HIGHER MOMENTS FOR THE STOCHASTIC CAHN - HILLIARD EQUATION WITH MULTIPLICATIVE FOURIER NOISE

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Abstract. We consider in dimensions $d = 1, 2, 3$ the ε-dependent stochastic Cahn-Hilliard equation with a multiplicative and sufficiently regular in space infinite dimensional Fourier noise with strength of order $\mathcal{O}(\varepsilon^\gamma)$, $\gamma > 0$. The initial condition is non-layered and independent from ε. Under general assumptions on the noise diffusion σ, we prove moment estimates in H^1 (and in L^∞ when $d = 1$). Higher H^2 regularity p-moment estimates are derived when σ is bounded, yielding as well space Hölder and L^∞ bounds for $d = 2, 3$, and path a.s. continuity in space. All appearing constants are expressed in terms of the small positive parameter ε. As in the deterministic case, in H^1, H^2, the bounds admit a negative polynomial order in ε. Finally, assuming layered initial data of initial energy uniformly bounded in ε, as proposed by X.F. Chen in [11], we use our H^1 2d-moment estimate and prove the stochastic solution's convergence to ± 1 as $\varepsilon \to 0$ a.s., when the noise diffusion has a linear growth.

1. Introduction

1.1. The stochastic equation. We consider the ε-dependent stochastic Cahn-Hilliard equation with multiplicative noise

$$
\begin{aligned}
&u_t = \Delta\Big(-\varepsilon\Delta u + \frac{1}{\varepsilon} f(u)\Big) + \varepsilon^\gamma \sigma(u)\dot{W}(x,t) \quad \text{in } \mathcal{D},\ \ t \in (0,T),\\
&\frac{\partial u}{\partial \eta} = \frac{\partial \Delta u}{\partial \eta} = 0 \quad \text{on } \partial\mathcal{D},\ \ t \in (0,T),\\
&u(x,0) = u_0(x) \quad \text{on } \mathcal{D}.
\end{aligned} \tag{1.1}
$$

Here, $\mathcal{D}$ is a bounded domain in $\mathbb{R}^d$ with $d = 1, 2, 3$ of sufficiently smooth boundary, $\gamma > 0$, and $\varepsilon > 0$ is a small positive parameter. The noise diffusion coefficient σ has at most a linear growth

$$|\sigma(x)| \leq c(1 + |x|^\alpha) \ \ \forall\, x \in \mathbb{R},$$

for $\alpha \in [0, 1]$. Along the boundary the standard Neumann conditions for u and Δu are imposed, where η is the outward normal vector. The function $f = f(u) = F'(u)$ is a balanced bistable nonlinearity, defined as the derivative of a double equal well potential F; a typical choice is

$$F(u) := \frac{1}{4}(u^2 - 1)^2, \quad f(u) = u^3 - u. \tag{1.2}$$

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For simplicity, we will assume (1.2). The noise $\dot{W}$ is the formal derivative of a $\mathcal{Q}$-Wiener process W which is given as a Fourier Brownian series, and is thus, non smooth in time; its required smoothness in space will be specified later.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be the filtered probability space where the $L^2(\mathcal{D})$-valued $\mathcal{Q}$-Wiener process $W(\cdot, t)$ is defined, cf. [14], for a symmetric, non-negative definite operator $\mathcal{Q}$. Let also $(e_i)_{i\in\mathbb{N}}$ be an induced complete $L^2(\mathcal{D})$-orthonormal basis of eigenfunctions corresponding to the non-negative eigenvalues a_i^2, satisfying thus,

$$\mathcal{Q}e_i = a_i^2 e_i.$$

W is then defined as the Fourier series

$$W(x,t) := \sum_{i=1}^{\infty} a_i \beta_i(t) e_i(x), \tag{1.3}$$

for a sequence of independent real-valued Brownian motions $\{\beta_i(t)\}_{t\geq 0}$, [14].

The stochastic Cahn-Hilliard equation is a model for the non-equilibrium dynamics of metastable states [12, 20, 21]. It describes the phase separation of a binary alloy which is forced to homogenization, where the solution u is the mass concentration of one of the alloy components. The parameter $\varepsilon > 0$ stands as the order of the width of the transition layers separating the two phases. The noise which is in general of a gaussian type may stem from external fields, impurities in the alloy, thermal fluctuations or external mass supply, see for example in [20, 12, 21, 19].

Various stochastic versions of the Cahn-Hilliard equation with general polynomial nonlinearity have been analyzed in [8, 9, 13, 15]. Via a convolution semigroup, for the case of unbounded noise diffusion with the non smooth in space and time noise of Walsh, [27], space-time Hölder estimates of mild solutions were proven when the initial and boundary value problem is posed on a rectangle, [6]. See also some more recent results in [25, 26], for additive and multiplicative Wiener noise and quite general double-well potentials, and for the stochastic viscous equation with potential of arbitrary growth at infinity. In [18], a wide class of equations including the stochastic Cahn-Hilliard equation was examined, with multiplicative finite dimensional Wiener noise of Stratonovich type; there, the authors investigated the density of projections of global mild solutions, if such solutions exist. In [4], existence of a density in dimension 1 was established for unbounded noise diffusion. When $\sigma(u) \equiv 1$ for the problem (1.1), layer dynamics have been derived in [3] when $d = 1$, and the sharp interface limit when $d = 2, 3$ in [2]. Considering reaction diffusion stochastic systems of second order with multiplicative noise, we refer to the interesting analysis in [10] on existence and uniqueness of mild solutions.

1.2. **Main results.** In Section 2, for ε-independent u_0, we derive up to $H^2(\mathcal{D})$ p-moments under sufficient conditions on the growth of σ and the regularity in space of the Fourier noise dW in $d = 1, 2, 3$.

The first part therein is devoted to the $H^1(\mathcal{D})$ 2d-moment derivation. In Lemma 2.1, we apply Itô formula and prove the fundamental identity (2.13) for the functional

$$\widetilde{F}(u) = \int_{\mathcal{D}} F(u)dx + \frac{\varepsilon^2}{2}\|\nabla u\|^2,$$

and the bound (2.8), then, in Main Theorem 2.1 we provide the moment estimate, when the noise is in $H^2(\mathcal{D})$ or $H^3(\mathcal{D})$ for $d = 1$ and $d = 2, 3$ respectively. The case of noise diffusion of linear growth is covered by the general assumptions on σ.

We proceed by proving p-moment estimates first in $H^1(\mathcal{D})$ and then in $H^2(\mathcal{D})$. As it seems higher regularity in $H^2(\mathcal{D})$ proven even for the $2d$ order through the p-moments in $H^1(\mathcal{D})$ (which involve the functional $\widetilde{F}(u)$, see Theorem 2.2 for the H^1 p-moments), restricts the noise diffusion growth to this of a bounded one. The technical proof of Main Theorem 2.3 establishes the H^2 p-moment estimates, after proper use of the Gagliardo-Nirenberg's inequality on various non-linear terms. All our bounds are expressed in terms of ε and have in general a negative polynomial order in ε. *A priori* estimates up to H^2 had been first introduced in the classic work of Elliott, Zheng Songmu, [16], for the deterministic problem where global existence was proven. In dimensions $d = 2, 3$ $H^2(\mathcal{D})$ regularity of the stochastic equation yields $L^\infty(\mathcal{D})$ regularity, and Hölder regularity which is essential for continuous paths in space for general domains. In a series of papers, the stochastic Cahn-Hilliard equation with multiplicative non smooth noise in space and in time was considered and path regularity in space and in time had been analyzed [8, 9, 5, 6]. However, there, the domains were rectangular and a semigroup approach with the Green's function estimates was applicable for deriving Hölder estimates in space and time. In a more general domain geometry such tools are not effective, due to the lack of knowledge of the bi-Laplacian eigenfunctions behaviour, even if the noise is smooth in space as in the current work.

Energy estimates can easily yield $H^1(\mathcal{D})$ bounds when the stochastic Cahn-Hilliard equation is considered, see for example in [13]; we also refer to [24] for some more recent p-moment estimates in $L^2(\mathcal{D})$ and $H^1(\mathcal{D})$ for the equation with degenerate mobility and logarithmic potential. In this paper, we prove p-moment regularity estimates for all $p \geq 1$ up to $H^2(\mathcal{D})$ which is the critical space for strong solutions in the spatial variables in dimensions $2, 3$. This is a fourth order nonlinear stochastic equation and many results are still missing from the literature for such problems even for the case of one only Brownian mode in the Fourier series.

