mathematics

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Numerical Approximation for a Stochastic Fractional Differential Equation Driven by Integrated Multiplicative Noise

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Abstract: We consider a numerical approximation for stochastic fractional differential equations driven by integrated multiplicative noise. The fractional derivative is in the Caputo sense with the fractional order $\alpha \in (0,1)$, and the non-linear terms satisfy the global Lipschitz conditions. We first approximate the noise with the piecewise constant function to obtain the regularized stochastic fractional differential equation. By applying Minkowski's inequality for double integrals, we establish that the error between the exact solution and the solution of the regularized problem has an order of $O(\Delta t^{\alpha})$ in the mean square norm, where Δt denotes the step size. To validate our theoretical conclusions, numerical examples are presented, demonstrating the consistency of the numerical results with the established theory.

Keywords: stochastic fractional differential equations; convergence order; regularit; Brownian motion

MSC: 60G22; 60H10; 60H50

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Citation: Hoult, J.; Yan, Y. Numerical Approximation for a Stochastic Fractional Differential Equation Driven by Integrated Multiplicative Noise. *Mathematics* **2024**, *12*, 365. https://doi.org/10.3390/math12030365

Academic Editor: Leonid Piterbarg

Received: 19 December 2023
Revised: 17 January 2024
Accepted: 22 January 2024
Published: 23 January 2024

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1. Introduction

Consider the following stochastic fractional differential equation driven by multiplicative white noise, with $\alpha \in (0,1)$ [1]:

$$\begin{cases} {}_0^C D_t^{\alpha} u(t) = f(t,u(t)) + \int_0^t g(t,u(s))dW(s), \quad t \in (0,T], \\ u(0) = u_0, \end{cases} \tag{1}$$

where $W(s)$ is Brownian motion defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and ${}_0^C D_t^{\alpha} v(t)$ denotes the Caputo fractional derivative defined by [2]

$$ {}_0^C D_t^{\alpha} v(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{v'(\tau)d\tau}{(t-\tau)^{\alpha}}.$$

The non-linear functions $f(t,x), g(t,x)$ satisfy the following globally Lipschitz conditions and the linear growth conditions with some suitable constant $C > 0$:

$$\begin{aligned} &|f(t,x) - f(t,y)| \le C|x-y|, \quad |g(t,x) - g(t,y)| \le C|x-y|, \quad x,y \in \mathbb{R}, \\ &|f(t,x)| \le C(1+|x|), \quad |g(t,x)| \le C(1+|x|), \quad x \in \mathbb{R}. \end{aligned}$$

It is well-known that (1) is equivalent to the following stochastic Volterra integral equations (SVIEs) with a weakly singular kernel of the form [1]

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\zeta, u(\zeta))}{(t-\zeta)^{1-\alpha}} d\zeta + \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^{\zeta} \frac{g(\zeta, u(s))}{(t-\zeta)^{1-\alpha}} dW(s) d\zeta. \tag{2}$$

It is evident that the solution to Equation (2) relies not only on the current states but also on past states. This characteristic makes stochastic Volterra integral equations (SVIEs) able to model the different problems involving memory and noise across various domains of science and technology. Examples include biological population models [3,4], mathematical finance models [5,6], and others [7]. In the realm of mathematics, numerous studies have been conducted, such as those by [8,9]. Ravichandran et al. [10] considered a fractional integrodifferential system with state-dependent delay in Banach spaces. They used Krasnoselskii's fixed-point theorem and the Leray–Schauder alternative theorem to consider the controllability and continuous dependence of these systems. Similarly, ref. [11] examined the conditions of a slightly different system to also find its controllability. Dhayal et al. [12] considered a second-order stochastic differential equation driven by fractional Brownian motion with many different Hurst parameters. Various Caputo-based fractional equations are also discussed in the current literature, such as the Caputo–Fabrizo fractional order differential equation with multiple lags. Zhang et al. [13] introduced the premise for finding the acquired difference form and obtained a solution by applying the fractional PCEC algorithm. These systems are widely used in control theory.

When $\alpha = 1$, the stochastic Volterra integral equations (SVIEs) (2), along with their numerical schemes, have been thoroughly investigated [14,15]. In contrast, the singular Volterra integral equations of the same form have received less attention. Some results concerning their existence and uniqueness have been established under a (global) Lipschitz condition and a linear growth condition, which can be found in [16–18] and the references therein.

When substituting $(t-s)^{\alpha-1}$ with alternative well-behaved functions, the exploration of numerical schemes for (regular) stochastic Volterra integral equations (SVIEs) has gained attention only in recent times. Tudor and Tudor [19] considered the strong convergence of one-step numerical approximations for Itô–Volterra equations, providing a convergence rate in the $L_p(\Omega)$ norm. Wen and Zhang [20] analyzed an enhanced version of the rectangular method for stochastic Volterra equations, demonstrating a convergence order of $O(\Delta t)$. Subsequently, Wang [21] approximated SVIE solutions using a class of stochastic differential equations (SDEs) and introduced two numerical methods: the stochastic theta method and the splitting method. Xiao et al. [22] presented a split-step collocation method for SVIEs, establishing its convergence with an order of $O(\Delta t^{\frac{1}{2}})$. Liang et al. [23] found that the Euler–Maruyama (EM) method achieves superconvergence on the order (Δt) if the kernel function in the diffusion term satisfies specific boundary conditions. More recently, research has extended to the Euler scheme for a broader class of equations, such as SVIEs with delay, stochastic Volterra integro-differential equations, and stochastic fractional integro-differential equations. For further exploration, we refer to [16,24–28] and the references therein.

The numerical treatment of stochastic Volterra integral equations (SVIEs) with a weakly singular kernel of the form (2) has been minimally explored in the existing literature. The primary challenge stems from the singularity of the integrand kernel. In such cases, the potent and essential Itô formula, commonly employed for stochastic differential equations (SDEs), is not applicable to SVIEs with a singular kernel.

Li et al. [29] addressed this issue by examining the Euler–Maruyama scheme for solving (2) with $\alpha > \frac{1}{2}$. They established that the convergence order of the scheme is $O(\Delta t^{\alpha-\frac{1}{2}})$ for $\alpha > \frac{1}{2}$. Another contribution by Kamrani [1] focused on the numerical approximation of (2) with additive noise when $g = 1$. Kamrani approximated the additive noise using a piecewise constant function, leading to a regularized equation. The study demonstrated that the error between the exact solution and the solution of the regularized problem is $O(\Delta t)$. This regularized equation was further approximated using the Jacobian method.

In this paper, we extend Kamrani's approach [1] to address (2) in the presence of multiplicative noise. We begin by examining the regularity of the solution of (2) and subsequently approximate the noise using piecewise constant functions to derive the regularized equation. The regularized equation is further approximated using an L1 scheme. Our analysis establishes that the error between the exact solution and the solution of the regularized problem is $O(\Delta t^{\alpha})$ for any $\alpha \in (0,1)$ by applying Minkowski's inequality for double integrals. This extends and improves upon the results presented in [29], where the authors only considered the case with $\alpha > \frac{1}{2}$ and achieved a convergence order of only $O(\Delta t^{\alpha-\frac{1}{2}})$ for $\alpha > \frac{1}{2}$.

