Commun Nonlinear Sci Numer Simulat 152 (2026) 109182

Contents lists available at ScienceDirect

Communications in Nonlinear Science and Numerical Simulation

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Numerical study for a stochastic semilinear subdiffusion equation driven by fractional Brownian motions

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ARTICLE INFO

Keywords:
Stochastic fractional partial differential equations
Fractional Brownian motion
Regularity
Finite element method
Grünwald–Letnikov scheme
Strong convergence

ABSTRACT

In this work, we consider the Galerkin finite element method for solving the stochastic semilinear subdiffusion equation driven by additive fractional Brownian motion with Hurst parameter $H \in (0,1)$, where the fractional Brownian motion has a Wiener integration representation. The existence and uniqueness of a mild solution are proved using the Banach fixed point theorem. The temporal and spatial regularity of the solution are studied via the semigroup approach. The finite element method is used to approximate the spatial variable. The Caputo fractional time derivative and the Riemann–Liouville integral are approximated using the Grünwald–Letnikov schemes, respectively, and the noise is discretized using the Euler method. The optimal error estimates of the fully discrete scheme are established using the discrete Laplace transform method. Under the assumption that the noise is in the trace class, we prove that the time convergence order is $O(\tau^{\min\{\alpha,H+\alpha+\gamma-1,1/2\}})$ when $H \in (0,1/2)$, and $O(\tau^{\min\{\alpha,H+\alpha+\gamma-1\}})$ when $H \in (1/2,1)$. Here, τ denotes the time step size. Numerical experiments are conducted to validate the theoretical results.

1. Introduction

Consider the following semilinear stochastic time fractional partial differential equation driven by fractionally integrated additive fractional Brownian motion (fBm), with $\alpha \in (0,1)$ and $\gamma \in [0,1]$,

$$\begin{aligned} &{}_C\mathrm{D}^{\alpha}_{0,t}u(t) + A^{\beta}u(t) = f(t,u(t)) + {}_{RL}\mathrm{D}^{-\gamma}_{0,t}\frac{\mathrm{d}W^H(t)}{\mathrm{d}t}, \quad 0 < t \le T, \\ &u(0) = u_0, \end{aligned} \tag{1}$$

where ${}_C\mathrm{D}^{\alpha}_{0,t}u(t)$ and ${}_{RL}\mathrm{D}^{-\gamma}_{0,t}u(t)$ denote the Caputo fractional derivative and the Riemann–Liouville fractional integral [1], respectively. Here $A = -\Delta$ with $D(A) = H_0^1(\mathcal{D}) \cap H^2(\mathcal{D})$, and $\mathcal{D} \subset \mathbb{R}^d, d = 1,2,3$ is a bounded domain with the smooth boundary, where Δ denotes the Laplacian, $H_0^1(\mathcal{D})$ and $H^2(\mathcal{D})$ are the standard Sobolev spaces. The term A^{β}, with $\beta \in (1/2,1]$, denotes the fractional power of the operator A, while $W^H(t)$ represents a Hilbert space-valued fractional Brownian motion with Hurst parameter $H \in (0,1)$. The nonlinear function f satisfies the properties outlined in Assumption 5. The initial value $u_0 \in U$ is a random variable, where $U = L_2(\mathcal{D})$ with the norm $\|\cdot\|$.

There are many theoretical and numerical results for integer order ($\alpha = 1$) stochastic partial differential equation driven by fractional Brownian motion (fBm). Boufoussi and Hajji [2] discussed the existence and uniqueness of mild solutions for nonlinear stochastic evolution equations driven by multiplicative fractional Brownian motion (fBm) with Hurst parameter $1/2 < H < 1$.

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https://doi.org/10.1016/j.cnsns.2025.109182
Received 14 April 2025; Received in revised form 10 July 2025; Accepted 24 July 2025
Available online 8 August 2025

Maslowski and Nualart [3] proposed a new approximation scheme based on the Taylor expansion for the solution of stochastic differential equations driven by multiplicative fBm with $1/2 < H < 1$. Kamrani and Jamshidi [4] constructed an implicit Euler approximation for stochastic evolution equations driven by multiplicative fBm with $1/2 < H < 1$. Hong et al. [5] investigated numerical schemes for stochastic differential equations driven by fBm with Hurst parameter $1/2 < H < 1$. Cao et al. [6] considered the Galerkin approximation for stochastic evolution equations driven by additive Gaussian noise, which is temporally white and spatially fractional, with Hurst parameter $0 < H < 1$. Ding and Wang [7] examined the regularity of stochastic evolution equations with Hurst parameter $0 < H < 1$ and the strong convergence of a fully discrete scheme. Erdoğan et al. [8] considered the numerical approximation of stochastic differential equation driven by fractional Brownian motion for all $H \in (0, 1)$ using the Wick–Itô–Skorohod integration. See further references in [9–15], etc.

There are also many theoretical and numerical results for stochastic time-fractional partial differential equations driven by standard Brownian motion with $H = 1/2$. Jin et al. [16] studied a numerical method for stochastic time-fractional diffusion driven by additive fractionally integrated Gaussian noise, deriving sharp strong and weak convergence rates. Sweilam et al. [17] investigated numerical methods for solving linear stochastic time-fractional partial differential equations. They employed the Galerkin finite element method for spatial approximation and the finite difference method for time discretization. The strong convergence error estimates of both the semi-discrete and fully discrete schemes were analyzed theoretically and verified numerically. Qi and Xu [18] proposed a spatio-temporal discretization method combining a Milstein exponential integrator scheme with the finite element method to solve stochastic time-fractional partial differential equations driven by multiplicative noise. Wu et al. [19] developed a fully discrete scheme for solving the stochastic subdiffusion problem driven by integrated space–time white noise and analyzed the strong convergence of this numerical approximation. Kang et al. [20] used the Galerkin finite element method to approximate semilinear stochastic space and time fractional subdiffusion problems driven by fractionally integrated additive noise, obtaining error estimates for spatial semi-discretization. Al-Maskari and Karaa [9] discussed the approximation of a stochastic time-fractional Allen–Cahn equation driven by additive fractionally integrated Gaussian noise, analyzing strong convergence rates for spatially semi-discrete and fully discrete schemes. Hu et al. [21] investigated semilinear subdiffusion and superdiffusion equations driven by fractionally integrated additive noise, and established optimal strong convergence error estimates. For more works, we refer readers to [11,22–27], etc.

Compared with stochastic time-fractional partial differential equations driven by standard Brownian motion, i.e., $H = 1/2$, studies on stochastic time-fractional partial differential equations driven by fractional Brownian motion (fBm) with $H \neq 1/2$ are relatively limited. The main difficulty lies in the fact that fBm is neither a semimartingale nor a Markov process, making traditional stochastic analysis methods inapplicable directly. Yan and Yin [28] focused on a class of stochastic fractional partial differential equations driven by fBm with $H \in (1/2, 1)$ and proved the optimal strong convergence rates for both semi-discrete and fully discrete schemes. Nie and Deng [29] considered a stochastic nonlinear fractional diffusion equation driven by fBm with $H \in (0, 1)$. They used the spectral Galerkin method and backward Euler convolution quadrature to discretize this equation and provided sharp error estimates. Ding & Jiang [30] studied semilinear stochastic Volterra evolution equations driven by additive fBm with $H \in (0, 1)$. They applied the spectral Galerkin method in space and the implicit Euler method combined with convolution quadrature in time, proving strong convergence for the fully discrete scheme. Lv [31] explored stochastic time-fractional evolution equations driven by fBm with $H \in (1/2, 1)$ and constructed a full approximation for such equations using the Galerkin finite element method in space and the backward Euler convolution quadrature formula in time, demonstrating strong convergence error estimates. For further reading, interested readers are referred to other works, see [7,27,32–41] and the references therein.

Inspired by the aforementioned research, in this paper, we develop a numerical algorithm for (1) with additive fractional Brownian motion (fBm) for all $H \in (0, 1)$, where the fBm admits a Wiener integral representation (see, for example, [7,33,42]). To the best of our knowledge, there is currently no work in the literature addressing numerical methods for semilinear stochastic time-fractional partial differential equations driven by fractional Brownian motion with a Wiener integration representation. To fill this gap, we begin by investigating the analytical properties of the mild solution to (1), including its existence, uniqueness, and regularity, which provide the foundation for the subsequent numerical analysis. We then derive a spatially semidiscrete scheme using the Galerkin linear finite element method and carry out a corresponding error analysis. Finally, we construct a fully discrete scheme by applying the Grünwald–Letnikov approximation in time and the Euler method for the noise, and we establish strong convergence rates for the proposed scheme.

Our main contributions in this work are summarized as follows:

1. We establish the existence and uniqueness of the mild solution to (1), along with its spatial and temporal regularity properties.
2. We derive optimal convergence rates for the numerical approximation of (1) using the Galerkin linear finite element method for spatial discretization, the Grünwald–Letnikov schemes for time discretization, and the Euler method for handling the stochastic noise.

The structure of the article is arranged as follows. In Section 2, we recall some preliminaries, including notations, assumptions, and important lemmas. Section 3 proves the existence and uniqueness of the mild solution to (1). In Section 4, the spatial and temporal regularity properties of the mild solution are studied. Section 5 proposes a spatially semidiscrete scheme using the Galerkin finite element method and provides an error estimate. By applying the Grünwald–Letnikov scheme, Section 6 establishes a fully discrete scheme for (1) and estimates the strong error bound of this approximation. In Section 7, numerical examples are provided to test the theoretical results. The conclusion is included in the final section.

Throughout the paper, we use C to denote a generic positive constant, which may vary at different occurrences but is independent of the step sizes τ and h.

2. Preliminaries

Let $(\lambda_j, e_j)_{j=1}^{\infty}$ be the eigenpairs of $A : D(A) \subset U \to U$. We denote $\dot{U}^r := D(A^{\frac{r}{2}})$, $r \in \mathbb{R}$ with the norm

$$|v|_r = \left(\sum_{j=1}^{\infty} \lambda_j^r (v, e_j)^2 \right)^{1/2} < \infty. \tag{2}$$

In particular, $|\cdot|_0 = \|\cdot\|$, where $\|\cdot\|$ denotes the norm in $L_2(D)$.

Let $L_p(\Omega; \dot{U}^r)$ with $p \geq 2$ be the Hilbert space of p-fold integrable random variables taking values in $\dot{U}^r$ such that $\|v\|^p_{L_p(\Omega;\dot{U}^r)} = \mathbb{E}|v|_r^p < \infty$, where $\mathbb{E}$ means the mathematical expectation.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be the complete probability space with filtration $(\mathscr{F}_t)_{t\in[0,T]}$, where the filtration $\mathscr{F}_t$ satisfies the usual conditions, i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets. Denote by $\mathcal{L}(U) = \mathcal{L}(U, U)$ the space of bounded linear operators from U to U. Let $Q \in \mathcal{L}(U)$ and define the space of Hilbert–Schmidt operators from $Q^{1/2}(U)$ to U by

$$\mathcal{L}_2^0 = \{\Phi : Q^{1/2}(U) \to U,\ \|\Phi\|^2_{\mathcal{L}_2^0} := \|\Phi Q^{1/2}\|^2_{\mathrm{HS}} = \sum_{j=1}^{\infty} \|\Phi Q^{1/2} e_j\|^2 < \infty\},$$

where $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis of U.

In order to ensure that the mild solution is well-defined, the following assumption is necessary.

Assumption 1 ([16]). The parameters $\alpha \in (0, 1)$ and $\gamma \in [0, 1]$ satisfy $\alpha + \gamma > 1/2$.

Assumption 2 ([14]). The linear operator $A : D(A) \subset U \to U$ is densely defined, self-adjoint and positive definite with compact inverse.

Assumption 3 ([14]). The covariance operator $Q \in \mathcal{L}(U)$ is self-adjoint and positive semidefinite, but it is not necessarily of trace class.

Assumption 4 ([14,43]). There exists a constant C such that, with $\sigma \in (0, 1]$,

$$\|A^{\frac{\sigma-1}{2}}\|_{\mathcal{L}_2^0} = \|A^{\frac{\sigma-1}{2}} Q^{1/2}\|_{\mathrm{HS}} < \infty.$$

Assumption 5 ([14]). Let $\alpha \in (0, 1)$, $\beta \in (1/2, 1]$, $\gamma \in [0, 1]$, and $\sigma \in (0, 1]$. Let $0 \leq r < \min\{2\beta + \sigma - 1, \frac{2\beta}{\alpha}(H + \alpha + \gamma - 1) + \sigma - 1\}$. The nonlinear function $f : [0, T] \times U \to U$ satisfies

$$\|f(t, u_1) - f(t, u_2)\| \leq C\|u_1 - u_2\| \tag{3}$$

for any $t \in [0, T]$ and $u_1, u_2 \in U$. Moreover, it holds

$$\|f(t_1, u) - f(t_2, u)\| \leq C(1 + \|u\|)|t_1 - t_2|^{\frac{\alpha r}{2\beta}} \tag{4}$$

for any $t_1, t_2 \in [0, T]$ and $u \in U$.

Remark 1. Suppose that $\|f(0, u)\| \leq C$ for all $u \in U$, then it is easy to show that from (4)

$$\|f(t, u)\| \leq C(1 + \|u\|) \tag{5}$$

for any $t \in [0, T]$ and $u \in U$.

Assumption 6 ([14]). Let r be defined as in Assumption 5. Let $\kappa \in [0, r]$ and $p \in [2, \infty)$. The initial value $u_0 : \Omega \to \dot{U}^\kappa$ is a random variable and satisfies

$$\|u_0\|_{L_p(\Omega;\dot{U}^\kappa)} := (\mathbb{E}[|u_0|_\kappa^p])^{\frac{1}{p}} < \infty. \tag{6}$$

Now we briefly introduce the fractional Brownian motion (fBm) and stochastic integrals for the fBm. More detailed descriptions can be found in [7,33,42,44,45].

Definition 1 ([44]). A real-valued fractional Brownian motion (fBm) $(W^H(t))_{t\geq 0}$ of Hurst index $H \in (0, 1)$ is a continuous and centered Gaussian process on the space $(\Omega, \mathscr{F}, \mathbb{P})$ with covariance function

$$\mathbb{E}[W^H(t) W^H(w)] = \frac{1}{2}\left(t^{2H} + w^{2H} - |t - w|^{2H}\right),\ t, w \geq 0. \tag{7}$$

If $H = 1/2$, then the fBm is the standard Brownian motion $W(t)$ whose covariance function is $\frac{1}{2}(t + w - |t - w|) = \min\{t, w\}$.

It is well-known that the fBm can be written via the Wiener integral

$$W^H(t) := \int_0^t K_H(t, w)\, \mathrm{d}W(w), \tag{8}$$

where the kernel function [33],

$$K_H(t,w)=\begin{cases} b_H\left[\left(\frac{t}{w}\right)^{H-\frac{1}{2}}(t-w)^{H-\frac{1}{2}}-(H-\frac{1}{2})w^{\frac{1}{2}-H}\right. \\ \quad\left.\times\int_w^t \zeta^{H-\frac{3}{2}}(\zeta-w)^{H-\frac{1}{2}}\mathrm{d}\zeta\right],\ 0<H<1/2, \\ c_H\int_w^t\left(\frac{\zeta}{w}\right)^{H-\frac{1}{2}}(\zeta-w)^{H-\frac{3}{2}}\mathrm{d}\zeta,\ 1/2<H<1, \end{cases} \tag{9}$$

where

$$b_H=\left(\frac{2H}{(1-2H)B(1-2H,H+1/2)}\right)^{1/2},\quad c_H=\left(\frac{H(2H-1)}{B(2-2H,H-1/2)}\right)^{1/2},$$

and $B(\cdot,\cdot)$ is the Euler-Beta function.

Assume that Q satisfy Assumption 3. We define the Q-fractional Brownian motion $W^H(t)$ (retaining this notation for simplicity) as follows:

$$W^H(t):=\sum_{j=1}^{\infty}\gamma_j^{1/2}e_jW_j^H(t),\quad \gamma_j>0, \tag{10}$$

where $\{W_j^H(t)\}_{j=1}^{\infty}$ is a sequence of stochastically independent real-valued fBms and $Qe_j=\gamma_je_j$.

Let $\mathcal{H}$ and $\mathcal{E}$ be the reproducing kernel Hilbert space of the fBm and the space of step functions on $[0,T]$, respectively. One may show that $\mathcal{H}$ is the closure of $\mathcal{E}$ with respect to the following scalar product [7,33],

$$\langle\chi_{[0,t]},\chi_{[0,w]}\rangle_{\mathcal{H}}=\frac{1}{2}\left(t^{2H}+w^{2H}-|t-w|^{2H}\right),\ 0\le w,t\le T,$$

where $\chi_{[a,b]}(t)$ denotes the characteristic function. Define a linear operator $K_{H,T}^*:\mathcal{E}\to L_2(0,T)$ such that

$$(K_{H,T}^*\varphi)(\zeta):=K_H(T,\zeta)\varphi(\zeta)+\int_\zeta^T(\varphi(w)-\varphi(\zeta))\frac{\partial K_H(w,\zeta)}{\partial w}\mathrm{d}w, \tag{11}$$

with

$$\frac{\partial K_H(w,\zeta)}{\partial w}=C_H\left(\frac{w}{\zeta}\right)^{H-\frac{1}{2}}(w-\zeta)^{H-\frac{3}{2}}, \tag{12}$$

where C_H is a constant only related to H. If $H\in(1/2,1)$, then $(K_{H,T}^*\varphi)(\zeta)$ becomes

$$(K_{H,T}^*\varphi)(\zeta)=\int_\zeta^T\varphi(w)\frac{\partial K_H(w,\zeta)}{\partial w}\mathrm{d}w. \tag{13}$$

We further remark that the linear operator $K_{H,T}^*$ can be defined from the Hilbert space $\mathcal{H}$ to $L_2(0,T)$.

Assume that $\psi(t),t\in[0,T]$ is a function taking values in $\mathcal{L}_2^0$. The stochastic integral with respect to Q-fBm $W^H(t)$ is represented by, with $t\in[0,T]$,

$$\int_0^t\psi(s)\mathrm{d}W^H(s)=\int_0^T\psi(s)\chi_{[0,t]}(s)\mathrm{d}W^H(s)=\int_0^T(K_{H,T}^*\psi\chi_{[0,t]})(s)\mathrm{d}W(s), \tag{14}$$

where $W(t)$ denotes the Q-fBm $W^H(t)$ with $H=1/2$.

Definition 2. A predictable stochastic process $u:[0,T]\to U$ is called a p-fold integrable mild solution to (1) with $p\ge2$ if

$$\sup_{t\in[0,T]}\|u(t)\|_{L_p(\Omega;U)}<\infty \tag{15}$$

and, for any $t\in[0,T]$, it holds that

$$u(t)=\mathbf{E}_{\alpha,1}^{(\beta)}(t)\,u_0+\int_0^t\mathbf{E}_{\alpha,\alpha}^{(\beta)}(t-\zeta)f(\zeta,u(\zeta))\mathrm{d}\zeta+\int_0^t\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta)}(t-\zeta)\mathrm{d}W^H(\zeta), \tag{16}$$

where

$$\mathbf{E}_{\alpha,1}^{(\beta)}(t)=E_{\alpha,1}(-A^\beta t^\alpha),\quad \mathbf{E}_{\alpha,\alpha}^{(\beta)}(t)=t^{\alpha-1}E_{\alpha,\alpha}(-A^\beta t^\alpha),$$
$$\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta)}(t)=t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-A^\beta t^\alpha),$$

in which $E_{\alpha,1}(\cdot)$, $E_{\alpha,\alpha}(\cdot)$ and $E_{\alpha,\alpha+\gamma}(\cdot)$ are well-known Mittag-Leffler functions, see [1].