Regularity estimates in norms higher than $H^2(\mathcal{D})$ in expectation are left for the interested reader. They can be derived for sufficiently smooth initial conditions, for a sufficiently smooth noise in space, by differentiating the stochastic equation and following the approach we proposed for deriving the H^2 estimates. The arguments of [16] on higher regularity in space are not directly applicable in the Fourier Brownian case due to the fact that the time integral of the noise term is only at most α-Hölder continuous in time, $\alpha < 1/2$; there see at pg. 345, the main argument was the time differentiability of the deterministic solution u, obviously not the case of non smooth noise in time as here. Moreover, for example p-moments estimates in $H^6(\mathcal{D})$ (in supremum in time) would yield $L^\infty(\mathcal{D})$ bounds in $d = 2, 3$ for the bi-Laplacian and all the lower order terms of the stochastic C-H, and thus, by integrating the equation in (t, s) for all $t < s$, α-Hölder continuity in time for some $\alpha < 1/2$, and so on.

In Section 3, we consider layered and thus ε-dependent u_0 where the initial energy (on u_0) is uniformly bounded in ε; this condition has been proposed by X.F. Chen in [11] for the scaling of (1.1). A direct application of the general H^1 2d-moment estimate establishes the stochastic solution's convergence to ± 1 as $\varepsilon \to 0$ a.s., when the noise diffusion has a linear growth, see Theorem 3.1. Additionally, we present some interesting simple cases for the noise where the solution is mass conservative and the estimates proven are valid.

We have also inserted an Appendix presenting the version of Burkholder-Davis-Gundy inequality used in our proofs, and the stochastic Gronwall's Lemma.

2. p-Moment estimates in ε for non layered initial data

2.1. **Preliminaries.** Let $(\cdot,\cdot)$ be the $L^2(\mathcal{D})$-inner product, $\|\cdot\|$ the induced $L^2(\mathcal{D})$-norm, and $H^k(\mathcal{D})$, $k\in\mathbb{N}$ the usual Sobolev spaces on $\mathcal{D}$. Let also $\|\cdot\|_{H^k}$ denote the $H^k(\mathcal{D})$-norm and $\|\cdot\|_\infty$ the $\|\cdot\|_{L^\infty(\mathcal{D})}$-norm.

Our aim, is to derive estimates for the higher moments in the $L^2(\mathcal{D})$, $H^1(\mathcal{D})$ and $H^2(\mathcal{D})$ norms for the solution u of (1.1), when the noise diffusion $\sigma(u)$ satisfies certain properties. Higher moments in the $L^\infty(\mathcal{D})$-norm will then follow by the H^1 and H^2 estimates in dimensions $d=1$ and $d=2,3$ respectively. These estimates will depend on ε and will be valid for u_0 independent of ε.

For the rest of this paper, the notation d will correspond to differentiation with respect to t, and c will be used for generic constants.

Equation (1.1) can be written as

$$du = L(u,\varepsilon)dt + \varepsilon^\gamma\sigma(u)dW, \tag{2.1}$$

for

$$L(u,\varepsilon) := \Delta\Big(-\varepsilon\Delta u + \frac{1}{\varepsilon}f(u)\Big). \tag{2.2}$$

By using the Itô calculus identities, see for example in [7]

$$d\beta_i d\beta_j = \delta_{ij}dt, \quad d\beta_i dt = dt d\beta_i = dt dt = 0dt, \tag{2.3}$$

and the noise definition (1.3), we easily obtain that

$$dW = \sum_{i=1}^\infty a_i d\beta_i(t)e_i(x), \quad dt dW = dW dt = 0dt. \tag{2.4}$$

Therefore, by taking the $L^2(\mathcal{D})$ inner product, (2.1) yields

$$(du,du) = \varepsilon^{2\gamma}\sum_{i=1}^\infty a_i^2\|\sigma(u)e_i\|^2 dt. \tag{2.5}$$

2.2. **L^2 and H^1 2d moment.** We define the functional

$$\widetilde{F}(v) := \int_{\mathcal{D}}\Big(F(v)+\frac{\varepsilon^2}{2}(\nabla v)^2\Big)dx = \int_{\mathcal{D}}F(v)dx + \frac{\varepsilon^2}{2}\|\nabla v\|^2, \tag{2.6}$$

for

$$F'(v) = f(v).$$

The next lemma presents a useful bound for $\widetilde{F}(u)$ when the Fourier noise is sufficiently regular.

Lemma 2.1. *Let u be the solution of* (1.1), *$\widetilde{F}(u)$ the functional defined by* (2.6), *and let the noise satisfy the next regularity assumptions*

$$\sum_{i=1}^\infty a_i^2\|e_i\|_{H^2}^2 < \infty \quad \textit{when} \;\; d=1, \quad \textit{and} \quad \sum_{i=1}^\infty a_i^2\|e_i\|_{H^3}^2 < \infty \quad \textit{when} \;\; d=2,3. \tag{2.7}$$

Then, it holds that

$$
\begin{aligned}
&\widetilde{F}(u(t))+\frac{1}{\varepsilon}\int_0^t \|\nabla[f(u)-\varepsilon^2\Delta u]\|^2 ds\\
&\quad\leq \int_{\mathcal{D}} F(u_0)dx+\frac{\varepsilon^2}{2}\|\nabla u_0\|^2+\frac{\varepsilon^{2\gamma}}{2}\int_0^t (f'(u)\sigma(u)dW,\sigma(u)dW)\\
&\qquad + c\varepsilon^{2+2\gamma}\int_0^t (\|\nabla(\sigma(u))\|^2+\|\sigma(u)\|^2)ds+\int_0^t (f(u)-\varepsilon^2\Delta u,\varepsilon^{\gamma}\sigma(u)dW).
\end{aligned}
\tag{2.8}
$$

Proof. Itô formula applied on the potential F yields

$$
d(F(u)) = F'(u)du+\frac{1}{2}F''(u)dudu. \tag{2.9}
$$

Indeed, observe that by the Fourier noise series and (2.3), (2.4), we get

$$
dWdW=\Big(\sum_{i=1}^{\infty} a_i^2 e_i^2(x)\Big)dt, \quad dWdWdt=0dt,
$$

and thus, by (2.1)

$$
dududu = dudududu = 0dt.
$$

So, it follows that

$$
\begin{aligned}
&d(u^2)=d(uu)=2udu+dudu,\\
&d(u^4)=d(u^2u^2)=2d(u^2)u^2+d(u^2)d(u^2)=2(2udu+dudu)u^2+4u^2dudu.
\end{aligned}
$$

Using that $F(u)=\frac{1}{4}(u^2-1)^2$, $F'(u)=u^3-u$, $F''(u)=3u^2-1$, and the above, we obtain (2.9).

Moreover, we have

$$
\begin{aligned}
&F'(u)=f(u), \quad F''(u)=f'(u),\\
&(d\nabla u,\nabla u)=-(du,\Delta u), \quad d(\nabla u\nabla u)=2(d\nabla u)\nabla u+(d\nabla u)(d\nabla u).
\end{aligned}
\tag{2.10}
$$

By (2.9) and (2.10), we get

$$
\begin{aligned}
d(\widetilde{F}(u)) =& d\Big(\int_{\mathcal{D}}\Big(F(u)+\frac{\varepsilon^2}{2}(\nabla u)^2\Big)dx\Big)=\int_{\mathcal{D}} d\Big(F(u)+\frac{\varepsilon^2}{2}(\nabla u)^2\Big)dx\\
=&\int_{\mathcal{D}}\Big[d(F(u))+2\frac{\varepsilon^2}{2}(d\nabla u)\nabla u+\frac{\varepsilon^2}{2}(d\nabla u)(d\nabla u)\Big]dx\\
=&\int_{\mathcal{D}}\Big[F'(u)du+\frac{1}{2}F''(u)dudu-\varepsilon^2du\Delta u+\frac{\varepsilon^2}{2}(d\nabla u)(d\nabla u)\Big]dx\\
=&\int_{\mathcal{D}}\Big[f(u)du+\frac{1}{2}f'(u)dudu-\varepsilon^2du\Delta u+\frac{\varepsilon^2}{2}(d\nabla u)(d\nabla u)\Big]dx\\
=&(f(u)-\varepsilon^2\Delta u,du)+\frac{\varepsilon^2}{2}(d\nabla u,d\nabla u)+\frac{1}{2}(f'(u)du,du).
\end{aligned}
\tag{2.11}
$$

Using (2.2), and (2.1), we arrive at

$$
\begin{aligned}
\frac{1}{2}(f'(u)du,du)=&\frac{1}{2}(f'(u)[L(u,\varepsilon)dt+\varepsilon^{\gamma}\sigma(u)dW],L(u,\varepsilon)dt+\varepsilon^{\gamma}\sigma(u)dW)\\
=&\frac{\varepsilon^{2\gamma}}{2}(f'(u)\sigma(u)dW,\sigma(u)dW).
\end{aligned}
\tag{2.12}
$$

Replacing (2.12) in (2.11), we derive the next fundamental identity

$$d(\widetilde{F}(u)) = (f(u) - \varepsilon^2\Delta u, du) + \frac{\varepsilon^2}{2}(d\nabla u, d\nabla u) + \frac{\varepsilon^{2\gamma}}{2}(f'(u)\sigma(u)dW, \sigma(u)dW). \tag{2.13}$$

In case of a logarithmic potential, an identity analogous to (2.13) is proven in [24]. We also refer to some general identities of [22] proven for $\mathcal{Q} := I$ i.e., when the trace is infinite since $\sum_{i=1}^{\infty} a_i^2\|e_i\|_{L^2}^2 = \sum_{i=1}^{\infty} 1 = \infty$ (case which is obviously not satisfying the regularity assumptions of this lemma).