Let us briefly review the main results obtained in this paper. Let $t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = T$ be a partition of $[0,T]$ and let Δt be the step size. We approximate the white noise $\frac{dW(t)}{dt}$ by a piecewise constant function $\frac{d\widehat{W}(t)}{dt}$ defined by [1]

$$\frac{d\widehat{W}(t)}{dt} = \begin{cases} \frac{W(t_1)-W(t_0)}{\Delta t} \approx \frac{\sqrt{\Delta t}\cdot\eta_1}{\Delta t}, & t \in [t_0, t_1), \\ \frac{W(t_2)-W(t_1)}{\Delta t} \approx \frac{\sqrt{\Delta t}\cdot\eta_2}{\Delta t}, & t \in [t_1, t_2), \\ \frac{W(t_3)-W(t_2)}{\Delta t} \approx \frac{\sqrt{\Delta t}\cdot\eta_3}{\Delta t}, & t \in [t_2, t_3), \\ \vdots & \vdots \\ \frac{W(t_n)-W(t_{n-1})}{\Delta t} \approx \frac{\sqrt{\Delta t}\cdot\eta_n}{\Delta t}, & t \in [t_{n-1}, t_n), \end{cases}$$

where $\eta_i \sim \mathcal{N}(0,1),\ i = 1,2,3,\ldots,n$ are the normally distributed random variables. For simplicity, we may write $\frac{d\widehat{W}(t)}{dt}$ as $\chi_i(t) = \begin{cases} 1, & [t_i, t_{i+1}), \\ 0, & \text{otherwise}, \end{cases}$

$$\frac{d\widehat{W}(t)}{dt} = \sum_{i=1}^{n} \frac{\sqrt{\Delta t}\chi_i(t)\eta_i}{\Delta t} = \sum_{i=1}^{n} \frac{\chi_i(t)\eta_i}{\sqrt{\Delta t}}.$$

We then obtain the following regularized stochastic fractional differential equation of (1):

$$\tilde{u}(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\zeta, \tilde{u}(\zeta))}{(t-\zeta)^{1-\alpha}} d\zeta + \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^\zeta \frac{g(\zeta, \tilde{u}(s))}{(t-\zeta)^{1-\alpha}} \frac{d\widehat{W}(t)}{dt} dt d\zeta. \quad (3)$$

In Theorem 2, we consider the error of convergence for the additive noise case and show that the convergence order is $O(\Delta t)$. In Theorem 4, we consider the error of convergence for the multiplicative noise case and show that the convergence order is $O(\Delta t^{\alpha})$, $\alpha \in (0,1)$.

The paper is organized as follows. In Section 2, we consider the approximation for the additive case. In Section 3, we consider the approximation for the multiplicative noise. In Section 4, we give some numerical simulations where the fractional derivatives are approximated using the L1 scheme. In the Appendix A, we include Minkowski's inequality for double integrals, which is the main tool used in the proofs of error estimates.

Throughout this paper, we denote C as a generic constant that is independent of the step size Δt, which could be different for different occurrences.

2. The Additive Noise Case

In this section, we will consider the approximation of (2) for the additive noise case, that is, $g(t, u(s)) = g_1(s)$ in (2), which is independent of u. We first study the regularity of (2).

The following Grönwall Lemma is used in this paper.

Lemma 1 (Grönwall Inequality ([1], Lemma 4.1))**.** *Let $z : \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying, for all $t \in [0,T]$, the inequality*

$$z(t) \leq a + K \int_0^t (t-s)^{\sigma} z(s) \mathrm{d}s,$$

with some constants $a \geq 0, K > 0$ and $\sigma > -1$. Then there exists a constant $C = C(\sigma, K, T)$ such that $z(t) \leq aC$ for all $t \in [0,T]$.

Lemma 2. *Let $u(t)$ be the solution of* (2) *with $g(t,u(s)) = g_1(s)$. Then, there exists a constant $C = C(T)$ such that*

$$\mathbb{E}|u(t_2) - u(t_1)|^2 \le C(t_2 - t_1)^{2\alpha}.$$

Proof. Note that

$$\begin{aligned} u(t_2) - u(t_1) &= \Big[\int_0^{t_2} (t_2-\zeta)^{\alpha-1} f(\zeta, u(\zeta)) d\zeta - \int_0^{t_1} (t_1-\zeta)^{\alpha-1} f(\zeta, u(\zeta)) d\zeta\Big] \\ &\quad + \Big[\int_0^{t_2}\int_0^{\zeta} (t_2-\zeta)^{\alpha-1} g_1(s) dW(s) d\zeta - \int_0^{t_1}\int_0^{\zeta} (t_1-\zeta)^{\alpha-1} g_1(s) dW(s) d\zeta\Big] \\ &= I + II. \end{aligned}$$

For I, we have

$$\begin{aligned} I &= \int_{t_1}^{t_2} (t_2-\zeta)^{\alpha-1} f(\zeta, u(\zeta)) d\zeta - \int_0^{t_1}\Big[(t_1-\zeta)^{\alpha-1} f(\zeta,u(\zeta)) - (t_2-\zeta)^{\alpha-1} f(\zeta,u(\zeta))\Big] d\zeta, \\ &= I_1 + I_2. \end{aligned}$$

For I_2, we obtain

$$\begin{aligned} \mathbb{E}|I_2|^2 &= \mathbb{E}\Big|\int_0^{t_1}\Big[(t_1-\zeta)^{\alpha-1} f(\zeta,u(\zeta)) - (t_2-\zeta)^{\alpha-1} f(\zeta,u(\zeta))\Big] d\zeta\Big|^2, \\ &= \mathbb{E}\Big|\int_0^{t_1}\Big[(t_1-\zeta)^{\alpha-1} - (t_2-\zeta)^{\alpha-1}\Big] f(\zeta,u(\zeta)) d\zeta\Big|^2, \\ &\le \mathbb{E}\Big[\int_0^{t_1}\Big((t_1-\zeta)^{\alpha-1} - (t_2-\zeta)^{\alpha-1}\Big) d\zeta\Big]\Big[\int_0^{t_1}\Big((t_1-\zeta)^{\alpha-1} - (t_2-\zeta)^{\alpha-1}\Big)\Big] f^2(\zeta,u(\zeta)) d\zeta, \\ &\le C\Big[\int_0^{t_1}\Big((t_1-\zeta)^{\alpha-1} - (t_2-\zeta)^{\alpha-1}\Big) d\zeta\Big]^2 \cdot \mathbb{E}\max_{0\le\zeta\le t} f^2(\zeta,u(\zeta)), \end{aligned}$$

which implies that

$$\mathbb{E}|I_2|^2 \le (t_2-t_1)^{2\alpha} \cdot \max_{0\le\zeta\le t} \mathbb{E} f^2(\zeta,u(\zeta)) \le C(1+\mathbb{E}|u(0)|^2)(t_2-t_1)^{2\alpha}.$$

For I_1, we obtain

$$\begin{aligned} \mathbb{E}|I_1|^2 &= \mathbb{E}\Big|\int_{t_1}^{t_2}(t_2-\zeta)^{\alpha-1} f(\zeta,u(\zeta)) d\zeta\Big|^2 \le \mathbb{E}\int_{t_1}^{t_2}(t_2-\zeta)^{\alpha-1} f^2(\zeta,u(\zeta)) d\zeta \cdot \Big[\int_{t_1}^{t_2}(t_2-\zeta)^{\alpha-1} d\zeta\Big], \\ &\le C\Big[\int_{t_1}^{t_2}(t_2-\zeta)^{\alpha-1} d\zeta\Big]^2\Big[\max_{0\le\zeta\le t}\mathbb{E} f^2(\zeta,u(\zeta))\Big] \le C(t_2-t_1)^{2\alpha}\Big[1+\mathbb{E}|u(0)|^2\Big] \le C\Delta t^{2\alpha}. \end{aligned}$$

Now we turn to II.

$$\begin{aligned} II &= \int_0^{t_2}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1} g_1(s) dW(s) d\zeta - \int_0^{t_1}\int_0^{\zeta}(t_1-\zeta)^{\alpha-1} g_1(s) dW(s) d\zeta, \\ &\le \Big[\int_0^{t_2}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1} g_1(s) dW(s) d\zeta - \int_0^{t_1}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1} g_1(s) dW(s) d\zeta\Big] \\ &\quad + \Big[\int_0^{t_1}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1} g_1(s) dW(s) d\zeta - \int_0^{t_1}\int_0^{\zeta}(t_1-\zeta)^{\alpha-1} g_1(s) dW(s) d\zeta\Big], \\ &= II_1 + II_2. \end{aligned}$$

For II_1, we have

$$\mathbb{E}|II_1|^2 = \mathbb{E}\bigg|\int_0^{t_2}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1}g_1(s)dW(s)d\zeta - \int_0^{t_1}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1}g_1(s)dW(s)d\zeta\bigg|^2,$$
$$\leq \mathbb{E}\bigg|\int_{t_1}^{t_2}(t_2-\zeta)^{\alpha-1}\Big[\int_0^{\zeta}(t_2-\zeta)^{\alpha-1}g_1(s)dW(s)\Big]d\zeta\bigg|^2.$$

