{N}_+. \tag{16}$$

Proof. Let $S_{\Delta t} := (I + \Delta t A)^{-1}$. The solution of scheme (15) can be written as follows:

$$u^n = S^n_{\Delta t} u_0 + \sum_{k=0}^{n-1} S^{n-k}_{\Delta t} r(u^k)F(u^k)\Delta t + \sum_{k=0}^{n-1} S^{n-k}_{\Delta t} \Delta W^k, \quad n \in \mathbb{N}_+. \tag{17}$$

Define the discrete stochastic convolution $\widetilde{W}^n := \sum_{k=0}^{n-1} S^{n-k}_{\Delta t}\Delta W^k$, with $\widetilde{W}^0 = 0$.

By $q_j = \pi^2 j^2$ and discrete Burkholder-Davis-Gundy inequality [23, Lemma 2.2], we obtain for any $p \ge 1$ and $\delta \in [0, 1/2)$,

$$\begin{aligned} \mathbb{E} \quad & \big[\|\widetilde{W}^n\|^{2p}_{\dot H^\delta}\big] = \mathbb{E}\Big[\Big\| \textstyle\sum_{k=0}^{n-1} A^{\frac{\delta}{2}} S^{n-k}_{\Delta t}\Delta W^k \Big\|^{2p}_H\Big] \le c\Big(\textstyle\sum_{k=0}^{n-1} \Big\| A^{\frac{\delta}{2}} S^{n-k}_{\Delta t}\Delta W^k\Big\|^2_{L^{2p}(\Omega,H)}\Big)^p \\ & \le c\Delta t^p \Big(\textstyle\sum_{k=0}^{n-1} \Big\| \sum_{j=1}^{\infty} \Big\| A^{\frac{\delta}{2}} S^{n-k}_{\Delta t}\phi_j\Big\|^2_H \Big\|_{L^p(\Omega)}\Big)^p \le c\Delta t^p \Big(\textstyle\sum_{k=0}^{n-1}\sum_{j=1}^{\infty} \big(q_j^{\frac{\delta}{2}}(1+\Delta t q_j)^{-(n-k)}\big)^2\Big)^p \\ & \le c\Delta t^p \Big(\textstyle\sum_{j=1}^{\infty} q_j^\delta \sum_{k=0}^{n-1}(1+\Delta t q_j)^{-2(n-k)}\Big)^p \le c\big(\textstyle\sum_{j=1}^{\infty} q_j^{\delta-1}\big)^p \le c, \quad \delta\in[0,1/2), \end{aligned} \tag{18}$$

where we used the estimate $\sum_{k=0}^{n-1}(1+\Delta t q_j)^{-2(n-k)} = \sum_{k=1}^{n}(\frac{1}{1+\Delta t q_j})^{2k} \le \frac{1}{\Delta t q_j(1+\Delta t q_j)} \le \frac{1}{\Delta t q_j}$.

Let $X^n := u^n - \widetilde{W}^n$, with $X^0 = u_0$. According to (17), we obtain

$$X^n = S^n_{\Delta t}u_0 + \sum_{k=0}^{n-1} S^{n-k}_{\Delta t} r(u^k)F(u^k)\Delta t,$$

which implies that X^n satisfies

$$(I + \Delta t A)X^n = X^{n-1} + r(u^{n-1})F(u^{n-1})\Delta t, \quad n \in \mathbb{N}_+.$$

Taking the $L^2(D)$-inner product of both sides with X^n, and using the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$, $a, b \in \mathbb{R}$, we arrive at

$$\|X^n\|^2_H - \|X^{n-1}\|^2_H + \|X^n - X^{n-1}\|^2_H + 2\Delta t\|\nabla X^n\|^2_H = \theta_1 + \theta_2 + \theta_3, \tag{19}$$

where

$$\begin{aligned} \theta_1 \quad & := 2r(u^{n-1})\Delta t\big(F(u^{n-1}), X^n - X^{n-1}\big), \\ \theta_2 \quad & := 2r(u^{n-1})\Delta t\big(F(u^{n-1}) - F(\widetilde{W}^{n-1}), X^{n-1}\big), \\ \theta_3 \quad & := 2r(u^{n-1})\Delta t\big(F(\widetilde{W}^{n-1}), X^{n-1}\big). \end{aligned}$$

We now estimate the terms θ_1, θ_2, and θ_3 separately. Firstly, employing $r(\cdot) \le 1$, and $r(u^{n-1})\|F(u^{n-1})\|^2_H\Delta t \le 1$, derives

$$\theta_1 \le \Delta t^2 r(u^{n-1})\|F(u^{n-1})\|^2_H + r(u^{n-1})\|X^n - X^{n-1}\|^2_H \le \Delta t + \|X^n - X^{n-1}\|^2_H.$$

The estimation of θ_2 follows by $r(\cdot) \le 1$ and condition (8)

$$\theta_2 \le 2\Delta t\|X^{n-1}\|^2_H.$$

For the estimate of θ_3, using $r(\cdot) \le 1$, Cauchy inequality, Sobolev embedding $L^1(D) \hookrightarrow \dot{H}^{-1}$ [2], and $F(v) = v - v^3$, $v \in \mathbb{R}$, we deduce

$$\theta_3 \le 2\Delta t \|A^{\frac{1}{2}} X^{n-1}\|_H \|A^{-\frac{1}{2}} F(\widetilde{W}^{n-1})\|_H \le \Delta t \|\nabla X^{n-1}\|_H^2 + c\Delta t \|F(\widetilde{W}^{n-1})\|_{\dot{H}^{-1}}^2$$
$$\le \Delta t \|\nabla X^{n-1}\|_H^2 + c\Delta t \|F(\widetilde{W}^{n-1})\|_{L^1(D)}^2 \le \Delta t \|\nabla X^{n-1}\|_H^2 + c\Delta t \big(\|\widetilde{W}^{n-1}\|_H^2 + \|\widetilde{W}^{n-1}\|_{L^6(D)}^6\big).$$

Substituting the estimation result of $\theta_i,\ i = 1,2,3$, into (19), we obtain

$$\|X^n\|_H^2 + 2\Delta t \|\nabla X^n\|_H^2 \le \|X^{n-1}\|_H^2 + \Delta t + 2\Delta t \|X^{n-1}\|_H^2 + \Delta t \|\nabla X^{n-1}\|_H^2 + c\Delta t \big(\|\widetilde{W}^{n-1}\|_H^2 + \|\widetilde{W}^{n-1}\|_{L^6(D)}^6\big). \tag{20}$$

Define the constant

$$c_0 := \frac{2\pi^2 \Delta t + 1}{1 + (2+\pi^2)\Delta t} > 1.$$

Through Poincaré inequality [3], we observe that

$$\big(c_0(1+2\Delta t) - 1\big)\|X^n\|_H^2 = (2 - c_0)\pi^2 \Delta t \|X^n\|_H^2 \le (2 - c_0)\Delta t \|\nabla X^n\|_H^2,$$

yielding

$$c_0\big((1+2\Delta t)\|X^n\|_H^2 + \Delta t \|\nabla X^n\|_H^2\big) \le \|X^n\|_H^2 + 2\Delta t \|\nabla X^n\|_H^2. \tag{21}$$

Combining (20) and (21), and applying the Sobolev embedding $\dot{H}^{\frac{1}{3}} \hookrightarrow L^6(D)$, we deduce

$$c_0\big((1+2\Delta t)\|X^n\|_H^2 + \Delta t \|\nabla X^n\|_H^2\big) \le \big((1+2\Delta t)\|X^{n-1}\|_H^2 + \Delta t \|\nabla X^{n-1}\|_H^2\big) + \Delta t + c\Delta t \big(\|\widetilde{W}^{n-1}\|_H^2 + \|\widetilde{W}^{n-1}\|_{\dot{H}^{\frac{1}{3}}}^6\big).$$

Taking expectations on both sides of the above inequality and using (18), we obtain

$$(1+2\Delta t)\|X^n\|_{L^2(\Omega,H)}^2 + \Delta t \|\nabla X^n\|_{L^2(\Omega,H)}^2 \le \frac{1}{c_0}\Big((1+2\Delta t)\|X^{n-1}\|_{L^2(\Omega,H)}^2 + \Delta t \|\nabla X^{n-1}\|_{L^2(\Omega,H)}^2\Big) + c\Delta t, \tag{22}$$

where $\frac{1}{c_0} < 1$ due to $c_0 > 1$. Recursively applying (22) gives

$$(1+2\Delta t)\|X^n\|_{L^2(\Omega,H)}^2 + \Delta t \|\nabla X^n\|_{L^2(\Omega,H)}^2 \le \frac{1}{c_0^n}\Big((1+2\Delta t)\|X^0\|_{L^2(\Omega,H)}^2 + \Delta t \|\nabla X^0\|_{L^2(\Omega,H)}^2\Big) + c\Delta t \sum_{k=0}^{n-1}\big(\tfrac{1}{c_0}\big)^k. \tag{23}$$