Using (2.13), and replacing du by its operator form, we have

$$d(\widetilde{F}(u)) = A_1 + A_2 + \frac{\varepsilon^{2\gamma}}{2}(f'(u)\sigma(u)dW, \sigma(u)dW), \tag{2.14}$$

for

$$A_1 := (f(u) - \varepsilon^2\Delta u, L(u,\varepsilon)dt + \varepsilon^\gamma\sigma(u)dW), \quad \text{and} \quad A_2 := \frac{\varepsilon^2}{2}(d\nabla u, d\nabla u).$$

Estimate of A_1:

Replacing the operator L, integrating by parts, and using the boundary conditions, we obtain the next identity

$$\begin{aligned} A_1 :=& (f(u) - \varepsilon^2\Delta u, L(u,\varepsilon)dt + \varepsilon^\gamma\sigma(u)dW) \\ =& \Big(f(u) - \varepsilon^2\Delta u, \Delta\Big(-\varepsilon\Delta u + \frac{1}{\varepsilon}f(u)\Big)dt + \varepsilon^\gamma\sigma(u)dW\Big) \\ =& -\frac{1}{\varepsilon}\|\nabla(f(u) - \varepsilon^2\Delta u)\|^2 dt + (f(u) - \varepsilon^2\Delta u, \varepsilon^\gamma\sigma(u)dW). \end{aligned} \tag{2.15}$$

Estimate of A_2:

We observe first that

$$\nabla dW = \sum_{i=1}^{\infty} a_i d\beta_i \nabla e_i,$$

while

$$d\nabla u = (\nabla L)dt + \varepsilon^\gamma\nabla(\sigma(u)dW) = (\nabla L)dt + \varepsilon^\gamma\nabla(\sigma(u))dW + \varepsilon^\gamma\sigma(u)\nabla dW,$$

and

$$dt\nabla(\sigma(u)dW) = dt\nabla dW = 0,$$

which yield

$$\begin{aligned} (d\nabla u, d\nabla u) &= \varepsilon^{2\gamma}(\nabla[\sigma(u)dW], \nabla[\sigma(u)dW]) \\ &= \varepsilon^{2\gamma}\sum_{i=1}^{\infty} a_i^2[\|\nabla\sigma(u)e_i\|^2 + \|\sigma(u)\nabla e_i\|^2 + 2(\nabla(\sigma(u))e_i, \sigma(u)\nabla e_i)]dt \\ &\leq c\varepsilon^{2\gamma}\sum_{i=1}^{\infty} a_i^2\|\nabla\sigma(u)e_i\|^2 dt + c\varepsilon^{2\gamma}\sum_{i=1}^{\infty} a_i^2\|\sigma(u)\nabla e_i\|^2 dt. \end{aligned}$$

Thus, we get

$$\begin{aligned} A_2 := \frac{\varepsilon^2}{2}(d\nabla u, d\nabla u) \leq & c\varepsilon^{2+2\gamma}\sum_{i=1}^{\infty} a_i^2\|\nabla\sigma(u)e_i\|^2 dt + c\varepsilon^{2+2\gamma}\sum_{i=1}^{\infty} a_i^2\|\sigma(u)\nabla e_i\|^2 dt \\ \leq & c\varepsilon^{2+2\gamma}\|\nabla\sigma(u)\|^2\sum_{i=1}^{\infty} a_i^2\|e_i\|_\infty^2 dt + c\varepsilon^{2+2\gamma}\|\sigma(u)\|^2\sum_{i=1}^{\infty} a_i^2\|\nabla e_i\|_\infty^2 dt. \end{aligned} \tag{2.16}$$

Due to the assumed noise regularity, we obtain for $d = 1, 2, 3$

$$\sum_{i=1}^{\infty} a_i^2\|e_i\|_\infty^2 < \infty, \quad \text{and} \quad \sum_{i=1}^{\infty} a_i^2\|\nabla e_i\|_\infty^2 < \infty, \tag{2.17}$$

and therefore,

$$A_2 := \frac{\varepsilon^2}{2}(d\nabla u, d\nabla u) \leq c[\|\sigma(u)\|^2 + \|\nabla\sigma(u)\|^2]\varepsilon^{2+2\gamma}dt. \tag{2.18}$$

Using (2.15), and (2.18) in (2.14) (integrating first in $[0, t]$), we get (2.8) as follows

$$\begin{aligned} &\widetilde{F}(u(t)) + \frac{1}{\varepsilon}\int_0^t \|\nabla[f(u) - \varepsilon^2\Delta u]\|^2 ds = \int_{\mathcal{D}} F(u(t))dx + \frac{\varepsilon^2}{2}\|\nabla u(t)\|^2 + \frac{1}{\varepsilon}\int_0^t \|\nabla[f(u) - \varepsilon^2\Delta u]\|^2 ds \\ &\leq \int_{\mathcal{D}} F(u_0)dx + \frac{\varepsilon^2}{2}\|\nabla u_0\|^2 + \int_0^t (f(u) - \varepsilon^2\Delta u, \varepsilon^\gamma\sigma(u)dW) \\ &\quad + c\varepsilon^{2+2\gamma}\int_0^t [\|\nabla(\sigma(u))\|^2 + \|\sigma(u)\|^2]ds + \frac{\varepsilon^{2\gamma}}{2}\int_0^t (f'(u)\sigma(u)dW, \sigma(u)dW). \end{aligned}$$

□

The next Main Theorem estimates in expectation the functional $\widetilde{F}(u)$, and so, the second moment in H^1 (resulting also to an L^2 second moment estimate).

Theorem 2.1. *Let u be the solution of the stochastic Cahn-Hilliard equation* (1.1)*. If the next conditions hold true*

$$|\sigma^2(v)f'(v)| \leq cF(v) + c, \tag{2.19}$$

and

$$\|\sigma(v)\|^2 + \|\nabla(\sigma(v))\|^2 \leq c\int_{\mathcal{D}} F(v)dx + c\|\nabla v\|^2 + c, \tag{2.20}$$

uniformly for any v, and

$$\sum_{i=1}^{\infty} a_i^2\|e_i\|_{H^2}^2 < \infty \quad \text{when} \quad d = 1, \quad \text{or} \quad \sum_{i=1}^{\infty} a_i^2\|e_i\|_{H^3}^2 < \infty \quad \text{when} \quad d = 2, 3, \tag{2.21}$$

then for $d = 1, 2, 3$, u satisfies for any $t > 0$

$$\begin{aligned} \mathbb{E}(\widetilde{F}(u(t))) \leq & \mathbb{E}\Big(\int_{\mathcal{D}} F(u(t))dx\Big) + \frac{\varepsilon^2}{2}\mathbb{E}(\|\nabla u(t)\|^2) + \frac{1}{\varepsilon}\mathbb{E}\Big(\int_0^t \|\nabla[f(u) - \varepsilon^2\Delta u]\|^2 ds\Big) \\ \leq & c\varepsilon^{2\gamma} + c\mathbb{E}\Big(\int_{\mathcal{D}} F(u_0)dx\Big) + c\varepsilon^2\mathbb{E}(\|\nabla u_0\|^2), \end{aligned} \tag{2.22}$$

and thus (since $(|u|-1)^2\leq 4F(u)$ *and therefore* $\|u\|^2\leq c\int_{\mathcal{D}}F(u)dx+c$*)*

$$\mathbb{E}(\|u(t)\|^2)\leq c+c\varepsilon^{2\gamma}+c\mathbb{E}\Big(\int_{\mathcal{D}}F(u_0)dx\Big)+c\varepsilon^2\mathbb{E}(\|\nabla u_0\|^2), \tag{2.23}$$

while

$$\mathbb{E}(\|\nabla u(t)\|^2)\leq c\frac{\varepsilon^{2\gamma}}{\varepsilon^2}+\frac{c}{\varepsilon^2}\mathbb{E}\Big(\int_{\mathcal{D}}F(u_0)dx\Big)+c\mathbb{E}(\|\nabla u_0\|^2). \tag{2.24}$$