Splitting $(t_2-\zeta)^{\alpha-1}$ into two parts yields

$$\mathbb{E}|II_1|^2 \leq \mathbb{E}\bigg|\int_{t_1}^{t_2}(t_2-\zeta)^{\frac{\alpha-1}{2}}\Big[\int_0^{\zeta}(t_2-\zeta)^{\frac{\alpha-1}{2}}g_1(s)dW(s)\Big]d\zeta\bigg|^2.$$

By applying the Cauchy–Schwarz inequality, we arrive at

$$\mathbb{E}|II_1|^2 \leq \mathbb{E}\bigg[\int_{t_1}^{t_2}(t_2-\zeta)^{\alpha-1}d\zeta\bigg]\cdot\int_{t_1}^{t_2}\bigg[\int_0^{\zeta}(t_2-\zeta)^{\frac{\alpha-1}{2}}g_1(s)dW(s)\bigg]^2 d\zeta,$$
$$\leq \Delta t^{\alpha}\int_{t_1}^{t_2}\mathbb{E}\bigg|\int_0^{\zeta}(t_2-\zeta)^{\frac{\alpha-1}{2}}g_1(s)dW(s)\bigg|^2 d\zeta.$$

Using the Ito isometry property, we obtain

$$\mathbb{E}|II_1|^2 \leq \Delta t^{\alpha}\cdot\int_{t_1}^{t_2}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1}\big|g_1(s)\big|^2 ds d\zeta.$$

Note that $\big|g_1(s)\big|^2$ is bounded so we arrive at

$$\mathbb{E}|II_1|^2 \leq C\Delta t^{\alpha}\cdot\int_{t_1}^{t_2}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1}ds d\zeta \leq C\Delta t^{\alpha}\cdot\int_{t_1}^{t_2}(t_2-\zeta)^{\alpha-1}d\zeta \leq C\Delta t^{2\alpha}.$$

For II_2, we obtain

$$\mathbb{E}|II_2|^2 = \mathbb{E}\bigg|\int_0^{t_1}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1}g_1(s)dW(s)d\zeta - \int_0^{t_1}\int_0^{\zeta}(t_1-\zeta)^{\alpha-1}g_1(s)dW(s)d\zeta\bigg|^2,$$
$$= \mathbb{E}\bigg|\int_0^{t_1}\int_0^{\zeta}\Big[(t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1}\Big]g_1(s)dW(s)d\zeta\bigg|^2.$$

Splitting $(t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1}$ into two yields

$$\mathbb{E}|II_2|^2 = \mathbb{E}\bigg|\int_0^{t_1}\Big[(t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1}\Big]^{\frac{1}{2}}$$
$$\cdot\int_0^{\zeta}\Big[(t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1}\Big]^{\frac{1}{2}}g_1(s)dW(s)d\zeta\bigg|^2.$$

By applying the Cauchy–Schwarz inequality, we have

$$\mathbb{E}|II_2|^2 \leq \bigg(\int_0^{t_1}(t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1}d\zeta\bigg)\bigg(\int_0^{t_1}(t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1}\bigg)$$
$$\cdot E\bigg|\int_0^{\zeta}g_1(s)dW(s)\bigg|^2 d\zeta.$$

Using the Ito isometry property, we yield

$$\mathbb{E}|II_2|^2 \leq \left(\int_0^{t_1} (t_2-\zeta)^{\alpha-1} - (t_1-\zeta)^{\alpha-1} d\zeta \right)^2 d\zeta \left| \int_0^{\zeta} |g_1(s)| \right|^2 ds.$$

Note that $g(s, u(s))$ is bounded so we obtain

$$\mathbb{E}|II_2|^2 \leq C\left(\int_0^{t_1} (t_2-\zeta)^{\alpha-1} - (t_1-\zeta)^{\alpha-1} d\zeta \right)^2 = C\left(\int_0^{t_1} \int_{t_1}^{t_2} (x-\zeta)^{\alpha-2} dx d\zeta \right)^2.$$

By interchanging the double integrals, we arrive at

$$\begin{aligned}\mathbb{E}|II_2|^2 &\leq C\left(\int_{t_1}^{t_2} \int_0^{t_1} (x-\zeta)^{\alpha-2} d\zeta dx \right)^2 \leq C\left(\int_{t_1}^{t_2} (x-t_1)^{\alpha-1} dx \right)^2, \\ &\leq C\left((t_2-t_1)^{\alpha} \right)^2 = C(t_2-t_1)^{2\alpha} = C\Delta t^{2\alpha}.\end{aligned}$$

Hence, we obtain

$$E|u(t_2) - u(t_1)|^2 \leq C(t_2-t_1)^{2\alpha},$$

which completes the proof of Lemma 2. □

We now introduce the following two theorems obtained in [1] for the additive noise case. Theorem 1 considers the stability of the solution of the regularized problem (3), and Theorem 2 considers the error estimates.

Theorem 1 ([1], Theorem 4.2)**.** *Let $\tilde{u}(t)$ be the solution of* (3)*. Then, we have*

$$\mathbb{E}\left(\int_0^T |\tilde{u}(t)dt|^2 \right) \leq C\big(1 + \mathbb{E}|\tilde{u}(0)|^2\big).$$

Theorem 2 ([1], Theorem 4.3)**.** *Let $u(t)$ and $\tilde{u}(t)$ be the solutions of* (2) *and* (3)*, respectively. Then, we have the following inequality:*

$$\mathbb{E} \int_0^1 |u(t) - \tilde{u}(t)|^2 \leq C(\Delta t)^2.$$

3. The Multiplicative Noise Case

In this section, we shall consider the approximation of (2) for the multiplicative noise case. We first consider the stability of the solution for the multiplicative noise case.

Theorem 3. *Let $\tilde{u}(t)$ be the solution of* (3)*. Then, we have*

$$\mathbb{E}\left(\int_0^T |\tilde{u}(t)dt|^2 \right) \leq C\big(1 + \mathbb{E}|\tilde{u}(0)|^2\big).$$

Proof. The proof of Theorem 3 is similar to the proof of Theorem 1, which can be found in [1]. For the length of the paper, we omit the proof here. □

Let us first consider the regularity of the solution of (2) when $g(t, u(s))$ is independent of t, i.e., $g(t, u(s)) = g_1(s, u(s))$ for some function g_1.

Lemma 3. *Let $u(t)$ be the solution of* (2) *with $g(t, u(s)) = g_1(s, u(s))$. Then, we have*

$$\mathbb{E}|u(t_2) - u(t_1)|^2 \leq C(t_2-t_1)^{2\alpha}.$$

Proof. Note that

$$
\begin{aligned}
u(t_2)-u(t_1) &= \left[\int_0^{t_2}(t_2-\zeta)^{\alpha-1}f(\zeta,u(\zeta))d\zeta-\int_0^{t_1}(t_1-\zeta)^{\alpha-1}f(\zeta,u(\zeta))d\zeta\right]\\
&\quad+\left[\int_0^{t_2}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1}g_1(s,u(s))dW(s)d\zeta-\int_0^{t_1}\int_0^{\zeta}(t_1-\zeta)^{\alpha-1}g_1(s,u(s))dW(s)d\zeta\right],\\
&= I+II.
\end{aligned}
$$

For I, we may establish the same result as in Lemma 1 using the same notion. We obtain

$$
\mathbb{E}|I|^2 \le C(t_2-t_1)^{2\alpha}.
$$

Now we turn to II.

$$
\begin{aligned}
II &= \int_0^{t_2}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1}g_1(s,u(s))dW(s)d\zeta-\int_0^{t_1}\int_0^{\zeta}(t_1-\zeta)^{\alpha-1}g_1(s,u(s))dW(s)d\zeta,\\
&\le \left[\int_0^{t_2}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1}g_1(s,u(s))dW(s)d\zeta-\int_0^{t_1}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1}g_1(s,u(s))dW(s)d\zeta\right]\\
&\quad+\left[\int_0^{t_1}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1}g_1(s,u(s))dW(s)d\zeta-\int_0^{t_1}\int_0^{\zeta}(t_1-\zeta)^{\alpha-1}g_1(s,u(s))dW(s)d\zeta\right],\\
&= II_1+II_2.
\end{aligned}
$$