Since $c_0 := \frac{2\pi^2\Delta t+1}{1+(2+\pi^2)\Delta t}$, $t_n = n\Delta t\ (n \in \mathbb{N}_+,\ \Delta t \in (0,1))$, $(\frac{1}{c_0})^n = \big(1 - \frac{(\pi^2-2)\Delta t}{2\pi^2\Delta t+1}\big)^n \le e^{-\frac{n\Delta t(\pi^2-2)}{1+2\pi^2\Delta t}}$, and $\sum_{k=0}^{n-1}(\frac{1}{c_0})^k \le \frac{c_0}{c_0-1}$, one readily deduces from (23) that

$$\|X^n\|_{L^2(\Omega,H)}^2 \le e^{-\frac{n\Delta t(\pi^2-2)}{1+2\pi^2\Delta t}}(1+2\Delta t)\|X^0\|_{L^2(\Omega,\dot{H}^1)}^2 + c\,\frac{2\pi^2\Delta t+1}{\pi^2-2}$$
$$\le c e^{-\frac{\pi^2-2}{1+2\pi^2}t_n}\|X^0\|_{L^2(\Omega,\dot{H}^1)}^2 + c, \quad n \in \mathbb{N}_+.$$

Finally, making use of the identity $X^n := u^n - \widetilde{W}^n$, $X^0 = u_0$, and the estimate (18), we conclude that

$$\Big\|u^n\Big\|_{L^2(\Omega,H)}^2 \le c e^{\left(-\frac{\pi^2-2}{1+2\pi^2}t_n\right)}\Big\|u_0\Big\|_{L^2(\Omega,\dot{H}^1)}^2 + c, \quad n \in \mathbb{N}_+,$$

where the constant c is independent of Δt. This completes the proof. □

Remark 2. From the inequality (22), we observe that the discrete variable X^n, which is associated with u^n, satisfies the following Lyapunov structure

$$V(X^n) \le \frac{1}{c_0}V(X^{n-1}) + c\Delta t, \quad n \in \mathbb{N}_+, \tag{24}$$

where $c_0 > 1$ and $V(X^n) := (1+2\Delta t)\|X^n\|_{L^2(\Omega,H)}^2 + \Delta t \|\nabla X^n\|_{L^2(\Omega,H)}^2$. This recursive inequality exhibits a dissipative structure and implies an exponential decay of the functional $V(X^n)$. It is precisely this Lyapunov structure that enables us to establish the long-time unconditional stability of the numerical solution u^n, uniformly with respect to the time step Δt.

Another noteworthy point is that the long-time stability properties are often closely linked to the analysis of the existence and uniqueness of the invariant measure associated with the Markov chain induced by the time discretization. A rigorous proof of the existence and uniqueness of an invariant measure for the Markov chain $\{u^n\}_{n\in\mathbb{N}_+}$ generated by the scheme (15) is beyond the scope of the present work. Nonetheless, under the assumption that such a measure exists, we provide a discussion on its uniqueness in the Appendix 7.

[2] $\dot{H}^{-1}$ is the dual space of $\dot{H}^1$.

[3] Poincaré inequality $\|\nabla v\|_H^2 \ge \pi^2\|v\|_H^2$, $\forall v \in \dot{H}^1$, see, e.g., [21, (2.2)].

4. Full discretization and strong error analysis

This section aims to analyze the strong convergence of a spatio-temporal fully discrete method that combines the proposed time-discretization scheme with a finite element spatial discretization. Recall that in the stability analysis of the time-discrete scheme (15), the time step size Δt is fixed, allowing us to examine the long-time stability of the scheme (15). In this section, we shall investigate the strong convergence order of the resulting fully discrete scheme on a finite-time interval $[0, T]$, where $T > 0$ is a fixed time.

Define the uniform time step size by $\Delta t := T/N$ for some positive integer N. Let $\mathcal{T}_h$ denote the uniform partition of the spatial domain $D = (0, 1)$, and define the corresponding finite element space by

$$V_h := \{v \in C(\bar{D}),\ v = 0 \text{ on } \partial D,\ v|_K \in \mathbb{P}_1(K) \text{ for all } K \in \mathcal{T}_h\},$$

where $\mathbb{P}_1(K)$ denotes the space of polynomials of degree at most one on the element K. Let $\mathcal{P}_h$ denote the orthogonal projection from H to V_h, and define the discrete Laplace operator $A_h : V_h \to V_h$ by

$$(A_h w, v) := (\nabla w, \nabla v), \quad \forall w, v \in V_h.$$

The resulting fully discrete scheme, called hereafter "drift-tamed Euler-Maruyama/finite element method", reads: find an $\mathcal{F}_{t_{n+1}}$-adapted, V_h-valued random variable u_h^{n+1} for $n = 0, \dots, N-1$, such that

$$\begin{aligned} (I + \Delta t A_h) u_h^{n+1} = &\quad u_h^n + r(u_h^n)\mathcal{P}_h F(u_h^n)\Delta t + \mathcal{P}_h \Delta W^n, \\ u_h^0 := &\quad \mathcal{P}_h u_0, \end{aligned} \tag{25}$$

where $r(u_h^n) := \frac{1}{1+\|F(u_h^{n+1})\|_H^2}$, and u_h^n denotes the fully discrete approximation to $u(t_n)$, with $t_n = n\Delta t$. It is worth mentioning that several numerical studies, see, e.g., [7,10], have investigated fully discrete schemes for SACEs incorporating the taming strategy same to that employed in the present work. However, the study in [10] utilizes a spectral method for spatial discretization, while [7] considers SACEs driven by multiplicative trace-class noise, rather than additive white noise. To the best of our knowledge, the present work appears to be the first to approximate SACEs driven by additive white noise using a finite element spatial discretization in combination with the tamed factor defined in (14).

We now focus on the strong error estimation of the fully discrete scheme (25). Our aim is to derive the error bound of the error $\|u(t_n) - u_h^n\|_{L^2(\Omega,H)}$, where $n = 1, \dots, N$.

To facilitate the analysis, we first introduce an auxiliary process $\mathcal{U}_h^n$ defined by

$$\begin{aligned} \mathcal{U}_h^{n+1} - \mathcal{U}_h^n + \Delta t A_h \mathcal{U}_h^{n+1} = &\quad r(u_h^n)\mathcal{P}_h F(u(t_n))\Delta t + \mathcal{P}_h \Delta W^n, \quad n = 0, \dots, N-1, \\ \mathcal{U}_h^0 = &\quad \mathcal{P}_h u_0. \end{aligned} \tag{26}$$

This auxiliary process helps us overcome the analytical challenges posed by the cubic, non-globally Lipschitz drift term. A similar technique was employed in [17] to analyze the strong convergence error of the backward Euler scheme.

Let $S_{h,\Delta t} := (I + \Delta t A_h)^{-1}$. Then the sequence $\mathcal{U}_h^n$ admits the following recursive representation

$$\begin{aligned} \mathcal{U}_h^n &= S_{h,\Delta t}^n \mathcal{P}_h u_0 + \sum_{k=0}^{n-1} r(u_h^k) S_{h,\Delta t}^{n-k}\mathcal{P}_h F(u(t_k))\Delta t + \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} S_{h,\Delta t}^{n-k}\mathcal{P}_h dW(\tau), \quad n = 1 \dots, N, \\ \mathcal{U}_h^0 &= \mathcal{P}_h u_0. \end{aligned} \tag{27}$$

The strong error can be decomposed into two components

$$\|u(t_n) - u_h^n\|_{L^2(\Omega,H)} \le \|u(t_n) - \mathcal{U}_h^n\|_{L^2(\Omega,H)} + \|\mathcal{U}_h^n - u_h^n\|_{L^2(\Omega,H)} =: \mathcal{E}_1 + \mathcal{E}_2. \tag{28}$$

To estimate the total error, we analyze the two terms $\mathcal{E}_1$ and $\mathcal{E}_2$ separately. Before proceeding with the detailed estimates, we present several preliminary results that will be used throughout the analysis.

- The following estimates for the operators A, A_h and $S_{h,\Delta t}$ hold

$$\|A^{-\frac{\delta_0}{2}} v\|_H \le c\|v\|_{L^1(D)}, \ \forall \delta_0 \in (\frac{3}{2}, 2),\ v \in L^1(D); \tag{29}$$

$$\|A^{-\frac{1}{2}} v\|_H \le c\|v\|_{L^{\frac{6}{5}}(D)}, \ v \in L^{\frac{6}{5}}(D); \tag{30}$$

$$c_1\|A_h^{\frac{\alpha}{2}}\mathcal{P}_h v\|_H \le \|A^{\frac{\alpha}{2}} v\|_H \le c_2\|A_h^{\frac{\alpha}{2}}\mathcal{P}_h v\|_H,\ v \in \dot{H}^\alpha,\ \alpha \in [-1, 1],\ c_1, c_2 > 0; \tag{31}$$

$$\left\|\left(S(t_m) - S_{h,\Delta t}^m \mathcal{P}_h\right) v\right\|_H \le c(h^\mu + \Delta t^{\frac{\mu}{2}}) t_m^{-\frac{\mu-\nu}{2}}\|A^{\frac{\nu}{2}} v\|_H,\ 0 \le \nu \le \mu \le 2,\ v \in \dot{H}^\nu,\ m \in \{1, \dots, N\}; \tag{32}$$

$$\|A_h^{\frac{\alpha}{2}} S_{h,\Delta t}^m \mathcal{P}_h v\|_H \le c t_m^{-\frac{\alpha}{2}}\|v\|_H,\ \alpha \in [0, 1],\ m \in \{1, \dots, N\},\ v \in H; \tag{33}$$

$$\Delta t \sum_{k=1}^m \|A_h^{\frac{1}{2}} S_{h,\Delta t}^k \mathcal{P}_h v\|_H^2 \le c\|v\|_H^2, \quad m \in \{1, \dots, N\},\ v \in H. \tag{34}$$

The estimates (29)-(34) are classical results frequently used in the numerical analysis of SPDEs; see, e.g., [17, (2.18), (2.19), (3.3), (4.7), (4.10), (4.11)].