Lemma 1 ([20]). *Let $\alpha\in(0,1)$, $\beta\in(1/2,1]$ and $\gamma\in[0,1]$. We then have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_{\alpha,1}^{(\beta)}(t)=-A^\beta\mathbf{E}_{\alpha,\alpha}^{(\beta)}(t), \tag{17}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_{\alpha,\alpha+\gamma-k}^{(\beta)}(t)=\mathbf{E}_{\alpha,\alpha+\gamma-1-k}^{(\beta)}(t),\ k=0,1,2,\ldots. \tag{18}$$

The following smoothing properties of solution operators are very useful, whose proofs can refer to [20,25].

Lemma 2. *Let $\alpha \in (0,1)$, $\beta \in (1/2,1]$ and $\gamma \in [0,1]$. For any $\mu \in \mathbb{R}, \nu \in \mathbb{R}$ and $t \geq 0$, we have*

$$\left|\mathbf{E}^{(\beta)}_{\alpha,1}(t)v\right|_{\mu} \leq Ct^{-\frac{\alpha}{2\beta}(\mu-\nu)}|v|_{\nu}, \qquad 0 \leq \mu-\nu \leq 2\beta, \tag{19}$$

$$\left|\mathbf{E}^{(\beta)}_{\alpha,\alpha-k}(t)v\right|_{\mu} \leq Ct^{\alpha-1-k-\frac{\alpha}{2\beta}(\mu-\nu)}|v|_{\nu}, \qquad 0 \leq \mu-\nu \leq 4\beta,\ k=0,1,2,\dots, \tag{20}$$

$$\left|\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma-k}(t)v\right|_{\mu} \leq Ct^{\alpha+\gamma-1-k-\frac{\alpha}{2\beta}(\mu-\nu)}|v|_{\nu}, \quad 0 \leq \mu-\nu \leq 2\beta,\ k=0,1,2,\dots, \tag{21}$$

where

$$\mathbf{E}^{(\beta)}_{\alpha,\alpha-k}(t) = t^{\alpha-1-k}E_{\alpha,\alpha-k}(-A^{\beta}t^{\alpha}),\ k=0,1,2,\dots,$$
$$\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma-k}(t) = t^{\alpha+\gamma-1-k}E_{\alpha,\alpha+\gamma-k}(-A^{\beta}t^{\alpha}),\ k=0,1,2,\dots.$$

We also need the following Burkholder–Davis–Gundy-type inequality.

Lemma 3 ([14,43]). *For any $\mathcal{L}^0_2$-valued predictable process $\psi(t)$ with $t \in [0,T]$, there holds, for $q \geq 1$,*

$$\mathbb{E}\Big(\sup_{s\in[0,t]}\Big|\int_0^s \psi(\zeta)\mathrm{d}W(\zeta)\Big|^{2q}\Big) \leq c_q \sup_{s\in[0,t]}\mathbb{E}\Big(\Big|\int_0^s \psi(\zeta)\mathrm{d}W(\zeta)\Big|^{2q}\Big)$$
$$\leq C_q\mathbb{E}\Big(\int_0^t \|\psi(s)\|^2_{\mathcal{L}^0_2}\mathrm{d}s\Big)^q,$$

where

$$c_q = \Big(\frac{2q}{2q-1}\Big)^{2q}, \quad C_q = (q(2q-1))^q\Big(\frac{2q}{2q-1}\Big)^{2q(q-1)}.$$

Finally, we present two integral identities that will be used frequently in the subsequent sections [30],

$$\int_0^t \big(\frac{t}{w}\big)^{\xi-1}(t-w)^{\eta-1}\mathrm{d}w = t^{\eta}B(2-\xi,\eta), \quad \xi<2,\ \eta>0, \tag{22}$$

$$\int_s^t (t-w)^{\xi-1}(w-s)^{\eta-1}\mathrm{d}w = (t-s)^{\xi+\eta-1}B(\xi,\eta), \quad \xi>0,\ \eta>0. \tag{23}$$

3. Existence and uniqueness

In this section, we use the Banach contraction mapping principle to establish the existence and uniqueness of the mild solution for (1). We first prove the spatial regularity of the stochastic integral in the mild solution.

Lemma 4. *Let $\alpha \in (0,1)$, $\beta \in (1/2,1]$, $\gamma \in [0,1]$, $\sigma \in (0,1]$, and $p \in [2,\infty)$. Let Assumptions 1–4 be fulfilled. Then there exists a constant $C>0$ such that*

$$\Big\|\int_0^t \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta)\mathrm{d}W^H(\zeta)\Big\|_{L_p(\Omega;\dot{U}^r)} \leq Ct^{H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}, \quad t\in[0,T], \tag{24}$$

where r is defined in Assumption 5.

Proof. According to the stochastic integral (14), Lemma 3 and Assumption 4, one has

$$\Big\|\int_0^t \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta)\mathrm{d}W^H(\zeta)\Big\|_{L_p(\Omega;\dot{U}^r)} = \Big\|\int_0^t A^{\frac{r}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta)\mathrm{d}W^H(\zeta)\Big\|_{L_p(\Omega;U)}$$
$$=\Big\|\int_0^t K^*_{H,t}\big(A^{\frac{r}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\cdot)\big)(\zeta)\mathrm{d}W(\zeta)\Big\|_{L_p(\Omega;U)}$$
$$\leq C\Big\|\Big(\int_0^t\Big\|K^*_{H,t}\big(A^{\frac{r}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\cdot)\big)(\zeta)\Big\|^2_{\mathcal{L}^0_2}\mathrm{d}\zeta\Big)^{\frac{1}{2}}\Big\|_{L_p(\Omega;\mathbb{R})}$$
$$=C\Big\|\Big(\int_0^t\Big\|K^*_{H,t}\big(A^{\frac{r-\sigma+1}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\cdot)A^{\frac{\sigma-1}{2}}Q^{1/2}\big)(\zeta)\Big\|^2_{\mathrm{HS}}\mathrm{d}\zeta\Big)^{\frac{1}{2}}\Big\|_{L_p(\Omega;\mathbb{R})}$$
$$\leq C\Big\|\Big(\int_0^t\Big\|K^*_{H,t}\big(A^{\frac{r-\sigma+1}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\cdot)\big)(\zeta)\Big\|^2\mathrm{d}\zeta\Big)^{\frac{1}{2}}\Big\|_{L_p(\Omega;\mathbb{R})}.$$

It follows from (11) that

$$\int_0^t\Big\|K^*_{H,t}\big(A^{\frac{r-\sigma+1}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\cdot)\big)(\zeta)\Big\|^2\mathrm{d}\zeta$$

$$\begin{aligned}
&\le \int_0^t \Big\| K_H(t,\zeta) A^{\frac{r-\sigma+1}{2}} \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta) \Big\|^2 \mathrm{d}\zeta \\
&\quad + \int_0^t \Big\| \int_\zeta^t A^{\frac{r-\sigma+1}{2}} \big(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-w) - \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta) \big) \frac{\partial K_H(w,\zeta)}{\partial w} \mathrm{d}w \Big\|^2 \mathrm{d}\zeta \\
&:= I + II.
\end{aligned}$$

Note that $K_H(t,\zeta)$ has different expressions for the cases $H \in (0,1/2)$ and $H \in (1/2,1)$, respectively. Therefore, we consider these two cases separately.

Case 1: $H \in (0,1/2)$. Based on the form of the kernel function $K_H(t,\zeta)$ in (9), it holds that

$$\begin{aligned}
I =& \int_0^t \Big\| K_H(t,\zeta) A^{\frac{r-\sigma+1}{2}} \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta) \Big\|^2 \mathrm{d}\zeta \\
\le& 2b_H^2 \int_0^t \Big\| \big(\frac{t}{\zeta}\big)^{H-\frac{1}{2}} (t-\zeta)^{H-\frac{1}{2}} A^{\frac{r-\sigma+1}{2}} \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta) \Big\|^2 \mathrm{d}\zeta \\
&+ 2b_H^2 (\frac{1}{2}-H)^2 \int_0^t \Big\| \zeta^{\frac{1}{2}-H} \Big(\int_\zeta^t w^{H-\frac{3}{2}} (w-\zeta)^{H-\frac{1}{2}} \mathrm{d}w \Big) A^{\frac{r-\sigma+1}{2}} \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta) \Big\|^2 \mathrm{d}\zeta \\
:=& I_1 + I_2.
\end{aligned}$$

Using the smoothing property (21) for the solution operator in Lemma 2 and the integral identity (22), I_1 can be estimated by

$$\begin{aligned}
I_1 =& 2b_H^2 \int_0^t \Big\| \big(\frac{t}{\zeta}\big)^{H-\frac{1}{2}} (t-\zeta)^{H-\frac{1}{2}} A^{\frac{r-\sigma+1}{2}} \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta) \Big\|^2 \mathrm{d}\zeta \\
\le& C \int_0^t \big(\frac{t}{\zeta}\big)^{2H-1} (t-\zeta)^{2H-1} (t-\zeta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} \mathrm{d}\zeta \\
=& C t^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} B\big(2-2H, 2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)\big),
\end{aligned}$$

where we require $r-\sigma+1<2\beta$ and $2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)>0$, i.e., $r<2\beta+\sigma-1$ and $r<\frac{2\beta}{\alpha}(H+\alpha+\gamma-1)+\sigma-1$.

Similarly, I_2 can be estimated as

$$\begin{aligned}
I_2 =& 2b_H^2 (\frac{1}{2}-H)^2 \int_0^t \Big\| \zeta^{\frac{1}{2}-H} \Big(\int_\zeta^t w^{H-\frac{3}{2}} (w-\zeta)^{H-\frac{1}{2}} \mathrm{d}w \Big) A^{\frac{r-\sigma+1}{2}} \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta) \Big\|^2 \mathrm{d}\zeta \\
\le& C \int_0^t \zeta^{1-2H} \Big(\int_\zeta^t w^{H-\frac{3}{2}} (w-\zeta)^{H-\frac{1}{2}} \mathrm{d}w \Big)^2 (t-\zeta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} \mathrm{d}\zeta \\
=& C \int_0^t \zeta^{2H-1} \Big(\int_{\zeta/t}^1 v^{-2H} (1-v)^{H-\frac{1}{2}} \mathrm{d}v \Big)^2 (t-\zeta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} \mathrm{d}\zeta \\
=& C t^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} \int_0^1 \theta^{2H-1} \Big(\int_\theta^1 v^{-2H} (1-v)^{H-\frac{1}{2}} \mathrm{d}v \Big)^2 \\
&\times (1-\theta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} \mathrm{d}\theta \\
=& C t^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} \int_0^1 \theta^{2H-1} \Big(\int_\theta^1 v^{-2H} (\frac{1-v}{1-\theta})^{H-\frac{1}{2}} \mathrm{d}v \Big)^2 \\
&\times (1-\theta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)+2H-1} \mathrm{d}\theta \\
=& C t^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} \int_0^1 \theta^{2H-1} \Big((1-\theta) \int_0^1 (1-s(1-\theta))^{-2H} s^{H-\frac{1}{2}} \mathrm{d}s \Big)^2 \\
&\times (1-\theta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)+2H-1} \mathrm{d}\theta \\
\le& C t^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} \Big(\int_0^1 (1-s)^{-2H} s^{H-\frac{1}{2}} \mathrm{d}s \Big)^2 \\
&\times \int_0^1 \theta^{2H-1} (1-\theta)^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)-1} \mathrm{d}\theta \\
=& C t^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} B^2\big(1-2H, H+1/2\big) \\
&\times B\big(2H, 2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)\big),
\end{aligned}$$

where we use the integral identity

$$\int_\zeta^t w^{H-\frac{3}{2}} (w-\zeta)^{H-\frac{1}{2}} \mathrm{d}w = \zeta^{2H-1} \int_{\zeta/t}^1 v^{-2H} (1-v)^{H-\frac{1}{2}} \mathrm{d}v$$

and the variable substitution $\zeta = t\theta$. Here we require $r-\sigma+1<2\beta$ and $2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)>0$, that is, $r<2\beta+\sigma-1$ and $r<\frac{2\beta}{\alpha}(H+\alpha+\gamma-1)+\sigma-1$.

Hence we obtain

$$I \le Ct^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}. \tag{25}$$

Next we estimate II. Applying Lemma 1, (21) in Lemma 2 and the elementary inequality

$$(x+y)^q \le 2^{q-1}(x^q+y^q) \tag{26}$$

for $0<q\le 1$ and for any $x,y\ge 0$, we arrive at

$$\begin{aligned}
&\left\|A^{\frac{r-\sigma+1}{2}}\left(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-w)-\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta)\right)\right\| = \left\|A^{\frac{r-\sigma+1}{2}}\int_{t-\zeta}^{t-w}\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\theta)\mathrm{d}\theta\right\|\\
&=\left\|A^{\frac{r-\sigma+1}{2}}\int_{t-\zeta}^{t-w}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma-1}(\theta)\mathrm{d}\theta\right\| \le \int_{t-w}^{t-\zeta}\left\|A^{\frac{r-\sigma+1}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma-1}(\theta)\right\|\mathrm{d}\theta\\
&\le C\int_{t-w}^{t-\zeta}\theta^{\alpha+\gamma-2-\frac{\alpha}{2\beta}(r-\sigma+1)}\mathrm{d}\theta = C\int_{\zeta}^{w}(t-v)^{\alpha+\gamma-2-\frac{\alpha}{2\beta}(r-\sigma+1)}\mathrm{d}v\\
&= C\int_{\zeta}^{w}(t-v)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}(t-v)^{-\frac{1}{2}}\mathrm{d}v.
\end{aligned} \tag{27}$$

If $\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-1/2<0$, then

$$\begin{aligned}
&\left\|A^{\frac{r-\sigma+1}{2}}\left(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-w)-\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta)\right)\right\|\\
&\le C(t-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}\int_{\zeta}^{w}(t-v)^{-\frac{1}{2}}\mathrm{d}v\\
&= 2C(t-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}[(t-\zeta)^{\frac{1}{2}}-(t-w)^{\frac{1}{2}}]\\
&\le C(t-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}(w-\zeta)^{\frac{1}{2}}.
\end{aligned} \tag{28}$$

Thanks to (12) and the fact $(w/\zeta)^{H-1/2}<1$, we arrive at

$$\begin{aligned}
II \le& CC_H^2\int_0^t\Big(\int_\zeta^t(t-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}\Big(\frac{w}{\zeta}\Big)^{H-\frac{1}{2}}(w-\zeta)^{H-\frac{3}{2}+\frac{1}{2}}\mathrm{d}w\Big)^2\mathrm{d}\zeta\\
\le& C\int_0^t\Big(\int_\zeta^t(t-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}(w-\zeta)^{H-1}\mathrm{d}w\Big)^2\mathrm{d}\zeta\\
=& C\int_0^t(t-\zeta)^{2[H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)]-1}B^2\Big(\alpha+\gamma-\frac{1}{2}-\frac{\alpha}{2\beta}(r-\sigma+1),H\Big)\mathrm{d}\zeta\\
=& Ct^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}B^2\Big(\alpha+\gamma-\frac{1}{2}-\frac{\alpha}{2\beta}(r-\sigma+1),H\Big),
\end{aligned} \tag{29}$$

where we require $\alpha+\gamma-1/2-\frac{\alpha}{2\beta}(r-\sigma+1)>0$, that is, $r<\frac{2\beta}{\alpha}(1/2+\alpha+\gamma-1)+\sigma-1$.

On the other hand, if $\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-1/2\ge 0$, then

$$\begin{aligned}
&\left\|A^{\frac{r-\sigma+1}{2}}\left(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-w)-\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta)\right)\right\|\\
\le& C(t-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}\int_\zeta^w(t-v)^{-\frac{1}{2}}\mathrm{d}v\\
\le& C(t-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}(w-\zeta)^{\frac{1}{2}}.
\end{aligned} \tag{30}$$

Consequently,

$$\begin{aligned}
II \le& CC_H^2\int_0^t\Big(\int_\zeta^t(t-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}\Big(\frac{w}{\zeta}\Big)^{H-\frac{1}{2}}(w-\zeta)^{H-\frac{3}{2}+\frac{1}{2}}\mathrm{d}w\Big)^2\mathrm{d}\zeta\\
\le& C\int_0^t(t-\zeta)^{2[\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)]-1}\Big(\int_\zeta^t(w-\zeta)^{H-1}\mathrm{d}w\Big)^2\mathrm{d}\zeta\\
=& C\int_0^t(t-\zeta)^{2[H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)]-1}\mathrm{d}\zeta = Ct^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}.
\end{aligned} \tag{31}$$

In both cases, one has by (29) and (31)

$$II \le Ct^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}. \tag{32}$$

Combining (25) and (32), we obtain for the case $H\in(0,1/2)$

$$\left\|\int_0^t\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta)\mathrm{d}W^H(\zeta)\right\|_{L_p(\Omega;\dot{U}^r)} \le Ct^{H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)},\quad t\in[0,T].$$

Case 2: $H \in (1/2, 1)$. In this case, by the kernel function $K_H(t,\zeta)$ in (9), (21) in Lemma 2 and the integral identity (22), we have

$$\begin{aligned} I \leq & Cc_H^2 \int_0^t \Big(\int_\zeta^t \big(\frac{w}{\zeta}\big)^{H-\frac{1}{2}} (w-\zeta)^{H-\frac{3}{2}} \mathrm{d}w \Big)^2 (t-\zeta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} \mathrm{d}\zeta \\ \leq & C \int_0^t \big(\frac{t}{\zeta}\big)^{2H-1} \Big(\int_\zeta^t (w-\zeta)^{H-\frac{3}{2}} \mathrm{d}w \Big)^2 (t-\zeta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} \mathrm{d}\zeta \\ = & C \int_0^t \big(\frac{t}{\zeta}\big)^{2H-1} (t-\zeta)^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)-1} \mathrm{d}\zeta \\ = & C t^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} B\big(2-2H, 2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)\big), \end{aligned} \tag{33}$$

where we require $r-\sigma+1 < 2\beta$ and $2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1) > 0$, i.e., $r < 2\beta+\sigma-1$ and $r < \frac{2\beta}{\alpha}(H+\alpha+\gamma-1)+\sigma-1$.

Furthermore, noting that (27) we obtain

$$\begin{aligned} & \Big\| A^{\frac{r-\sigma+1}{2}} \big(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-w) - \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta) \big) \Big\| \leq C \int_\zeta^w (t-v)^{\alpha+\gamma-2-\frac{\alpha}{2\beta}(r-\sigma+1)} \mathrm{d}v \\ & = \frac{C}{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)} [(t-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)} - (t-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}]. \end{aligned}$$

If $\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1) \geq 0$, using the fact $(w/\zeta)^{H-1/2} < (t/\zeta)^{H-1/2}$ it follows that

$$\begin{aligned} II \leq & CC_H^2 \int_0^t \Big\| \int_\zeta^t (t-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)} \big(\frac{w}{\zeta}\big)^{H-\frac{1}{2}} (w-\zeta)^{H-\frac{3}{2}} \mathrm{d}w \Big\|^2 \mathrm{d}\zeta \\ \leq & C \int_0^t \big(\frac{t}{\zeta}\big)^{2H-1} (t-\zeta)^{2[\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)]} \Big\| \int_\zeta^t (w-\zeta)^{H-\frac{3}{2}} \mathrm{d}w \Big\|^2 \mathrm{d}\zeta \\ = & C \int_0^t \big(\frac{t}{\zeta}\big)^{2H-1} (t-\zeta)^{2[H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)]-1} \mathrm{d}\zeta \\ = & C t^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} B\big(2-2H, 2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)\big). \end{aligned} \tag{34}$$

where we require $2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1) > 0$, that is, $r < \frac{2\beta}{\alpha}(H+\alpha+\gamma-1)+\sigma-1$.