For II_1, we have

$$
\begin{aligned}
\mathbb{E}|II_1|^2 &= \mathbb{E}\left|\int_0^{t_2}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1}g_1(s,u(s))dW(s)d\zeta-\int_0^{t_1}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1}g_1(s,u(s))dW(s)d\zeta\right|^2,\\
&\le \mathbb{E}\left|\int_{t_1}^{t_2}(t_2-\zeta)^{\alpha-1}\left[\int_0^{\zeta}(t_2-\zeta)^{\alpha-1}g_1(s,u(s))dW(s)\right]d\zeta\right|^2.
\end{aligned}
$$

Splitting $(t_2-\zeta)^{\alpha-1}$ into two yields

$$
\mathbb{E}|II_1|^2 \le \mathbb{E}\left|\int_{t_1}^{t_2}(t_2-\zeta)^{\frac{\alpha-1}{2}}\left[\int_0^{\zeta}(t_2-\zeta)^{\frac{\alpha-1}{2}}g_1(s,u(s))dW(s)\right]d\zeta\right|^2.
$$

By applying the Cauchy–Schwarz inequality (3), we arrive at

$$
\begin{aligned}
\mathbb{E}|II_1|^2 &\le \mathbb{E}\left[\int_{t_1}^{t_2}(t_2-\zeta)^{\alpha-1}d\zeta\right]\cdot\int_{t_1}^{t_2}\left[\int_0^{\zeta}(t_2-\zeta)^{\frac{\alpha-1}{2}}g_1(s,u(s))dW(s)\right]^2d\zeta,\\
&\le \Delta t^{\alpha}\int_{t_1}^{t_2}\mathbb{E}\left|\int_0^{\zeta}(t_2-\zeta)^{\frac{\alpha-1}{2}}g_1(s,u(s))dW(s)\right|^2d\zeta.
\end{aligned}
$$

Using the Ito isometry property, we obtain

$$
\mathbb{E}|II_1|^2 \le \Delta t^{\alpha}\cdot\int_{t_1}^{t_2}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1}\big|g_1(s,u(s))\big|^2dsd\zeta.
$$

Note that $\big|g_1(s,u(s))\big|^2$ is bounded such that we have

$$
\mathbb{E}|II_1|^2 \le C\Delta t^{\alpha}\cdot\int_{t_1}^{t_2}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1}dsd\zeta \le C\Delta t^{\alpha}\cdot\int_{t_1}^{t_2}(t_2-\zeta)^{\alpha-1}d\zeta \le C\Delta t^{2\alpha}.
$$

For II_2, we obtain

$$\mathbb{E}|II_2|^2 = \mathbb{E}\bigg|\int_0^{t_1}\int_0^{\zeta}(t_2-\zeta)^{\alpha-1}g_1(s,u(s))dW(s)d\zeta - \int_0^{t_1}\int_0^{\zeta}(t_1-\zeta)^{\alpha-1}g_1(s,u(s))dW(s)d\zeta\bigg|^2,$$
$$= \mathbb{E}\bigg|\int_0^{t_1}\int_0^{\zeta}\Big[(t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1}\Big]g_1(s,u(s))dW(s)d\zeta\bigg|^2.$$

Splitting $(t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1}$ into two yields

$$\mathbb{E}|II_2|^2 = \mathbb{E}\bigg|\int_0^{t_1}\Big[(t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1}\Big]^{\frac{1}{2}}$$
$$\cdot\int_0^{\zeta}\Big[(t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1}\Big]^{\frac{1}{2}}g_1(s,u(s))dW(s)d\zeta\bigg|^2.$$

By applying the Cauchy–Schwarz inequality, it follows that

$$\mathbb{E}|II_2|^2 \le \bigg(\int_0^{t_1}(t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1}d\zeta\bigg)\bigg(\int_0^{t_1}(t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1}\bigg)$$
$$\cdot E\bigg|\int_0^{\zeta}g_1(s,u(s))dW(s)\bigg|^2 d\zeta.$$

Using the Ito isometry property, we obtain

$$\mathbb{E}|II_2|^2 \le \bigg(\int_0^{t_1}(t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1}d\zeta\bigg)^2 d\zeta\bigg|\int_0^{\zeta}|g_1(s,u(s))\bigg|^2 ds.$$

Note that $g_1(s,u(s))$ is bounded; hence, we have

$$\mathbb{E}|II_2|^2 \le C\bigg(\int_0^{t_1}(t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1}d\zeta\bigg)^2 = C\bigg(\int_0^{t_1}\int_{t_1}^{t_2}(x-\zeta)^{\alpha-2}dxd\zeta\bigg)^2.$$

By interchanging the double integrals, we arrive at

$$\mathbb{E}|II_2|^2 \le C\bigg(\int_{t_1}^{t_2}\int_0^{t_1}(x-\zeta)^{\alpha-2}d\zeta dx\bigg)^2 \le C\bigg(\int_{t_1}^{t_2}(x-t_1)^{\alpha-1}dx\bigg)^2,$$
$$\le C\Big((t_2-t_1)^{\alpha}\Big)^2 = C(t_2-t_1)^{2\alpha} = C\Delta t^{2\alpha}.$$

Hence, we obtain
$$E|u(t_2)-u(t_1)|^2 \le C(t_2-t_1)^{2\alpha},$$
which completes the proof of Lemma 3. □

Remark 1. *The difference between Lemma 2 and Lemma 3 are as follows. Lemma 2 considers the case for (2) driven by additive noise with $g(t,u(s)) = g_1(s)$, whereas Lemma 3 considers the case for (2) driven by multiplicative noise with $g(t,u(s)) = g_1(s,u(s))$. Both cases yield the same regularity order $O(\Delta t^{\alpha}), \alpha \in (0,1)$.*

Now, we introduce the main theorem in this section.

Theorem 4. *Let $u(t)$ and $\tilde{u}(t)$ be the solutions of (2) and (3), respectively. Then, we have*

$$\mathbb{E}\int_0^1 \big|u(t)-\tilde{u}(t)\big|^2 dt \le C\Delta t^{2\alpha}.$$

Proof. Note that the solution $\tilde{u}(t)$ of the regularized stochastic fractional differential equation takes the following form:

$$\tilde{u}(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1} f(\zeta, \tilde{u}(\zeta)) d\zeta + \frac{1}{\Gamma(\alpha)}\int_0^t \left[\int_0^\zeta (t-\zeta)^{\alpha-1} g(\zeta, \tilde{u}(s)) \frac{d\widehat{W}(s)}{ds} ds\right] d\zeta.$$

Denote

$$e(t) = u(t) - \tilde{u}(t).$$

Then, $e(t)$ satisfies the following equation:

$$\begin{aligned}
e(t) =& \frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1}\Big[f(\zeta, u(\zeta)) - f(\zeta, \tilde{u}(\zeta))\Big] d\zeta + \frac{1}{\Gamma(\alpha)}\int_0^t \left[\int_0^\zeta (t-\zeta)^{\alpha-1} g(\zeta, u(s)) \frac{dW(s)}{ds} ds\right.\\
&\left. - \int_0^\zeta (t-\zeta)^{\alpha-1} g(\zeta, u(s)) \frac{d\widehat{W}(s)}{ds} ds\right] d\zeta \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t \int_0^\zeta (t-\zeta)^{\alpha-1}\Big[g(\zeta, u(s)) - g(\zeta, \tilde{u}(s))\Big] \frac{d\widehat{W}(s)}{ds} ds d\zeta,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\mathbb{E}\int_0^T |e(t)|^2 dt \\
&\leq C\mathbb{E}\int_0^T \left(\int_0^t (t-\zeta)^{\alpha-1}\Big[f(\zeta, u(\zeta)) - f(\zeta, \tilde{u}(\zeta)\Big] d\zeta\right)^2 dt \\
&\quad + C\mathbb{E}\int_0^T \left[\int_0^t \int_0^\zeta (t-\zeta)^{\alpha-1} g(\zeta, u(s)) \frac{dW(s)}{ds} ds d\zeta - \int_0^t \int_0^\zeta (t-\zeta)^{\alpha-1} g(\zeta, u(s)) \frac{d\widehat{W}(s)}{ds} ds d\zeta\right]^2 dt \\
&\quad + C\mathbb{E}\int_0^T \left(\frac{1}{\Gamma(\alpha)}\int_0^t \int_0^\zeta (t-\zeta)^{\alpha-1}\Big[g(\zeta, u(s)) - g(\zeta, \tilde{u}(s))\Big] \frac{d\widehat{W}(s)}{ds} ds d\zeta\right)^2 dt \\
&= I + II + III.
\end{aligned}$$