- Suppose that $u_0 \in L^{2p}(\Omega, \dot{H}^1)$ is an $\mathcal{F}_0$-measurable random variable. Then, the auxiliary process $\mathcal{U}_h^n$ satisfies the following moment estimate

$$\|\mathcal{U}_h^n\|_{L^{2p}(\Omega,L^6(D))} \le c, \quad p \ge 1, \quad n = 1, \dots, N. \tag{35}$$

Indeed, to derive this bound, we first apply the Burkholder-Davis-Gundy inequality [25, Proposition 2.6], the Sobolev embedding $\dot H^{\frac{1}{3}} \hookrightarrow L^6(D)$, the estimates [17, (4.19)], (34), and condition (9) to obtain

$$\begin{aligned}\Big\| \sum_{k=0}^{n-1} S_{h,\Delta t}^{n-k}\mathcal{P}_h \Delta W^k \Big\|_{L^{2p}(\Omega, L^6(D))} &\le c \Big\| \sum_{k=0}^{n-1} S_{h,\Delta t}^{n-k}\mathcal{P}_h \Delta W^k \Big\|_{L^{2p}(\Omega, \dot H^{\frac{1}{3}})} \le c\Big(\Delta t \sum_{k=0}^{n-1} \Big\| A_h^{\frac{1}{6}} S_{h,\Delta t}^{n-k}\mathcal{P}_h \Big\|_{\mathcal{L}_2^0}^2\Big)^{\frac{1}{2}} \\ &\le c\|A^{-\frac{1}{3}}\|_{\mathcal{L}_2^0} \le c. \end{aligned} \tag{36}$$

Then using the representation (27), the fact that $r(u_h^n) \le 1$, the Sobolev embeddings $\dot H^1 \hookrightarrow \dot H^{\frac{1}{3}} \hookrightarrow L^6(D)$, along with the bounds (31), (33), (13), and (36), we deduce

$$\begin{aligned}\|\mathscr{U}_h^n\|_{L^{2p}(\Omega, L^6(D))} &\le c\|S_{h,\Delta t}^n \mathcal{P}_h u_0\|_{L^{2p}(\Omega, \dot H^{\frac{1}{3}})} + c\Delta t \sum_{k=0}^{n-1} \Big\| S_{h,\Delta t}^{n-k}\mathcal{P}_h F(u(t_k))\Big\|_{L^{2p}(\Omega, \dot H^{\frac{1}{3}})} \\ &\quad + \Big\| \sum_{k=0}^{n-1} S_{h,\Delta t}^{n-k}\mathcal{P}_h \Delta W^k \Big\|_{L^{2p}(\Omega, L^6(D))} \\ &\le c\|u_0\|_{L^{2p}(\Omega, \dot H^1)} + c\sup_{0\le t\le T} \|F(u(t))\|_{L^{2p}(\Omega,H)} \Delta t \sum_{k=0}^{n-1} \Big\| A^{\frac{1}{6}} S_{h,\Delta t}^{n-k}\mathcal{P}_h \Big\|_{\mathcal{L}(H)} + c \\ &\le c + c\sup_{0\le t\le T} \|F(u(t))\|_{L^{2p}(\Omega,H)} \Delta t \sum_{k=0}^{n-1} t_{n-k}^{-\frac{1}{6}} \le c. \end{aligned}$$

This completes the proof of (35).

We now begin the analysis of the term $\mathcal{E}_1$, as stated in the following lemma.

Lemma 1 (Estimate of $\mathcal{E}_1$). *Let $u(t_n)$ and $\mathscr{U}_h^n$ be the solutions to (5) and (27), respectively, where $n = 1, \dots, N$. Suppose that $u_0 \in L^{2p}(\Omega, \dot H^1)$ for $p \ge 1$. Furthermore, assume that the spatial mesh size h and time step size Δt are coupled such that $\Delta t = \mathcal{O}(h^2)$[2] Then, there exists a constant $c > 0$, independent of h and Δt, such that*

$$\Big\| u(t_n) - \mathscr{U}_h^n \Big\|_{L^{2p}(\Omega,H)} \le c(\Delta t^{\frac{\gamma}{2}} + h^\gamma), \quad n = 1, \dots, N, \tag{37}$$

where the regularity parameter $\gamma \in [\frac{1}{3}, \frac{1}{2})$.

Proof. Subtracting (27) from the mild solution (5) yields

$$\|u(t_n) - \mathscr{U}_h^n\|_{L^{2p}(\Omega,H)} \le e_1 + e_2 + e_3, \tag{38}$$

where e_1, e_2, e_3 are defined as follows

$$\begin{aligned} e_1 &:= \|S(t_n)u_0 - S_{h,\Delta t}^n \mathcal{P}_h u_0\|_{L^{2p}(\Omega,H)}, \\ e_2 &:= \Big\| \sum_{k=0}^{n-1} \Big(\int_{t_k}^{t_{k+1}} S(t_n - \tau) F(u(\tau)) d\tau - r(u_h^k) S_{h,\Delta t}^{n-k}\mathcal{P}_h F(u(t_k))\Delta t \Big) \Big\|_{L^{2p}(\Omega,H)}, \\ e_3 &:= \Big\| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \big(S(t_n-\tau) - S_{h,\Delta t}^{n-k}\mathcal{P}_h\big) dW(\tau) \Big\|_{L^{2p}(\Omega,H)}. \end{aligned}$$

We estimate each term e_1, e_2, and e_3 separately.

Applying (32) with $\mu = \nu = 1$, we obtain

$$e_1 = \Big\| \big(S(t_n) - S_{h,\Delta t}^n \mathcal{P}_h\big) u_0 \Big\|_{L^{2p}(\Omega,H)} \le c(h + \Delta t^{\frac{1}{2}})\|A^{\frac{1}{2}} u_0\|_{L^{2p}(\Omega,H)} \le c(h + \Delta t^{\frac{1}{2}}). \tag{39}$$

Next we estimate e_2. By the triangle inequality, we split e_2 as follows

$$e_2 \le e_{2,1} + e_{2,2}, \tag{40}$$

where

$$\begin{aligned} e_{2,1} &:= \Big\| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \big(S(t_n-\tau) - S_{h,\Delta t}^{n-k}\mathcal{P}_h\big) r(u_h^k) F(u(t_k)) d\tau \Big\|_{L^{2p}(\Omega,H)}, \\ e_{2,2} &:= \Big\| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} S(t_n-\tau)\big(F(u(\tau)) - r(u_h^k)F(u(t_k))\big) d\tau \Big\|_{L^{2p}(\Omega,H)}. \end{aligned}$$

We first estimate $e_{2,1}$. Using (32) with $\mu = 2$, $\nu = 0$, the boundedness $\|S(\cdot)\|_{\mathcal{L}(H)} \le 1$, and the inequalities (6) and (7), we derive

$$\begin{aligned} &\|S(t_n - t_k) - S_{h,\Delta t}^{n-k}\mathcal{P}_h\|_{\mathcal{L}(H)} \le c\frac{\Delta t + h^2}{t_n - t_k}, \quad \|S(t_n-\tau) - S(t_n - t_{n-1})\|_{\mathcal{L}(H)} \le c, \\ &\|S(t_n-\tau) - S(t_n - t_k)\|_{\mathcal{L}(H)} \le \|AS(t_n-\tau)\|_{\mathcal{L}(H)} \|A^{-1}(I - S(\tau - t_k))\|_{\mathcal{L}(H)} \\ &\qquad \le c\frac{\tau - t_k}{t_n - \tau} \le c\frac{\Delta t}{t_n - t_{k+1}}, \quad \tau \in [t_k, t_{k+1}), \quad k = 0, \dots, n-2. \end{aligned} \tag{41}$$

Utilizing the bound $r(u_h^k) \le 1$, (41), (13), and the fact that $n \le \frac{T}{\Delta t}$, we obtain

$$\begin{aligned} e_{2,1} &\le c\Big(\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}\Big[\Big\|S(t_n-\tau) - S(t_n - t_k)\Big\|_{\mathcal{L}(H)}^{2p} \Big\|F(u(t_k))\Big\|_H^{2p}\Big]^{\frac{1}{2p}} d\tau \\ &\quad + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}\Big[\Big\|S(t_n - t_k) - S_{h,\Delta t}^{n-k}\mathcal{P}_h\Big\|_{\mathcal{L}(H)}^{2p} \Big\|F(u(t_k))\Big\|_H^{2p}\Big]^{\frac{1}{2p}} d\tau \Big) \\ &\le c\sup_{0\le t\le T} \|F(u(t))\|_{L^{2p}(\Omega,H)} \Big(\Delta t + \sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} \frac{\Delta t}{t_n - t_{k+1}} d\tau + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{\Delta t + h^2}{t_n - t_k} d\tau \Big) \\ &\le c\Big(\Delta t + \Delta t \sum_{k=1}^{n} \frac{1}{k} + (\Delta t + h^2) \sum_{k=0}^{n-1} \frac{1}{n-k} \Big) \le c\big(\Delta t + \Delta t \ln(\Delta t^{-1}) + (\Delta t + h^2)\ln(\Delta t^{-1})\big). \end{aligned}$$

[2] This is the Courant-Friedrichs-Lewy (CFL) type condition, which is commonly imposed to balance the spatial and temporal discretization errors in the numerical analysis of SPDEs; see, e.g., [12,26].