If $\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1) < 0$, then

$$\begin{aligned} II \leq & CC_H^2 \int_0^t \Big\| \int_\zeta^t (t-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)} \big(\frac{w}{\zeta}\big)^{H-\frac{1}{2}} (w-\zeta)^{H-\frac{3}{2}} \mathrm{d}w \Big\|^2 \mathrm{d}\zeta \\ \leq & C \int_0^t \big(\frac{t}{\zeta}\big)^{2H-1} \Big\| \int_\zeta^t (t-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)} (w-\zeta)^{H-\frac{3}{2}} \mathrm{d}w \Big\|^2 \mathrm{d}\zeta \\ = & C \int_0^t \big(\frac{t}{\zeta}\big)^{2H-1} (t-\zeta)^{2[H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)]-1} \mathrm{d}\zeta \\ & \times B^2\big((\alpha+\gamma-1)-\frac{\alpha}{2\beta}(r-\sigma+1)+1, H-1/2\big) \\ = & C t^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} B\big(2-2H, 2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)\big) \\ & \times B^2\big((\alpha+\gamma-1)-\frac{\alpha}{2\beta}(r-\sigma+1)+1, H-1/2\big), \end{aligned} \tag{35}$$

where we require $2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1) > 0$ and $(\alpha+\gamma-1)-\frac{\alpha}{2\beta}(r-\sigma+1)+1 > 0$, that is, $r < \frac{2\beta}{\alpha}(H+\alpha+\gamma-1)+\sigma-1$ and $r < \frac{2\beta}{\alpha}(1+\alpha+\gamma-1)+\sigma-1$.

Together with (33)–(35), we also get for the case $H \in (1/2, 1)$

$$\Big\| \int_0^t \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta) \mathrm{d}W^H(\zeta) \Big\|_{L_p(\Omega;\dot{U}^r)} \leq C t^{H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}, \quad t \in [0,T].$$

The proof is thus completed. □

In view of the Banach contraction mapping principle, the existence and uniqueness of the mild solution to (1) is provided in the following theorem.

Theorem 1. *Let $\alpha \in (0,1)$, $\beta \in (1/2, 1]$, $\gamma \in [0,1]$, $\sigma \in (0,1]$, and $p \in [2, \infty)$. Let Assumptions 1–6 be satisfied. Then there exists a unique p-fold integrable mild solution $u : [0,T] \to U$ to (1) such that*

$$\sup_{t\in[0,T]} \|u(t)\|_{L_p(\Omega;U)} < \infty \tag{36}$$

for every $t \in [0,T]$.

Proof. We will use the contraction mapping principle to prove this result. For convenience, we let

$$\mathcal{U}_p = \{u : [0,T] \to U, \ \sup_{t\in[0,T]} \mathbb{E}[\|u(t)\|^p] < \infty\}$$

equipped with the norm

$$\|u\|^p_{\mathcal{U}_p} := \sup_{t\in[0,T]} \mathbb{E}[\|u(t)\|^p].$$

It is easy to verify that $(\mathcal{U}_p, \|\cdot\|_{\mathcal{U}_p})$ is a Banach space.

Now we define a mapping $\mathcal{F} : \mathcal{U}_p \to \mathcal{U}_p$ such that

$$\begin{aligned}\mathcal{F}(u)(t) :=& \mathbf{E}^{(\beta)}_{\alpha,1}(t)u_0 + \int_0^t \mathbf{E}^{(\beta)}_{\alpha,\alpha}(t-\zeta)f(\zeta,u(\zeta))\mathrm{d}\zeta + \int_0^t \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta)\mathrm{d}W^H(\zeta)\\ :=& \mathcal{F}_0(t) + \mathcal{F}_1(u)(t) + \mathcal{F}_2(t)\end{aligned} \tag{37}$$

for $t \in [0,T]$ and $u \in \mathcal{U}_p$.

We need to state that the mapping $\mathcal{F}$ is well-defined. First of all, according to (19) ($\mu = \nu = 0$) in Lemma 2 and Assumption 6, it follows that

$$\|\mathcal{F}_0(t)\|_{L_p(\Omega;U)} = \left\|\mathbf{E}^{(\beta)}_{\alpha,1}(t)u_0\right\|_{L_p(\Omega;U)} \le C\left(\mathbb{E}\left[\|u_0\|^p\right]\right)^{\frac{1}{p}} = C\|u_0\|_{L_p(\Omega;U)}, \tag{38}$$

which indicates $\mathcal{F}_0 \in \mathcal{U}_p$ for any $t \in [0,T]$.

For $\mathcal{F}_1(u)(t)$, we take $\mu = 0$ and $\nu = 0$ in (20) of Lemma 2 and apply (5) to derive

$$\begin{aligned}\|\mathcal{F}_1(u)(t)\|_{L_p(\Omega;U)} &= \left(\mathbb{E}\left[\|\mathcal{F}_1(u)(t)\|^p\right]\right)^{\frac{1}{p}} = \left(\mathbb{E}\left[\left\|\int_0^t \mathbf{E}^{(\beta)}_{\alpha,\alpha}(t-\zeta)f(\zeta,u(\zeta))\mathrm{d}\zeta\right\|^p\right]\right)^{\frac{1}{p}}\\ &\le \left(\mathbb{E}\left[\left(\int_0^t \left\|\mathbf{E}^{(\beta)}_{\alpha,\alpha}(t-\zeta)f(\zeta,u(\zeta))\right\|\mathrm{d}\zeta\right)^p\right]\right)^{\frac{1}{p}}\\ &\le C\left(\mathbb{E}\left[\left(\int_0^t (t-\zeta)^{\alpha-1}\|f(\zeta,u(\zeta))\|\mathrm{d}\zeta\right)^p\right]\right)^{\frac{1}{p}}\\ &\le C\int_0^t (t-\zeta)^{\alpha-1}\|f(\zeta,u(\zeta))\|_{L_p(\Omega;U)}\mathrm{d}\zeta\\ &\le C\int_0^t (t-\zeta)^{\alpha-1}\mathrm{d}\zeta\left(1 + \sup_{t\in[0,T]}\|u(t)\|_{L_p(\Omega;U)}\right)\\ &\le \frac{C}{\alpha}t^\alpha(1+\|u\|_{\mathcal{U}_p}).\end{aligned}$$

Hence, for any $u \in \mathcal{U}_p$ and $t \in [0,T]$, it holds

$$\sup_{t\in[0,T]}\|\mathcal{F}_1(u)(t)\|_{L_p(\Omega;U)} \le C(1+\|u\|_{\mathcal{U}_p}) < \infty. \tag{39}$$

As for $\mathcal{F}_2(t)$, by means of Lemma 4 with $r = 0$, we have

$$\|\mathcal{F}_2(t)\|_{L_p(\Omega;U)} \le Ct^{H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(1-\sigma)},$$

which gives

$$\sup_{t\in[0,T]}\|\mathcal{F}_2(t)\|_{L_p(\Omega;U)} < \infty \tag{40}$$

for any $t \in [0,T]$.

Next, we prove that the mapping $\mathcal{F}$ is contractive. To this end, we introduce an equivalent norm on the space $\mathcal{U}_p$. For $\rho \in \mathbb{R}$, let

$$\|u\|_{\mathcal{U}_p,\rho} := \sup_{t\in[0,T]} e^{-\rho t}\|u\|_{L_p(\Omega;U)}.$$

It is easy to verify that the norms $\|\cdot\|_{\mathcal{U}_p,\rho}$ and $\|\cdot\|_{\mathcal{U}_p}$ are equivalent.

For any $u, v \in \mathcal{U}_p$, according to (37), (20) ($\mu = 0, \nu = 0$) of Lemma 2 and (3) of Assumption 5, one can get

$$\begin{aligned}&\|\mathcal{F}(u)(t) - \mathcal{F}(v)(t)\|_{L_p(\Omega;U)} \le \|\mathcal{F}_1(u)(t) - \mathcal{F}_1(v)(t)\|_{L_p(\Omega;U)}\\ \le& \int_0^t \left\|\mathbf{E}^{(\beta)}_{\alpha,\alpha}(t-\zeta)(f(\zeta,u(\zeta)) - f(\zeta,v(\zeta)))\right\|_{L_p(\Omega;U)}\mathrm{d}\zeta\\ \le& C\int_0^t (t-\zeta)^{\alpha-1}\|f(\zeta,u(\zeta)) - f(\zeta,v(\zeta))\|_{L_p(\Omega;U)}\mathrm{d}\zeta\\ \le& C\int_0^t (t-\zeta)^{\alpha-1}\|u(\zeta) - v(\zeta)\|_{L_p(\Omega;U)}\mathrm{d}\zeta \le C\int_0^t (t-\zeta)^{\alpha-1}e^{\rho\zeta}\mathrm{d}\zeta\|u-v\|_{\mathcal{U}_p,\rho},\end{aligned}$$

that is,

$$\|\mathcal{F}(u)(t)-\mathcal{F}(v)(t)\|_{L_p(\Omega;U)} \le C\int_0^t (t-\zeta)^{\alpha-1}e^{\rho\zeta}\mathrm{d}\zeta\,\|u-v\|_{\mathcal{U}_p,\rho}.$$

An application of Hölder inequality with $\frac{1}{\mu}+\frac{1}{\nu}=1$ implies that

$$\begin{aligned}\int_0^t (t-\zeta)^{\alpha-1}e^{\rho\zeta}\mathrm{d}\zeta \le& \Big(\int_0^t (t-\zeta)^{(\alpha-1)\nu}\mathrm{d}\zeta\Big)^{\frac{1}{\nu}}\Big(\int_0^t e^{\mu\rho\zeta}\mathrm{d}\zeta\Big)^{\frac{1}{\mu}}\\ \le& \Big(\frac{T^{(\alpha-1)\nu+1}}{(\alpha-1)\nu+1}\Big)^{\frac{1}{\nu}}\Big(\frac{1}{\mu\rho}\big(e^{\mu\rho t}-1\big)\Big)^{\frac{1}{\mu}},\end{aligned}$$

where we can always choose $1<\nu<\frac{1}{1-\alpha}$ such that $(\alpha-1)\nu>-1$.

As a result, for $\rho>0$, there holds

$$\begin{aligned}&e^{-\rho t}\|\mathcal{F}(u)(t)-\mathcal{F}(v)(t)\|_{L_p(\Omega;U)} \le C\Big(\frac{T^{(\alpha-1)\nu+1}}{(\alpha-1)\nu+1}\Big)^{\frac{1}{\nu}}\Big(\frac{1}{\mu\rho}\big(1-e^{-\mu\rho t}\big)\Big)^{\frac{1}{\mu}}\|u-v\|_{\mathcal{U}_p,\rho}\\ &\le C\Big(\frac{T^{(\alpha-1)\nu+1}}{(\alpha-1)\nu+1}\Big)^{\frac{1}{\nu}}(\mu\rho)^{-\frac{1}{\mu}}\|u-v\|_{\mathcal{U}_p,\rho}.\end{aligned}$$

We can choose $\rho>0$ large enough so that the mapping $\mathcal{F}$ is contractive on the space $\mathcal{U}_p$ with respect to the norm $\|\cdot\|_{\mathcal{U}_{p,\rho}}$. Therefore, there is a unique fixed point $u\in\mathcal{U}_p$ and it is a unique mild solution to (1). The proof of this theorem is now completed. □

4. Regularities of the solution

This section is concerned with spatial and temporal regularities of the mild solution to (1). These results will be used in the error analysis of semidiscrete and fully discrete approximation. Let us begin by proving the regularity property of the mild solution in space.

Theorem 2. *Let $\alpha\in(0,1)$, $\beta\in(1/2,1]$, $\gamma\in[0,1]$, $\sigma\in(0,1]$, and $p\in[2,\infty)$. Let Assumptions 1–6 be satisfied. Then the mild solution u of (1) satisfies for any $t\in[0,T]$*

$$\sup_{t\in[0,T]}\|u(t)\|_{L_p(\Omega;\dot{U}^r)}<\infty,$$

where r is defined in Assumption 5.

Proof. According to Theorem 1, (1) has the unique mild solution

$$\begin{aligned}u(t)=&\mathbf{E}^{(\beta)}_{\alpha,1}(t)u_0+\int_0^t\mathbf{E}^{(\beta)}_{\alpha,\alpha}(t-\zeta)f(\zeta,u(\zeta))\mathrm{d}\zeta+\int_0^t\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\zeta)\mathrm{d}W^H(\zeta)\\ =&S_0(t)+S_1(u)(t)+S_2(t)\end{aligned}\tag{41}$$

for any $t\in[0,T]$ and $u\in\mathcal{U}_p$.

Applying (19) in Lemma 2 with $\mu=\nu=r$ and Assumption 6 leads to

$$\begin{aligned}\|S_0(t)\|_{L_p(\Omega;\dot{U}^r)}=&\Big\|\mathbf{E}^{(\beta)}_{\alpha,1}(t)u_0\Big\|_{L_p(\Omega;\dot{U}^r)}=\Big(\mathbb{E}\Big[\Big|\mathbf{E}^{(\beta)}_{\alpha,1}(t)u_0\Big|_r^p\Big]\Big)^{\frac{1}{p}}\\ \le& C\left(\mathbb{E}\left[|u_0|_r^p\right]\right)^{\frac{1}{p}}=C\|u_0\|_{L_p(\Omega;\dot{U}^r)}.\end{aligned}\tag{42}$$

Similarly, we use (20) ($\mu=r,\nu=0$) in Lemma 2 and (5) to obtain

$$\begin{aligned}&\|S_1(u)(t)\|_{L_p(\Omega;\dot{U}^r)}=\left(\mathbb{E}\left[|S_1(u)(t)|_r^p\right]\right)^{\frac{1}{p}}\le\left(\mathbb{E}\Big[\Big(\int_0^t\Big|\mathbf{E}^{(\beta)}_{\alpha,\alpha}(t-\zeta)f(\zeta,u(\zeta))\Big|_r\mathrm{d}\zeta\Big)^p\Big]\right)^{\frac{1}{p}}\\ &\le\left(\mathbb{E}\Big[\Big(\int_0^t(t-\zeta)^{\alpha-1-\frac{\alpha}{2\beta}r}\|f(\zeta,u(\zeta))\|\mathrm{d}\zeta\Big)^p\Big]\right)^{\frac{1}{p}}\le\frac{C}{\alpha-\frac{\alpha}{2\beta}r}t^{\alpha-\frac{\alpha}{2\beta}r}(1+\|u(t)\|_{\mathcal{U}_p}),\end{aligned}$$

which implies

$$\sup_{t\in[0,T]}\|S_1(u)(t)\|_{L_p(\Omega;\dot{U}^r)}\le C(1+\|u\|_{\mathcal{U}_p})<\infty.\tag{43}$$

Moreover, Lemma 4 immediately indicates

$$\|S_2(t)\|_{L_p(\Omega;\dot{U}^r)}\le Ct^{H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}<C.\tag{44}$$

Collecting estimates (42)–(44) we complete the proof of this theorem. □

In order to show temporal regularity of the mild solution, we first investigate regularity property of stochastic integral in time.

Lemma 5. *Let $\alpha \in (0,1)$, $\beta \in (1/2,1]$, $\gamma \in [0,1]$, $\sigma \in (0,1]$, and $p \in [2,\infty)$. Let Assumptions 1–4 be fulfilled. Let r be defined as Assumption 5. Then there exists a constant $C>0$ such that*

$$\begin{aligned}&\Big\| \int_0^{t_2} \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_2-\zeta)\mathrm{d}W^H(\zeta) - \int_0^{t_1} \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_1-\zeta)\mathrm{d}W^H(\zeta)\Big\|_{L_p(\Omega;U)}\\ \le& C(t_2-t_1)^{\frac{\alpha r}{2\beta}}, \quad 0\le t_1<t_2\le T. \end{aligned} \tag{45}$$

Proof. It is obvious that

$$\begin{aligned}&\Big\| \int_0^{t_2} \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_2-\zeta)\mathrm{d}W^H(\zeta) - \int_0^{t_1} \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_1-\zeta)\mathrm{d}W^H(\zeta)\Big\|_{L_p(\Omega;U)}\\ =&\Big\| \int_{t_1}^{t_2} \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_2-\zeta)\mathrm{d}W^H(\zeta)\Big\|_{L_p(\Omega;U)}\\ &+ \Big\| \int_0^{t_1} \big(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_2-\zeta) - \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_1-\zeta)\big)\mathrm{d}W^H(\zeta)\Big\|_{L_p(\Omega;U)}\\ :=&I+II. \end{aligned}$$

It follows from Lemma 4 that for $0\le t_1<t_2\le T$

$$\begin{aligned} I =&\Big\| \int_{t_1}^{t_2} \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_2-\zeta)\mathrm{d}W^H(\zeta)\Big\|_{L_p(\Omega;U)} \le C(t_2-t_1)^{H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(1-\sigma)}\\ =&C(t_2-t_1)^{H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}(t_2-t_1)^{\frac{\alpha r}{2\beta}} \le C(t_2-t_1)^{\frac{\alpha r}{2\beta}}. \end{aligned} \tag{46}$$

We now estimate II. By the stochastic integral (14), Lemma 3 and Assumption 4, there holds

$$\begin{aligned} II =&\Big\| \int_0^{t_1} \big(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_2-\zeta) - \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_1-\zeta)\big)\mathrm{d}W^H(\zeta)\Big\|_{L_p(\Omega;U)}\\ =&\Big\| \int_0^{t_1} K^*_{H,t_1}\big(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_2-\cdot) - \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_1-\cdot)\big)(\zeta)\mathrm{d}W(\zeta)\Big\|_{L_p(\Omega;U)}\\ \le&C\Big\|\Big(\int_0^{t_1}\Big\|K^*_{H,t_1}\Big(A^{\frac{1-\sigma}{2}}\big(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_2-\cdot) - \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_1-\cdot)\big)(\zeta)\Big)\Big\|^2\mathrm{d}\zeta\Big)^{1/2}\Big\|_{L_p(\Omega;\mathbb{R})}. \end{aligned}$$

In the following, we will divide two cases $H\in(0,1/2)$ and $H\in(1/2,1)$ to show II. We first consider the case $H\in(0,1/2)$. Denote

$$\Phi(t_2,t_1,\zeta) = \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_2-\zeta) - \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_1-\zeta).$$

Then it follows by (11) that

$$\begin{aligned}&\int_0^{t_1}\Big\|K^*_{H,t_1}\Big(A^{\frac{1-\sigma}{2}}\big(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_2-\cdot) - \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_1-\cdot)\big)(\zeta)\Big)\Big\|^2\mathrm{d}\zeta\\ \le&\int_0^{t_1}\Big\|K_H(t_1,\zeta)A^{\frac{1-\sigma}{2}}\Phi(t_2,t_1,\zeta)\Big\|^2\mathrm{d}\zeta\\ &+\int_0^{t_1}\Big\|\int_\zeta^{t_1} A^{\frac{1-\sigma}{2}}(\Phi(t_2,t_1,w)-\Phi(t_2,t_1,\zeta))\frac{\partial K_H(w,\zeta)}{\partial w}\mathrm{d}w\Big\|^2\mathrm{d}\zeta\\ :=&II_1+II_2. \end{aligned}$$

In terms of Lemma 1, (21) in Lemma 2, one has the following estimate

$$\begin{aligned}&\Big\|A^{\frac{1-\sigma}{2}}\Phi(t_2,t_1,\zeta)\Big\| = \Big\|A^{\frac{1-\sigma}{2}}\big(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_2-\zeta) - \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_1-\zeta)\big)\Big\|\\ =&\Big\|A^{\frac{1-\sigma}{2}}\int_{t_1-\zeta}^{t_2-\zeta}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma-1}(\theta)\mathrm{d}\theta\Big\| \le C\int_{t_1}^{t_2}(v-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}(v-\zeta)^{\frac{\alpha r}{2\beta}-1}\mathrm{d}v. \end{aligned} \tag{47}$$

If $\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)<0$, then the elementary inequality (26) yields

$$\begin{aligned}\Big\|A^{\frac{1-\sigma}{2}}\Phi(t_2,t_1,\zeta)\Big\| \le&C(t_1-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}\int_{t_1}^{t_2}(v-\zeta)^{\frac{\alpha r}{2\beta}-1}\mathrm{d}v\\ \le&C(t_2-t_1)^{\frac{\alpha r}{2\beta}}(t_1-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}. \end{aligned} \tag{48}$$

Thus, by (48), the first equality in (9), (22) and (23), II_1 is estimated by

$$\begin{aligned} II_1 \le&C\int_0^{t_1}\Big(\frac{t_1}{\zeta}\Big)^{2H-1}(t_1-\zeta)^{2H-1}(t_1-\zeta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}(t_2-t_1)^{\frac{\alpha r}{\beta}}\mathrm{d}\zeta\\ &+C\int_0^{t_1}\zeta^{1-2H}\Big(\int_\zeta^{t_1} w^{H-\frac{3}{2}}(w-\zeta)^{H-\frac{1}{2}}\mathrm{d}w\Big)^2(t_1-\zeta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}(t_2-t_1)^{\frac{\alpha r}{\beta}}\mathrm{d}\zeta \end{aligned}$$

$$\begin{aligned}
&\le C(t_2-t_1)^{\frac{\alpha r}{\beta}} t_1^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} B\Big(2-2H, 2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)\Big)\\
&\quad + C(t_2-t_1)^{\frac{\alpha r}{\beta}} t_1^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} B^2\big(1-2H, H+1/2\big)\\
&\quad \times B\Big(2H, 2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)\Big) \le C(t_2-t_1)^{\frac{\alpha r}{\beta}},
\end{aligned}$$

where we require $2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)>0$, i.e., $r<\frac{2\beta}{\alpha}(\alpha+\gamma-1)+\sigma+\frac{\beta}{\alpha}2H-1$.