For I, using a variable change $\nu = t - \zeta$, we have

$$\begin{aligned}
I &= \mathbb{E}\int_0^T \left(\int_0^t (t-\zeta)^{\alpha-1}\Big[f(\zeta, u(\zeta)) - f(\zeta, \tilde{u}(\zeta)\Big] d\zeta\right)^2 dt, \\
&= \mathbb{E}\int_0^T \left(\int_0^t \nu^{\alpha-1}\Big[f(t-\nu, u(t-\nu)) - f(t-\nu, \tilde{u}(t-\nu)\Big] d\nu\right)^2 dt, \\
&= \mathbb{E}\int_0^T \left(\int_0^T \chi_{[0,t]}(\nu)\cdot \nu^{\alpha-1}\Big[f(t-\nu, u(t-\nu)) - f(t-\nu, \tilde{u}(t-\nu)\Big] d\nu\right)^2 dt,
\end{aligned}$$

where $\chi_{[0,t]}(\nu)$ is defined as

$$\chi_{[0,t]}(\nu) = \begin{cases} 1, & T \geq t \leq \nu, \\ 0, & 0 \geq t \leq \nu, \end{cases} = \begin{cases} 1, & 0 \leq \nu \leq t, \\ 0, & t \leq \nu \leq T. \end{cases}$$

Applying Minkowski's inequality for the double integrals of Lemma A1, we arrive at

$$
\begin{aligned}
I &= \mathbb{E}\int_0^T \left(\int_0^T \chi_{[0,t]}(\nu)\cdot \nu^{\alpha-1}\Big[f(t-\nu,u(t-\nu)) - f(t-\nu,\tilde{u}(t-\nu)\Big]d\nu\right)^2 dt,\\
&\leq \left[\int_0^T \left(\mathbb{E}\int_0^T \Big|\chi_{[0,t]}(\nu)\cdot \nu^{\alpha-1}\big[f(t-\nu,u(t-\nu)) - f(t-\nu,\tilde{u}(t-\nu))\big]\Big|^2 dt\right)^{\frac{1}{2}} d\nu\right]^2,\\
&= \left[\int_0^T \nu^{\alpha-1}\left(\mathbb{E}\int_0^T \Big|\chi_{[0,t]}(\nu)\big[f(t-\nu,u(t-\nu)) - f(t-\nu,\tilde{u}(t-\nu))\big]\Big|^2 dt\right)^{\frac{1}{2}} d\nu\right]^2,
\end{aligned}
$$

which implies that

$$
I \leq C\left[\int_0^T \nu^{\alpha-1}\left(\mathbb{E}\int_\nu^T \Big|\big[f(t-\nu,u(t-\nu)) - f(t-\nu,\tilde{u}(t-\nu))\big]\Big|^2 dt\right)^{\frac{1}{2}} d\nu\right]^2.
$$

Using the Lipschitz condition for f, we obtain

$$
I \leq C\left[\int_0^T \nu^{\alpha-1}\left(\mathbb{E}\int_\nu^T e^2(t-\nu)dt\right)^{\frac{1}{2}} d\nu\right]^2 \leq C\left[\int_0^T \nu^{\alpha-1}\left(\mathbb{E}\int_0^{T-\nu} e^2(\zeta)d\zeta\right)^{\frac{1}{2}} d\nu\right]^2.
$$

By applying a variable change $\nu = T - \tilde{\nu}$, we arrive at

$$
I \leq C\left[\int_0^T (T-\tilde{\nu})^{\alpha-1}\left(\mathbb{E}\int_0^{\tilde{\nu}} e^2(\zeta)d\zeta\right)^{\frac{1}{2}} d\tilde{\nu}\right]^2 = C\left[\int_0^T (T-\nu)^{\alpha-1}\left(\mathbb{E}\int_0^{\nu} e^2(\zeta)d\zeta\right)^{\frac{1}{2}} d\nu\right]^2.
$$

Now, we turn to II. We have

$$
II = C\mathbb{E}\int_0^T \left[\int_0^t \int_0^\zeta \frac{g(\zeta,u(s))}{(t-\zeta)^{1-\alpha}} dW(s)d\zeta - \int_0^t \int_0^\zeta \frac{g(\zeta,u(s))}{(t-\zeta)^{1-\alpha}}\left(\frac{d\widehat{W}(s)}{ds}\right) ds\, d\zeta\right]^2 dt.
$$

Applying a change of variable $\nu = t - \zeta$, we have

$$
II = C\mathbb{E}\int_0^T \left[\int_0^t \int_0^{t-\nu} \frac{g(t-\nu,u(s))}{\nu^{1-\alpha}} dW(s)d\nu - \int_0^t \int_0^{t-\nu} \frac{g(t-\nu,u(s))}{\nu^{1-\alpha}}\left(\frac{d\widehat{W}(s)}{ds}\right) ds\, d\nu\right]^2 dt,
$$

which implies that

$$
II \leq C\mathbb{E}\int_0^T \left[\int_0^T \chi_{[0,t]} \int_0^{t-\nu} \frac{g(t-\nu,u(s))}{\nu^{1-\alpha}} dW(s)d\nu - \int_0^T \chi_{[0,t]} \int_0^{t-\nu} \frac{g(t-\nu,u(s))}{\nu^{1-\alpha}}\left(\frac{d\widehat{W}(s)}{ds}\right) ds\, d\nu\right]^2 dt.
$$

Applying Minkowski's inequality for the double integrals of Lemma A1, we arrive at

$$
\begin{aligned}
II &\leq \left(\int_0^T \left[\mathbb{E}\int_0^T \left|\chi_{[0,t]} \int_0^{t-\nu} \frac{g(t-\nu,u(s))}{\nu^{1-\alpha}}\left[\frac{dW(s)}{ds} - \frac{d\widehat{W}(s)}{ds}\right] ds\right|^2 dt\right]^{\frac{1}{2}} d\nu\right)^2,\\
&= \left(\int_0^T \left[\mathbb{E}\int_\nu^T \left| \int_0^{t-\nu} \frac{g(t-\nu,u(s))}{\nu^{1-\alpha}}\left[\frac{dW(s)}{ds} - \frac{d\widehat{W}(s)}{ds}\right] ds\right|^2 dt\right]^{\frac{1}{2}} d\nu\right)^2,\\
&= \left(\int_0^T \nu^{\alpha-1}\left[\mathbb{E}\int_\nu^T \left| \int_0^{t-\nu} g(t-\nu,u(s))\left[\frac{dW(s)}{ds} - \frac{d\widehat{W}(s)}{ds}\right] ds\right|^2 dt\right]^{\frac{1}{2}} d\nu\right)^2.
\end{aligned}
$$