Proof. We observe first, that

$$(\sigma(u)f'(u)dW,\sigma(u)dW)=\sum_{i=1}^{\infty}a_i^2(e_i^2,\sigma^2(u)f'(u))dt.$$

So, using that $f(u)=u^3-u$, we have $f'(u)=3u^2-1$, and thus

$$\sigma^2(u)f'(u)=3\sigma^2(u)u^2-\sigma^2(u).$$

Therefore, we have

$$|\sigma^2(u)f'(u)|\leq 3\sigma^2(u)u^2+\sigma^2(u)=\sigma^2(u)(3u^2-1)+2\sigma^2(u).$$

The above and the noise regularity yield

$$\begin{aligned}(\sigma(u)f'(u)dW,\sigma(u)dW)=&\sum_{i=1}^{\infty}a_i^2(e_i^2,\sigma(u)^2f'(u))dt\\ \leq&\sum_{i=1}^{\infty}a_i^2\|e_i\|_\infty^2\int_{\mathcal{D}}|\sigma^2(u)f'(u)|dxdt\\ =&c\Big[\int_{\mathcal{D}}|\sigma^2(u)f'(u)|dx\Big]dt.\end{aligned} \tag{2.25}$$

This inequality motivates the condition (2.19) $|\sigma^2(u)f'(u)|\leq cF(u)+c$, since this term will be hidden at the left-hand side of our next estimate, together with the term $c\varepsilon^{2+2\gamma}\int_0^t(\|\nabla(\sigma(u))\|^2+\|\sigma(u)\|^2)ds$ of (2.8) which motivated the condition (2.20) $\|\sigma(u)\|^2+\|\nabla\sigma(u)\|^2\leq c\int_{\mathcal{D}}F(u)dx+c\|\nabla u\|^2+c$.

We use (2.8), which is valid due to the assumed noise regularity, and take expectation. Then the assumptions (2.19), (2.20) on σ, and relation (2.25), yield

$$\begin{aligned}&\mathbb{E}\Big(\int_{\mathcal{D}}F(u(t))dx+\frac{\varepsilon^2}{2}\|\nabla u(t)\|^2+\frac{1}{\varepsilon}\int_0^t\|\nabla[f(u)-\varepsilon^2\Delta u]\|^2ds\Big)\\ &\quad\leq\mathbb{E}\Big(\int_{\mathcal{D}}F(u_0)dx+\frac{\varepsilon^2}{2}\|\nabla u_0\|^2\Big)+\frac{\varepsilon^{2\gamma}}{2}\mathbb{E}\Big(\int_0^t(\sigma(u)f'(u)dW,\sigma(u)dW)\Big)\\ &\qquad+c\varepsilon^{2+2\gamma}\mathbb{E}\Big(\int_0^t\Big[\int_{\mathcal{D}}F(u(t))dx+\|\nabla u\|^2\Big]ds\Big)+c\varepsilon^{2+2\gamma}+0\\ &\quad\leq\mathbb{E}\Big(\int_{\mathcal{D}}F(u_0)dx+\frac{\varepsilon^2}{2}\|\nabla u_0\|^2\Big)+c\varepsilon^{2\gamma}\Big(\int_0^t\Big[\int_{\mathcal{D}}F(u(t))dx+\varepsilon^2\|\nabla u\|^2\Big]ds\Big)+c\varepsilon^{2+2\gamma}+c\varepsilon^{2\gamma},\end{aligned}$$

and therefore, by Gronwall's lemma, we get the result

$$\begin{aligned}\mathbb{E}\Big(\int_{\mathcal{D}} F(u(t))dx + &\varepsilon^2\|\nabla u(t)\|^2 + \frac{1}{\varepsilon}\int_0^t \|\nabla[f(u) - \varepsilon^2\Delta u]\|^2 ds\Big)\\ &\leq c\mathbb{E}\Big(\int_{\mathcal{D}} F(u_0)dx + c\varepsilon^2\|\nabla u_0\|^2\Big) + c\varepsilon^{2\gamma},\end{aligned}$$

since $\exp(\varepsilon^{2\gamma t})$ is bounded as ε tends to zero. □ □

Remark 2.2. *We note that* (2.19)

$$|\sigma^2(v)f'(v)| \leq cF(v) + c,$$

can be implemented, as f is a polynomial of third order, when for example σ has a linear growth, i.e., satisfies the growth condition

$$|\sigma(v)| \leq c|v|^\alpha + c, \quad \alpha \leq 1, \tag{2.26}$$

uniformly for any v, since $|v|^3 \leq cv^4 + c_1 = c(v^4 - 2v^2 + 1) + 2cv^2 - c + c_1 \leq c_2F(v) + \frac{1}{2}|v|^3 + c_2$.

Moreover, (2.20)

$$\|\sigma(v)\|^2 + \|\nabla(\sigma(v))\|^2 \leq c\int_{\mathcal{D}} F(v)dx + c\|\nabla v\|^2 + c,$$

is satisfied when for example σ has the growth (2.26)*, and additionally the derivative of σ satisfies*

$$|\sigma'(v)| \leq c, \tag{2.27}$$

uniformly for any v.

Remark 2.3. *In the proof of the previous Theorem 2.1 we only needed*

$$\sum_{i=1}^{\infty} a_i^2\|e_i\|_\infty^2 < \infty,$$

for which sufficient are the conditions in (2.21)*.*

Remark 2.4. *The H^1 estimate provided from the previous Theorem yields an L^∞ estimate in dimensions $d = 1$.*

2.3. **Higher moments in L^2, H^1, and H^2.** In order to derive second and higher moments estimates for the supremum in time in H^1 and thus in L^2 norm (in dimensions $2, 3$), we will avoid Itô calculus and will use directly Burkholder-Davis-Gundy inequality. Moreover, we will use the time integral of $\|u\|_{H^2}^2$ norm appearing in Lemma 2.5. Due to the power 2 there a bounded noise diffusion assumption

$$|\sigma(u)| < \text{const},$$

will be essential for deriving H^2 bounds in $d = 2, 3$ and so L^∞ bounds for this case. We point out that the same term restricted the result of a.s. continuous solutions, when non-smooth in space and in time noise was used in [6], in dimensions only $d = 1$, when the problem involved unbounded noise diffusion.

An analogous argument will be applied when treating estimates in the H^2 norm, where the time integral of $\|\Delta^2 u\|^2$ will appear at the left-hand side of our estimates and will bound some terms stemming from the noise.

We present first the following useful technical Lemma.

Lemma 2.5. *Let u be the solution of the stochastic Cahn-Hilliard equation* (1.1)*. If the next conditions hold true*

$$|\sigma(v)| \leq c|v|^{\alpha} + c, \quad \alpha \leq \frac{1}{2}, \tag{2.28}$$

uniformly for any v, and

$$\sum_{i=1}^{\infty} a_i^2 \|e_i\|_{\infty}^2 < \infty, \tag{2.29}$$

then in dimensions $d = 1, 2, 3$, u satisfies for any $t > 0$ and any $p \geq 1$

$$\begin{aligned} &\mathbb{E}\Big(\sup_{0\leq t\leq T} \|u\|^{2p}\Big) + \varepsilon^p \mathbb{E}\Big(\Big(\int_0^T \|u\|_{H^2}^2 dt\Big)^p\Big) \\ \leq & c\varepsilon^{-3p}\mathbb{E}\Big(\Big(\int_0^T \|u\|^2 dt\Big)^p\Big) + c(\varepsilon^{p\gamma} + \varepsilon^{2\gamma p - p}) + c\mathbb{E}(\|u_0\|^{2p}), \end{aligned} \tag{2.30}$$

for some $c > 0$.