Similarly, if $\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)\ge 0$, then

$$\begin{aligned}
\Big\|A^{\frac{1-\sigma}{2}}\boldsymbol{\Phi}(t_2,t_1,\zeta)\Big\| &\le C(t_2-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}\int_{t_1}^{t_2}(v-\zeta)^{\frac{\alpha r}{2\beta}-1}\mathrm{d}v\\
&\le C(t_2-t_1)^{\frac{\alpha r}{2\beta}}(t_2-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}.
\end{aligned}\tag{49}$$

Therefore, we also have

$$\begin{aligned}
II_1 \le& C\int_0^{t_1}\Big(\frac{t_1}{\zeta}\Big)^{2H-1}(t_1-\zeta)^{2H-1}(t_2-\zeta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}(t_2-t_1)^{\frac{\alpha r}{\beta}}\mathrm{d}\zeta\\
&+ C\int_0^{t_1}\zeta^{1-2H}\Big(\int_\zeta^{t_1} w^{H-\frac{3}{2}}(w-\zeta)^{H-\frac{1}{2}}\mathrm{d}w\Big)^2(t_2-\zeta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}(t_2-t_1)^{\frac{\alpha r}{\beta}}\mathrm{d}\zeta\\
\le& C(t_2-t_1)^{\frac{\alpha r}{\beta}} t_2^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} B\big(2-2H,2H\big)\\
&+ C(t_2-t_1)^{\frac{\alpha r}{\beta}} t_2^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)} B^2\big(1-2H,H+1/2\big)\\
\le& C(t_2-t_1)^{\frac{\alpha r}{\beta}}.
\end{aligned}$$

We continue to estimate II_2. Note that

$$\begin{aligned}
&\Big\|A^{\frac{1-\sigma}{2}}(\boldsymbol{\Phi}(t_2,t_1,w)-\boldsymbol{\Phi}(t_2,t_1,\zeta))\Big\|\\
=&\Big\|A^{\frac{1-\sigma}{2}}\big(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_2-w)-\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_1-w)-\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_2-\zeta)+\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_1-\zeta)\big)\Big\|\\
=&\Big\|A^{\frac{1-\sigma}{2}}\int_{t_1}^{t_2}\big(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma-1}(\eta-w)-\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma-1}(\eta-\zeta)\big)\mathrm{d}\eta\Big\|\\
\le&\int_{t_1}^{t_2}\int_{\eta-w}^{\eta-\zeta}\Big\|A^{\frac{1-\sigma}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma-2}(\theta)\Big\|\mathrm{d}\theta\mathrm{d}\eta \le C\int_{t_1}^{t_2}\int_\zeta^w(\eta-v)^{\alpha+\gamma-3-\frac{\alpha}{2\beta}(1-\sigma)}\mathrm{d}v\mathrm{d}\eta\\
=&C\int_{t_1}^{t_2}\int_\zeta^w(\eta-v)^{\alpha+\gamma-2-\frac{\alpha}{2\beta}(1-\sigma)-\frac{1}{2}}(\eta-v)^{-\frac{1}{2}}\mathrm{d}v\mathrm{d}\eta\\
\le&C\int_{t_1}^{t_2}(\eta-w)^{\alpha+\gamma-2-\frac{\alpha}{2\beta}(1-\sigma)-\frac{1}{2}}\int_\zeta^w(\eta-v)^{-\frac{1}{2}}\mathrm{d}v\mathrm{d}\eta\\
\le&C(w-\zeta)^{\frac{1}{2}}\int_{t_1}^{t_2}(\eta-w)^{\alpha+\gamma-2-\frac{\alpha}{2\beta}(1-\sigma)-\frac{1}{2}}\mathrm{d}\eta\\
=&C(w-\zeta)^{\frac{1}{2}}\int_{t_1}^{t_2}(\eta-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}(\eta-w)^{\frac{\alpha r}{2\beta}-1}\mathrm{d}\eta.
\end{aligned}\tag{50}$$

If $\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-1/2<0$, then we have

$$\begin{aligned}
&\Big\|A^{\frac{1-\sigma}{2}}(\boldsymbol{\Phi}(t_2,t_1,w)-\boldsymbol{\Phi}(t_2,t_1,\zeta))\Big\|\\
\le&C(w-\zeta)^{\frac{1}{2}}(t_1-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}\int_{t_1}^{t_2}(\eta-w)^{\frac{\alpha r}{2\beta}-1}\mathrm{d}\eta\\
\le&C(t_2-t_1)^{\frac{\alpha r}{2\beta}}(t_1-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-1/2}(w-\zeta)^{\frac{1}{2}}.
\end{aligned}\tag{51}$$

Applying (51), (12), (23), and the fact $(w/\zeta)^{H-1/2}<1$ for $H<1/2$, we arrive at

$$\begin{aligned}
II_2 =& \int_0^{t_1}\Big\|\int_\zeta^{t_1}A^{\frac{1-\sigma}{2}}(\boldsymbol{\Phi}(t_2,t_1,w)-\boldsymbol{\Phi}(t_2,t_1,\zeta))\frac{\partial K_H(w,\zeta)}{\partial w}\mathrm{d}w\Big\|^2\mathrm{d}\zeta\\
\le& C(t_2-t_1)^{\frac{\alpha r}{\beta}}\int_0^{t_1}\Big(\int_\zeta^{t_1}(t_1-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}\Big(\frac{w}{\zeta}\Big)^{H-\frac{1}{2}}(w-\zeta)^{H-\frac{3}{2}+\frac{1}{2}}\mathrm{d}w\Big)^2\mathrm{d}\zeta\\
\le& C(t_2-t_1)^{\frac{\alpha r}{\beta}}\int_0^{t_1}\Big(\int_\zeta^{t_1}(t_1-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}(w-\zeta)^{H-1}\mathrm{d}w\Big)^2\mathrm{d}\zeta\\
\le& CB^2\Big(\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)+\frac{1}{2},H\Big)(t_2-t_1)^{\frac{\alpha r}{\beta}}t_1^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}\\
\le& C(t_2-t_1)^{\frac{\alpha r}{\beta}},
\end{aligned}$$

where we require $\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)+1/2>0$ and $2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)>0$, that is, $r<\frac{2\beta}{\alpha}(\alpha+\gamma-1)+\sigma+\frac{\beta}{\alpha}-1$ and $r<\frac{2\beta}{\alpha}(\alpha+\gamma-1)+\sigma+\frac{2\beta}{\alpha}H-1$.

In addition, if $\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-1/2\geq 0$, then

$$\begin{aligned}&\left\|A^{\frac{1-\sigma}{2}}(\boldsymbol{\Phi}(t_2,t_1,w)-\boldsymbol{\Phi}(t_2,t_1,\zeta))\right\|\\ \leq& C(t_2-t_1)^{\frac{\alpha r}{2\beta}}(t_2-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}(w-\zeta)^{\frac{1}{2}}.\end{aligned}\tag{52}$$

As a result,

$$\begin{aligned}II_2=&\int_0^{t_1}\Big\|\int_\zeta^{t_1}A^{\frac{1-\sigma}{2}}(\boldsymbol{\Phi}(t_2,t_1,w)-\boldsymbol{\Phi}(t_2,t_1,\zeta))\frac{\partial K_H(w,\zeta)}{\partial w}\mathrm{d}w\Big\|^2\mathrm{d}\zeta\\ \leq& C(t_2-t_1)^{\frac{\alpha r}{\beta}}\int_0^{t_1}\Big(\int_\zeta^{t_1}(t_2-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}\big(\frac{w}{\zeta}\big)^{H-\frac{1}{2}}(w-\zeta)^{H-\frac{3}{2}+\frac{1}{2}}\mathrm{d}w\Big)^2\mathrm{d}\zeta\\ \leq& C(t_2-t_1)^{\frac{\alpha r}{\beta}}t_2^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)-1}\int_0^{t_1}\Big(\int_\zeta^{t_1}(w-\zeta)^{H-1}\mathrm{d}w\Big)^2\mathrm{d}\zeta\\ \leq& C(t_2-t_1)^{\frac{\alpha r}{\beta}}.\end{aligned}$$

Based on the above estimates II_1 and II_2, we obtain for the case $H\in(0,1/2)$

$$II\leq C(t_2-t_1)^{\frac{\alpha r}{2\beta}}.\tag{53}$$

For the case $H\in(1/2,1)$, with the help of (48), (13), and (22), we get

$$\begin{aligned}II=&\Big\|\int_0^{t_1}\big(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_2-\zeta)-\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_1-\zeta)\big)\mathrm{d}W^H(\zeta)\Big\|_{L_p(\Omega;U)}\\ \leq& C\Big\|\Big(\int_0^{t_1}\Big\|K^*_{H,t_1}\Big(A^{\frac{1-\sigma}{2}}\big(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_2-\cdot)-\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_1-\cdot)\big)\Big)(\zeta)\Big\|^2\mathrm{d}\zeta\Big)^{\frac{1}{2}}\Big\|_{L_p(\Omega;\mathbb{R})}\\ \leq& C\Big\|\Big(\int_0^{t_1}(t_2-t_1)^{\frac{\alpha r}{2\beta}}(t_1-\zeta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}\\ &\times\Big(\int_\zeta^{t_1}\big(\frac{w}{\zeta}\big)^{H-\frac{1}{2}}(w-\zeta)^{H-\frac{3}{2}}\mathrm{d}w\Big)^2\mathrm{d}\zeta\Big)^{\frac{1}{2}}\Big\|_{L_p(\Omega;\mathbb{R})}\\ \leq& C(t_2-t_1)^{\frac{\alpha r}{2\beta}}\Big\|\Big(\int_0^{t_1}(t_1-\zeta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}\big(\frac{t_1}{\zeta}\big)^{2H-1}\\ &\times\Big(\int_\zeta^{t_1}(w-\zeta)^{H-\frac{3}{2}}\mathrm{d}w\Big)^2\mathrm{d}\zeta\Big)^{\frac{1}{2}}\Big\|_{L_p(\Omega;\mathbb{R})}\\ \leq& C(t_2-t_1)^{\frac{\alpha r}{2\beta}}\Big\|\Big(\int_0^{t_1}(t_1-\zeta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}\big(\frac{t_1}{\zeta}\big)^{2H-1}(t_1-\zeta)^{2H-1}\mathrm{d}\zeta\Big)^{\frac{1}{2}}\Big\|_{L_p(\Omega;\mathbb{R})}\\ \leq& C(t_2-t_1)^{\frac{\alpha r}{2\beta}}t_1^{H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}B^{1/2}\big(2-2H,2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)\big)\\ \leq& C(t_2-t_1)^{\frac{\alpha r}{2\beta}},\end{aligned}\tag{54}$$

where we require $2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)>0$, that is, $r<\frac{2\beta}{\alpha}(\alpha+\gamma-1)+\sigma+\frac{2\beta}{\alpha}H-1$.

In the same way we derive by (49)

$$\begin{aligned}II=&\Big\|\int_0^{t_1}\big(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_2-\zeta)-\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t_1-\zeta)\big)\mathrm{d}W^H(\zeta)\Big\|_{L_p(\Omega;U)}\\ \leq& C\Big\|\Big(\int_0^{t_1}(t_2-t_1)^{\frac{\alpha r}{2\beta}}(t_2-\zeta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}\\ &\times\Big(\int_\zeta^{t_1}\big(\frac{w}{\zeta}\big)^{H-\frac{1}{2}}(w-\zeta)^{H-\frac{3}{2}}\mathrm{d}w\Big)^2\mathrm{d}\zeta\Big)^{\frac{1}{2}}\Big\|_{L_p(\Omega;\mathbb{R})}\\ \leq& C(t_2-t_1)^{\frac{\alpha r}{2\beta}}t_2^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}\Big\|\Big(\int_0^{t_1}\big(\frac{t_1}{\zeta}\big)^{2H-1}\Big(\int_\zeta^{t_1}(w-\zeta)^{H-\frac{3}{2}}\mathrm{d}w\Big)^2\mathrm{d}\zeta\Big)^{\frac{1}{2}}\Big\|_{L_p(\Omega;\mathbb{R})}\\ \leq& C(t_2-t_1)^{\frac{\alpha r}{2\beta}}t_2^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}t_1^H B^{1/2}(2-2H,2H)\\ \leq& C(t_2-t_1)^{\frac{\alpha r}{2\beta}}.\end{aligned}\tag{55}$$

Combining the estimates (46), (53)–(55) we obtain the desired result. The proof is completed. □

Now we are in a position to present temporal regularity of the mild solution to (1)

Theorem 3. *Let $\alpha \in (0,1)$, $\beta \in (1/2,1]$, $\gamma \in [0,1]$, $\sigma \in (0,1]$, and $p \in [2,\infty)$. Let Assumptions 1–6 be satisfied. Then the mild solution u of (1) satisfies*

$$\|u(t_2)-u(t_1)\|_{L_p(\Omega;U)} < C(t_2-t_1)^{\min\{\alpha,\frac{ar}{2\beta}\}} \tag{56}$$

for all $0 \le t_1 < t_2 \le T$.

Proof. In view of the mild solution formula defined in (41), we have

$$\begin{aligned}\|u(t_2)-u(t_1)\|_{L_p(\Omega;U)} \le& \|S_0(t_2)-S_0(t_1)\|_{L_p(\Omega;U)} + \|S_1(u)(t_2)-S_1(u)(t_1)\|_{L_p(\Omega;U)}\\ &+ \|S_2(t_2)-S_2(t_1)\|_{L_p(\Omega;U)}.\end{aligned} \tag{57}$$

By using (17), (20) in Lemma 2 with $\mu = 2\beta$ and $\nu = r$ and Assumption 6, it has

$$\begin{aligned}&\|S_0(t_2)-S_0(t_1)\|_{L_p(\Omega;U)} = \left\|\mathbf{E}^{(\beta)}_{\alpha,1}(t_2)u_0 - \mathbf{E}^{(\beta)}_{\alpha,1}(t_1)u_0\right\|_{L_p(\Omega;U)}\\ \le& \int_{t_1}^{t_2}\left\| - A^{\beta}\mathbf{E}^{(\beta)}_{\alpha,\alpha}(\zeta)u_0\right\|_{L_p(\Omega;U)}\mathrm{d}\zeta \le C\int_{t_1}^{t_2}\zeta^{\alpha-1-\frac{\alpha}{2\beta}(2\beta-r)}\mathrm{d}\zeta\|u_0\|_{L_p(\Omega;\dot{U}^r)}\\ \le& C(t_2-t_1)^{\frac{ar}{2\beta}}\|u_0\|_{L_p(\Omega;\dot{U}^r)}.\end{aligned} \tag{58}$$

Obviously there holds

$$\begin{aligned}&\|S_1(u)(t_2)-S_1(u)(t_1)\|_{L_p(\Omega;U)}\\ =&\left\|\int_0^{t_2}\mathbf{E}^{(\beta)}_{\alpha,\alpha}(t_2-\zeta)f(\zeta,u(\zeta))\mathrm{d}\zeta - \int_0^{t_1}\mathbf{E}^{(\beta)}_{\alpha,\alpha}(t_1-\zeta)f(\zeta,u(\zeta))\mathrm{d}\zeta\right\|_{L_p(\Omega;U)}\\ \le&\left\|\int_{t_1}^{t_2}\mathbf{E}^{(\beta)}_{\alpha,\alpha}(t_2-\zeta)f(\zeta,u(\zeta))\mathrm{d}\zeta\right\|_{L_p(\Omega;U)}\\ &+\left\|\int_0^{t_1}\left(\mathbf{E}^{(\beta)}_{\alpha,\alpha}(t_2-\zeta)-\mathbf{E}^{(\beta)}_{\alpha,\alpha}(t_1-\zeta)\right)f(\zeta,u(\zeta))\mathrm{d}\zeta\right\|_{L_p(\Omega;U)}.\end{aligned}$$

For the first term, it follows from (20) ($\mu = 0, \nu = -1$) in Lemma 2 and (5) that

$$\begin{aligned}&\left\|\int_{t_1}^{t_2}\mathbf{E}^{(\beta)}_{\alpha,\alpha}(t_2-\zeta)f(\zeta,u(\zeta))\mathrm{d}\zeta\right\|_{L_p(\Omega;U)} \le \int_{t_1}^{t_2}\left\|\mathbf{E}^{(\beta)}_{\alpha,\alpha}(t_2-\zeta)f(\zeta,u(\zeta))\right\|_{L_p(\Omega;U)}\mathrm{d}\zeta\\ \le& C\int_{t_1}^{t_2}(t_2-\zeta)^{\alpha-1}\|f(\zeta,u(\zeta))\|_{L_p(\Omega;U)}\mathrm{d}\zeta \le \frac{C}{\alpha}(t_2-t_1)^{\alpha}(1+\|u\|_{\mathcal{V}_p}).\end{aligned}$$

For the second term, by (18), (20) ($\mu = 0, \nu = -1$) in Lemma 2, and (5), one can get

$$\begin{aligned}&\left\|\int_0^{t_1}\left(\mathbf{E}^{(\beta)}_{\alpha,\alpha}(t_2-\zeta)-\mathbf{E}^{(\beta)}_{\alpha,\alpha}(t_1-\zeta)\right)f(\zeta,u(\zeta))\mathrm{d}\zeta\right\|_{L_p(\Omega;U)}\\ =&\left\|\int_0^{t_1}\int_{t_1}^{t_2}\mathbf{E}^{(\beta)}_{\alpha,\alpha-1}(w-\zeta)f(\zeta,u(\zeta))\mathrm{d}w\mathrm{d}\zeta\right\|_{L_p(\Omega;U)}\\ \le&\int_{t_1}^{t_2}\int_0^{t_1}\left\|\mathbf{E}^{(\beta)}_{\alpha,\alpha-1}(w-\zeta)f(\zeta,u(\zeta))\right\|_{L_p(\Omega;U)}\mathrm{d}\zeta\mathrm{d}w\\ \le&C\int_{t_1}^{t_2}\int_0^{t_1}(w-\zeta)^{\alpha-2}\|f(\zeta,u(\zeta))\|_{L_p(\Omega;U)}\mathrm{d}\zeta\mathrm{d}w\\ \le&\frac{C}{(1-\alpha)\alpha}(t_2-t_1)^{\alpha}(1+\|u\|_{\mathcal{V}_p}).\end{aligned}$$

Thus, the above two estimates result in

$$\|S_1(u)(t_2)-S_1(u)(t_1)\|_{L_p(\Omega;U)} \le C(t_2-t_1)^{\alpha} \tag{59}$$

for all $0 \le t_1 < t_2 \le T$.