Finally, under the assumption $\Delta t = \mathcal{O}(h^2)$, and using the fact that $\Delta t^{-\epsilon_0}$ dominates $\ln(\Delta t^{-1})$ for any arbitrarily small $\epsilon_0 > 0$, we conclude that

$$e_{2,1} \le c(\Delta t^{1-\epsilon_0} + h^2 \Delta t^{-\epsilon_0}) \le c(\Delta t^{1-\epsilon_0} + h^{2-2\epsilon_0}). \tag{42}$$

To estimate the error term $e_{2,2}$, we first observe that

$$1 - r(u_h^k) = \frac{\Delta t \|F(u_h^k)\|_H^2}{1 + \Delta t \|F(u_h^k)\|_H^2} \le \Delta t \|F(u_h^k)\|_H^2.$$

Also note that, for almost every sample point, we have $\|F(u_h^k)\|_H \le c(\|u_h^k\|_H + \|u_h^k\|_{L^6(D)}^3) \le c(\|u_h^k\|_H + \|u_h^k\|_{\dot H^1}^3) < \infty$ almost surely, since $u_h^k \in V_h \subset \dot H^1$ and $\dot H^1 \hookrightarrow L^6(D)$. Consequently, $\|F(u_h^k)\|_{L^{2p}(\Omega,H)} \le c(\|u_h^k\|_{L^{2p}(\Omega,H)} + \|u_h^k\|^3_{L^{6p}(\Omega,L^6(D))}) \le c(\|u_h^k\|_{L^{2p}(\Omega,H)} + \|u_h^k\|^3_{L^{6p}(\Omega,\dot H^1)}) < \infty$ for $k = 0, \dots, N$, and thus we obtain

$$\begin{aligned} e_{2,2} &\le \Big\| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} S(t_n-\tau)(F(u(\tau)) - F(u(t_k)))d\tau \Big\|_{L^{2p}(\Omega,H)} + \Big\| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \big(1 - r(u_h^k)\big) S(t_n-\tau)F(u(t_k))d\tau \Big\|_{L^{2p}(\Omega,H)} \\ &\le \Big\| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} S(t_n-\tau)(F(u(\tau)) - F(u(t_k)))d\tau \Big\|_{L^{2p}(\Omega,H)} + c\Delta t \Big\| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} S(t_n-\tau)F(u(t_k))d\tau \Big\|_{L^{4p}(\Omega,H)} \\ &\quad \times \sup_{k=0,\dots,n-1} \|F(u_h^k)\|^2_{L^{8p}(\Omega,L^2(D))} \\ &\le c\| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} S(t_n-\tau)(F(u(\tau)) - F(u(t_k)))d\tau \Big\|_{L^{2p}(\Omega,H)} + c\Delta t \Big\| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} S(t_n-\tau)F(u(t_k))d\tau \Big\|_{L^{4p}(\Omega,H)}. \end{aligned} \tag{43}$$

Furthermore, we recall that for $\tau \in [t_k, t_{k+1})$, the mild solution satisfies

$$u(\tau) - u(t_k) = (S(\tau - t_k) - I)u(t_k) + \int_{t_k}^{\tau} S(\tau - r)F(u(r))dr + \int_{t_k}^{\tau} S(\tau - r)dW(r),$$

which allows to expand the nonlinear difference $\big(F(u(t_k)) - F(u(\tau))\big)$ via a first-order Taylor expansion with remainder

$$\begin{aligned} F(u(t_k)) &- F(u(\tau)) = F'(u(\tau))\big(u(t_k) - u(\tau)\big) + R_F(u(\tau), u(t_k)) = -F'(u(\tau))(S(\tau - t_k) - I)u(t_k) \\ &- F'(u(\tau)) \int_{t_k}^{\tau} S(\tau - r)F(u(r))dr - F'(u(\tau)) \int_{t_k}^{\tau} S(\tau - r)dW(r) + \mathcal{R}_F(u(\tau), u(t_k)), \end{aligned} \tag{44}$$

where the remainder term is given by

$$\mathcal{R}_F(u(\tau), u(t_k)) := \int_0^1 F''\big(u(\tau) + \eta(u(t_k) - u(\tau))\big)(u(t_k) - u(\tau))^2 (1-\eta) d\eta.$$

Substituting (44) into (43) yields

$$e_{2,2} \le c(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5),$$

where

$$\mathcal{I}_1 := \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \Big\| S(t_n - \tau)F'(u(\tau))(S(\tau - t_k) - I)u(t_k) \Big\|_{L^{2p}(\Omega,H)} d\tau,$$

$$\mathcal{I}_2 := \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \Big\| S(t_n - \tau)F'(u(\tau)) \int_{t_k}^{\tau} S(\tau - r)F(u(r))dr \Big\|_{L^{2p}(\Omega,H)} d\tau,$$

$$\mathcal{I}_3 := \Big\| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} S(t_n - \tau)F'(u(\tau)) \int_{t_k}^{\tau} S(\tau - r)dW(r)d\tau \Big\|_{L^{2p}(\Omega,H)},$$

$$\mathcal{I}_4 := \Big\| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} S(t_n - \tau)\mathcal{R}_F(u(\tau), u(t_k))d\tau \Big\|_{L^{2p}(\Omega,H)},$$

$$\mathcal{I}_5 := \Delta t \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \Big\| S(t_n - \tau)F(u(t_k)) \Big\|_{L^{4p}(\Omega,H)} d\tau.$$

We now proceed to estimate the terms $\mathcal{I}_1$ through $\mathcal{I}_5$ individually.

To estimation $\mathcal{I}_1$, we apply (29) and (7), yielding

$$\begin{aligned} \mathcal{I}_1 &\le c \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \Big\| A^{\frac{\delta_0}{2}} S(t_n - \tau) \Big\|_{\mathcal{L}(H)} \Big\| A^{-\frac{\delta_0}{2}} F'(u(\tau))(S(\tau - t_k) - I)u(t_k) \Big\|_{L^{2p}(\Omega,H)} d\tau \\ &\le c \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - \tau)^{-\frac{\delta_0}{2}} \Big\| F'(u(\tau))(S(\tau - t_k) - I)u(t_k) \Big\|_{L^{2p}(\Omega,L^1(D))} d\tau, \quad \forall\, \delta_0 \in (\tfrac{3}{2}, 2). \end{aligned}$$

Recall that $F(v) = v - v^3$, $v \in \mathbb{R}$. Using (6), the Sobolev embedding $\dot H^{\frac{1}{3}} \hookrightarrow L^6(D) \hookrightarrow L^4(D)$, the regularity estimate (11), and the Hölder's inequality, we obtain for $\tau \in (t_k, t_{k+1})$,

$$\begin{aligned} \Big\| F'(u(\tau))(S(\tau &- t_k) - I)u(t_k) \Big\|_{L^{2p}(\Omega,L^1(D))} \le c(1 + \|u(\tau)\|^2_{L^{8p}(\Omega,L^4(D))}) \|(S(\tau - t_k) - I)u(t_k)\|_{L^{4p}(\Omega,H)} \\ &\le c(1 + \|u(\tau)\|^2_{L^{8p}(\Omega,L^4(D))}) \mathbb{E}\Big[\Big\| A^{-\frac{\gamma}{2}}(S(\tau - t_k) - I) \Big\|^{4p}_{\mathcal{L}(H)} \Big\| A^{\frac{\gamma}{2}} u(t_k) \Big\|^{4p}_H \Big]^{\frac{1}{4p}} \\ &\le c\Delta t^{\frac{\gamma}{2}} \big(1 + \sup_{0\le t\le T} \|u(t)\|^2_{L^{8p}(\Omega,L^4(D))}\big) \sup_{0\le t\le T} \|u(t)\|_{L^{4p}(\Omega,\dot H^{\gamma})} \le c\Delta t^{\frac{\gamma}{2}}, \quad \gamma \in [\tfrac{1}{3}, \tfrac{1}{2}). \end{aligned}$$

Therefore, for $\delta_0 \in (\frac{3}{2}, 2)$, we conclude

$$\mathcal{I}_1 \leq c\Delta t^{\frac{\gamma}{2}} \int_0^{t_n} (t_n-\tau)^{-\frac{\delta_0}{2}} d\tau \leq c\Delta t^{\frac{\gamma}{2}}, \quad \gamma \in [\tfrac{1}{3}, \tfrac{1}{2}).$$