Proof. We write (1.1) as

$$du = L(u, \varepsilon)dt + \varepsilon^{\gamma}\sigma(u)dW,$$

and derive after integration by parts

$$\begin{aligned} d(u, u) =& 2(du, u) + (du, du) = 2(du, u) + \varepsilon^{2\gamma} \sum_{i=1}^{\infty} a_i^2 \|\sigma(u)e_i\|^2 dt \\ =& -2\varepsilon\|\Delta u\|^2 dt - \frac{2}{\varepsilon}(f'(u)\nabla u, \nabla u)dt + 2\varepsilon^{\gamma}(\sigma(u)dW, u) + \varepsilon^{2\gamma} \sum_{i=1}^{\infty} a_i^2 \|\sigma(u)e_i\|^2 dt. \end{aligned} \tag{2.31}$$

Thus, using (2.31), we obtain for $c_0 > 0$ as small we want

$$\begin{aligned} d(u, u) + 2\varepsilon\|\Delta u\|^2 dt =& -\frac{2}{\varepsilon}(f'(u)\nabla u, \nabla u)dt + \varepsilon^{2\gamma} \sum_{i=1}^{\infty} a_i^2 \|\sigma(u)e_i\|^2 dt + 2\varepsilon^{\gamma}(\sigma(u)u, dW) \\ \leq & c_0\varepsilon\|\Delta u\|^2 dt + \frac{c}{\varepsilon^3}\|u\|^2 dt + c\varepsilon^{2\gamma}\|\sigma(u)\|^2 dt + 2\varepsilon^{\gamma}(\sigma(u)u, dW). \end{aligned} \tag{2.32}$$

For the derivation of the above, we used that by Young's inequality

$$\begin{aligned} -\frac{2}{\varepsilon}(f'(u)\nabla u, \nabla u)dt =& -\frac{2}{\varepsilon}((3u^2 - 1)\nabla u, \nabla u)dt \\ \leq & \frac{2}{\varepsilon}(\nabla u, \nabla u)dt = -\frac{2}{\varepsilon}(u, \Delta u)dt \\ \leq & c_0\varepsilon\|\Delta u\|^2 dt + \frac{c}{\varepsilon^3}\|u\|^2 dt, \end{aligned}$$

and the relation

$$\sum_{i=1}^{\infty} a_i^2 \|\sigma(u)e_i\|^2 dt \leq \sum_{i=1}^{\infty} a_i^2 \|\sigma(u)\|^2 \|e_i\|_{\infty}^2 dt < c\|\sigma(u)\|^2,$$

which is valid due to condition (2.29).

By (2.28), and since $\alpha \leq \frac{1}{2} \leq 1$, we get

$$\|\sigma(u)\|^2 \leq c + c\|u\|^2.$$

We use the previous inequality in (2.32), and arrive at

$$\begin{aligned} d(u,u)+\varepsilon\|\Delta u\|^2 dt \leq & \frac{c}{\varepsilon^3}\|u\|^2 dt + c\varepsilon^{2\gamma}\|u\|^2 dt + c\varepsilon^{2\gamma}dt + c\varepsilon^{\gamma}(\sigma(u)u, dW) \\ \leq & \frac{c}{\varepsilon^3}\|u\|^2 dt + c\varepsilon^{2\gamma}dt + c\varepsilon^{\gamma}(\sigma(u)u, dW). \end{aligned} \tag{2.33}$$

Integration in $[0,t]$ for $0 \leq t \leq T$ yields

$$\|u\|^2 + \varepsilon\int_0^t \|\Delta u\|^2 ds \leq \frac{c}{\varepsilon^3}\int_0^t \|u\|^2 ds + c\varepsilon^{\gamma}\int_0^t (\sigma(u)u, dW) + \|u_0\|^2 + cT\varepsilon^{2\gamma}. \tag{2.34}$$

So, adding in both sides the term $\varepsilon\int_0^t \|u\|^2 ds$, using that $c\varepsilon^{-3}+c\varepsilon \leq c\varepsilon^{-3}$, and the fact that due to the boundary conditions of u, $\|u\|+\|\Delta u\|$ is a norm equivalent to $\|u\|_{H^2}$, we obtain

$$\|u\|^2 + \varepsilon\int_0^t \|u\|^2_{H^2} ds \leq \frac{c}{\varepsilon^3}\int_0^t \|u\|^2 ds + c\varepsilon^{\gamma}\int_0^t (\sigma(u)u, dW) + c\|u_0\|^2 + c\varepsilon^{2\gamma}.$$

Taking supremum for $0 \leq t \leq T$, we have

$$\sup_{0\leq t\leq T}\|u\|^2 + \varepsilon\int_0^T \|u\|^2_{H^2} ds \leq \frac{c}{\varepsilon^3}\int_0^T \|u\|^2 ds + c\varepsilon^{\gamma}\sup_{0\leq t\leq T}\Big|\int_0^t (\sigma(u)u, dW)\Big| + c\|u_0\|^2 + c\varepsilon^{2\gamma}. \tag{2.35}$$

We use the main inequality (2.35) as follows: first, we take p-powers at both sides, and then expectation, and also apply Burkholder-Davis-Gundy inequality for the stochastic integral (note that by (2.29), we have as e_i form an orthonormal basis, that $\sum_{i=1}^{\infty} a_i^2 \leq c\sum_{i=1}^{\infty} a_i^2\|e_i\|^2_{\infty} < \infty$). This yields

$$\begin{aligned} &\mathbb{E}\Big(\sup_{0\leq t\leq T}\|u\|^{2p}\Big) + \varepsilon^p\mathbb{E}\Big(\Big(\int_0^T \|u\|^2_{H^2} dt\Big)^p\Big) \\ &\leq c\varepsilon^{-3p}\mathbb{E}\Big(\Big(\int_0^T \|u\|^2 dt\Big)^p\Big) + c\varepsilon^{p\gamma}\mathbb{E}\Big(\sup_{0\leq t\leq T}\Big|\int_0^t (\sigma(u)u, dW)\Big|^p\Big) + c\mathbb{E}(\|u_0\|^{2p}) + c\varepsilon^{2\gamma p} \\ &\leq c\varepsilon^{-3p}\mathbb{E}\Big(\Big(\int_0^T \|u\|^2 dt\Big)^p\Big) + c\varepsilon^{p\gamma}\mathbb{E}\Big(\Big(\int_0^T \|\sigma(u)u\|^2 ds)\Big)^{p/2}\Big) + c\mathbb{E}(\|u_0\|^{2p}) + c\varepsilon^{2\gamma p}. \end{aligned} \tag{2.36}$$

We observe now that since $\alpha \leq \frac{1}{2}$,

$$\|\sigma(u)u\|^2 \leq c\|u\|^2_{H^2} + c\sup_{0\leq t\leq T}\|u\|^4 + c. \tag{2.37}$$

Indeed, since $1+2\alpha \leq 2$, and $\|u\|_\infty \leq c\|u\|_{H^2}$, we have

$$\begin{aligned} \|\sigma(u)u\|^2 \leq & \int_{\mathcal{D}} |u|^{2+2\alpha} dx + c \leq \|u\|_{H^2}\int_{\mathcal{D}} |u|^{1+2\alpha} dx + c \\ \leq & c\|u\|^2_{H^2} + c\Big(\int_{\mathcal{D}} |u|^{1+2\alpha} dx\Big)^2 + c \\ \leq & c\|u\|^2_{H^2} + c\|u\|^4 + c \leq c\|u\|^2_{H^2} + c\sup_{0\leq t\leq T}\|u\|^4 + c. \end{aligned}$$

Hence, since $4p/2 = 2p$, we get for $c_0 > 0$ as small

$$
\begin{aligned}
&c\varepsilon^{p\gamma}\mathbb{E}\Big(\Big(\int_0^T \|\sigma(u)u\|^2 dt\Big)^{p/2}\Big)\\
&\leq c\varepsilon^{p\gamma}\mathbb{E}\Big(\Big(\int_0^T \|u\|_{H^2}^2 dt\Big)^{p/2}\Big) + c\varepsilon^{p\gamma}T\mathbb{E}\Big(\sup_{0\leq t\leq T}\|u\|^{2p}\Big) + c\varepsilon^{p\gamma}\\
&= c\varepsilon^{p\gamma-p/2}\varepsilon^{p/2}\mathbb{E}\Big(\Big(\int_0^T \|u\|_{H^2}^2 dt\Big)^{p/2}\Big) + c\varepsilon^{p\gamma}T\mathbb{E}\Big(\sup_{0\leq t\leq T}\|u\|^{2p}\Big) + c\varepsilon^{p\gamma}\\
&\leq c_0\varepsilon^{p}\mathbb{E}\Big(\Big(\int_0^T \|u\|_{H^2}^2 dt\Big)^{p}\Big) + c_0\mathbb{E}\Big(\sup_{0\leq t\leq T}\|u\|^{2p}\Big) + c(\varepsilon^{p\gamma} + \varepsilon^{2\gamma p-p}),
\end{aligned}
$$

where we used the fact that ε is small enough, so that $c\varepsilon^{p\gamma}T \leq c_0$.