In addition, Lemma 5 immediately gives

$$\|S_2(t_2)-S_2(t_1)\|_{L_p(\Omega;U)} \le C(t_2-t_1)^{\frac{ar}{2\beta}}. \tag{60}$$

Together estimates (58), (59) with (60) completes the proof of Theorem 3. □

5. The semidiscrete FEM

Let $\mathcal{T}_h$ be a shape regular and quasi-uniform triangulation of the domain $D \subset R^d, d = 1, 2, 3$ and h_K denote the diameter of the finite element K in the triangulation $\mathcal{T}_h$. Let the mesh size $h = \max_{K\in\mathcal{T}_h} h_K$ denote the maximal diameter of all element K. Let $V_h \subset \dot{H}^\beta, 1/2 < \beta \le 1$ be the piecewise linear finite element space of associated with the triangulation $\mathcal{T}_h$,

$$V_h := \{v_h \in \dot{H}^\beta(D) : v_h|_K \in P_1(K), \forall K \in \mathcal{T}_h\},$$

where $P_1(K)$ is the space of linear functions on the finite element K.

Next, we define L_2-projection P_h, the fractional Ritz projection R_h and the fractional discrete Laplacian A_h^β on the space V_h.

The L_2-projection $P_h : L_2(D) \to V_h$ is defined by, for any $v \in L_2(D)$,

$$(P_h v, \chi) = (v, \chi), \ \forall \chi \in V_h.$$

The fractional Ritz projection $R_h : \dot{H}^\beta \to V_h$ with $1/2 < \beta \le 1$ is defined by, for any $v \in \dot{H}^\beta$,

$$(A^{\beta/2} R_h v, A^{\beta/2}\chi) = (A^{\beta/2} v, A^{\beta/2}\chi), \ \forall \chi \in V_h.$$

The operators P_h and R_h satisfy the following approximation properties:

$$\|P_h v - v\| + h^\beta \|A^{\beta/2}(P_h v - v)\| \le C h^s |v|_s, \ \forall v \in \dot{H}^s, \ s \in [\beta, 2\beta],$$

and

$$\|R_h v - v\| + h^\beta \|A^{\beta/2}(R_h v - v)\| \le C h^s |v|_s, \ \forall v \in \dot{H}^s, \ s \in [\beta, 2\beta].$$

The discrete Laplacian $A_h : V_h \to V_h$ is defined by

$$(A_h v, \chi) = (A^{1/2} v, A^{1/2}\chi), \ \forall v, \chi \in V_h,$$

and further denoting λ_j^h and ϕ_j^h are the eigenvalues and eigenfunctions of the discrete Laplacian A_h, i.e.,

$$A_h \phi_j^h = \lambda_j^h \phi_j^h, \ j = 1, 2, \dots, M_h,$$

where M_h is the dimension of the finite element space V_h.

Similarly, the fractional discrete Laplacian $A_h^\beta : V_h \to V_h$ with $1/2 < \beta \le 1$ is defined by

$$(A_h^\beta v, \chi) = (A^{\beta/2} v, A^{\beta/2}\chi), \ \forall v, \chi \in V_h.$$

By using the eigenvalues λ_j^h and eigenfunctions ϕ_j^h, we can introduce the discrete norm on the finite element space V_h by, for $s \in \mathbb{R}$,

$$|\chi|_{s,h}^2 = \sum_{j=1}^{M_h} (\lambda_j^h)^s (\chi, \phi_j^h)^2, \ \forall \chi \in V_h. \tag{61}$$

Based on the above preparation, the semidiscrete finite element approximation for (1) reads as

$${}_C\mathrm{D}_{0,t}^\alpha u_h(t) + A_h^\beta u_h(t) = P_h f(t, u_h(t)) + P_h\Big({}_{RL}\mathrm{D}_{0,t}^{-\gamma} \frac{\mathrm{d}W^H(t)}{\mathrm{d}t}\Big), \quad 0 < t \le T, \tag{62}$$

where $u_h(0) = P_h u_0$.

As for the continuous problem (1), there exists a unique mild solution $u_h(t)$ of the semidiscrete problem (62) which is expressed by

$$\begin{aligned} u_h(t) = \mathbf{E}_{\alpha,1}^{(\beta,h)}(t)\, P_h u_0 &+ \int_0^t \mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t-\zeta) P_h f(\zeta, u_h(\zeta))\mathrm{d}\zeta \\ &+ \int_0^t \mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t-\zeta) P_h \mathrm{d}W^H(\zeta), \end{aligned} \tag{63}$$

where the operators $\mathbf{E}_{\alpha,1}^{(\beta,h)}(t)$, $\mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t)$ and $\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t)$ are defined from V_h to V_h by

$$\mathbf{E}_{\alpha,1}^{(\beta,h)}(t)\chi := \sum_{j=1}^{M_h} E_{\alpha,1}((\lambda_j^h)^\beta t^\alpha)(\chi, \phi_j^h)\phi_j^h,$$

$$\mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t)\chi := \sum_{j=1}^{M_h} t^{\alpha-1} E_{\alpha,\alpha}((\lambda_j^h)^\beta t^\alpha)(\chi, \phi_j^h)\phi_j^h,$$

$$\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t)\chi := \sum_{j=1}^{M_h} t^{\alpha+\gamma-1} E_{\alpha,\alpha+\gamma}((\lambda_j^h)^\beta t^\alpha)(\chi, \phi_j^h)\phi_j^h,$$

We remark that the semidiscrete solution operators in (63) have also the similar smoothing properties as Lemma 2 and their proofs can be referred to [20].

Lemma 6. *Let $\alpha \in (0,1)$, $\beta \in (1/2,1]$ and $\gamma \in [0,1]$. Let $|\cdot|_{s,h}$, $s \in \mathbb{R}$ be the discrete norm given by (61). Then for any $\mu \in \mathbb{R}, \nu \in \mathbb{R}$ and $t \geq 0$, one has*

$$\left\|\mathbf{E}_{\alpha,1}^{(\beta,h)}(t)\chi\right\|_{\mu,h} \leq Ct^{-\frac{\alpha}{2\beta}(\mu-\nu)}|\chi|_{\nu,h}, \quad 0 \leq \mu-\nu \leq 2\beta, \tag{64}$$

$$\left\|\mathbf{E}_{\alpha,\alpha-k}^{(\beta,h)}(t)\chi\right\|_{\mu,h} \leq Ct^{\alpha-1-k-\frac{\alpha}{2\beta}(\mu-\nu)}|\chi|_{\nu,h}, \quad 0 \leq \mu-\nu \leq 4\beta,\ k=0,1,2,\ldots, \tag{65}$$

$$\left\|\mathbf{E}_{\alpha,\alpha+\gamma-k}^{(\beta)}(t)\chi\right\|_{\mu,h} \leq Ct^{\alpha+\gamma-1-k-\frac{\alpha}{2\beta}(\mu-\nu)}|\chi|_{\nu,h}, \quad 0 \leq \mu-\nu \leq 2\beta,\ k=0,1,2,\ldots. \tag{66}$$

We demonstrate the following results which will play an important role in the error estimates of the semidiscrete approximation, see [20].

Lemma 7. *Let $0 \leq \nu \leq \mu \leq 2\beta$. Then there hold*

$$\left\|\left(\mathbf{E}_{\alpha,1}^{(\beta)}(t)-\mathbf{E}_{\alpha,1}^{(\beta,h)}(t)P_h\right)x\right\| \leq Ch^{\mu}t^{-\frac{\alpha}{2\beta}(\mu-\nu)}|x|_{\nu}, \quad \forall x \in \dot{H}^{\nu}, \tag{67}$$

$$\left\|\left(\mathbf{E}_{\alpha,\alpha}^{(\beta)}(t)-\mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t)P_h\right)x\right\| \leq Ch^{\mu}t^{\alpha-1-\frac{\alpha}{2\beta}(\mu-\nu)}|x|_{\nu}, \quad \forall x \in \dot{H}^{\nu}, \tag{68}$$

$$\left\|A^{\frac{\nu}{2}}\left(\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta)}(t)-\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t)P_h\right)\right\| \leq Ch^{\mu-\nu}t^{\alpha+\gamma-1-\frac{\alpha}{2\beta}\mu}, \quad 0 \leq \nu \leq 1. \tag{69}$$

Theorem 4. *Let $\alpha \in (0,1)$, $\beta \in (1/2,1]$, $\gamma \in [0,1]$, $\sigma \in (0,1]$, and $p \in [2,\infty)$. Let Assumptions 1–6 be satisfied and $\kappa \in [0,r]$, where r is defined in Assumption 5. Then there exists a constant C such that*

$$\begin{aligned}\|u(t)-u_h(t)\|_{L_p(\Omega;U)} \leq& Ch^{2\beta}t^{-\frac{\alpha}{2\beta}(2\beta-\kappa)}\|u_0\|_{L_p(\Omega;\dot{U}^{\kappa})} + Ch^{2\beta-\epsilon}\\ &+ Ch^{r}t^{H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}, \quad t \in (0,T].\end{aligned} \tag{70}$$

Proof. Subtracting (63) from (16), it follows that

$$\begin{aligned}\|u(t)-u_h(t)\|_{L_p(\Omega;U)} \leq& \left\|\left(\mathbf{E}_{\alpha,1}^{(\beta)}(t)-\mathbf{E}_{\alpha,1}^{(\beta,h)}(t)P_h\right)u_0\right\|_{L_p(\Omega;U)}\\ &+ \left\|\int_0^t\left(\mathbf{E}_{\alpha,\alpha}^{(\beta)}(t-\zeta)f(\zeta,u(\zeta))-\mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t-\zeta)P_hf(\zeta,u_h(\zeta))\right)\mathrm{d}\zeta\right\|_{L_p(\Omega;U)}\\ &+ \left\|\int_0^t\left(\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta)}(t-\zeta)-\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t-\zeta)P_h\right)\mathrm{d}W^H(\zeta)\right\|_{L_p(\Omega;U)} := I_1+I_2+I_3.\end{aligned} \tag{71}$$

The first term I_1 is estimated by using (67) and Assumption 6

$$I_1 = \left\|\left(\mathbf{E}_{\alpha,1}^{(\beta)}(t)-\mathbf{E}_{\alpha,1}^{(\beta,h)}(t)P_h\right)u_0\right\|_{L_p(\Omega;U)} \leq Ch^{2\beta}t^{-\frac{\alpha}{2\beta}(2\beta-\kappa)}\|u_0\|_{L_p(\Omega;\dot{U}^{\kappa})}. \tag{72}$$

For the term I_2, we apply (68) in Lemma 7, (3) in Assumption 5 and (5) to derive

$$\begin{aligned}I_2 =& \left\|\int_0^t\left(\mathbf{E}_{\alpha,\alpha}^{(\beta)}(t-\zeta)f(\zeta,u(\zeta))-\mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t-\zeta)P_hf(\zeta,u_h(\zeta))\right)\mathrm{d}\zeta\right\|_{L_p(\Omega;U)}\\ \leq& \left\|\int_0^t\left(\mathbf{E}_{\alpha,\alpha}^{(\beta)}(t-\zeta)-\mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t-\zeta)P_h\right)f(\zeta,u(\zeta))\mathrm{d}\zeta\right\|_{L_p(\Omega;U)}\\ &+ \left\|\int_0^t\mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t-\zeta)P_h\left(f(\zeta,u(\zeta))-f(\zeta,u_h(\zeta))\right)\mathrm{d}\zeta\right\|_{L_p(\Omega;U)}\\ \leq& Ch^{2\beta-\epsilon}\int_0^t(t-\zeta)^{\alpha-1-\frac{\alpha}{2\beta}(2\beta-\epsilon)}\|f(\zeta,u(\zeta))\|_{L_p(\Omega;U)}\mathrm{d}\zeta\\ &+ C\int_0^t(t-\zeta)^{\alpha-1}\|f(\zeta,u(\zeta))-f(\zeta,u_h(\zeta))\|_{L_p(\Omega;U)}\mathrm{d}\zeta\\ \leq& C\left(1+\sup_{t\in[0,T]}\|u(t)\|_{L_p(\Omega;U)}\right)Ch^{2\beta-\epsilon}\int_0^t(t-\zeta)^{-1+\epsilon}\mathrm{d}\zeta\\ &+ C\int_0^t(t-\zeta)^{\alpha-1}\|u(\zeta)-u_h(\zeta)\|_{L_p(\Omega;U)}\mathrm{d}\zeta\\ \leq& Ch^{2\beta-\epsilon}+C\int_0^t(t-\zeta)^{\alpha-1}\|u(\zeta)-u_h(\zeta)\|_{L_p(\Omega;U)}\mathrm{d}\zeta.\end{aligned} \tag{73}$$

It remains to estimate the term I_3, which is similar to the proof of Lemma 4. In fact, using stochastic integral (14), Lemma 3, Assumption 4, and (69) ($\mu = r-\sigma+1, \nu = 1-\sigma$) in Lemma 7 we obtain

$$\begin{aligned}I_3 =& \left\|\int_0^t\left(\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta)}(t-\zeta)-\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t-\zeta)P_h\right)\mathrm{d}W^H(\zeta)\right\|_{L_p(\Omega;U)}\\ =& \left\|\int_0^t K_{H,t}^{*}\left(\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta)}(t-\cdot)-\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t-\cdot)P_h\right)(\zeta)\mathrm{d}W(\zeta)\right\|_{L_p(\Omega;U)}\end{aligned}$$

$$\leq C\Big\|\Big(\int_0^t \Big\|K^*_{H,t}\Big(A^{\frac{1-\sigma}{2}}\big(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(t-\cdot)-\mathbf{E}^{(\beta,h)}_{\alpha,\alpha+\gamma}(t-\cdot)P_h\big)\Big)(\zeta)\Big\|^2 \mathrm{d}\zeta\Big)^{\frac{1}{2}}\Big\|_{L_p(\Omega;\mathbb{R})}$$

$$\leq Ch^r\Big\|\Big(\int_0^t \Big\|K^*_{H,t}(t-\cdot)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}(\zeta)\Big\|^2 \mathrm{d}\zeta\Big)^{\frac{1}{2}}\Big\|_{L_p(\Omega;\mathbb{R})}.$$

Note that

$$\int_0^t \Big\|K^*_{H,t}(t-\cdot)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}(\zeta)\Big\|^2 \mathrm{d}\zeta \leq \int_0^t \Big\|K_H(t,\zeta)(t-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}\Big\|^2 \mathrm{d}\zeta$$
$$+ \int_0^t \Big\|\int_\zeta^t \Big((t-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)} - (t-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}\Big)\frac{\partial K_H(w,\zeta)}{\partial w}\mathrm{d}w\Big\|^2 \mathrm{d}\zeta$$
$$:= I_3^1 + I_3^2.$$

In the following, we separately consider cases $H \in (0,1/2)$ and $H \in (1/2,1)$. For the case $H \in (0,1/2)$, I_3^1 can be estimated by

$$I_3^1 = \int_0^t \Big\|K_H(t,\zeta)(t-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}\Big\|^2 \mathrm{d}\zeta$$
$$\leq 2b_H^2\int_0^t \Big\|\big(\frac{t}{\zeta}\big)^{H-\frac{1}{2}}(t-\zeta)^{H-\frac{1}{2}}(t-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}\Big\|^2 \mathrm{d}\zeta$$
$$+ 2b_H^2(\frac{1}{2}-H)^2\int_0^t \Big\|\zeta^{\frac{1}{2}-H}\Big(\int_\zeta^t w^{H-\frac{3}{2}}(w-\zeta)^{H-\frac{1}{2}}\mathrm{d}w\Big)(t-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}\Big\|^2 \mathrm{d}\zeta$$
$$\leq Ct^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}.$$

Meanwhile, one finds

$$(t-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)} - (t-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}$$
$$= \int_\zeta^w (t-v)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}(t-v)^{-\frac{1}{2}}\mathrm{d}v.$$

Analogue to the discussion of (28) and (30) we thus arrive at

$$I_3^2 = \int_0^t \Big\|\int_\zeta^t \Big((t-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)} - (t-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}\Big)$$
$$\times \frac{\partial K_H(w,\zeta)}{\partial w}\mathrm{d}w\Big\|^2 \mathrm{d}\zeta \leq Ct^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}.$$

For the case $H \in (1/2,1)$, it is evident that

$$I_3^1 = \int_0^t \Big\|K_H(t,\zeta)(t-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}\Big\|^2 \mathrm{d}\zeta$$
$$\leq c_H^2\int_0^t \Big(\int_\zeta^t \big(\frac{w}{\zeta}\big)^{H-\frac{1}{2}}(w-\zeta)^{H-\frac{3}{2}}\mathrm{d}w\Big)^2 (t-\zeta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}\mathrm{d}\zeta$$
$$= Ct^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)},$$

and

$$I_3^2 = \int_0^t \Big\|\int_\zeta^t \Big((t-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)} - (t-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}\Big)$$
$$\times \frac{\partial K_H(w,\zeta)}{\partial w}\mathrm{d}w\Big\|^2 \mathrm{d}\zeta \leq Ct^{2H+2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}.$$

Together these estimates we deduce that

$$I_3 \leq Ch^r t^{H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}. \tag{74}$$

A combination of the estimates (72)–(74) yields

$$\|u(t)-u_h(t)\|_{L_p(\Omega;U)} \leq Ch^{2\beta}t^{-\frac{\alpha}{2\beta}(2\beta-\kappa)}\|u_0\|_{L_p(\Omega;\dot{U}^\kappa)}$$
$$+ Ch^{2\beta-\epsilon} + Ch^r t^{H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}$$
$$+ C\int_0^t (t-\zeta)^{\alpha-1}\|u(\zeta)-u_h(\zeta)\|_{L_p(\Omega;U)}\mathrm{d}\zeta. \tag{75}$$

Applying the Gronwall inequality ([14], Lemma A.2.) to the above inequality which completes the proof of the theorem. □

6. Fully discrete scheme

Let $0 = t_0 < t_1 < \cdots < t_N$ be the uniform partition on the time interval $[0, T]$ and denote by $t_n = n\tau$ with $\tau = T/N$ and $0 \le n \le N$. Let $U_h^n \approx u_h(t_n)$ be the approximation of $u_h(t_n)$. Let $f^n = f(t_{n-1}, U_h^{n-1})$ and $f^0 = 0$. Let $g^k = \frac{W^H(t_k) - W^H(t_{k-1})}{\tau}$, $k = 1, 2, \ldots, N$ and $g^0 = 0$. Then, the fully discrete scheme for (1) is to seek $U_h^n, n = 1, 2, \ldots, N$ such that

$$\begin{aligned} &\tau^{-\alpha} \sum_{k=0}^{n} \omega_{n-k}^{(\alpha)} (U_h^k - U_h^0) + A_h^\beta U_h^n = P_h f^n + \tau^\gamma \sum_{k=0}^{n} \omega_{n-k}^{(-\gamma)} P_h g^k, \\ &U_h^0 = P_h u_0, \end{aligned} \tag{76}$$

where the weights $\omega_n^{(\alpha)}$ and $\omega_n^{(-\gamma)}$, $n = 0, 1, 2, \ldots$, are generated by

$$(1 - z)^\alpha = \sum_{n=0}^{\infty} \omega_n^{(\alpha)} z^n, \quad (1 - z)^{-\gamma} = \sum_{n=0}^{\infty} \omega_n^{(-\gamma)} z^n.$$

Let $\widetilde{U}_h^n$ denote the solution of the homogeneous problems of (76), that is,

$$\begin{aligned} &\tau^{-\alpha} \sum_{k=0}^{n} \omega_{n-k}^{(\alpha)} (\widetilde{U}_h^k - \widetilde{U}_h^0) + A_h^\beta \widetilde{U}_h^n = 0, \\ &\widetilde{U}_h^0 = P_h u_0. \end{aligned} \tag{77}$$

For error estimate of the fully discrete scheme (77), we can refer to Theorem 3.5 in [46] to obtain the following result.

