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A tabular method for performing Fourier analysis of complex biological shape

Stephen Lewis

Abstract

Whilst linear dimensions are easily measured and analysed numerically, curvilinear forms are difficult both to define and to compare and are frequently left unexplored. A method of describing curved or non-uniform shapes, which has become popular among a number of biological workers, is Fourier analysis — a numerical analytical technique with an established mathematical background.

Of the three stages followed when using this technique to describe biological shape - the construction of a wave-like curve from the shape being studied, the numerical (Fourier) analysis and the use of the Fourier coefficients to perform statistical analyses - that of how the Fourier analysis is performed is largely unreported. This leaves many unclear about how to perform a technique which they may otherwise find useful.

A tabular method, which allows the computational steps of Fourier analysis to be monitored throughout, is described. This procedure can be readily performed, using a computer spreadsheet or on paper. The original curve may also be reconstructed from the Fourier coefficients, allowing one to check the success and accuracy of the method and to determine the number of coefficients necessary to define the shape to the required precision.

Introduction

Whilst linear dimensions are easily measured and analysed numerically, curvilinear forms are difficult to define and compare. Frequently, the shape and form of an object are left unexplored, relegated to being merely qualitative features. However, a method of defining curved and non-uniform shape which has proved popular among some biological workers is Fourier analysis (Lestrel 1982). This is a numerical technique based upon the function:

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Eq. 1

This technique was first developed by Jean Baptiste Fourier (1768-1830) and now has an established mathematical background. Today, it is a technique frequently used by electrical engineers, for example, for the decomposition of

complex waveforms into simple component waves.   These component waves, if re-combined by the simple addition of their amplitudes, would reconstitute the original waveform. The principle is illustrated in Figure 1 where the waveform (a) may be considered as being the composite of two component waves (b) and (c).

An analogy may be drawn with white light passed through a prism where it is divided into its component spectral colours (Figure 2).

There are usually three stages in the use of Fourier analysis to describe biological shape:
1. the construction of a wave-like curve derived from the shape to be analysed
2. the numerical steps of Fourier analysis
3. the use of the Fourier coefficients to perform statistical tests.

Morphological work by biologists who have used Fourier analysis usually contains detailed descriptions of the first step as this tends to be specific to a given study. Generally, all such procedures are aimed at converting the outline of a given shape into a series of linear measurements which serve as the amplitudes of the waveform to be analysed. In the example (Figure 3), the outline of a lateral view of a human skull (Figure 3 a) has been characterised using radii, at equi-angular separations, to produce a waveform (Figure 3b) when the lengths of these radii are plotted upright and side by side. The criteria for determination of the point from which the radii are drawn differs between workers. Some favour siting such a point on a bony landmark whilst others choose the centre of an area. (Note - Figure 3 a shows just 12 radii whilst Figure 3b was drawn from many more.)

The way in which Fourier analysis is performed has gone largely unreported by biological workers since this is a widely described mathematical procedure. The procedure tends, however, to elude many who might otherwise be interested in using the technique. Although computer programs may be written to perform Fourier analysis, a means whereby one can monitor the computational steps at all stages of the procedure is useful — certainly when first applying the technique.

Procedure
Fourier analysis may be performed quite easily by a tabular method using a computer spreadsheet or even on paper (Stroud 1984). A spreadsheet example is shown in Table 1. For illustrative purposes, the analysis performed in Table 1 is based upon the 12 skull measurements in Figure 3. In practice, the number of radii used is influenced by, for example, the complexity of the shape to be analysed such that small changes in outline are not missed. Table 2 gives the relevant formulae associated with the cells — as the same formulae recur horizontally and vertically, only a selection is shown.

Using this spreadsheet to perform Fourier analysis, the 'amplitude' values are entered in column E, cells E8-E19, from whence they are echoed in cells E30-E41. Each amplitude is multiplied by a sine or cosine function as defined in columns F to J (cosine functions - upper block; sine functions - lower block).

The value of each Fourier coefficient, a_1-a_n and b_1-b_n, is twice the average of specified groups of cells e.g. F8-F19 for a_1, F30-F41 for b_1 etc. The value which is twice the average of the amplitude values is also calculated to give coefficient a_0. Coefficients a_1-a_n, and b_1-b_n are the main descriptors of shape whilst a_0 is a measure of the vertical displacement of the waveform.