Note that, for $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = t - \nu$,

$$\begin{aligned}
&\mathbb{E}\bigg|\int_0^{t-\nu} g(t-\nu, u(s))\left[\frac{dW(s)}{ds} - \frac{d\widehat{W}(s)}{ds}\right]ds\bigg|^2,\\
&= \mathbb{E}\bigg|\sum_{i=1}^{m+1}\left[\int_{t_{i-1}}^{t_i} g(t_{m+1}, u(s))dW(s) - \int_{t_{i-1}}^{t_i} g(t_{m+1}, u(\tau))\frac{W(t_i) - W(t_{i-1})}{\Delta t}d\tau\right]\bigg|^2,\\
&= \mathbb{E}\bigg|\sum_{i=1}^{m+1}\left[\int_{t_{i-1}}^{t_i} g(t_{m+1}, u(s))dW(s) - \int_{t_{i-1}}^{t_i}\left(\frac{1}{\Delta t}\int_{t_{i-1}}^{t_i} g(t_{m+1}, u(\tau))d\tau\right)dW(s)\right]\bigg|^2,\\
&= \mathbb{E}\bigg|\sum_{i=1}^{m+1}\left[\int_{t_{i-1}}^{t_i}\left(\frac{1}{\Delta t}\int_{t_{i-1}}^{t_i}\Big[g(t_{m+1}, u(s)) - g(t_{m+1}, u(\tau))\Big]d\tau\right)dW(s)\right]\bigg|^2,\\
&= \mathbb{E}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\left(\frac{1}{\Delta t}\int_{t_{i-1}}^{t_i}\Big[g(t_{m+1}, u(s)) - g(t_{m+1}, u(\tau))\Big]d\tau\right)^2 ds,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\mathbb{E}\bigg|\int_0^{t-\nu} g(t-\nu, u(s))\left[\frac{dW(s)}{ds} - \frac{d\widehat{W}(s)}{ds}\right]ds\bigg|^2,\\
&\le \mathbb{E}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\left[\frac{1}{\Delta t^2}\left(\int_{t_{i-1}}^{t_i} 1^2 dt\right)\cdot\left(\int_{t_{i-1}}^{t_i}\Big[g(t_{m+1}, u(s)) - g(t_{m+1}, u(\tau))\Big]^2 d\tau\right)\right]ds,\\
&\le \mathbb{E}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\left[\frac{1}{\Delta t}\int_{t_{i-1}}^{t_i}\Big[g(t_{m+1}, u(s)) - g(t_{m+1}, u(\tau))\Big]^2 d\tau\right]ds.
\end{aligned}$$

Applying the Lipschitz condition for g and Lemma 3, we have

$$\mathbb{E}\Big|g(t_{m+1}, u(s)) - g(t_{m+1}, u(\tau))\Big| \le \mathbb{E}|u(s) - u(\tau)|^2 \le C|s-\tau|^2,$$

which implies that

$$\begin{aligned}
&\mathbb{E}\bigg|\int_0^{t-\nu} g(t-\nu, u(s))\left[\frac{dW(s)}{ds} - \frac{d\widehat{W}(s)}{ds}\right]ds\bigg|^2,\\
&\le \mathbb{E}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\left[\frac{1}{\Delta t}\int_{t_{i-1}}^{t_i}\Big[g(t_{m+1}, u(s)) - g(t_{m+1}, u(\tau))\Big]^2 d\tau\right]ds \le C\Delta t^{2\alpha}.
\end{aligned}$$

Thus, we obtain

$$II \le \left[\int_0^T \nu^{\alpha-1}\left[\int_0^T \Delta t^2 dt\right]^{\frac{1}{2}} d\nu\right]^2 \le C\Delta t^{2\alpha}\left[\int_0^T \nu^{\alpha-1}d\nu\right]^2 \le C\Delta t^{2\alpha}.$$

For III, using a change of variable $\nu = t - \zeta$, it follows that

$$\begin{aligned}
III &= \mathbb{E}\int_0^T\left(\int_0^t\int_0^\zeta (t-\zeta)^{\alpha-1}\Big[g(\zeta, u(s)) - g(\zeta, \tilde{u}(s))\Big]\frac{d\widehat{W}(s)}{ds}dsd\zeta\right)^2 dt,\\
&= C\mathbb{E}\int_0^T\left(\int_0^t\int_0^{t-\nu} \nu^{\alpha-1}\Big[g(\zeta, u(s)) - g(\zeta, \tilde{u}(s))\Big]\frac{d\widehat{W}(s)}{ds}dsd\nu\right)^2 dt,\\
&= C\mathbb{E}\int_0^T\left(\int_0^T \chi_{[0,t]}(\nu)\int_0^{t-\nu} \nu^{\alpha-1}\Big[g(\zeta, u(s)) - g(\zeta, \tilde{u}(s))\Big]\frac{d\widehat{W}(s)}{ds}dsd\nu\right)^2 dt.
\end{aligned}$$

Applying Minkowski's inequality for the double integrals of Lemma A1, we have

$$
\begin{aligned}
III &\leq \left[\int_0^T \left(\mathbb{E}\int_0^T \left|\chi_{[0,t]}(\nu)\int_0^{t-\nu}\nu^{\alpha-1}\left[g(\zeta,u(s))-g(\zeta,\tilde{u}(s))\right]\frac{d\widehat{W}(s)}{ds}ds\right|^2 dt\right)^{\frac{1}{2}} d\nu\right]^2, \\
&= \left[\int_0^T \left(\mathbb{E}\int_0^T \left|\int_0^{t-\nu}\nu^{\alpha-1}\left[g(\zeta,u(s))-g(\zeta,\tilde{u}(s))\right]\frac{d\widehat{W}(s)}{ds}ds\right|^2 dt\right)^{\frac{1}{2}} d\nu\right]^2, \\
&= \left[\int_0^T \nu^{\alpha-1}\left(\int_\nu^T \left|\int_0^{t-\nu}\left[g(\zeta,u(s))-g(\zeta,\tilde{u}(s))\right]\frac{d\widehat{W}(s)}{ds}ds\right|^2 dt\right)^{\frac{1}{2}} d\nu\right]^2.
\end{aligned}
$$

Following the same argument as for the estimate of the term II, we obtain

$$
\mathbb{E}\left|\int_0^{t-\nu}\left[g(\zeta,u(s))-g(\zeta,\tilde{u}(s))\right]\frac{d\widehat{W}(s)}{ds}ds\right|^2 \leq C\mathbb{E}\int_0^{t-\nu}|e(s)|^2 ds.
$$

Therefore, using the variable change $\zeta = t-\nu$,

$$
III \leq \left[\int_0^T \nu^{\alpha-1}\left(\int_\nu^T\left[\int_0^{t-\nu}\mathbb{E}|e(s)|^2 ds\right]dt\right)^{\frac{1}{2}} d\nu\right]^2 \leq \left[\int_0^T \nu^{\alpha-1}\left(\int_0^{T-\nu}\left[\int_0^{\zeta}\mathbb{E}|e(s)|^2 ds\right]d\zeta\right)^{\frac{1}{2}} d\nu\right]^2,
$$

Using the variable change $\nu = T-\tilde{\nu}$, we arrive at

$$
\begin{aligned}
III &\leq \left[\int_0^T (T-\tilde{\nu})^{\alpha-1}\left(\int_0^{\tilde{\nu}}\left[\int_0^{\zeta}\mathbb{E}|e(s)|^2 ds\right]d\zeta\right)^{\frac{1}{2}} d\tilde{\nu}\right]^2, \\
&= \left[\int_0^T (T-\nu)^{\alpha-1}\left(\int_0^{\nu}\left[\int_0^{\zeta}\mathbb{E}|e(s)|^2 ds\right]d\zeta\right)^{\frac{1}{2}} d\nu\right]^2, \\
&\leq \left[\int_0^T (T-\nu)^{\alpha-1}\left(\int_0^{\nu}\left[\int_0^{\nu}\mathbb{E}|e(s)|^2 ds\right]d\zeta\right)^{\frac{1}{2}} d\nu\right]^2, \\
&\leq C\left[\int_0^T (T-\nu)^{\alpha-1}\left(\mathbb{E}\int_0^{\nu} e^2(s)ds\right)^{\frac{1}{2}} d\nu\right]^2.
\end{aligned}
$$

Hence, we obtain

$$
\mathbb{E}\left(\int_0^T e^2(t)dt\right) \leq C\left[\int_0^T (T-\nu)^{\alpha-1}\left(\mathbb{E}\int_0^{\nu} e^2(\zeta)d\zeta\right)^{\frac{1}{2}} d\nu\right]^2 + C\Delta t^{2\alpha}.
$$

Denote $e_1(\nu) = \left[\mathbb{E}\left(\int_0^{\nu} e^2(\zeta)d\zeta\right)\right]^{\frac{1}{2}}$. We therefore obtain

$$
e_1(T) \leq \int_0^T (T-\nu)^{\alpha-1}e_1(\nu)d\nu + C\Delta t^{\beta}.
$$

By applying the Grönwall Lemma 1, we have

$$
e_1(T) \leq C\Delta t^{\alpha},
$$

which implies that

$$
\mathbb{E}\int_0^1 \left|u(t)-\tilde{u}(t)\right|^2 \leq C\Delta t^{2\alpha}.
$$