Lemma 8. *Let $u(t_n)$ and $\widetilde{U}_h^n$ be the solution of homogeneous problem of (1) and its fully discrete approximation (77), respectively. Then there holds for $0 \le q \le 2\beta$,*

$$\|u(t_n) - \widetilde{U}_h^n\| \le C\left(t_n^{-1+\frac{\alpha}{2\beta}q}\tau + t_n^{-\frac{\alpha}{2\beta}(2\beta-q)} h^{2\beta}\right) |u_0|_q. \tag{78}$$

Applying the discrete Laplace transform on both sides of (76) we can obtain

$$U_h^n = \mathcal{E}_h^n P_h u_0 + \tau \sum_{k=0}^{n-1} R_{n-k} P_h f^{k+1} + \tau \sum_{k=0}^{n-1} Q_{n-k} P_h g^{k+1}, \; n \ge 1, \tag{79}$$

where $\mathcal{E}_h^n P_h u_0 = \widetilde{U}_h^n$, and

$$\sum_{n=0}^{\infty} R_n z^n = I + z\tau^{-1} \left(\tau^{-\alpha}(1 - z)^\alpha + A_h^\beta\right)^{-1}, \tag{80}$$

$$\sum_{n=0}^{\infty} Q_n z^n = I + z\tau^{-1} \left(\tau^{-\alpha}(1 - z)^\alpha + A_h^\beta\right)^{-1} \tau^\gamma (1 - z)^{-\gamma}. \tag{81}$$

For the aforementioned R_n and Q_n, it is easy to prove the following two lemmas, or see Lemmas 4.4 and 4.6 in [16].

Lemma 9. *For any $0 \le s \le 1$, it holds*

$$\left\| A_h^{\frac{s}{2}} R_n \right\| \le t_{n+1}^{\alpha - 1 - \frac{\alpha}{2\beta}s} \quad \text{and} \quad \left\| A_h^{\frac{s}{2}} \left(\mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t_n) - R_n\right) P_h \right\| \le C t_{n+1}^{\alpha - 2 - \frac{\alpha}{2\beta}s} \tau.$$

Lemma 10. *For any $0 \le s \le 1$, it holds*

$$\left\| A_h^{\frac{s}{2}} Q_n \right\| \le t_{n+1}^{\alpha + \gamma - 1 - \frac{\alpha}{2\beta}s} \quad \text{and} \quad \left\| A_h^{\frac{s}{2}} \left(\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t_n) - Q_n\right) P_h \right\| \le C t_{n+1}^{\alpha + \gamma - 2 - \frac{\alpha}{2\beta}s} \tau.$$

As in the continuous case, the semidiscrete solution also satisfies the same regularity properties in time and space, and their proofs are analogue to Theorems 2 and 3.

Lemma 11. *Let $\alpha \in (0, 1)$, $\beta \in (1/2, 1]$, $\gamma \in [0, 1]$, $\sigma \in (0, 1]$, and $p \in [2, \infty)$. Let Assumptions 1–6 be satisfied. Then the mild solution $u_h(t)$ of the semidiscrete problem (62) satisfies for any $t \in [0, T]$*

$$\sup_{t \in [0,T]} \|u_h(t)\|_{L_p(\Omega; \dot{U}^r)} < \infty,$$

where r is defined in Assumption 5.

Lemma 12. *Let $\alpha \in (0, 1)$, $\beta \in (1/2, 1]$, $\gamma \in [0, 1]$, $\sigma \in (0, 1]$, and $p \in [2, \infty)$. Let Assumptions 1–6 be satisfied. Then the mild solution $u_h(t)$ of the semidiscrete problem (62) satisfies*

$$\|u_h(t_2) - u_h(t_1)\|_{L_p(\Omega; U)} < C(t_2 - t_1)^{\min\{\alpha, \frac{\alpha r}{2\beta}\}} \tag{82}$$

for all $0 \le t_1 < t_2 \le T$. Here r is defined in Assumption 5.

Finally, we give error estimate of the fully discrete scheme (76) in $L_p(\Omega;U)$ with $p \geq 2$, which is our main result in this section.

Theorem 5. *Let $\alpha \in (0,1)$, $\beta \in (1/2,1]$, $\gamma \in [0,1]$, $\sigma \in (0,1]$, and $p \in [2,\infty)$. Let Assumptions 1–6 be satisfied and $\kappa \in [0,r]$, where r is defined in Assumption 5. Let $u(t_n)$ and U_h^n be the solutions of (1) and (76), respectively.*

1. *If $H \in (0,1/2)$, then*

$$\begin{aligned}\|u(t_n)-U_h^n\|_{L_p(\Omega;U)} \leq & Ch^{2\beta}t_n^{-\frac{\alpha}{2\beta}(2\beta-\kappa)}\|u_0\|_{L_p(\Omega;\dot{U}^\kappa)} + Ct_n^{-1+\frac{\alpha}{2\beta}\kappa}\tau\|u_0\|_{L_p(\Omega;\dot{U}^\kappa)} \\ & + Ch^{2\beta-\epsilon} + Ch^r t_n^{H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)} + C\tau^{\min\{\alpha,\frac{\alpha r}{2\beta},\frac{1}{2}\}};\end{aligned} \tag{83}$$

2. *If $H \in (1/2,1)$, then*

$$\begin{aligned}\|u(t_n)-U_h^n\|_{L_p(\Omega;U)} \leq & Ch^{2\beta}t_n^{-\frac{\alpha}{2\beta}(2\beta-\kappa)}\|u_0\|_{L_p(\Omega;\dot{U}^\kappa)} + Ct_n^{-1+\frac{\alpha}{2\beta}\kappa}\tau\|u_0\|_{L_p(\Omega;\dot{U}^\kappa)} \\ & + Ch^{2\beta-\epsilon} + Ch^r t_n^{H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)} + C\tau^{\min\{\alpha,\frac{\alpha r}{2\beta}\}}.\end{aligned} \tag{84}$$

Proof. We insert $u_h(t_n)$ and split the error into

$$u(t_n)-U_h^n = u(t_n)-u_h(t_n)+u_h(t_n)-U_h^n.$$

By using Theorem 4 one get

$$\begin{aligned}\|u(t_n)-u_h(t_n)\|_{L_p(\Omega;U)} \leq & Ch^{2\beta}t_n^{-\frac{\alpha}{2\beta}(2\beta-\kappa)}\|u_0\|_{L_p(\Omega;\dot{U}^\kappa)} + Ch^{2\beta-\epsilon} \\ & + Ch^r t_n^{H+\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}.\end{aligned} \tag{85}$$

Subtracting (76) from (63), we derive

$$\begin{aligned}\|u_h(t_n)-U_h^n\|_{L_p(\Omega;U)} \leq & \left\|\left(\mathbf{E}_{\alpha,1}^{(\beta,h)}(t_n)-\mathcal{E}_h^n\right)P_hu_0\right\|_{L_p(\Omega;U)} \\ & + \left\|\int_0^{t_n}\mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t_n-\zeta)P_hf(\zeta,u_h(\zeta))\mathrm{d}\zeta - \tau\sum_{k=0}^{n-1}R_{n-k}P_hf^{k+1}\right\|_{L_p(\Omega;U)} \\ & + \left\|\int_0^{t_n}\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t_n-\zeta)P_h\mathrm{d}W^H(\zeta) - \tau\sum_{k=0}^{n-1}Q_{n-k}P_hg^{k+1}\right\|_{L_p(\Omega;U)} := I+II+III.\end{aligned} \tag{86}$$

The first term I in (86) is bounded by Lemma 8

$$I = \left\|\left(\mathbf{E}_{\alpha,1}^{(\beta,h)}(t_n)-\mathcal{E}_h^n\right)P_hu_0\right\|_{L_p(\Omega;U)} \leq Ct_n^{-1+\frac{\alpha}{2\beta}\kappa}\tau\|u_0\|_{L_p(\Omega;\dot{U}^\kappa)}. \tag{87}$$

The second term II in (86) is dominated by five additional terms as follows

$$\begin{aligned}II =& \left\|\int_0^{t_n}\mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t_n-\zeta)P_hf(\zeta,u_h(\zeta))\mathrm{d}\zeta - \tau\sum_{k=0}^{n-1}R_{n-k}P_hf^{k+1}\right\|_{L_p(\Omega;U)} \\ =& \left\|\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t_n-\zeta)P_hf(\zeta,u_h(\zeta)) - R_{n-k+1}P_hf(t_{k-1},U_h^{k-1})\right)\mathrm{d}\zeta\right\|_{L_p(\Omega;U)} \\ \leq & \left\|\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t_n-\zeta)P_h\left(f(\zeta,u_h(\zeta))-f(\zeta,u_h(t_{k-1}))\right)\mathrm{d}\zeta\right\|_{L_p(\Omega;U)} \\ & + \left\|\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t_n-\zeta)P_h\left(f(\zeta,u_h(t_{k-1}))-f(t_{k-1},u_h(t_{k-1}))\right)\mathrm{d}\zeta\right\|_{L_p(\Omega;U)} \\ & + \left\|\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t_n-\zeta)-\mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t_n-t_{k-1})\right)P_hf(t_{k-1},u_h(t_{k-1}))\mathrm{d}\zeta\right\|_{L_p(\Omega;U)} \\ & + \left\|\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t_n-t_{k-1})-R_{n-k+1}\right)P_hf(t_{k-1},u_h(t_{k-1}))\mathrm{d}\zeta\right\|_{L_p(\Omega;U)} \\ & + \left\|\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}R_{n-k+1}P_h\left(f(t_{k-1},u_h(t_{k-1}))-f(t_{k-1},U_h^{k-1})\right)\mathrm{d}\zeta\right\|_{L_p(\Omega;U)} \\ :=& II_1+II_2+II_3+II_4+II_5.\end{aligned} \tag{88}$$

Now we estimate each term separately. For II_1, applying (65) ($\mu = 0, \nu = 0$) in Lemma 6, (3) in Assumption 5 and Lemma 12 lead to

$$\begin{aligned} II_1 \le & \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left\| \mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t_n-\zeta) P_h (f(\zeta, u_h(\zeta)) - f(\zeta, u_h(t_{k-1}))) \right\|_{L_p(\Omega;U)} \mathrm{d}\zeta \\ \le & \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n-\zeta)^{\alpha-1} \| u_h(\zeta) - u_h(t_{k-1}) \|_{L_p(\Omega;U)} \mathrm{d}\zeta \\ \le & C \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n-\zeta)^{\alpha-1} (\zeta - t_{k-1})^{\frac{\alpha r}{2\beta}} \mathrm{d}\zeta \le C\tau^{\frac{\alpha r}{2\beta}} \int_0^{t_n} (t_n-\zeta)^{\alpha-1} \mathrm{d}\zeta \le C\tau^{\frac{\alpha r}{2\beta}}. \end{aligned} \tag{89}$$

The term II_2 is bounded by using (65) ($\mu = 0, \nu = 0$) in Lemma 6 and (4) in Assumption 5

$$\begin{aligned} II_2 \le & \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left\| \mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t_n-\zeta) P_h \left(f(\zeta, u_h(t_{k-1})) - f(t_{k-1}, u_h(t_{k-1})) \right) \right\|_{L_p(\Omega;U)} \mathrm{d}\zeta \\ \le & C \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n-\zeta)^{\alpha-1} (1 + \| u_h(t_{k-1}) \|_{L_p(\Omega;U)}) (\zeta - t_{k-1})^{\frac{\alpha r}{2\beta}} \mathrm{d}\zeta \\ \le & C(1 + \sup_{t\in[0,T]} \| u_h(t) \|_{L_p(\Omega;U)}) \tau^{\frac{\alpha r}{2\beta}} \int_0^{t_n} (t_n-\zeta)^{\alpha-1} \mathrm{d}\zeta \\ = & \frac{C}{\alpha} t_n^{\alpha} (1 + \sup_{t\in[0,T]} \| u_h(t) \|_{L_p(\Omega;U)}) \tau^{\frac{\alpha r}{2\beta}} \le C\tau^{\frac{\alpha r}{2\beta}}. \end{aligned} \tag{90}$$

For the estimate of II_3 we use (65) ($\mu = 0, \nu = 0$) in Lemma 6, the linear growth condition (5) and the elementary inequality (26) to get

$$\begin{aligned} II_3 \le & \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left\| \int_{t_n - t_{k-1}}^{t_n - \zeta} \mathbf{E}_{\alpha,\alpha-1}^{(\beta,h)}(\theta) \mathrm{d}\theta P_h f(t_{k-1}, u_h(t_{k-1})) \right\|_{L_p(\Omega;U)} \mathrm{d}\zeta \\ \le & C(1 + \sup_{t\in[0,T]} \| u_h(t) \|_{L_p(\Omega;U)}) \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{\zeta} (t_n - w)^{\alpha-2} \mathrm{d}w \mathrm{d}\zeta \\ \le & C(1 + \sup_{t\in[0,T]} \| u_h(t) \|_{L_p(\Omega;U)}) \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - \zeta)^{\alpha-1-\frac{\alpha r}{2\beta}} \int_{t_{k-1}}^{\zeta} (t_n - w)^{-1+\frac{\alpha r}{2\beta}} \mathrm{d}w \mathrm{d}\zeta \\ \le & C(1 + \sup_{t\in[0,T]} \| u_h(t) \|_{L_p(\Omega;U)}) \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - \zeta)^{\alpha-1-\frac{\alpha r}{2\beta}} (\zeta - t_{k-1})^{\frac{\alpha r}{2\beta}} \mathrm{d}\zeta \\ \le & \frac{C}{\alpha - \frac{\alpha r}{2\beta}} t_n^{\alpha - \frac{\alpha r}{2\beta}} (1 + \sup_{t\in[0,T]} \| u_h(t) \|_{L_p(\Omega;U)}) \tau^{\frac{\alpha r}{2\beta}} \le C\tau^{\frac{\alpha r}{2\beta}}. \end{aligned} \tag{91}$$

For II_4, Lemma 9 and the linear growth condition (5) yields, noting $\alpha > \frac{\alpha r}{2\beta}$,

$$\begin{aligned} II_4 \le & \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left\| \left(\mathbf{E}_{\alpha,\alpha}^{(\beta,h)}(t_n - t_{k-1}) - R_{n-k+1} \right) P_h f(t_{k-1}, u_h(t_{k-1})) \right\|_{L_p(\Omega;U)} \mathrm{d}\zeta \\ \le & C(1 + \sup_{t\in[0,T]} \| u_h(t) \|_{L_p(\Omega;U)}) \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} t_{n-k+1}^{\alpha-2} \tau \mathrm{d}\zeta \\ \le & C(1 + \sup_{t\in[0,T]} \| u_h(t) \|_{L_p(\Omega;U)}) \Big(\sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} t_{n-k+1}^{\alpha-2} \tau \mathrm{d}\zeta + \int_{t_{n-1}}^{t_n} \tau^{\alpha-2} \tau \mathrm{d}\zeta \Big) \\ \le & C(1 + \sup_{t\in[0,T]} \| u_h(t) \|_{L_p(\Omega;U)}) \Big(\int_{\tau}^{t_n} \zeta^{\alpha-2} \tau \mathrm{d}\zeta + \tau^{\alpha} \Big) \\ \le & C(1 + \sup_{t\in[0,T]} \| u_h(t) \|_{L_p(\Omega;U)}) \tau^{\alpha} \le C\tau^{\alpha}. \end{aligned} \tag{92}$$

In view of Lemma 9 and (3) in Assumption 5, the final term II_5 can be estimated by

$$\begin{aligned} II_5 \le & \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left\| R_{n-k+1} P_h \big(f(t_{k-1}, u_h(t_{k-1})) - f(t_{k-1}, U_h^{k-1}) \big) \right\|_{L_p(\Omega;U)} \mathrm{d}\zeta \\ \le & C \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} t_{n-k+1}^{\alpha-1} \mathrm{d}\zeta \left\| u_h(t_{k-1}) - U_h^{k-1} \right\|_{L_p(\Omega;U)} \\ \le & C\tau \sum_{k=1}^{n} t_{n-k+1}^{\alpha-1} \left\| u_h(t_{k-1}) - U_h^{k-1} \right\|_{L_p(\Omega;U)}. \end{aligned} \tag{93}$$

To summarize our estimates (89)–(93) we show that

$$II \le C\tau^{\frac{\alpha r}{2\beta}} + C\tau^{\alpha} + C\tau \sum_{k=1}^{n} t_{n-k+1}^{\alpha-1} \left\| u_h(t_{k-1}) - U_h^{k-1} \right\|_{L_p(\Omega;U)}. \tag{94}$$

It remains to estimate the term III. Let

$$Q_{n-k+1}(\zeta) = Q_{n-k+1},\ \zeta \in (t_{k-1}, t_k].$$

It is clear that

$$\begin{aligned}
III &= \Big\| \int_0^{t_n} \mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t_n-\zeta) P_h \mathrm{d}W^H(\zeta) - \tau \sum_{k=0}^{n-1} Q_{n-k} P_h g^{k+1} \Big\|_{L_p(\Omega;U)} \\
&= \Big\| \int_0^{t_n} \mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t_n-\zeta) P_h \mathrm{d}W^H(\zeta) - \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} Q_{n-k+1}(\zeta) P_h \mathrm{d}W^H(\zeta) \Big\|_{L_p(\Omega;U)} \\
&= \Big\| \int_0^{t_n} \big(\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t_n-\zeta) - Q_{n-k+1}(\zeta) \big) P_h \mathrm{d}W^H(\zeta) \Big\|_{L_p(\Omega;U)} \\
&= \Big\| \int_0^{t_n} K_{H,t_n}^{*} \big(\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t_n-\cdot) - Q_{n-k+1}(\cdot) \big) P_h(\zeta) \mathrm{d}W(\zeta) \Big\|_{L_p(\Omega;U)} \\
&\le C \Big\| \Big(\int_0^{t_n} \Big\| K_{H,t_n}^{*} \big[A_h^{\frac{1-\sigma}{2}} \big(\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t_n-\cdot) - Q_{n-k+1}(\cdot) \big) P_h \big](\zeta) \Big\|^2 \mathrm{d}\zeta \Big)^{\frac{1}{2}} \Big\|_{L_p(\Omega;\mathbb{R})} \\
&= C \Big\| \Big(\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big\| K_{H,t_n}^{*} \big[A_h^{\frac{1-\sigma}{2}} \big(\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t_n-\cdot) - Q_{n-k+1} \big) P_h \big](\zeta) \Big\|^2 \mathrm{d}\zeta \Big)^{\frac{1}{2}} \Big\|_{L_p(\Omega;\mathbb{R})} \\
&\le C \Big\| \Big(\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big\| K_{H,t_n}^{*} \big[A_h^{\frac{1-\sigma}{2}} \big(\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t_n-\cdot) - \mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t_n-t_{k-1}) \big) P_h \big](\zeta) \Big\|^2 \mathrm{d}\zeta \Big)^{\frac{1}{2}} \Big\|_{L_p(\Omega;\mathbb{R})} \\
&\quad + C \Big\| \Big(\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big\| K_{H,t_n}^{*} \big[A_h^{\frac{1-\sigma}{2}} \big(\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t_n-t_{k-1}) - Q_{n-k+1} \big) P_h \big](\zeta) \Big\|^2 \mathrm{d}\zeta \Big)^{\frac{1}{2}} \Big\|_{L_p(\Omega;\mathbb{R})} \\
&:= III_1 + III_2.
\end{aligned} \tag{95}$$

In the following, we treat III_1 and III_2, respectively. Indeed, the estimate of III_1 works similarly as Lemma 5 and III_2 as Lemma 4. Denote

$$\Psi(t_n, t_{k-1}, \zeta) = \mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t_n-\zeta) - \mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t_n-t_{k-1}).$$

Then III_1 is transformed into

$$\begin{aligned}
III_1 &= C \Big\| \Big(\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big\| K_{H,t_n}^{*} \big(A_h^{\frac{1-\sigma}{2}} \Psi(t_n, t_{k-1}, \cdot) \big) P_h(\zeta) \Big\|^2 \mathrm{d}\zeta \Big)^{\frac{1}{2}} \Big\|_{L_p(\Omega;\mathbb{R})} \\
&\le C \Big\| \Big(\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big\| K_H(t_n,\zeta) A_h^{\frac{1-\sigma}{2}} \Psi(t_n, t_{k-1}, \zeta) \Big\|^2 \mathrm{d}\zeta \Big)^{\frac{1}{2}} \Big\|_{L_p(\Omega;\mathbb{R})} \\
&\quad + C \Big\| \Big(\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big\| \int_{\zeta}^{t_k} A_h^{\frac{1-\sigma}{2}} (\Psi(t_n, t_{k-1}, w) - \Psi(t_n, t_{k-1}, \zeta)) \frac{\partial K_H(w,\zeta)}{\partial w} \mathrm{d}w \Big\|^2 \mathrm{d}\zeta \Big)^{\frac{1}{2}} \Big\|_{L_p(\Omega;\mathbb{R})} \\
&:= III_1^1 + III_1^2.
\end{aligned} \tag{96}$$

We separately consider cases $H \in (0, 1/2)$ and $H \in (1/2, 1)$ for the term III_1.