Statistically, coefficients a_1-a_n, and b_1-b_n may be analysed as pairs using canonical analysis (O'Higgins and Williams 1987) or, in certain cases where one series can be removed, using discriminant analysis (Lestrel et al. 1977).

Reconstruction of the original waveform may be performed using the function:

$$f(x) = \tfrac{1}{2}a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x +$$
$$b_2 \sin 2x + \ldots + a_n \cos nx + b_n \sin nx$$

Eq. 2

The ability to reconstruct the original waveform from Fourier coefficients allows a check on the success and accuracy of the method. Applying predetermined criteria about how close an approximation the reconstructed shape should be to the original, allows the number of coefficients necessary to achieve this to be determined by progressively increasing the number of coefficient pairs:

$$f(x) = \tfrac{1}{2}a_0 + a_1 \cos x + b_1 \sin x$$
$$f(x) = \tfrac{1}{2}a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x$$
$$f(x) = \tfrac{1}{2}a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x +$$
$$a_3 \cos 3x + b_3 \sin 3x$$

Eq. 3

Fourier analysis gives a method whereby shape may be described and analysed, albeit using a mathematical function. Since it is performed on raw data, it is important that, should shape be the sole point of interest, amplitudes are scaled to accommodate for size. Some workers scale for the area enclosed by shape outlines whilst Thompson (1912), pre-empting the use of Fourier analysis in this form, gave an alternative method of converting skull contours into wavelike curves using transverse diameters scaled against vertical height.

References
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Lestrel, P. E., Kimbel, W. H., Prior, F. W. and Fleischmann, M. L. 1977 'Size and shape of the hominid distal femur: Fourier analysis', *American Journal of Physical Anthropology* 46, 281-90.

O'Higgins, P. and Williams, N. W. 1987 'An investigation into the use of Fourier coefficients in characterizing cranial shape in primates', *J. Zool. Lond* 211, 409-30.

Stroud, K. A. 1984 *Fourier series and harmonic analysis*. Stanley Thornes, Cheltenham, pp. 83-103.

Thompson, D. W. 1912 'Contour diagrams of human crania', *Nature* 88, 513-15.

Figure 1 – Waveform (a) may be considered to be the composite of (b) and (c)
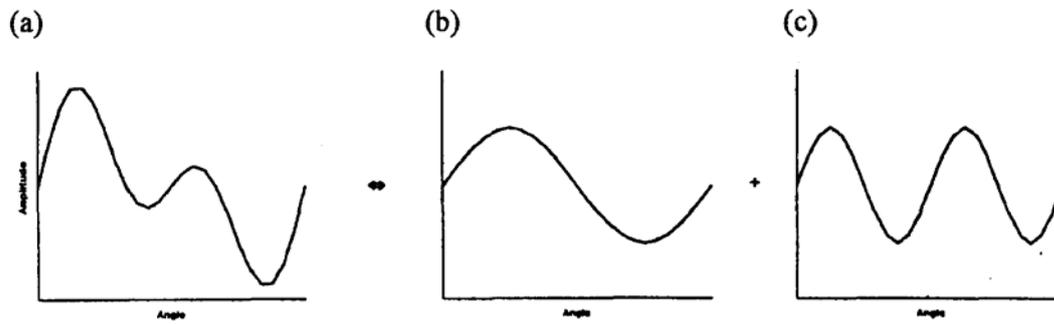
(a)   (b)   (c)

Figure 2 – Division of light waves passing through a prism
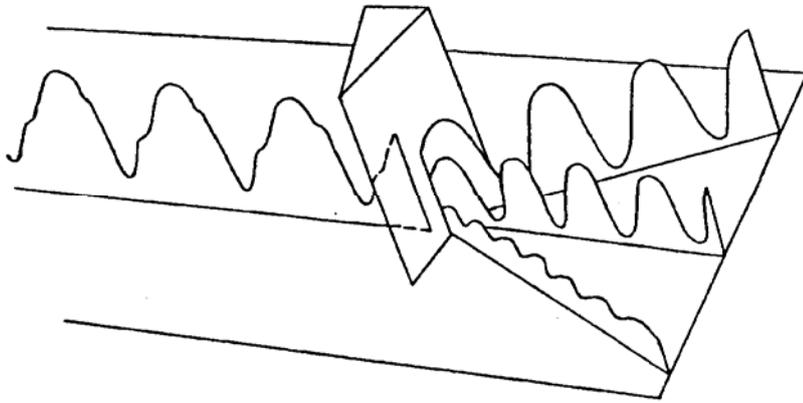
Figure 3 – (a) Characterisation of a human skull outline using radii at equi-angular separations, to produce a waveform (b) (After O'Higgins and Williams 1987)