Using now the above estimate in (2.36), we obtain

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leq t\leq T}\|u\|^{2p}\Big) + \varepsilon^p\mathbb{E}\Big(\Big(\int_0^T \|u\|_{H^2}^2 dt\Big)^{p}\Big)\\
&\leq c\varepsilon^{-3p}\mathbb{E}\Big(\Big(\int_0^T \|u\|^2 dt\Big)^{p}\Big) + c\varepsilon^{p\gamma}\mathbb{E}\Big(\Big(\int_0^T \|\sigma(u)u\|^2 ds)\Big)^{p/2}\Big) + c\mathbb{E}(\|u_0\|^{2p}) + c\varepsilon^{2\gamma p}\\
&\leq c\varepsilon^{-3p}\mathbb{E}\Big(\Big(\int_0^T \|u\|^2 dt\Big)^{p}\Big) + c_0\varepsilon^{p}\mathbb{E}\Big(\Big(\int_0^T \|u\|_{H^2}^2 dt\Big)^{p}\Big) + c_0\mathbb{E}\Big(\sup_{0\leq t\leq T}\|u\|^{2p}\Big)\\
&\quad + c(\varepsilon^{p\gamma} + \varepsilon^{2\gamma p-p}) + c\mathbb{E}(\|u_0\|^{2p}) + c\varepsilon^{2\gamma p},
\end{aligned}
$$

which yields the result, since c_0 is as small we want, i.e.,

$$
\mathbb{E}\Big(\sup_{0\leq t\leq T}\|u\|^{2p}\Big) + \varepsilon^p\mathbb{E}\Big(\Big(\int_0^T \|u\|_{H^2}^2 dt\Big)^{p}\Big) \leq c\varepsilon^{-3p}\mathbb{E}\Big(\Big(\int_0^T \|u\|^2 dt\Big)^{p}\Big) + c(\varepsilon^{p\gamma} + \varepsilon^{2\gamma p-p}) + c\mathbb{E}(\|u_0\|^{2p}).
$$

□

The next lemma involves the p-moments of the functional $\widetilde{F}(u) = \int_{\mathcal{D}} F(u)dx + \frac{\varepsilon^2}{2}\|\nabla u\|^2$ in supremum.

Lemma 2.6. *Under the assumptions of Theorem 2.1, it holds that*

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leq t\leq T}\Big(\int_{\mathcal{D}} F(u(t))dx + \frac{\varepsilon^2}{2}\|\nabla u(t)\|^2\Big)^p\Big) + \frac{1}{\varepsilon^p}\mathbb{E}\Big(\Big(\int_0^T \|\nabla[f(u) - \varepsilon^2\Delta u]\|^2 ds\Big)^p\Big)\\
&\leq c\mathbb{E}\Big(\Big(\int_{\mathcal{D}} F(u_0)dx\Big)^p\Big) + c\varepsilon^{2p}\mathbb{E}\Big(\|\nabla u_0\|^{2p}\Big)\\
&\quad + \varepsilon^{p\gamma}\mathbb{E}\Big(\sup_{0\leq t\leq T}\Big|\int_0^t (f(u),\sigma(u)dW)\Big|^p\Big) + c\varepsilon^{2p+p\gamma}\mathbb{E}\Big(\sup_{0\leq t\leq T}\Big|\int_0^t (\Delta u,\sigma(u)dW)\Big|^p\Big).
\end{aligned}
\tag{2.38}
$$

Proof. We return to (2.8), take absolute values, then p power, and then supremum. In details, we obtain

$$\begin{aligned}
&\Big(\int_{\mathcal{D}}F(u(t))dx+\frac{\varepsilon^2}{2}\|\nabla u(t)\|^2\Big)^p+\varepsilon^{-p}\Big(\int_0^t\|\nabla[f(u)-\varepsilon^2\Delta u]\|^2ds\Big)^p\\
\le& c\Big(\int_{\mathcal{D}}F(u_0)dx\Big)^p+c\varepsilon^{2p}\|\nabla u_0\|^{2p}+c\varepsilon^{2\gamma p}\Big|\int_0^t(\sigma(u)f'(u)dW,\sigma(u)dW)ds\Big|^p\\
&+c\varepsilon^{2p+2p\gamma}\Big(\int_0^t[\|\nabla(\sigma(u))\|^2+\|\sigma(u)\|^2]ds\Big)^p+c\Big|\int_0^t(f(u),\varepsilon^\gamma\sigma(u)dW)\Big|^p\\
&+c\Big|\int_0^t(\Delta u,\varepsilon^{\gamma+2}\sigma(u)dW)\Big|^p,
\end{aligned}$$

and so

$$\begin{aligned}
&\Big(\int_{\mathcal{D}}F(u(t))dx+\frac{\varepsilon^2}{2}\|\nabla u(t)\|^2\Big)^p+\varepsilon^{-p}\Big(\int_0^t\|\nabla[f(u)-\varepsilon^2\Delta u]\|^2ds\Big)^p\\
\le& c\Big(\int_{\mathcal{D}}F(u_0)dx\Big)^p+\varepsilon^{2p}\|\nabla u_0\|^{2p}+c\varepsilon^{2\gamma p}T^p\Big(\sup_{0\le t\le T}\int_{\mathcal{D}}|\sigma^2(u)f'(u)|dx\Big)^p\\
&+c\varepsilon^{p(2+2\gamma)}T^p\Big(\sup_{0\le t\le T}(\|\nabla(\sigma(u))\|^2+\|\sigma(u)\|^2)\Big)^p\\
&+\varepsilon^{p\gamma}\sup_{0\le t\le T}\Big|\int_0^t(f(u),\sigma(u)dW)\Big|^p+c\varepsilon^{2p+p\gamma}\sup_{0\le t\le T}\Big|\int_0^t(\Delta u,\sigma(u)dW)\Big|^p.
\end{aligned}$$

We take again supremum, then by hiding terms at the left side and taking expectation the result follows. ∎

Remark 2.7. *In view of the bound provided by the previous lemma, we see that in order to derive higher moment estimates in L^2 and H^1, in supremum, we need to control the p-moments of the noise terms at the right. This is achieved by the next Main Theorem.*

Theorem 2.2. *Let u be the solution of the stochastic Cahn-Hilliard* (1.1)*. If the next conditions hold true*

$$|\sigma(v)|\le c, \tag{2.39}$$

and

$$\|\sigma(v)\|_{H^2}^2\le c\int_{\mathcal{D}}F(v)dx+c\|\nabla v\|^2+c, \tag{2.40}$$

uniformly for any v, and

$$\sum a_i^2\|e_i\|_{H^2}^2<\infty\quad\text{for}\quad d=1,\quad\text{or}\quad\sum a_i^2\|e_i\|_{H^3}^2<\infty\quad\text{for}\quad d=2,3,$$

then for any $p \geq 1$, and $d = 1,2,3$, u satisfies for any $t > 0$ and some integer $k = k(p) > 0$

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leq t\leq T}\|u(t)\|^{2p}\Big) + \varepsilon^{2p}\mathbb{E}\Big(\sup_{0\leq t\leq T}\|\nabla u(t)\|^{2p}\Big)\\
&+\mathbb{E}\Big(\sup_{0\leq t\leq T}\Big(\int_{\mathcal{D}}F(u(t))dx + \frac{\varepsilon^2}{2}\|\nabla u(t)\|^2\Big)^p\Big) + \frac{1}{\varepsilon^p}\mathbb{E}\Big(\Big(\int_0^T\|\nabla[f(u)-\varepsilon^2\Delta u]\|^2 ds\Big)^p\Big)\\
\leq& c\mathbb{E}\Big(\Big|\int_{\mathcal{D}}F(u_0)dx\Big|^p\Big) + c\varepsilon^{2p}\mathbb{E}\Big(\|\nabla u_0\|^{2p}\Big) + \frac{c}{\varepsilon^{k(p)}}\mathbb{E}\Big(\|u_0\|^{2p}\Big) + \frac{c}{\varepsilon^{k(p)}}.
\end{aligned} \tag{2.41}$$