The boundedness of $\mathcal{I}_2$ follows directly from Hölder's inequality, the semigroup inequality $\|S(\cdot)\|_{\mathcal{L}(H)} \leq 1$, together with (29), (7), (13), the embedding $\dot{H}^{\frac{1}{3}} \hookrightarrow L^4(D)$, and (11). Specifically, for $\delta \in (\frac{3}{2}, 2)$, we have

$$\begin{aligned}
\mathcal{I}_2 &\leq c \textstyle\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{\tau} \left\| A^{\frac{\delta_0}{2}} S(t_n-\tau) \right\|_{\mathcal{L}(H)} \left\| A^{-\frac{\delta_0}{2}} F'(u(\tau)) S(\tau-r) F(u(r)) \right\|_{L^{2p}(\Omega,H)} dr d\tau \\
&\leq c \textstyle\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{\tau} (t_n-\tau)^{-\frac{\delta_0}{2}} \left\| F'(u(\tau)) S(\tau-r) F(u(r)) \right\|_{L^{2p}(\Omega,L^1(D))} dr d\tau \\
&\leq c \textstyle\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{\tau} (t_n-\tau)^{-\frac{\delta_0}{2}} (1 + \|u(\tau)\|^2_{L^{8p}(\Omega,L^4(D))}) \left\| F(u(r)) \right\|_{L^{4p}(\Omega,H)} dr d\tau \\
&\leq c\Delta t (1 + \sup_{0\leq t\leq T} \|u(t)\|^2_{L^{8p}(\Omega,\dot{H}^{\frac{1}{3}})}) \sup_{0\leq t\leq T} \|F(u(t))\|_{L^{4p}(\Omega,H)} \int_0^{t_n} (t_n-\tau)^{-\frac{\delta_0}{2}} d\tau \leq c\Delta t.
\end{aligned}$$

To estimate $\mathcal{I}_3$, let $\chi_{[a,b)}(\cdot)$ denote the characteristic function of the interval $[a,b)$ for $a, b \in \mathbb{R}$. By swapping the order of integration as in [17, (4.34)], and applying the Burkholder-Davis-Gundy inequality, along with Hölder's inequality, we obtain

$$\begin{aligned}
\mathcal{I}_3 &= \left\| \textstyle\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \chi_{[t_k,\tau)}(r) S(t_n-\tau) F'(u(\tau)) S(\tau-r) d\tau dW(r) \right\|_{L^{2p}(\Omega,H)} \\
&\leq c \Big(\textstyle\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left\| \int_{t_k}^{t_{k+1}} S(t_n-\tau) F'(u(\tau)) S(\tau-r) d\tau \right\|^2_{L^{2p}(\Omega,\mathcal{L}_0^2)} dr \Big)^{\frac{1}{2}} \\
&\leq c\Delta t^{\frac{1}{2}} \Big(\textstyle\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \left\| S(t_n-\tau) F'(u(\tau)) S(\tau-r) \right\|^2_{L^{2p}(\Omega,\mathcal{L}_0^2)} d\tau dr \Big)^{\frac{1}{2}}.
\end{aligned} \tag{45}$$

Using the boundedness of the semigroup $\|S(\cdot)\|_{\mathcal{L}(H)} \leq c$, the Sobolev embedding $\dot{H}^{\frac{1}{3}} \hookrightarrow L^4(D)$, the inequality (7), the condition (9), and regularity estimate (10), there exists an arbitrarily small $\epsilon_0 > 0$ such that

$$\begin{aligned}
&\left\| S(t_n-\tau) F'(u(\tau)) S(\tau-r) \right\|^2_{L^{2p}(\Omega,\mathcal{L}_0^2)} \leq c \|F'(u(\tau))\|^2_{L^{2p}(\Omega,L^4(D))} \Big(\textstyle\sum_{j=1}^{\infty} \|S(\tau-r)\phi_j\|^2_{L^4(D)} \Big) \\
&\leq c \|F'(u(\tau))\|^2_{L^{2p}(\Omega,L^4(D))} \Big(\textstyle\sum_{j=1}^{\infty} \left\| S(\tau-r)\phi_j \right\|^2_{\dot{H}^{\frac{1}{3}}} \Big) \leq c \|F'(u(\tau))\|^2_{L^{2p}(\Omega,L^4(D))} \left\| A^{\frac{1}{2}-\epsilon_0} S(\tau-r) \right\|^2_{\mathcal{L}(H)} \left\| A^{-\frac{1}{3}+\epsilon_0} \right\|^2_{\mathcal{L}_0^2} \\
&\leq c \Big(1 + \sup_{0\leq t\leq T} \left\| u(t) \right\|^4_{L^{4p}(\Omega,C(D))} \Big) (\tau-r)^{-1+2\epsilon_0} \leq c(\tau-r)^{-1+2\epsilon_0}.
\end{aligned}$$

Therefore the term $\mathcal{I}_3$ can be bounded by

$$\mathcal{I}_3 \leq c\Delta t^{\frac{1}{2}} \Big(\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} (\tau-r)^{-1+2\epsilon_0} d\tau dr \Big)^{\frac{1}{2}} \leq c\Delta t^{\frac{1}{2}} \Big(\sum_{k=0}^{n-1} \Delta t^{1+2\epsilon_0} \Big)^{\frac{1}{2}} \leq c\Delta t^{\frac{1}{2}+\epsilon_0}.$$

For the estimation of $\mathcal{I}_4$, we employ the inequality (6) with $\alpha = \frac{\delta_0}{2}$ ($\delta_0 \in (\frac{3}{2}, 2)$), together with the inequality (29), the estimate (10), and the temporal Hölder regularity result (12). This yields

$$\begin{aligned}
\mathcal{I}_4 &\leq c \textstyle\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n-\tau)^{-\frac{\delta_0}{2}} \left\| A^{-\frac{\delta_0}{2}} \mathcal{R}_F(u(\tau), u(t_k)) \right\|_{L^{2p}(\Omega,H)} d\tau \\
&\leq c \textstyle\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n-\tau)^{-\frac{\delta_0}{2}} \|\mathcal{R}_F(u(\tau), u(t_k))\|_{L^{2p}(\Omega,L^1(D))} d\tau \\
&\leq c \textstyle\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n-\tau)^{-\frac{\delta_0}{2}} \int_0^1 \Big(\left\| \big((1-\eta)\|u(\tau)\|_{C(D)} + \eta\|u(t_k)\|_{C(D)} \big) \|u(\tau)-u(t_k)\|^2_H \right\|_{L^{2p}(\Omega)} \Big) d\eta d\tau \\
&\leq c \sup_{0\leq t\leq T} \|u(t)\|_{L^{4p}(\Omega,C(D))} \textstyle\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n-\tau)^{-\frac{\delta_0}{2}} \|u(\tau)-u(t_k)\|^2_{L^{8p}(\Omega,H)} d\tau \\
&\leq c\Delta t^{\gamma} \int_0^{t_n} (t_n-\tau)^{-\frac{\delta_0}{2}} d\tau \leq c\Delta t^{\gamma}.
\end{aligned}$$

For $\mathcal{I}_5$, by virtue of $\|S(\cdot)\|_{\mathcal{L}(H)} \leq 1$ and the estimate (13), we arrive at

$$\mathcal{I}_5 = \Delta t \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left\| S(t_n-\tau) F(u(t_k)) \right\|_{L^{4p}(\Omega,H)} d\tau \leq c \sup_{0\leq t\leq T} \|F(u(t))\|_{L^{4p}(\Omega,H)} \Delta t \leq c\Delta t.$$

Combining all the estimates for $\mathcal{I}_1$ through $\mathcal{I}_5$, we conclude that

$$e_{2,2} \leq c\Delta t^{\frac{\gamma}{2}}. \tag{46}$$

Together with (40), (42), and (46), we finally obtain

$$e_2 \leq c(\Delta t^{\frac{\gamma}{2}} + h^{2-2\epsilon_0}), \ \gamma \in [\frac{1}{3}, \frac{1}{2}), \tag{47}$$

where $\epsilon_0 > 0$ is arbitrarily small.

It remains to analyze the term e_3. In fact, Qi et al. [17, (4.39)] have established the optimal estimate for e_3, which reads as

$$e_3 \leq c(h^{\gamma} + \Delta t^{\frac{\gamma}{2}}) \|A^{\frac{\gamma-1}{2}}\|_{\mathcal{L}_0^2}.$$

By further applying the regularity condition (9), we deduce that

$$e_3 \le c(h^\gamma + \Delta t^{\frac{\gamma}{2}}). \tag{48}$$

Finally, collecting (38), (39), (47), (48) completes the proof. □

We next analyze the error term $\mathcal{E}_2$, as demonstrated in the following lemma.