(a)

(b)
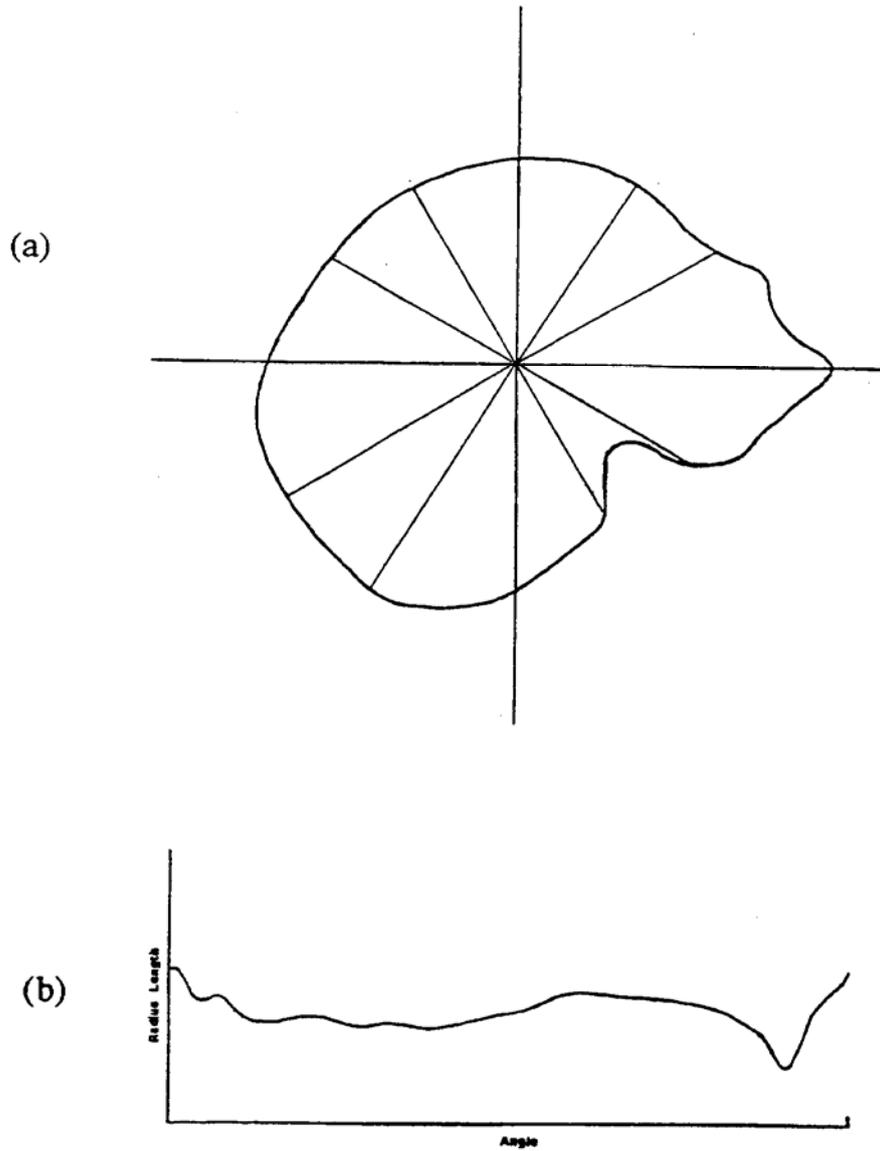
Table 1 – Example of Fourier analysis using a standard computer spreadsheet

Fourier Analysis (Example)

Cosine Components		Input Amplitudes						
x(deg)	x(rad)	y	yCosx	yCos2x	yCos3x	yCos4x	yCos5x	...
0	0.00	4.25	4.25	4.25	4.25	4.25	4.25	...
30	0.52	3.20	2.77	1.60	-0.00	-1.60	-2.77	...
60	1.05	3.00	1.50	-1.50	-3.00	-1.50	1.50	...
90	1.57	2.85	0.00	-2.85	0.00	2.85	0.00	...
120	2.09	2.80	-1.40	-1.40	2.80	-1.40	-1.40	...
150	2.62	2.95	-2.55	1.47	0.00	-1.48	2.55	...
180	3.14	3.40	-3.40	3.40	-3.40	3.40	-3.40	...
210	3.67	3.70	-3.20	1.85	-0.00	-1.85	3.20	...
240	4.19	3.70	-1.85	-1.85	3.70	-1.85	-1.85	...
270	4.71	3.10	0.00	-3.10	-0.00	3.10	0.00	...
300	5.24	2.35	1.17	-1.18	-2.35	-1.17	1.18	...
330	5.76	2.65	2.29	1.32	-0.00	-1.33	-2.29	...
		6.33						
			-0.07	0.34	0.33	0.24	0.16	...
			$a0$	$a1$	$a2$	$a3$	$a4$	$a5$	...