Proof. Observe first that the assumptions of Theorem 2.1 hold true. Then by (2.38) and Burkholder inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leq t\leq T}\Big(\int_{\mathcal{D}}F(u(t))dx + \frac{\varepsilon^2}{2}\|\nabla u(t)\|^2\Big)^p\Big) + \frac{1}{\varepsilon^p}\mathbb{E}\Big(\Big(\int_0^T\|\nabla[f(u)-\varepsilon^2\Delta u]\|^2 ds\Big)^p\Big)\\
\leq& c\mathbb{E}\Big(\Big(\int_{\mathcal{D}}F(u_0)dx\Big)^p\Big) + c\varepsilon^{2p}\mathbb{E}\Big(\|\nabla u_0\|^{2p}\Big)\\
&+ c\varepsilon^{p\gamma}\mathbb{E}\Big[\Big(\int_0^T\|f(u)\sigma(u)\|^2\Big)^{p/2}\Big] + c\varepsilon^{2p+p\gamma}\mathbb{E}\Big[\Big(\int_0^T\|\Delta u\sigma(u)\|^2 ds\Big)^{p/2}\Big]\\
\leq& c\mathbb{E}\Big(\Big(\int_{\mathcal{D}}F(u_0)dx\Big)^p\Big) + c\varepsilon^{2p}\mathbb{E}\Big(\|\nabla u_0\|^{2p}\Big)\\
&+ c\varepsilon^{p\gamma}\mathbb{E}\Big[\Big(\int_0^T\|f(u)\sigma(u)\|^2\Big)^{p/2}\Big] + c\varepsilon^{2p+p\gamma}\mathbb{E}\Big[\sup_{0\leq t\leq T}\|\sigma(u)\|^p_{L^\infty(\mathcal{D})}\Big(\int_0^T\|\Delta u\|^2 ds\Big)^{p/2}\Big]\\
\leq& c\mathbb{E}\Big(\Big(\int_{\mathcal{D}}F(u_0)dx\Big)^p\Big) + c\varepsilon^{2p}\mathbb{E}\Big(\|\nabla u_0\|^{2p}\Big) + c\varepsilon^{p\gamma}\mathbb{E}\Big[\Big(\int_0^T\|f(u)\sigma(u)\|^2\Big)^{p/2}\Big]\\
&+ c\varepsilon^{2p+p\gamma}\mathbb{E}\Big[\sup_{0\leq t\leq T}\|\sigma(u)\|^{2p}_{H^2(\mathcal{D})}\Big] + c\varepsilon^{2p+p\gamma}\mathbb{E}\Big[\Big(\int_0^T\|u\|^2_{H^2(\mathcal{D})}ds\Big)^p\Big].
\end{aligned} \tag{2.42}$$

Here, we remind that by Lemma 2.5, the assumptions of which are satisfied, we have

$$\mathbb{E}\Big(\Big(\int_0^T\|u\|^2_{H^2}dt\Big)^p\Big) \leq c\varepsilon^{-4p}\mathbb{E}\Big(\Big(\int_0^T\|u\|^2 dt\Big)^p\Big) + c\varepsilon^{-p}(\varepsilon^{p\gamma}+\varepsilon^{2\gamma p-p}) + c\varepsilon^{-p}\mathbb{E}(\|u_0\|^{2p}). \tag{2.43}$$

Due to the presence of the term $\|f(u)\sigma(u)\|$ at (2.42), the noise diffusion growth is assumed reduced to $\alpha = 0$, and we considered $|\sigma(u)| \leq c$, for the purposes of this theorem. More analytically, by using Young's inequality, we get

$$\begin{aligned}
\|f(u)\sigma(u)\|^2 =& \|(u^3-u)\sigma(u)\|^2 \leq c\int_{\mathcal{D}}(u^3-u)^2 dx \leq c\int_{\mathcal{D}}u^6 dx + c\\
\leq& c\|u\|^2_{L^\infty(\mathcal{D})}\int_{\mathcal{D}}u^4 dx + c \leq c\|u\|^2_{H^2}\|u\|_4^4 + c \leq c\|u\|^2_{H^2}\sup_{0\leq t\leq T}\|u\|_4^4 + c,
\end{aligned}$$

which yields

$$\int_0^T\|f(u)\sigma(u)\|^2 ds \leq c\sup_{0\leq t\leq T}\|u\|_4^4\int_0^T\|u\|^2_{H^2}ds + c,$$

and thus

$$\begin{aligned}\Big(\int_0^T \|f(u)\sigma(u)\|^2 ds\Big)^{p/2} \le & c \sup_{0\le t\le T} \|u\|_4^{2p}\Big(\int_0^T \|u\|_{H^2}^2 ds\Big)^{p/2} + c \\ \le & c \sup_{0\le t\le T} \|u\|_4^{4p} + c\Big(\int_0^T \|u\|_{H^2}^2 ds\Big)^{p} + c.\end{aligned} \tag{2.44}$$

By (2.43) and (2.44), we obtain

(2.45)

$$\begin{aligned}\varepsilon^{p\gamma}\mathbb{E}\Big[\Big(\int_0^T \|f(u)\sigma(u)\|^2\Big)^{p/2}\Big] \le & c\varepsilon^{p\gamma}\mathbb{E}\Big(\sup_{0\le t\le T} \|u\|_4^{4p}\Big) + c\varepsilon^{p\gamma}\mathbb{E}\Big(\Big(\int_0^T \|u\|_{H^2}^2 ds\Big)^{p}\Big) + c\varepsilon^{p\gamma} \\ \le & c\varepsilon^{p\gamma}\mathbb{E}\Big(\sup_{0\le t\le T} \Big(\int_{\mathcal{D}} F(u)dx\Big)^p + c\Big) + c\varepsilon^{p\gamma}\mathbb{E}\Big(\Big(\int_0^T \|u\|_{H^2}^2 ds\Big)^{p}\Big) + c\varepsilon^{p\gamma} \\ \le & c\varepsilon^{p\gamma}\mathbb{E}\Big(\sup_{0\le t\le T} \Big(\int_{\mathcal{D}} F(u)dx\Big)^p + c\Big) + c\varepsilon^{p\gamma-4p}\mathbb{E}\Big(\Big(\int_0^T \|u\|^2 ds\Big)^{p}\Big) \\ & + c\varepsilon^{p\gamma-p}(\varepsilon^{p\gamma} + \varepsilon^{2\gamma p-p}) + c\varepsilon^{p\gamma-p}\mathbb{E}(\|u_0\|^{2p}) + c\varepsilon^{p\gamma},\end{aligned}$$

where we used that $\int_{\mathcal{D}} u^4 dx \le c\int_{\mathcal{D}} F(u)dx + c$.

Moreover, using once again (2.43), we get

$$\begin{aligned}c\varepsilon^{2p+p\gamma}\mathbb{E}\Big(\Big(\int_0^T \|u\|_{H^2}^2 dt\Big)^{p}\Big) \le & c\varepsilon^{2p+p\gamma}\varepsilon^{-4p}\mathbb{E}\Big(\Big(\int_0^T \|u\|^2 dt\Big)^{p}\Big) \\ & + c\varepsilon^{2p+p\gamma}\varepsilon^{-p}(\varepsilon^{p\gamma} + \varepsilon^{2\gamma p-p}) + c\varepsilon^{2p+p\gamma}\varepsilon^{-p}\mathbb{E}(\|u_0\|^{2p}).\end{aligned} \tag{2.46}$$

Using that $u^4 \le cF(u) + c$ and that $\varepsilon^{\gamma-4}u^2 \le \varepsilon^{-a_1} + \varepsilon^{a_2}u^4$ for some $a_1, a_2 > 0$, we obtain

$$c\varepsilon^{p\gamma-4p}\mathbb{E}\Big(\Big(\int_0^T \|u\|^2 ds\Big)^{p}\Big) \le \mathbb{E}\Big(\sup_{0\le t\le T} \Big(\int_{\mathcal{D}} F(u(t))dx\Big)^p\Big) + c\varepsilon^{-a_3}, \tag{2.47}$$

for some $a_3 > 0$.

Considering also

$$\|\sigma(u)\|_{H^2}^2 \le c\int_{\mathcal{D}} F(u)dx + c\|\nabla u\|^2 + c$$

and bounding the right-hand side of (2.42) by using (2.45), (2.46) and (2.47), we derive the result. □

Our aim is to consider in the sequel H^2 bounds. Gagliardo-Nirenberg's inequality will control the non-linear terms in the estimates. We present a useful general lemma in dimensions $d = 2, 3$ estimating various norms thereof when the Neumann boundary conditions are satisfied; see also in [16] for some analogous, yet not the same, results.