The proof of Theorem 4 is now complete. □

Remark 2. *The difference between Theorem 2 and Theorem 4 are as follows. Theorem 2 considers the convergence order for (2) driven by additive noise, whereas Theorem 4 considers the convergence order for (2) driven by multiplicative noise with $g(t,u(s))$. The additive case yields a convergence order of $O(\Delta t), \alpha \in (0,1)$, whereas the multiplicative case achieves a convergence order of $O(\Delta t^{\alpha})$, $\alpha \in (0,1)$.*

4. Numerical Simulations

In this section, we shall consider the numerical simulations for the following problem with different values of f and g.

$$ {}_0^C D_0^{\alpha} u(t) = f(t,u(t)) + \int_0^t g(t,u(s)) dW(s), \tag{4} $$

$$ u(0) = u_0. \tag{5} $$

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be the partition of $[0,T]$ and let Δt be the step size. At $t = t_n$, we have

$$ {}_0^C D_0^{\alpha} u(t)\Big|_{t=t_n} = f(t_n, u(t_n)) + \int_0^{t_n} g(t_n, u(s)) dW(s). $$

We shall approximate the Caputo fractional derivative ${}_0^C D_0^{\alpha} u(t)\Big|_{t=t_n}$ with the L1 scheme [30]

$$ {}_0^C D_0^{\alpha} u(t)\Big|_{t=t_n} \approx \Delta t^{-\alpha} \sum_{j=0}^{n} w_{j,n} u(t_{n-j}), $$

where the weights $w_{j,n}$ are defined by

$$ \Gamma(2-\alpha) w_{j,n} = \begin{cases} 1, & j = 0, \\ 2^{1-\alpha} - 2, & j = 1, \\ (j-1)^{1-\alpha} + (j+1)^{1-\alpha} - 2j^{1-\alpha}, & j = 2,3,\ldots,n-1, \\ (j-1)^{1-\alpha} - (\alpha-1)j^{-\alpha} - j^{1-\alpha}, & j = n. \end{cases} $$

Further, we will approximate the integral $\int_0^{t_n} g(t_n,u(s))dW(s)$ by the following rectangular integration formula:

$$ \int_0^{t_n} g(t_n, u(s)) dW(s) \approx \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} g(t_n, u(t_{j-1})) \eta_j = \sum_{j=1}^{n} \Delta t g(t_n, u(t_{j-1})) \eta_j, $$

where $\eta_j = \sqrt{\Delta t}\mathcal{N}(0,1)$. Here $\mathcal{N}(0,1)$ denotes the standard normally distributed random variable calculated by the MATLAB function "randn".

Let $U^n \approx u(t_n)$ be the approximate solution. We may obtain the following numerical method for $U^n, n = 1,2,\ldots,N$ with $U^0 = u_0$:

$$ \Delta t^{-\alpha} \sum_{j=0}^{n} w_{j,n} U^{n-j} = f(t_n, U^n) + \sum_{j=1}^{n} \Delta t g(t_n, U^{j-1}) \eta_j. \tag{6} $$

In our numerical simulations, we chose $T = 1$ and the different step sizes $h_1 = \frac{1}{16}$, $h_2 = \frac{1}{32}, h_3 = \frac{1}{64}$, and $h_4 = \frac{1}{128}$. Since there are no exact solutions for our problems, we shall use a reference solution calculated with a sufficiently small step size $h = \Delta t = 2^{-12}$.

Since we did not obtain the convergence order of the scheme defined in (6), we shall use the following method to compute the experimentally determined order of convergence (EOC) $p > 0$.

We shall calculate the error at $T = t_N = Nh = 1$. Assume that we have the following error estimate, which depends on the step size $h = \Delta t$; that is, if $p > 0$,

$$error(h) = ||U^N - u(t_N)|| \leq Ch^p, \quad p > 0.$$

By choosing $t_{N_i} = N_i h_i = T = 1$ with the different step sizes $h_i = \frac{1}{2^i}$ for $i = 4, 5, 6, 7$, we have

$$error(h_i) = \mathbb{E}\Big[||U^{N_i} - u(T)||^2\Big]^{\frac{1}{2}} \approx Ch_i^p,$$

which implies that the convergence order $p > 0$ satisfies, with $i = 4, 5, 6$,

$$\frac{error(h_i)}{error(h_{i+1})} \approx \left(\frac{h_i}{h_{i+1}}\right)^p,$$

or

$$p \approx \left(\frac{\log_2\left(\frac{error(h_i)}{error(h_{i+1})}\right)}{\log_2\left(\frac{h_i}{h_{i+1}}\right)}\right).$$

We obtain 3 different EOCs with 4 step sizes $h_i, i = 4, 5, 6, 7$ and take the average of the three EOCs, which are found in the EOC column of Tables 1–5.

In Tables 1–5, we provide the approximation results by using the following different f and g.

$$f(t, u(t)) = -u(t) + t + \frac{\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha}, \quad g(t, u(t)) = 1, \tag{7}$$

$$f(t, u(t)) = -u(t) + t^2 + 2\frac{t^{1.5}}{\Gamma(2.5)} t^{1-\alpha}, \quad g(t, u) = t, \tag{8}$$

$$f(t, u(t)) = -u(t) + t^2 + 2\frac{t^{1.5}}{\Gamma(2.5)} t^{1-\alpha}, \quad g(t, u) = u, \tag{9}$$

$$f(t, u(t)) = -u(t) + t^2 + 2\frac{t^{1.5}}{\Gamma(2.5)} t^{1-\alpha}, \quad g(t, u) = \sin(u), \tag{10}$$

$$f(t, u(t)) = -u(t) + t^2 + 2\frac{t^{1.5}}{\Gamma(2.5)} t^{1-\alpha}, \quad g(t, u) = u^3 - u. \tag{11}$$

We note that, across all cases, the experimentally determined convergence orders are nearly $O(\Delta t)$ for various $\alpha \in (0, 1)$. These observed orders outperform the theoretical order $\Delta t^\alpha, \alpha \in (0, 1)$ in the context of multiplicative noise cases. In future investigations, we will consider the factors contributing to the superior performance of experimentally determined convergence orders compared to their theoretical counterparts in the presence of multiplicative noise.

Table 1. The convergence orders for Equations (4) and (5) defined by (7).

α	$h_1 = \frac{1}{16}$	$h_2 = \frac{1}{32}$	$h_3 = \frac{1}{64}$	$h_4 = \frac{1}{128}$	EOC
0.2	1.1869×10^{-3}	5.8082×10^{-4} 1.0310	2.8542×10^{-4} 1.0250	1.3972×10^{-4} 1.0305	1.03
0.4	2.5771×10^{-3}	1.2469×10^{-3} 1.0473	6.0768×10^{-4} 1.0370	2.9570×10^{-4} 1.0392	1.04
0.6	4.4683×10^{-3}	2.1448×10^{-3} 1.0589	1.0357×10^{-3} 1.0502	4.9966×10^{-4} 1.0516	1.05
0.8	6.9502×10^{-3}	3.3953×10^{-3} 1.0335	1.6528×10^{-3} 1.0386	7.9956×10^{-4} 1.0476	1.04
1	8.0788×10^{-3}	4.0749×10^{-3} 0.9874	2.0344×10^{-3} 1.0021	1.0043×10^{-3} 1.0184	1.00

Table 2. The convergence orders for Equations (4) and (5) defined by (8).

α	$h_1 = \frac{1}{16}$	$h_2 = \frac{1}{32}$	$h_3 = \frac{1}{64}$	$h_4 = \frac{1}{128}$	EOC
0.2	3.6254×10^{-2}	1.7819×10^{-2}	8.9154×10^{-3}	4.4166×10^{-3}	1.01
		1.0247	0.99904	1.0134	
0.4	2.9412×10^{-2}	1.4521×10^{-2}	7.2194×10^{-3}	3.5814×10^{-3}	1.01
		1.0183	1.0082	1.0114	
0.6	3.1514×10^{-2}	1.5556×10^{-2}	7.6321×10^{-3}	3.7425×10^{-3}	1.02
		1.0185	1.0273	1.0281	
0.8	3.2311×10^{-2}	1.6120×10^{-2}	7.9141×10^{-3}	3.8351×10^{-3}	1.02
		1.0032	1.0264	1.0452	
1	4.9803×10^{-2}	2.4976×10^{-2}	1.2433×10^{-2}	6.1283×10^{-3}	1.01
		1.0057	1.0116	1.0232	

Table 3. The convergence orders for Equations (4) and (5) defined by (9).