Lemma 2 (Estimate of $\mathcal{E}_2$). *Let u_h^n and $\mathscr{U}_h^n$ denote the numerical solutions given by (25) and (26), respectively. Then there exists a constant c, independent of h and Δt, such that*

$$\|\mathscr{U}_h^n - u_h^n\|_{L^2(\Omega,H)} \le c(\Delta t^{\frac{\gamma}{2}} + h^\gamma), \quad n = 1, \dots, N.$$

Proof. Let $\theta_h^n := \mathscr{U}_h^n - u_h^n$. Clearly, $\theta_h^0 = 0$ and $\theta_h^n \in V_h$. Subtracting Eq. (25) from (26) yields

$$\theta_h^{n+1} - \theta_h^n = -\Delta t A_h \theta_h^{n+1} + r(u_h^n)\mathcal{P}_h\big(F(u(t_n)) - F(u_h^n)\big)\Delta t.$$

Taking the H-inner product of both sides with θ_h^{n+1}, we obtain

$$\begin{aligned}(\theta_h^{n+1} - \theta_h^n, \theta_h^{n+1}) + \Delta t\left\|\nabla\theta_h^{n+1}\right\|_H^2 &= r(u_h^n)\Delta t\Big(\big(F(u(t_n)) - F(\mathscr{U}_h^n), \theta_h^{n+1}\big) + \big(F(\mathscr{U}_h^n) - F(u_h^n), \theta_h^n\big)\\ &\quad + \big(F(\mathscr{U}_h^n) - F(u_h^n), \theta_h^{n+1} - \theta_h^n\big)\Big).\end{aligned} \tag{49}$$

Applying the bound $r(u_h^n) \le 1$, (8), (30), $\|F(v)\|_H^2 \le c(\|v\|_H^2 + \|v\|_{L^6(D)}^6), \forall v \in L^6(D)$, and Young's inequality, we estimate the right-hand side of (49) as follows

$$\begin{aligned}& r(u_h^n)\Delta t\Big(\big(F(u(t_n)) - F(\mathscr{U}_h^n), \theta_h^{n+1}\big) + \big(F(\mathscr{U}_h^n) - F(u_h^n), \theta_h^n\big) + \big(F(\mathscr{U}_h^n) - F(u_h^n), \theta_h^{n+1} - \theta_h^n\big)\Big)\\ &\le c\Delta t\|A^{-\frac{1}{2}}(F(u(t_n)) - F(\mathscr{U}_h^n))\|_H\|\nabla\theta_h^{n+1}\|_H + \Delta t\|\theta_h^n\|_H^2 + \Delta t^2\|F(\mathscr{U}_h^n) - F(u_h^n)\|_H^2\\ &\quad + \tfrac{1}{4}\|\theta_h^{n+1} - \theta_h^n\|_H^2\\ &\le c\Delta t\|F(u(t_n)) - F(\mathscr{U}_h^n)\|^2_{L^{\frac{6}{5}}(D)} + \Delta t\|\nabla\theta_h^{n+1}\|_H^2 + \Delta t\|\theta_h^n\|_H^2\\ &\quad + c\Delta t^2\Big(\|\mathscr{U}_h^n\|_H^2 + \|\mathscr{U}_h^n\|_{L^6(D)}^6 + \|u_h^n\|_H^2 + \|u_h^n\|_{L^6(D)}^6\Big) + \tfrac{1}{2}\left\|\theta_h^{n+1} - \theta_h^n\right\|_H^2.\end{aligned} \tag{50}$$

Employing Hölder's inequality allows to deduce

$$\left\|F(u(t_n)) - F(\mathscr{U}_h^n)\right\|^2_{L^{\frac{6}{5}}(D)} \le c\left\|u(t_n) - \mathscr{U}_h^n\right\|_H^2\big(1 + \|u(t_n)\|_{L^6(D)}^4 + \|\mathscr{U}_h^n\|_{L^6(D)}^4\big). \tag{51}$$

Taking expectation on both sides of (49), and applying (50), (51), along with the identity $a(a-b) = \frac{1}{2}(a^2 - b^2) + \frac{1}{2}(a-b)^2$ for any $a, b \in \mathbb{R}$, we arrive at

$$\begin{aligned}\tfrac{1}{2}\mathbb{E}\Big[\left\|\theta_h^{n+1}\right\|_H^2 - \|\theta_h^n\|_H^2\Big] &\le c\big(1 + \mathbb{E}[\|u(t_n)\|_{L^6(D)}^8] + \mathbb{E}[\|\mathscr{U}_h^n\|_{L^6(D)}^8]\big)^{\frac{1}{2}}\left\|u(t_n) - \mathscr{U}_h^n\right\|^2_{L^4(\Omega,H)} + \Delta t\mathbb{E}[\|\theta_h^n\|_H^2]\\ &\quad + c\Delta t^2\mathbb{E}\big[\|\mathscr{U}_h^n\|_H^2 + \|\mathscr{U}_h^n\|_{L^6(D)}^6 + \|u_h^n\|_H^2 + \|u_h^n\|_{L^6(D)}^6\big].\end{aligned}$$

By virtue of (36), (11), the Sobolev embedding $\dot{H}^{\frac{1}{3}} \hookrightarrow L^6(D)$, and estimate (37), we obtain

$$\frac{1}{2}\left\|\theta_h^{n+1}\right\|^2_{L^2(\Omega,H)} \le \Big(\frac{1}{2} + \Delta t\Big)\left\|\theta_h^n\right\|^2_{L^2(\Omega,H)} + c(\Delta t^{\frac{\gamma}{2}} + h^\gamma)^2.$$

Denote $\rho(\Delta t) := 1 + 2\Delta t$, so the inequality becomes

$$\left\|\theta_h^{n+1}\right\|^2_{L^2(\Omega,H)} \le \rho(\Delta t)\left\|\theta_h^n\right\|^2_{L^2(\Omega,H)} + c(\Delta t^{\frac{\gamma}{2}} + h^\gamma)^2.$$

Applying this inequality recursively yields

$$\left\|\theta_h^n\right\|^2_{L^2(\Omega,H)} \le \rho(\Delta t)^n\left\|\theta_h^0\right\|^2_{L^2(\Omega,H)} + c(\Delta t^{\frac{\gamma}{2}} + h^\gamma)^2\textstyle\sum_{k=0}^{n-1}\rho(\Delta t)^k.$$

Note that $\rho(\Delta t) := 1 + \frac{2T}{N}$, a straightforward calculation gives

$$\lim_{N\to\infty}\rho(\Delta t)^n = \lim_{N\to\infty}\Big(1 + \frac{2T}{N}\Big)^n \le \mathrm{e}^{2T}, \quad \forall n \le N.$$

This implies $\sum_{k=0}^{n-1}\rho(\Delta t)^k \le c$. Moreover, since $\theta_h^0 = 0$, it follows that

$$\|\theta_h^n\|^2_{L^2(\Omega,H)} \le c(\Delta t^{\frac{\gamma}{2}} + h^\gamma)^2.$$

This completes the proof. □

We are now in a position to derive the strong error bound for the fully discrete scheme, utilizing the estimates established in Lemmas 1 and 2. The result is summarized in the following theorem.

Theorem 1 (Full discretization error estimate). *Let $u(t)$ and u_h^n be the mild solution and numerical solution given by (5) and (25), respectively. Under the assumptions stated in Lemma 1, there exists a constant c independent of Δt and h, such that for $\gamma \in [\frac{1}{3}, \frac{1}{2})$, the following strong error estimate holds*

$$\left\| u(t_n) - u_h^n \right\|_{L^2(\Omega,H)} \le c\left(\Delta t^{\frac{\gamma}{2}} + h^\gamma\right), \quad n = 1, \dots, N.$$

Proof. The result follows directly from Lemmas 1 and 2, combined with the triangle inequality

$$\left\| u(t_n) - u_h^n \right\|_{L^2(\Omega,H)} \le c\left(\Delta t^{\frac{\gamma}{2}} + h^\gamma\right), \; n = 1, \dots, N.$$

This ends the proof. □

5. Numerical experiments

Since the primary focus of this work is on temporal discretization, this section only presents numerical experiments to examine the performance of the proposed scheme in the time direction, including its stability, computational efficiency, and convergence accuracy.

As a benchmark problem, we consider the following stochastic Allen-Cahn equation

$$\begin{aligned} du(x,t) &= \left(\partial_{xx}u + \tfrac{(u-u^3)}{\varepsilon^2}\right)dt + \sigma dW(x,t), \; 0 < t \le T, \; x \in (0,1), \\ u(0,t) &= u(1,t) = 0, \quad 0 \le t \le T, \\ u(x,0) &= u_0(x), \; x \in [0,1], \end{aligned} \tag{52}$$

where ε denotes the thickness parameter, σ is the noise intensity, and $W(x,t)$ is defined by (4).

We first compare the computational stability of the proposed fully discrete scheme (25) with that of the "pure (untamed) semi-implicit Euler-Maruyama/ finite element" method, given by

$$\begin{aligned} (I + \Delta t A_h)u_h^{n+1} = &\quad u_h^n + \mathcal{P}_h F(u_h^n) + \mathcal{P}_h \Delta W^n, \\ u_h^0 = &\quad \mathcal{P}_h u_0. \end{aligned} \tag{53}$$

To this end, we fix a random seed and examine a single sample path using a spatial mesh size $h = 1/64$, final time $T = 1$, $\varepsilon = 0.01$, and $\sigma = 0.5$. Both schemes (25) and (53) are implemented with various time step sizes, and we monitor the evolution of the discrete solution $\|u_h^n\|_{L^2(D)}$. If the computed solution returns "*NaN*", it is interpreted as a numerical blow-up. As shown in Fig. 1 (a), the scheme (53) becomes unstable and blows up at time $t = 3.08 \times 10^{-3}$ when $\Delta t \ge 2.2 \times 10^{-4}$. In contrast, the proposed scheme (25) remains stable under the same conditions, as illustrated in Fig. 1 (b). Remarkably, even for a large time step size $\Delta t = 0.01$, the scheme (25) retains numerical stable. These observations confirm that the proposed scheme (25) offers significantly improved computational stability compared to the pure semi-implicit Euler-Maruyama scheme (53).