Case 1: $H \in (0, 1/2)$. Note that

$$\begin{aligned}
&\Big\| A_h^{\frac{1-\sigma}{2}} \Psi(t_n, t_{k-1}, \zeta) \Big\| = \Big\| A_h^{\frac{1-\sigma}{2}} \big(\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t_n-\zeta) - \mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta,h)}(t_n-t_{k-1}) \big) \Big\| \\
&= \Big\| A_h^{\frac{1-\sigma}{2}} \int_{t_n-t_{k-1}}^{t_n-\zeta} \mathbf{E}_{\alpha,\alpha+\gamma-1}^{(\beta,h)}(\theta) \mathrm{d}\theta \Big\| \\
&\le C \int_{t_{k-1}}^{\zeta} (t_n-v)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)} (t_n-v)^{\frac{\alpha r}{2\beta}-1} \mathrm{d}v.
\end{aligned}$$

If $\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)<0$, then

$$\left\|A_h^{\frac{1-\sigma}{2}}\Psi(t_n,t_{k-1},\zeta)\right\|\le C(\zeta-t_{k-1})^{\frac{\alpha r}{2\beta}}(t_n-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)},$$

and if $\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)\ge 0$, then

$$\left\|A_h^{\frac{1-\sigma}{2}}\Psi(t_n,t_{k-1},\zeta)\right\|\le C(\zeta-t_{k-1})^{\frac{\alpha r}{2\beta}}(t_n-t_{k-1})^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)}.$$

Hence, in two cases, III_1^1 can be evaluated by

$$III_1^1=C\Big\|\Big(\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left\|K_H(t_n,\zeta)A_h^{\frac{1-\sigma}{2}}\Psi(t_n,t_{k-1},\zeta)\right\|^2\mathrm{d}\zeta\Big)^{\frac{1}{2}}\Big\|_{L_p(\Omega;\mathbb{R})}\le C\tau^{\frac{\alpha r}{2\beta}}. \tag{97}$$

Further, owing to the fact

$$\begin{aligned}
&\left\|A_h^{\frac{1-\sigma}{2}}(\Psi(t_n,t_{k-1},w)-\Psi(t_n,t_{k-1},\zeta))\right\|\\
=&\left\|A_h^{\frac{1-\sigma}{2}}\left(\mathbf{E}^{(\beta,h)}_{\alpha,\alpha+\gamma}(t_n-w)-\mathbf{E}^{(\beta,h)}_{\alpha,\alpha+\gamma}(t_n-t_{k-1})-\mathbf{E}^{(\beta,h)}_{\alpha,\alpha+\gamma}(t_n-\zeta)+\mathbf{E}^{(\beta,h)}_{\alpha,\alpha+\gamma}(t_n-t_{k-1})\right)\right\|\\
=&\left\|A_h^{\frac{1-\sigma}{2}}\left(\mathbf{E}^{(\beta,h)}_{\alpha,\alpha+\gamma}(t_n-w)-\mathbf{E}^{(\beta,h)}_{\alpha,\alpha+\gamma}(t_n-\zeta)\right)\right\|\le C\int_{\zeta}^{w}(t_n-v)^{\alpha+\gamma-2-\frac{\alpha}{2\beta}(1-\sigma)}\mathrm{d}v\\
=&C\int_{\zeta}^{w}(t_n-v)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r+1-\sigma)-\frac{1}{2}}(t_n-v)^{\frac{\alpha r}{2\beta}-\frac{1}{2}}\mathrm{d}v.
\end{aligned}$$

We need to investigate the cases $\frac{\alpha r}{2\beta}\le 1/2$ and $\frac{\alpha r}{2\beta}>1/2$, respectively.

When $\frac{\alpha r}{2\beta}\le 1/2$, one has

$$\left\|A_h^{\frac{1-\sigma}{2}}(\Psi(t_n,t_{k-1},w)-\Psi(t_n,t_{k-1},\zeta))\right\|\le C\tau^{\frac{\alpha r}{2\beta}}(t_n-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}(w-\zeta)^{\frac{1}{2}}$$

if $\alpha+\gamma-1-\frac{\alpha}{2\beta}(r+1-\sigma)-1/2<0$, and

$$\left\|A_h^{\frac{1-\sigma}{2}}(\Psi(t_n,t_{k-1},w)-\Psi(t_n,t_{k-1},\zeta))\right\|\le C\tau^{\frac{\alpha r}{2\beta}}(t_n-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}(w-\zeta)^{\frac{1}{2}},$$

if $\alpha+\gamma-1-\frac{\alpha}{2\beta}(r+1-\sigma)-1/2\ge 0$.

Consequently, for the case $\frac{\alpha r}{2\beta}\le 1/2$, III_1^2 is estimated as

$$\begin{aligned}
III_1^2=&C\Big\|\Big(\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\Big\|\int_{\zeta}^{t_k}A_h^{\frac{1-\sigma}{2}}(\Psi(t_n,t_{k-1},w)-\Psi(t_n,t_{k-1},\zeta))\\
&\times\frac{\partial K_H(w,\zeta)}{\partial w}\mathrm{d}w\Big\|^2\mathrm{d}\zeta\Big)^{\frac{1}{2}}\Big\|_{L_p(\Omega;\mathbb{R})}\le C\tau^{\frac{\alpha r}{2\beta}}.
\end{aligned} \tag{98}$$

When $\frac{\alpha r}{2\beta}>1/2$, it holds

$$\begin{aligned}
&\left\|A_h^{\frac{1-\sigma}{2}}(\Psi(t_n,t_{k-1},w)-\Psi(t_n,t_{k-1},\zeta))\right\|\\
\le&C\int_{\zeta}^{w}(t_n-v)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r+1-\sigma)-\frac{1}{2}}(t_n-v)^{\frac{\alpha r}{2\beta}-\frac{1}{2}}\mathrm{d}v\\
\le&C(t_n-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}\int_{\zeta}^{w}(t_n-v)^{\frac{\alpha r}{2\beta}-\frac{1}{2}}\mathrm{d}v\\
\le&C(t_n-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}(t_n-\zeta)^{\frac{\alpha r}{2\beta}-\frac{1}{2}}(w-\zeta)\\
=&Ct_n^{\frac{\alpha r}{2\beta}-\frac{1}{2}}(w-\zeta)^{\frac{1}{2}}(t_n-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}(w-\zeta)^{\frac{1}{2}}\\
\le&CT^{\frac{\alpha r}{2\beta}-\frac{1}{2}}\tau^{\frac{1}{2}}(t_n-w)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}(w-\zeta)^{\frac{1}{2}}
\end{aligned}$$

for $\alpha+\gamma-1-\frac{\alpha}{2\beta}(r+1-\sigma)-\frac{1}{2}<0$, and

$$\begin{aligned}
&\left\|A_h^{\frac{1-\sigma}{2}}(\Psi(t_n,t_{k-1},w)-\Psi(t_n,t_{k-1},\zeta))\right\|\\
\le&CT^{\frac{\alpha r}{2\beta}-\frac{1}{2}}\tau^{\frac{1}{2}}(t_n-\zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r-\sigma+1)-\frac{1}{2}}(w-\zeta)^{\frac{1}{2}}
\end{aligned}$$

for $\alpha+\gamma-1-\frac{\alpha}{2\beta}(r+1-\sigma)-\frac{1}{2}\ge 0$.

Thus, for the case $\frac{\alpha r}{2\beta} > 1/2$, III_1^2 is estimated as

$$III_1^2 = C\Big\| \Big(\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big\| \int_{\zeta}^{t_k} A_h^{\frac{1-\sigma}{2}} (\Psi(t_n, t_{k-1}, w) - \Psi(t_n, t_{k-1}, \zeta)) \times \frac{\partial K_H(w,\zeta)}{\partial w} \mathrm{d}w \Big\|^2 \mathrm{d}\zeta \Big)^{\frac{1}{2}} \Big\|_{L_p(\Omega;\mathbb{R})} \le C\tau^{\frac{1}{2}}. \tag{99}$$

Case 2: $H \in (1/2, 1)$. In this case, III_1 can be directly estimated by

$$III_1 = C\Big\| \Big(\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big\| K^*_{H,t_n} \big(A_h^{\frac{1-\sigma}{2}} \Psi(t_n, t_{k-1}, \cdot) \big)(\zeta) \Big\|^2 \mathrm{d}\zeta \Big)^{\frac{1}{2}} \Big\|_{L_p(\Omega;\mathbb{R})} \le C\tau^{\frac{\alpha r}{2\beta}}. \tag{100}$$

Therefore, in both cases, we are able to get by (97)–(100),

$$III_1 \le \begin{cases} C\tau^{\min\{\frac{\alpha r}{2\beta}, \frac{1}{2}\}}, & \text{if } H \in (0, 1/2), \\ C\tau^{\frac{\alpha r}{2\beta}}, & \text{if } H \in (1/2, 1). \end{cases} \tag{101}$$

Next we show the estimate for III_2 as done by Lemma 4. Firstly, III_2 is turned into

$$\begin{aligned} III_2 =& C\Big\| \Big(\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big\| K^*_{H,t_n} \big[A_h^{\frac{1-\sigma}{2}} \big(\mathbf{E}^{(\beta,h)}_{\alpha,\alpha+\gamma}(t_n - t_{k-1}) - Q_{n-k+1} \big) P_h \big](\zeta) \Big\|^2 \mathrm{d}\zeta \Big)^{\frac{1}{2}} \Big\|_{L_p(\Omega;\mathbb{R})} \\ \le& C\Big\| \Big(\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big\| K_H(t_n, \zeta) A_h^{\frac{1-\sigma}{2}} \big(\mathbf{E}^{(\beta,h)}_{\alpha,\alpha+\gamma}(t_n - t_{k-1}) - Q_{n-k+1} \big) P_h \Big\|^2 \mathrm{d}\zeta \Big)^{\frac{1}{2}} \Big\|_{L_p(\Omega;\mathbb{R})} \end{aligned} \tag{102}$$

We again discuss separately cases $H \in (0, 1/2)$ and $H \in (1/2, 1)$ for III_2.

Case 1: $H \in (0, 1/2)$. In terms of Lemma 10 one has

$$\begin{aligned} & \int_{t_{k-1}}^{t_k} \Big\| K_H(t_n, \zeta) A_h^{\frac{1-\sigma}{2}} \big(\mathbf{E}^{(\beta,h)}_{\alpha,\alpha+\gamma}(t_n - t_{k-1}) - Q_{n-k+1} \big) P_h \Big\|^2 \mathrm{d}\zeta \\ \le& 2b_H^2 \int_{t_{k-1}}^{t_k} \Big\| \big(\frac{t_n}{\zeta} \big)^{H-\frac{1}{2}} (t_n - \zeta)^{H-\frac{1}{2}} A_h^{\frac{1-\sigma}{2}} \big(\mathbf{E}^{(\beta,h)}_{\alpha,\alpha+\gamma}(t_n - t_{k-1}) - Q_{n-k+1} \big) P_h \Big\|^2 \mathrm{d}\zeta \\ & + 2b_H^2 (\frac{1}{2} - H)^2 \int_{t_{k-1}}^{t_k} \Big\| \zeta^{\frac{1}{2}-H} \Big(\int_{\zeta}^{t_n} w^{H-\frac{3}{2}} (w - \zeta)^{H-\frac{1}{2}} \mathrm{d}w \Big) \\ & \times A_h^{\frac{1-\sigma}{2}} \big(\mathbf{E}^{(\beta,h)}_{\alpha,\alpha+\gamma}(t_n - t_{k-1}) - Q_{n-k+1} \big) P_h \Big\|^2 \mathrm{d}\zeta \\ \le& C \int_{t_{k-1}}^{t_k} \Big\| \big(\frac{t_n}{\zeta} \big)^{H-\frac{1}{2}} (t_n - \zeta)^{H-\frac{1}{2}} (t_n - t_{k-1})^{\alpha+\gamma-2-\frac{\alpha}{2\beta}(1-\sigma)} \tau \Big\|^2 \mathrm{d}\zeta \\ & + C \int_{t_{k-1}}^{t_k} \Big\| \zeta^{\frac{1}{2}-H} \Big(\int_{\zeta}^{t_n} w^{H-\frac{3}{2}} (w - \zeta)^{H-\frac{1}{2}} \mathrm{d}w \Big) (t_n - t_{k-1})^{\alpha+\gamma-2-\frac{\alpha}{2\beta}(1-\sigma)} \tau \Big\|^2 \mathrm{d}\zeta \\ \le& C \int_{t_{k-1}}^{t_k} \Big\| \big(\frac{t_n}{\zeta} \big)^{H-\frac{1}{2}} (t_n - \zeta)^{H-\frac{1}{2}} (t_n - \zeta)^{\alpha+\gamma-2-\frac{\alpha}{2\beta}(1-\sigma)} \tau^{1-\frac{\alpha r}{2\beta}} \tau^{\frac{\alpha r}{2\beta}} \Big\|^2 \mathrm{d}\zeta \\ & + C \int_{t_{k-1}}^{t_k} \Big\| \zeta^{\frac{1}{2}-H} \Big(\int_{\zeta}^{t_n} w^{H-\frac{3}{2}} (w - \zeta)^{H-\frac{1}{2}} \mathrm{d}w \Big) (t_n - \zeta)^{\alpha+\gamma-2-\frac{\alpha}{2\beta}(1-\sigma)} \tau^{1-\frac{\alpha r}{2\beta}} \tau^{\frac{\alpha r}{2\beta}} \Big\|^2 \mathrm{d}\zeta \\ \le& C\tau^{\frac{\alpha r}{\beta}} \int_{t_{k-1}}^{t_k} \Big\| \big(\frac{t_n}{\zeta} \big)^{H-\frac{1}{2}} (t_n - \zeta)^{H-\frac{1}{2}} (t_n - \zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r+1-\sigma)} \Big\|^2 \mathrm{d}\zeta \\ & + C\tau^{\frac{\alpha r}{\beta}} \int_{t_{k-1}}^{t_k} \Big\| \zeta^{\frac{1}{2}-H} \Big(\int_{\zeta}^{t_n} w^{H-\frac{3}{2}} (w - \zeta)^{H-\frac{1}{2}} \mathrm{d}w \Big) (t_n - \zeta)^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(r+1-\sigma)} \Big\|^2 \mathrm{d}\zeta. \end{aligned}$$

Thus there holds

$$III_2 \le C\tau^{\frac{\alpha r}{2\beta}}. \tag{103}$$

Case 2: $H \in (1/2, 1)$. By using again Lemma 10 which yields

$$\begin{aligned} III_2 \le& C\Big\| \Big(\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big\| K_H(t_n, \zeta) A_h^{\frac{1-\sigma}{2}} \big(\mathbf{E}^{(\beta,h)}_{\alpha,\alpha+\gamma}(t_n - t_{k-1}) - Q_{n-k+1} \big) P_h \Big\|^2 \mathrm{d}\zeta \Big)^{\frac{1}{2}} \Big\|_{L_p(\Omega;\mathbb{R})} \\ \le& Cc_H \Big\| \Big(\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big(\int_{\zeta}^{t_k} \big(\frac{w}{\zeta} \big)^{H-\frac{1}{2}} (w - \zeta)^{H-\frac{3}{2}} \mathrm{d}w (t_n - t_{k-1})^{\alpha+\gamma-2-\frac{\alpha}{2\beta}(1-\sigma)} \tau \Big)^2 \mathrm{d}\zeta \Big)^{\frac{1}{2}} \Big\|_{L_p(\Omega;\mathbb{R})} \end{aligned}$$

$$\leq C\tau^{\frac{\alpha r}{2\beta}}\Big(\int_0^{t_n}\big(\frac{t_n}{\zeta}\big)^{2H-1}\Big(\int_\zeta^{t_n}(w-\zeta)^{H-\frac{3}{2}}\mathrm{d}w\Big)^2(t_n-\zeta)^{2(\alpha+\gamma-1)-\frac{\alpha}{\beta}(r-\sigma+1)}\mathrm{d}\zeta\Big)^{\frac{1}{2}}$$
$$= C\tau^{\frac{\alpha r}{2\beta}}. \tag{104}$$

Therefore, from (103) and (104) one has

$$III_2 \leq C\tau^{\frac{\alpha r}{2\beta}}. \tag{105}$$

Gathering the estimates (101) and (105) for III_1 and III_2 results in

$$III \leq \begin{cases} C\tau^{\min\{\frac{\alpha r}{2\beta},\frac{1}{2}\}}, & \text{if } H\in(0,1/2),\\ C\tau^{\frac{\alpha r}{2\beta}}, & \text{if } H\in(1/2,1). \end{cases} \tag{106}$$

Together the above estimates (87), (94) and (106) for I, II and III, respectively, we conclude that

1. If $H\in(0,1/2)$, then

$$\|u_h(t_n)-U_h^n\|_{L_p(\Omega;U)} \leq Ct_n^{-1+\frac{\alpha}{2\beta}\kappa}\tau\|u_0\|_{L_p(\Omega;\dot{U}^\kappa)} + C\tau^{\min\{\alpha,\frac{\alpha r}{2\beta},\frac{1}{2}\}}$$
$$+ C\tau\sum_{k=1}^{n} t_{n-k+1}^{\alpha-1}\left\|u_h(t_{k-1})-U_h^{k-1}\right\|_{L_p(\Omega;U)}$$
$$= Ct_n^{-1+\frac{\alpha}{2\beta}\kappa}\tau\|u_0\|_{L_p(\Omega;\dot{U}^\kappa)} + C\tau^{\min\{\alpha,\frac{\alpha r}{2\beta},\frac{1}{2}\}} + C\tau\sum_{k=0}^{n-1} t_{n-k}^{\alpha-1}\left\|u_h(t_k)-U_h^k\right\|_{L_p(\Omega;U)}.$$

2. If $H\in(1/2,1)$, then

$$\|u_h(t_n)-U_h^n\|_{L_p(\Omega;U)} \leq Ct_n^{-1+\frac{\alpha}{2\beta}\kappa}\tau\|u_0\|_{L_p(\Omega;\dot{U}^\kappa)} + C\tau^{\min\{\alpha,\frac{\alpha r}{2\beta}\}}$$
$$+ C\tau\sum_{k=1}^{n} t_{n-k+1}^{\alpha-1}\left\|u_h(t_{k-1})-U_h^{k-1}\right\|_{L_p(\Omega;U)}$$
$$= Ct_n^{-1+\frac{\alpha}{2\beta}\kappa}\tau\|u_0\|_{L_p(\Omega;\dot{U}^\kappa)} + C\tau^{\min\{\alpha,\frac{\alpha r}{2\beta}\}} + C\tau\sum_{k=0}^{n-1} t_{n-k}^{\alpha-1}\left\|u_h(t_k)-U_h^k\right\|_{L_p(\Omega;U)}.$$