α	$h_1 = \frac{1}{16}$	$h_2 = \frac{1}{32}$	$h_3 = \frac{1}{64}$	$h_4 = \frac{1}{128}$	EOC
0.2	3.8205×10^{-2}	1.9042×10^{-2}	9.4368×10^{-3}	4.6378×10^{-3}	1.01
		1.0046	1.0128	1.0249	
0.4	3.3152×10^{-2}	1.6205×10^{-2}	7.9161×10^{-3}	3.8499×10^{-3}	1.04
		1.0326	1.0336	1.0400	
0.6	3.1321×10^{-2}	1.4878×10^{-2}	7.0823×10^{-3}	3.3676×10^{-3}	1.07
		1.0740	1.0708	1.0725	
0.8	3.4886×10^{-2}	1.6429×10^{-2}	7.7244×10^{-3}	3.6156×10^{-3}	1.09
		1.0864	1.0888	1.0952	
1	4.7068×10^{-2}	2.3335×10^{-2}	1.1548×10^{-2}	5.6754×10^{-3}	1.02
		1.0123	1.0148	1.0248	

Table 4. The convergence orders for Equations (4) and (5) defined by (10).

α	$h_1 = \frac{1}{16}$	$h_2 = \frac{1}{32}$	$h_3 = \frac{1}{64}$	$h_4 = \frac{1}{128}$	EOC
0.2	3.1570×10^{-2}	1.5542×10^{-2}	7.6456×10^{-3}	3.7412×10^{-3}	1.03
		1.0224	1.0234	1.0312	
0.4	2.9524×10^{-2}	1.4249×10^{-2}	6.8956×10^{-3}	3.3313×10^{-3}	1.05
		1.0510	1.0471	1.0496	
0.6	2.9624×10^{-2}	1.3939×10^{-2}	6.5793×10^{-3}	3.1055×10^{-3}	1.08
		1.0876	1.0832	1.0831	
0.8	3.4298×10^{-2}	1.6103×10^{-2}	7.5450×10^{-3}	3.5191×10^{-3}	1.09
		1.0908	1.0937	1.1003	
1	4.7021×10^{-2}	2.3320×10^{-2}	1.1543×10^{-2}	5.6735×10^{-3}	1.02
		1.0117	1.0146	1.0247	

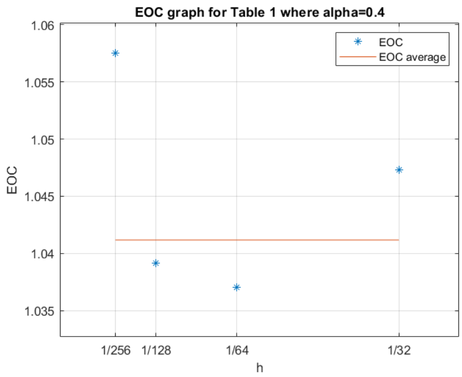
Table 5. The convergence orders for Equations (4) and (5) defined by (11).

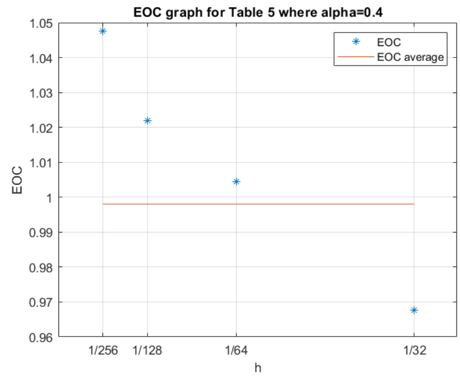
α	$h_1 = \frac{1}{16}$	$h_2 = \frac{1}{32}$	$h_3 = \frac{1}{64}$	$h_4 = \frac{1}{128}$	EOC
0.2	5.8169×10^{-2}	2.7932×10^{-2}	1.3664×10^{-2}	6.6809×10^{-3}	1.04
		1.0583	1.0315	1.0323	
0.4	3.8562×10^{-2}	1.9719×10^{-2}	9.8286×10^{-3}	4.8399×10^{-3}	1.00
		0.9676	1.0045	1.0220	
0.6	1.9561×10^{-2}	9.7425×10^{-3}	4.7290×10^{-3}	2.2623×10^{-3}	1.04
		1.0056	1.0427	1.0638	
0.8	2.3815×10^{-2}	1.1518×10^{-2}	5.7530×10^{-3}	2.8175×10^{-3}	1.03
		1.0480	1.0015	1.0299	
1	4.5648×10^{-2}	2.2806×10^{-2}	1.1329×10^{-2}	5.5780×10^{-3}	1.01
		1.0011	1.0094	1.0221	

Table 1 presents the approximation results in the presence of additive noise with $G(t, u(t)) = 1$. Three experimental orders of convergence are computed using step sizes h_i

for $i = 4, 5, 6, 7$. The experimentally determined order of convergence in the EOC column is obtained by averaging the EOC values. Across various fractional orders α, an EOC of approximately 1 is consistently achieved.

Tables 2–5 examine additional cases, and they exhibit a comparable average EOC of approximately 1. This observation suggests that the experimental orders of convergence for both additive and multiplicative noise scenarios are similar.





The left graph illustrates the experimental orders of convergence corresponding to Table 1, where $\alpha = 0.4$, and various step sizes are considered. The EOC line represents the average of the EOC values, clearly indicating an EOC of approximately 1 for the additive noise case.

On the right, the graph displays the experimental orders of convergence for Table 5, with $\alpha = 0.4$ and different step sizes. The EOC line, representing the average of the EOC values, distinctly shows an EOC of approximately 1 for the multiplicative noise case.

5. Conclusions

This paper considers the numerical approximation of stochastic fractional differential equations driven by integrated multiplicative noise. The approach involves employing a piecewise constant function to approximate the noise, leading to the derivation of a regularized stochastic fractional differential equation. We establish the regularity of the solution and analyze the convergence order of the proposed approximation scheme. To conduct numerical simulations, we employ the L1 scheme for approximating the Caputo fractional derivative. The results of our numerical simulations reveal convergence orders of nearly $O(\Delta t)$ for cases involving multiplicative noise. Surprisingly, these experimentally determined convergence orders outperform the expected theoretical orders of $O(\Delta t)^{\alpha}$, $\alpha \in (0, 1)$. In future research, we will investigate the reasons behind this discrepancy and further explore the implications of our findings.

Author Contributions: We have both contributed equal amounts towards this paper. J.H. conducted the theoretical analysis, wrote the original draft, and carried out the numerical simulations. Y.Y. introduced and provided guidance in this research area. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The data are contained within the article.

Conflicts of Interest: The authors declare that they have no competing interest.

Appendix A

In the Appendix A, we will provide Minkowski's inequality for double integrals.

Lemma A1 (Minkowski's inequality for double integrals)**.** *Suppose that we have the following σ-finite measure spaces: $(E_1, \mathcal{A}, \lambda)$ and $(E_2, \mathcal{B}, \mu)$, and suppose that $f : E_1 \times E_2 \to \mathbb{R}$ is $\mathcal{A} \bigotimes \mathcal{B}$-measurable. Then, we have*

$$\left[\int_{E_1} \left(\int_{E_2} |f(x,y)| \, \mu(dy) \right)^p \lambda(dx) \right]^{\frac{1}{p}} \le \int_{E_2} \left(\int_{E_1} |f(x,y)|^p \, \lambda(dx) \right)^{\frac{1}{p}} \mu(dy),$$

which is satisfied $\forall\, p, r \in [1, \infty)$. An equality occurs if $p = 1$.

Proof. We omit the proof here. For more details, please see Theorem 13.14 in [31]. □

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