Next, we numerically explore the ergodic behavior of the proposed time-discrete scheme, although a rigorous theoretical proof of its ergodicity is not provided in this work. As noted in existing studies (see, e.g., [27, Section 5.2]; [21, Section 4]), if the scheme (15) is uniquely ergodic, then for any test function $f \in C_b(H)$, the limiting time average quantity

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \mathbb{E}[f(u^k)] \tag{54}$$

converges to the common ergodic value, irrespective of the choice of initial condition $u_0(x)$. Here u^k denotes the numerical solution at k-th time step generated by the scheme (15). In our simulation, the computation is carried out up to time $t = 1000$ to approximate long-time behavior [2] The expectation $\mathbb{E}[f(u^k)]$ is approximated by averaging over 500 independent sample paths. Specifically, with a fixed time step $\Delta t = 0.1$, the limiting time average in (54) is numerically approximated by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \mathbb{E}[f(u^k)] \approx \frac{1}{5,000,000} \sum_{k=1}^{10,000} \sum_{j=1}^{500} f(u_j^k),$$

where u_j^k denotes the j-th sample of the numerical solution u^k.

Define $E_n := \frac{1}{n}\sum_{k=0}^{n} \mathbb{E}[f(u^k)]$, and fix the other parameters as $h = 1/128$, $\varepsilon = 0.01$, $\sigma = 0.1$. We then track the evolution of the sequence $\{E_n\}_{n=1}^{t/\Delta t}$ under different initial conditions and test functions, as shown in Fig. 2. To be specific, we consider the test functions $f(\cdot) = e^{-\|\cdot\|_H^2}$ and $f(\cdot) = \|\cdot\|_H^2$, both of which have been adopted in [21,27]. It is clearly observed that as n increases (i.e., as time evolves), the values of E_n corresponding to different initial conditions converge toward the common limiting value. This numerically demonstrates the unique ergodicity of the scheme (15).

We now compare the computational efficiency of the proposed scheme (25) with that of the backward Euler scheme introduced in [17, (4.1)], focusing on a single sample path. All numerical experiments were performed on a machine equipped with 32 GB of RAM and a Core Ultra 7 155 CPU. Fixing the terminal time at $T = 1$, and setting the other parameters as $\varepsilon = 0.01$, $h = 1/64$, and $\sigma = 1$, we

[2] This simulation horizon follows the precedent set in [21,27], where $t = 200$ was deemed sufficiently long for ergodic investigations.

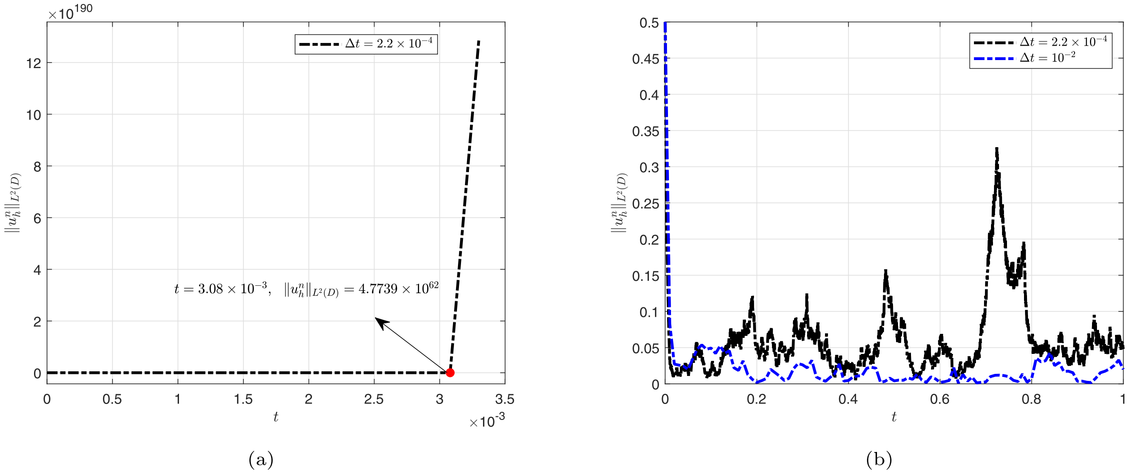

Fig. 1. The temporal evolution of the numerical solutions $\|u_h^n\|_{L^2(D)}$ obtained from different schemes. (a): Results from scheme (53). (b): Results from scheme (25).

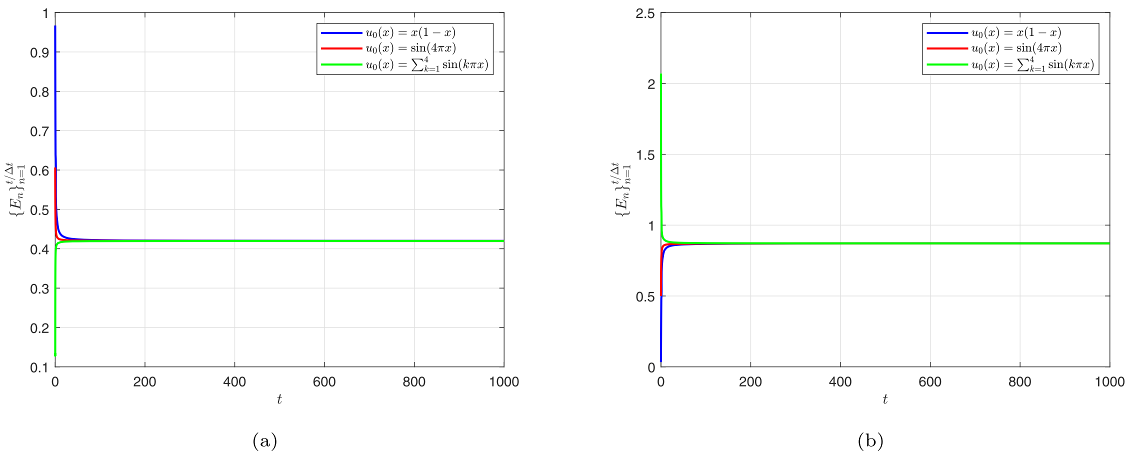

Fig. 2. Evolution of $E_n := \frac{1}{n}\sum_{k=0}^{n}\mathbb{E}[f(u^k)]$ under different test functions. (a): $f(\cdot) = e^{-\|\cdot\|_H^2}$. (b) $f(\cdot) = \|\cdot\|_H^2$. The results are shown for different initial conditions to illustrate convergence toward the common ergodic limit.

record the CPU time required by both methods for various time step sizes. The results, summarized in Table 1, clearly indicate that the proposed scheme (25) achieves superior computational efficiency compared to the backward Euler method.

Finally, we numerically test the temporal convergence accuracy of the proposed scheme (25). The strong convergence rate in time is evaluated by computing the mean-square approximation error at the final times $T = 0.1$ and $T = 1$, respectively. Since the exact solution to problem (52) is not available, we employ a reference solution computed on a sufficiently fine space-time grid as a surrogate. Specifically, the reference solution is generated using the spatial mesh size $h = 1/128$ and a time step $\Delta t = 10^{-6}$. The expectation in the mean-square error, $\left(\mathbb{E}[\|u(T) - u_h^N\|_H^2]\right)^{\frac{1}{2}}$, is approximated by the empirical mean over 200 independent realizations

$$\left(\mathbb{E}\left[\|u(T) - u_h^N\|_H^2\right]\right)^{\frac{1}{2}} \approx \left(\frac{1}{200}\sum_{j=1}^{200}\|u_j^{\mathrm{ref}} - u_{j,h}^N\|_H^2\right)^{\frac{1}{2}} =: u_{\mathrm{error}},$$

where u_j^{ref} and $u_{j,h}^N$ denote respectively the reference and fully discrete numerical solutions of the j-th sample.

With parameters set to $\varepsilon = 1$, $\sigma = 0.5$, $\gamma = 0.4995$, the numerical error u_{error} corresponding to different final times T and time step sizes is shown in Fig. 3. According to the theoretical analysis presented in Theorem 1, the optimal temporal convergence order achievable as $\gamma \to \frac{1}{2}$ is $\mathcal{O}(\Delta t^{\frac{1}{4}-\varepsilon_0})$, where ε_0 denotes an arbitrarily small positive constant. The numerical results reported in Fig. 3 are consistent with this theoretical prediction.

Table 1
CPU time consumption for different schemes and time steps.