Hence, the discrete Gronwall inequality ([14], Lemma A.4.) implies the require result and the proof of the theorem is finished. □

7. Numerical simulations

In this section, we shall consider the numerical scheme (76) for solving the following stochastic time fractional PDEs: with $\alpha\in(0,1)$, $\gamma\in[0,1]$,

$${}_C\mathrm{D}_{0,t}^{\alpha}u(t,x) - \Delta u(t,x) = f(u(x,t)) + g(t,x), \quad 0\leq t\leq T,\ 0<x<1, \tag{107}$$
$$u(0,x) = u_0(x), \tag{108}$$
$$u(t,0) = u(t,1) = 0. \tag{109}$$

Here $f(u)=\sin(u)$, $u_0(x)=x(1-x)$ and

$$g(t,x) := {}_{RL}\mathrm{D}_{0,t}^{-\gamma}\frac{dW^H(t,x)}{dt} = {}_{RL}\mathrm{D}_{0,t}^{-\gamma}\sum_{m=1}^{\infty}\gamma_m^{1/2}e_m(x)\frac{d\beta_m^H(t)}{dt}, \tag{110}$$

where $\beta_m^H(t), m=1,2,\ldots$ are the fractional Brownian motions with Hurst number $H\in(0,1)$. When $H=1/2$, $\beta_m^H(t), m=1,2,\ldots$ are standard Brownian motion. The functions $e_m(x)=\sqrt{2}\sin m\pi x$ denote the eigenfunctions of the operator $A=-\frac{\partial^2}{\partial x^2}$ with $D(A)=H_0^1(0,1)\cap H^2(0,1)$. Further $\gamma_m, m=1,2,\ldots$ are the eigenvalues of the covariance operator Q of the stochastic process $W(t)$, that is

$$Qe_m = \gamma_m e_m.$$

The two cases below are especially interesting:
Case 1: the white noise case, e.g., $\gamma_m = m^{-\beta_1}$ with $\beta_1=0$ which implies that

$$\mathrm{tr}(Q) = \sum_{m=1}^{\infty}\gamma_m = \sum_{m=1}^{\infty}m^{-\beta_1} = \sum_{m=1}^{\infty}1 = \infty. \tag{111}$$

Case 2: The trace class case, e.g., $\gamma_m = m^{-\beta_1}$ with $\beta_1 > 1$, which implies that

$$\mathrm{tr}(Q) = \sum_{m=1}^{\infty} \gamma_m = \sum_{m=1}^{\infty} m^{-\beta_1} < \infty.$$

Let $v = u - u_0$. Then (107)–(109) can be written as the following

$$ {}_C\mathrm{D}^{\alpha}_{0,t} v(t,x) - \Delta v(t,x) = \Delta u_0(x) + f\big(v(t,x) + u_0(x)\big) + g(t,x), \tag{112}$$

$$v(0,x) = 0, \tag{113}$$

$$u(t,0) = u(t,1) = 0. \tag{114}$$

Since the initial value $v(0,x) = 0$ in (112)–(114), it is easier to consider the numerical analysis for the time discretization scheme of (112)–(114). From now on, we shall consider the fully discrete schemes for solving (112)–(114).

The system of (112)–(114) can be written as the following abstract form

$$ {}_C\mathrm{D}^{\alpha}_{0,t} u + Au = -Au_0 + f\big(v(t) + u_0\big) + g(t), \quad v(0) = 0. \tag{115}$$

Let $0 < t_0 < t_1 < \cdots < t_N = T$ be a partition of the time interval $[0,T]$ and τ the time step size. Let $0 = x_0 < x_1 < \cdots < x_M = 1$ be a partition of the space interval $[0,1]$ and h the space step size.

Let $S_h \subset H_0^1(0,1)$ be the linear finite element space which is piecewise on the partition $\{x_i\}_{i=0}^{M}$ defined by

$$S_h = \{\chi \in C[0,1] : \chi \text{ is the piecewise linear function on } [0,1] \text{ and } \chi(0) = \chi(1) = 0\}.$$

The finite element method of (112)–(114) is to find $v_h(t) \in S_h$ such that, for any $\chi \in S_h$,

$$\Big({}_C\mathrm{D}^{\alpha}_{0,t} v_h(t), \chi \Big) + \big(\nabla v_h(t), \nabla \chi\big) = -\big(\nabla u_0, \nabla \chi\big) + \big(f(v_h(t) + u_0), \chi\big) + \big(g(t), \chi\big), \tag{116}$$

$$u_h(0) = P_h u_0, \tag{117}$$

where $P_h : L_2(0,1) \to S_h$ denotes the L_2 projection operator defined by

$$\big(P_h v, \chi\big) = (v, \chi), \quad \forall \chi \in S_h.$$

Let $V^n \approx v_h\left(t_n\right), n = 0, 1, \ldots, N$ be the approximation of $v_h\left(t_n\right)$. The fully discrete scheme is to find $V^n \in S_h$, with $n = 1, 2, \ldots, N$, such that

$$\begin{aligned} &\Big(\tau^{-\alpha} \sum_{j=1}^{n} w_{n-j}^{(\alpha)} V^j, \chi\Big) + (\nabla V^n, \nabla \chi) \\ &= -\left(\nabla u_0, \nabla \chi\right) + \left(f\left(V^{n-1} + u_0\right), \chi\right) + \left(g\left(t_n\right), \chi\right), \quad \forall \chi \in S_h, \end{aligned} \tag{118}$$

$$V^0 = 0, \tag{119}$$

where the weights $w_l^{(\alpha)}$ and $g(t_n)$ are defined in (76).

Let $A_h : S_h \to S_h$ be the discrete analogue of the operator A defined by

$$\big(A_h \psi, \chi\big) = (\nabla \psi, \nabla \chi), \quad \forall \chi \in S_h. \tag{120}$$

Then we have

$$\tau^{-\alpha} \sum_{j=1}^{n} w_{n-j}^{(\alpha)} V^j + A_h V^n = -A_h u_0 + f\left(V^{n-1} + u_0\right) + g\left(t_n\right), \quad V^0 = 0. \tag{121}$$

Let $\varphi_1(x), \varphi_2(x), \ldots, \varphi_{M-1}(x)$ be the linear finite element basis functions defined by, with $j = 1, 2, \ldots, M-1$,

$$\varphi_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}}, & x_{j-1} < x < x_j, \\ \frac{x - x_{j+1}}{x_j - x_{j+1}}, & x_j < x < x_{j+1}, \\ 0, & \text{otherwise.} \end{cases}$$

To find the solution $V^n \in S_h, n = 0, 1, \ldots, N$, we assume that

$$V^n = \sum_{m=1}^{M-1} \alpha_m^n \varphi_m$$

for some coefficients $\alpha_k^n, k = 1, 2, \ldots, M-1$, which are to be found. Choose $\chi = \varphi_l, l = 1, 2, \ldots, M-1$ in (118), we have, with $n = 1, 2, \ldots, N$,

$$\tau^{-\alpha} \sum_{j=1}^{n} w_{n-j} \left[\sum_{m=1}^{M-1} \left(\varphi_m, \varphi_l\right) \alpha_m^j\right] + \sum_{m=1}^{M-1} \left(\nabla \varphi_m, \nabla \varphi_l\right) \alpha_m^n$$

$$= -\sum_{m=1}^{M-1} \left(\nabla\varphi_m, \nabla\varphi_l\right) \alpha_m^0 + \left(f\left(V^{n-1} + u_0\right), \varphi_l\right) + \left(g\left(t_n\right), \varphi_l\right), \tag{122}$$

where $V^{n-1} = \sum_{m=1}^{M-1} \alpha_m^{n-1} \varphi_m$.

To get $\alpha_m^n, n = 1, 2, \ldots, N$ from (122), we also need the initial α_m^0 which can be obtained by

$$u_0 \approx P_h u_0 = \sum_{m=1}^{M-1} \alpha_m^0 \varphi_m.$$

To solve (122) by MATLAB, we need to write (122) into the matrix form which we shall do now.

Denote

$$\alpha^n = \begin{pmatrix} \alpha_1^n \\ \alpha_2^n \\ \vdots \\ \alpha_{M-1}^n \end{pmatrix}_{(M-1)\times 1}, \quad \mathbf{f}^n = \begin{pmatrix} \left(f\left(V^{n-1}(x_1) + u_0(x_1)\right), \varphi_1\right) \\ \left(f\left(V^{n-1}(x_2) + u_0(x_2)\right), \varphi_2\right) \\ \vdots \\ \left(f\left(V^{n-1}(x_{M-1}) + u_0(x_{M-1})\right), \varphi_{M-1}\right) \end{pmatrix}_{(M-1)\times 1},$$

and

$$\mathbf{g}^n = \begin{pmatrix} \left(g\left(t_n\right), \varphi_1\right) \\ \left(g\left(t_n\right), \varphi_2\right) \\ \vdots \\ \left(g\left(t_n\right), \varphi_{M-1}\right) \end{pmatrix}_{(M-1)\times 1}.$$

After some simple calculations, we may get the following mass and stiffness metrics,

$$\mathbf{M} = \left(\left(\varphi_m, \varphi_l\right)\right)_{m,l=1}^{M-1} = h \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & & 0 \\ \frac{1}{6} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{1}{6} \\ 0 & & \frac{1}{6} & \frac{2}{3} \end{pmatrix}_{(M-1)\times(M-1)},$$

and

$$\mathbf{S} = \left(\left(\nabla\varphi_m, \nabla\varphi_l\right)\right)_{m,l=1}^{M-1} = \frac{1}{h} \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}_{(M-1)\times(M-1)},$$

respectively. Then (122) can be written as the following matrix form, $n = 1, 2, \ldots, N$,

$$\tau^{-\alpha} \sum_{j=1}^{n} w_{n-j} \mathbf{M} \alpha^j + \mathbf{S}\alpha^n = -\mathbf{S}\alpha^0 + \mathbf{f}^n + \mathbf{g}^n, \quad \alpha^0 \text{ given.} \tag{123}$$

Denote $\mathbf{A}_h = \mathbf{M}^{-1}\mathbf{S}$. Then (123) can be written as, with $n = 1, 2, \ldots, N$,

$$\tau^{-\alpha} \sum_{j=1}^{n} w_{n-j} \alpha^j + \mathbf{A}_h \alpha^n = -\mathbf{A}_h \alpha^0 + \mathbf{M}^{-1}\mathbf{f}^n + \mathbf{M}^{-1}\mathbf{g}^n, \quad \alpha^0 \text{ given,} \tag{124}$$

which is the matrix approximation form of (121). Hence $\alpha^n, n = 1, 2, \ldots, N$ can be calculated by the following formula

$$\alpha^n = \left(w_0 + \tau^\alpha \mathbf{A}_h\right)^{-1} \left(-\tau^\alpha \mathbf{A}_h \alpha^0 + \tau^\alpha \mathbf{M}^{-1}\mathbf{f}^n + \tau^\alpha \mathbf{M}^{-1}\mathbf{g}^n - \sum_{j=1}^{n-1} w_{n-j} \alpha^{n-j}\right), \tag{125}$$

where α^0 is given.

By Theorem 5, we see that

$$\text{the temporal convergence order} = \begin{cases} C\tau^{\min\{\alpha, \frac{\alpha r}{2\beta}, \frac{1}{2}\}}, & \text{if } H \in (0, \frac{1}{2}), \\ C\tau^{\min\{\alpha, \frac{\alpha r}{2\beta}\}}, & \text{if } H \in (\frac{1}{2}, 1), \end{cases}$$

where $r = \min\{2\beta + \sigma - 1, \frac{2\beta}{\alpha}(H + \alpha + \gamma - 1) + \sigma - 1\}$. In particular, when $\sigma = 1$ (the trace class), we obtain

$$\text{the temporal convergence order} = \begin{cases} C\tau^{\min\{\alpha, H+\alpha+\gamma-1, \frac{1}{2}\}}, & \text{if } H \in (0, \frac{1}{2}), \\ C\tau^{\min\{\alpha, H+\alpha+\gamma-1\}}, & \text{if } H \in (\frac{1}{2}, 1). \end{cases}$$

In our numerical simulations, we set $\sigma = 1$ (the trace class), which is equivalent to choosing $\beta_1 > 1$ in (111). Specifically, we take $\beta_1 = 2$. We also consider different parameter sets to estimate the convergence rates toward the solution of the equation.

Table 1
The convergence orders with $H = 0.3$.

α	γ	$\tau = 1/4$	1/8	1/16	1/32	Order
0.6	0.4	4.2604e − 1	3.4743e − 1	3.1338e − 1	2.1560e − 1	0.32(0.30)
0.7	0.4	2.3381e − 1	1.7264e − 1	1.4263e − 1	9.3493e − 2	0.42(0.40)
0.8	0.4	1.2193e − 1	8.2402e − 2	6.1490e − 2	3.9531e − 2	0.51(0.50)
0.9	0.4	1.0633e − 1	7.1027e − 2	5.2168e − 2	3.3629e − 2	0.52(0.50)

Table 2
The convergence orders with $H = 0.4$.

α	γ	$\tau = 1/4$	1/8	1/16	1/32	Order
0.6	0.5	1.3627e − 1	9.6224e − 2	7.8592e − 2	4.9649e − 2	0.48(0.50)
0.7	0.5	1.1663e − 1	8.0523e − 2	6.4457e − 2	4.0222e − 2	0.51(0.50)
0.8	0.5	9.9692e − 2	6.7280e − 2	5.2663e − 2	3.2469e − 2	0.53(0.50)
0.9	0.5	9.2118e − 2	6.1469e − 2	4.7537e − 2	2.9138e − 2	0.55(0.50)

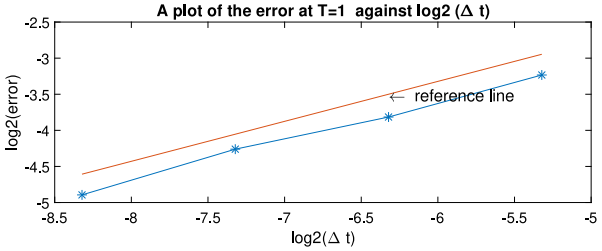

Fig. 1. The experimentally determined orders of convergence with $\alpha = 0.9$ and $\gamma = 0.4$ and $H = 0.3$ in Table 1.

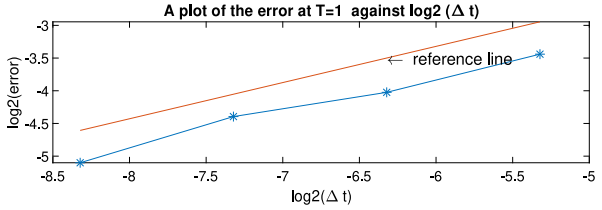

Fig. 2. The experimentally determined orders of convergence with $\alpha = 0.9$ and $\gamma = 0.5$ and $H = 0.4$ in Table 2.

Table 3
The convergence orders with $H = 0.5$.

α	γ	$\tau = 1/4$	1/8	1/16	1/32	Order
0.6	0.2	2.9601e − 1	2.4981e − 1	2.1691e − 1	1.4711e − 1	0.31(0.30)
0.7	0.2	1.8143e − 1	1.3707e − 1	1.0986e − 1	7.0327e − 2	0.42(0.40)
0.8	0.2	1.0923e − 1	7.2998e − 2	5.3337e − 2	3.2405e − 2	0.53(0.50)
0.9	0.2	9.8312e − 2	6.4115e − 2	4.5953e − 2	2.7639e − 2	0.61(0.60)

Furthermore, we examine cases with varying Hurst parameters to assess whether the numerically observed convergence orders align with the theoretical predictions stated in Theorem 5. The values in brackets indicate the theoretical orders.

We set $N_{\text{ref}} = 2^7$ and $T = 0.1$, and define the reference time step size as $dt_{\text{ref}} = T/N_{\text{ref}}$. To compute the reference solution v_{ref}, we choose a spatial step size of $h = 2^{-6}$ and a time step size of $dt_{\text{ref}} = 2^{-7}$. To examine the temporal convergence behavior for various values of H, α, and γ, we consider a range of time step sizes given by $\tau = \kappa \cdot dt_{\text{ref}}$, where $\kappa \in \{2^5, 2^4, 2^3, 2^2\}$. As shown in Tables 1–4, the numerically observed convergence orders align well with the theoretical predictions in all cases. The numerical results also indicate that the convergence orders improve as the parameters α, γ, and H increase, reflecting the enhanced regularity of the solution with larger values of these parameters. (See Figs. 1–4).

8. Conclusion

In this paper, we study numerical approximation methods for a stochastic semilinear subdiffusion equation driven by fractionally integrated additive fractional Brownian motion (fBm). We establish the existence and uniqueness of the mild solution using the Banach contraction mapping principle and conduct a thorough analysis of its spatial and temporal regularity. A semidiscrete scheme

Table 4
The convergence orders with $H = 0.6$.

α	γ	$\tau = 1/4$	1/8	1/16	1/32	Order
0.6	0.5	5.9732e − 2	3.8982e − 2	2.8460e − 2	1.6682e − 2	0.61(0.60)
0.7	0.5	3.7565e − 2	2.1594e − 2	1.3684e − 2	7.5050e − 3	0.72(0.70)
0.8	0.5	2.2088e − 2	1.1822e − 2	6.7937e − 3	3.5575e − 3	0.82(0.80)
0.9	0.5	8.9305e − 3	4.5422e − 3	2.8387e − 3	1.3431e − 3	0.91(0.90)

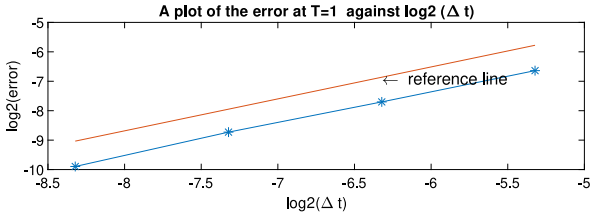

Fig. 3. The experimentally determined orders of convergence with $\alpha = 0.9$ and $\gamma = 0.2$ and $H = 0.5$ in Table 3.

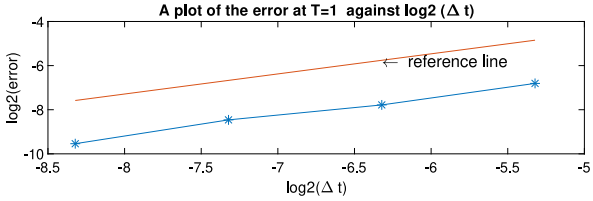

Fig. 4. The experimentally determined orders of convergence with $\alpha = 0.9$ and $\gamma = 0.5$ and $H = 0.6$ in Table 4.

is then constructed using the Galerkin finite element method (FEM), and corresponding error estimates are derived. Furthermore, we propose a fully discrete scheme by approximating the time-fractional derivative with a convolution quadrature based on the Grünwald–Letnikov scheme and analyze its convergence. Optimal convergence rates for the fully discrete scheme are established. Finally, numerical experiments are presented to confirm the consistency of the simulation results with the theoretical predictions.

CRediT authorship contribution statement

Zhiqiang Li: Writing – original draft, Validation, Methodology, Formal analysis. **George Dewhirst:** Software. **Yubin Yan:** Writing – review & editing, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The work was supported in part by the key research and development project of Lvliang city in Shanxi Province (No. 2022RC11) and fund program for the scientific activities of selected returned overseas professionals in Shanxi Province (No. 20240037).

Data availability

No data was used for the research described in the article.

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