Sine Components		Input Amplitudes						
x(deg)	x(rad)	y	ySinx	ySin2x	ySin3x	ySin4x	ySin5x	...
0	0.00	4.25	0.00	0.00	0.00	0.00	0.00	...
30	0.52	3.20	1.60	2.77	3.20	2.77	1.60	...
60	1.05	3.00	2.60	2.60	-0.00	-2.60	-2.60	...
90	1.57	2.85	2.85	0.00	-2.85	0.00	2.85	...
120	2.09	2.80	2.42	-2.42	0.00	2.42	-2.42	...
150	2.62	2.95	1.48	-2.55	2.95	-2.55	1.47	...
180	3.14	3.40	0.00	0.00	-0.00	0.00	0.00	...
210	3.67	3.70	-1.85	3.20	-3.70	3.20	-1.85	...
240	4.19	3.70	-3.20	3.20	0.00	-3.20	3.20	...
270	4.71	3.10	-3.10	-0.00	3.10	0.00	-3.10	...
300	5.24	2.35	-2.04	-2.04	0.00	2.04	2.04	...
330	5.76	2.65	-1.33	-2.29	-2.65	-2.29	-1.32	...
			-0.09	0.41	0.01	-0.04	-0.02	...
			$b1$	$b2$	$b3$	$b4$	$b5$	...

Table 2 – Formulae associated with spreadsheet cells

	B	C	D	E	F	G	H	I	J	K	L
1	**Fourier Analysis** *(Formulæ)*										
2											
3											
4	Cosine Components			Input							
5				Amplitudes							
6	x(deg)	x(rad)		y	yCosx	yCos2x	yCos3x	yCos4x	yCos5x		
7											
8	0	0.00		y1	=(D8xCOS(B8))		=(D8xCOS(3xB8))		=(D8xCOS(5xB8))		
9	30	0.52		y2	...		...		...		
10	60	1.05		y3	...		...		...		
11	90	1.57		y4	...		...		...		
12	120	2.09		y5	...		...		...		
13	150	2.62		y6	...		...		...		
14	180	3.14		y7	...		...		...		
15	210	3.67		y8	...		...		...		
16	240	4.19		y9	...		...		...		
17	270	4.71		y10	...		...		...		
18	300	5.24		y11	...		...		...		
19	330	5.76		y12	=(D19xCOS(B19))		=(D8xCOS(3xB19))		=(D19xCOS(5xB19))		
21				=(2xAVERAGE(D8..D19))			=(2xAVERAGE(G8..G19))		=(2xAVERAGE(I8..I19))		
22					=(2xAVERAGE(E8..E19))						
23											
24				a0	a1	...	a3	...	a5		
25											
26	Sine Components			Input							
27				Amplitudes							
28	x(deg)	x(rad)		y	ySinx	ySin2x	ySin3x	ySin4x	ySin5x		
29											
30	0	0.00		=E8	=(D30xSIN(B30))		=(D30xSIN(3xB30))		=(D30xSIN(5xB30))		
31	30	0.52		=E9	...		...		...		
32	60	1.05		=E10	...		...		...		
33	90	1.57		=E11	...		...		...		
34	120	2.09		=E12	...		...		...		
35	150	2.62		=E13	...		...		...		
36	180	3.14		=E14	...		...		...		
37	210	3.67		=E15	...		...		...		
38	240	4.19		=E16	...		...		...		
39	270	4.71		=E17	...		...		...		
40	300	5.24		=E18	...		...		...		
41	330	5.76		=E19	=(D41xSIN(B41))		=(D41xSIN(3xB30))		=(D41xSIN(5xB41))		
43											
44					=(2xAVERAGE(E30..E41))		=(2xAVERAGE(G30..G41))		=(2xAVERAGE(I30..I41))		
45											
46					b1	...	b3	...	b5		
47											
48											