Time step Δt	Tamed scheme (25) CPU time (s)	Backward Euler scheme CPU time (s)	Speed-up factor
1.00E-2	0.048312	1.566181	32.4181
5.00E-3	0.073324	2.529613	34.4991
2.00E-3	0.177648	4.281687	24.1021
1.00E-3	0.346869	5.178428	14.9291

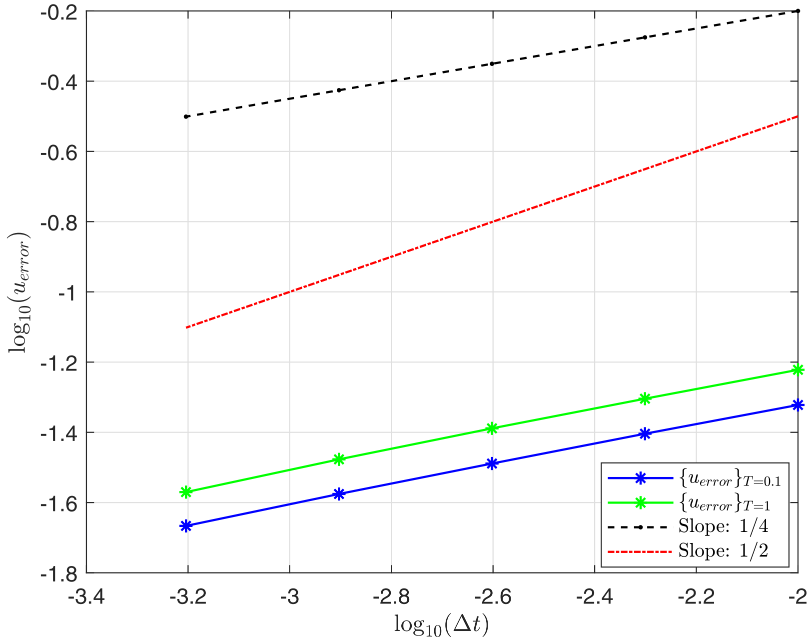

Fig. 3. The error u_{error} is plotted on a log-log scale for two final times, $T = 0.1$ and $T = 1$, across a range of time step sizes: $\Delta t = 10^{-2}$, 5×10^{-3}, 2.5×10^{-3}, 1.25×10^{-3}, and 6.25×10^{-4}.

6. Conclusion

In this study, we proposed an efficient temporal discretization scheme for the stochastic Allen-Cahn equation driven by additive white noise and rigorously analyzed the spatio-temporal strong convergence of the resulting fully discrete method within the Galerkin finite element framework. In particular, we established the long-time stability of the semi-discrete scheme by constructing and analyzing an appropriate discrete Lyapunov functional. Moreover, the strong convergence rates in both time and space for the fully discrete approximation were derived through a suitable error-decomposition strategy. Finally, the theoretical findings were verified by numerical experiments.

7. Appendix

The existence of an invariant measure for the sequence $\{u^n\}_{n\in\mathbb{N}_+}$ *is not established in this work*. This section is intended to show that, if the sequence $\{u^n\}_{n\in\mathbb{N}_+}$ exists an invariant measure in the space H, then that measure is unique.

Let $\mathcal{B}(H)$ denote the Borel σ-algebra on H, and define the transition kernel associated with the sequence $\{u^n\}_{n\in\mathbb{N}_+}$, generated by scheme (15), as

$$P(\eta, B) := \mathbb{P}(u^{n+1} \in B | u^n = \eta),\ \eta \in H, B \in \mathcal{B}(H). \tag{55}$$

A probability measure μ on $\mathcal{B}(H)$ is said to be an *invariant measure* of the sequence $\{u^n\}_{n\in\mathbb{N}_+}$ if it satisfies

$$\int_H P(\eta, B)\mu(d\eta) = \mu(B),\quad \forall B \in \mathcal{B}(H). \tag{56}$$

In what follows, we will show that, assuming the existence of such an invariant measure, its uniqueness can be justified, as stated in Proposition 2.

Proposition 2. *For any* $n \in \mathbb{N}_+$ *and* $\Delta t \in (0, 1)$*, the transition kernel* $P(\cdot, \cdot)$ *defined in* (55) *is irreducible and regular. Consequently, there exists at most, if it exists, one invariant measure of* $\{u^n\}_{n\in\mathbb{N}_+}$.

Proof. Motivated by the arguments presented in [13, Proposition 3.1] and [21, Proposition 3.2], we observe that the uniqueness of the invariant measure can be deduced if the transition kernel $P(\cdot,\cdot)$ defined in (55) is both *irreducible* and *regular*.[2] According to Doob's theorem [29, Theorem 1.12], these two properties are sufficient to guarantee uniqueness. We therefore aim to verify that the transition kernel $P(\cdot,\cdot)$ satisfies both conditions.

We first define operator $\Psi(v) := (I + \Delta t A)v$, for $v \in H$. Then we will show that for any open set $B \in \mathcal{B}(H)$, $\Psi(B)$ is also an open set in $\mathcal{B}(H)$. This property will later be used to establish the irreducibility of the Markov chain $\{u^n\}_{n\in\mathbb{N}_+}$. Applying Poincaré inequality gives

$$\begin{aligned}\big(\Psi(v_1) - \Psi(v_2), v_1 - v_2\big) &= \Delta t\|\nabla(v_1 - v_2)\|_H^2 + \|v_1 - v_2\|_H^2\\ &\geq (1 + c\Delta t)\|v_1 - v_2\|_H^2, \quad \forall v_1, v_2 \in H,\end{aligned}$$

and thus

$$\|\Psi(v_1) - \Psi(v_2)\|_H \geq c_1\|v_1 - v_2\|_H.$$

This shows, see, e.g., [21, (2.19)], that there exists a constant $r > 0$ such that

$$\mathrm{b}(\Psi(v), r) \subset \Psi\big(\mathrm{b}(v, r/c_1)\big), \quad v \in H, \tag{57}$$

where $\mathrm{b}(z, \rho)$ denotes the open ball centered at z with radius ρ.

Now, let $B \in \mathcal{B}(H)$ be an arbitrary open set. This means that for every $f \in B$, there exists a radius $r_f > 0$ such that the open ball $\mathrm{b}(f, r_f) \subset B$. We next show that $\Psi(B)$ is also open. To this end, it suffices to prove that for any $f \in B$, one can find an radius $r_1 > 0$ such that the open ball $\mathrm{b}(\Psi(f), r_1) \in \Psi(B)$. Indeed, by (57) and $\mathrm{b}(f, r_f) \subset B$, we can choose such an r_1 satisfying

$$\mathrm{b}(\Psi(f), r_1) \subset \Psi\big(\mathrm{b}(f, r_1/c_1)\big) \subset \Psi(B),$$

which confirms that $\Psi(B)$ is open.

Next, we prove that the transition kernel $P(\cdot,\cdot)$ defined in (55) is both irreducible and regular. To this end, we first denote by $\mu_{m,C}$ the Gaussian measure on H with mean m and variance operator C. Consider a non-empty open set $B \in \mathcal{B}(H)$. By (15), we have

$$P(\eta, B) \quad = \mathbb{P}(u^{n+1} \in B | u^n = \eta) = \mathbb{P}\big((\eta + r(\eta)F(\eta)\Delta t + \Delta W^n) \in \Psi(B)\big) = \mu_{\eta+r(\eta)F(\eta)\Delta t, \Delta t I}(\Psi(B)), \quad \eta \in H,$$

where we used the fact that $\eta + r(\eta)F(\eta)\Delta t + \Delta W^n$ is normally distributed with mean $\eta + r(\eta)F(\eta)\Delta t$ and variance operator $\Delta t I$.

It has been shown that $\Psi(B)$ is a non-empty open set. Moreover, we observed that the Gaussian measure $\mu_{\eta+r(\eta)F(\eta)\Delta t,\Delta t I}$ is non-degenerate[2] due to the presence of the variance operator $\Delta t I$. It is known (see, e.g., [21, Proof of Proposition 3.2]) that any non-degenerate Gaussian measure in separable Banach space measures any non-empty open set positive. Therefore, we conclude that $\mu_{\eta+r(\eta)F(\eta)\Delta t,\Delta t I}(\Psi(B)) > 0$, i.e., the transition kernel satisfies $P(\eta, B) > 0$. Consequently, the Markov chain $\{u^n\}_{n\in\mathbb{N}_+}$ is irreducible in H.

To establish the regular property, we invoke the Feldman-Hajek theorem [28, Theorem 2.25], which ensures that all non-degenerate Gaussian measures $\{\mu_{\xi,\Delta t I}, \xi \in H\}$ are mutually equivalent. This equivalence implies that the transition kernel $P(\cdot,\cdot)$ defined by (55) is regular. Consequently, by Doob's theorem [29, Theorem 1.12], the Markov chain $\{u^n\}_{\mathbb{N}_+}$ admits at most one invariant measure. This completes the proof. □

CRediT authorship contribution statement

Xiao Qi: Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Data curation; **Yubin Yan:** Writing – review & editing, Supervision, Resources, Project administration, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Xiao Qi reports financial support was provided by the China Scholarship Council and the Research Fund of Jianghan University under Grant No. 2024JCYJ04. The authors declare that they have no competing interests.

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[2] For precise definitions of irreducibility and regularity of Markov transition kernels, we refer the reader to [28, Section 11.2.2].
[2] See, e.g., [27, Assumption 3(i), Remark 4(ii)] for further details on the non-degeneracy of the measure.

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