Lemma 2.8. *Let v satisfying the b.c.*

$$\frac{\partial v}{\partial \eta} = \frac{\partial \Delta v}{\partial \eta} = 0 \quad on\ \partial\mathcal{D}.$$

Then, in dimensions $d = 2$, the next estimates hold true for some $k_1 > 0$ large

$$\|v^2\Delta v\| \le c\|\Delta^2 v\|^{7/10}\Big[\|v\|_{H^1}^{23/10} + 1\Big] + c\|v\|_{H^1}^{35/4}, \tag{2.48}$$

$$\|v|\nabla v|^2\| \leq c\|\Delta^2 v\|^{13/30}\Big[\|v\|_{H^1}^{77/30}+1\Big]+c\|v\|_{H^1}^{k_1}+c, \tag{2.49}$$

while in dimensions $d=3$ for some $k_2>0$ large

$$\|v^2\Delta v\| \leq c\|\Delta^2 v\|^{5/6}\Big[\|v\|_{H^1}^{13/6}+1\Big]+c\|v\|_{H^1}^{25/4}, \tag{2.50}$$

$$\|v|\nabla v|^2\| \leq c\|\Delta^2 v\|^{2/3}\Big[\|v\|_{H^1}^{7/3}+1\Big]+c\|v\|_{H^1}^{k_2}+c. \tag{2.51}$$

Proof. According to the Gagliardo-Nirenberg's inequality, [1],

$$\begin{aligned}&\|D^j v\|_{L^p} \leq c\|D^m v\|_{L^r}^{\alpha}\|v\|_{L^q}^{1-\alpha}+c\|v\|_{L^q},\\ &\frac{j}{m}\leq\alpha\leq 1,\ \ \frac{1}{p}=\frac{j}{d}+\alpha\Big(\frac{1}{r}-\frac{m}{d}\Big)+(1-\alpha)\frac{1}{q}.\end{aligned} \tag{2.52}$$

Let us consider first the case $d=2$.

Taking (2.52) for $p=\infty,\ j=0,\ m=4,\ r=2,\ q=6$ and $\alpha=1/10$, we obtain

$$\|v\|_{L^\infty} \leq c\|\Delta^2 v\|^{1/10}\|v\|_{L^6}^{9/10}+c\|v\|_{L^6}, \tag{2.53}$$

while taking (2.52) for $p=6,\ j=0,\ m=1,\ r=q=2$ and $\alpha=2/3$, we obtain

$$\|v\|_{L^6} \leq c\|\nabla v\|^{2/3}\|v\|^{1/3}+c\|v\| \leq c\|v\|_{H^1}. \tag{2.54}$$

Moreover, taking (2.52) for v replaced by $\nabla v,\ p=4,\ j=0,\ m=3,\ r=q=2$ and $\alpha=1/6$, we have

$$\|\nabla v\|_{L^4} \leq c\|\Delta^2 v\|^{1/6}\|\nabla v\|^{5/6}+c\|\nabla v\| \leq c\|\Delta^2 v\|^{1/6}\|v\|_{H^1}^{5/6}+c\|v\|_{H^1}. \tag{2.55}$$

We also note that due to the b.c. we have

$$(\Delta v,\Delta v)=(\Delta^2 v,v),$$

which yields

$$\|\Delta v\| \leq \|\Delta^2 v\|^{1/2}\|v\|^{1/2}. \tag{2.56}$$

Using (2.53), (2.54), (2.56), and Young's inequality we arrive at

$$\begin{aligned}\|v^2\Delta v\| \leq \|v\|_\infty^2\|\Delta v\| &\leq c\Big(\|\Delta^2 v\|^{2/10}\|v\|_{L^6}^{18/10}+\|v\|_{L^6}^2\Big)\|\Delta^2 v\|^{1/2}\|v\|^{1/2}\\ &\leq c\|\Delta^2 v\|^{7/10}\|v\|_{H^1}^{23/10}+c\|\Delta^2 v\|^{7/10}+c\|v\|_{H^1}^{35/4}\\ &\leq c\|\Delta^2 v\|^{7/10}\Big[\|v\|_{H^1}^{23/10}+1\Big]+c\|v\|_{H^1}^{35/4},\end{aligned}$$

i.e. (2.48).

By (2.53) and (2.54), we get

$$\|v\|_\infty \leq c\|\Delta^2 v\|^{1/10}\|v\|_{H^1}^{9/10}+c\|v\|_{H^1}. \tag{2.57}$$

Hence, we use (2.57) and (2.55), and Young's inequality, and obtain

$$\begin{aligned}\|v|\nabla v|^2\| \leq \|v\|_\infty\|\nabla v\|_{L^4}^2 &\leq c\|\Delta^2 v\|^{1/10}\|v\|_{H^1}^{9/10}\|\nabla v\|_{L^4}^2+c\|v\|_{H^1}\|\nabla v\|_{L^4}^2\\ &\leq c\|\Delta^2 v\|^{13/30}\Big[\|v\|^{77/30}+1\Big]+c\|v\|_{H^1}^{k_1}+c\end{aligned}$$

i.e. (2.49).

We consider now the case $d=3$.

Using (2.52) for $p=\infty,\ j=0,\ m=4,\ r=2,\ q=6$ and $\alpha=1/6$, we obtain

$$\|v\|_\infty \le c\|\Delta^2 v\|^{1/6}\|v\|_{L^6}^{5/6} + c\|v\|_{L^6}, \tag{2.58}$$

while taking (2.52) for $p=6,\ j=0,\ m=1,\ r=q=2$ and $\alpha=1$, we obtain

$$\|v\|_{L^6} \le c\|\nabla v\| + c\|v\| \le c\|v\|_{H^1}. \tag{2.59}$$

Taking now (2.52) for v replaced by $\nabla v,\ p=4,\ j=0,\ m=3,\ r=q=2$ and $\alpha=1/4$, we have

$$\|\nabla v\|_{L^4} \le c\|\Delta^2 v\|^{1/4}\|\nabla v\|^{3/4} + c\|\nabla v\| \le c\|\Delta^2 v\|^{1/4}\|v\|_{H^1}^{3/4} + c\|v\|_{H^1}. \tag{2.60}$$

Using (2.58), (2.59), (2.56), and Young's inequality we have

$$\begin{aligned}\|v^2\Delta v\| \le \|v\|_\infty^2\|\Delta v\| &\le c\Big(\|\Delta^2 v\|^{2/6}\|v\|_{L^6}^{10/6} + \|v\|_{L^6}^2\Big)\|\Delta^2 v\|^{1/2}\|v\|^{1/2}\\ &\le c\|\Delta^2 v\|^{5/6}\Big[\|v\|_{H^1}^{13/6}+1\Big] + c\|u\|_{H^1}^{25/4},\end{aligned}$$

i.e. (2.50).

We use (2.58) and (2.60), and once again Young's inequality, and get

$$\begin{aligned}\|v|\nabla v|^2\| \le \|v\|_\infty\|\nabla v\|_{L^4}^2 &\le c\Big[\|\Delta^2 v\|^{1/6}\|v\|_{L^6}^{5/6} + \|v\|_{L^6}\Big]\|\nabla v\|_{L^4}^2\\ &\le c\|\Delta^2 v\|^{2/3}\Big[\|v\|_{H^1}^{7/3}+1\Big] + c\|v\|_{H^1}^{k_2} + c,\end{aligned}$$

i.e. (2.51). □ □

Remark 2.9. *In* [16]*, the authors derived more elegant estimates analogous to these presented in the previous lemma, under the assumption that* $\int_{\mathcal{D}} v dx = 0$*. The general strategy applied in* [16] *for the derivation of* H^2 *a priori estimates for the deterministic Cahn-Hilliard equation, significantly inspired our approach towards the derivation of the* p*-moments in* H^2 *norm for the stochastic problem as well.*

We now proceed to an H^2 (and thus $L^\infty(\mathcal{D})$) higher moment estimate.

Theorem 2.3. *Let the noise diffusion satisfy*

$$\|\sigma(v)\|_{H^2} \le c, \tag{2.61}$$

uniformly for any v*. Moreover, assume that the noise is sufficiently regular in space, in particular let*

$$\sum_{i=1}^\infty a_i^2\|e_i\|_{H^4}^2 < \infty, \tag{2.62}$$

and that $\|u_0\|_{H^2}$ *has bounded* p*-moments.*

Then for the solution u *of* (1.1)*, for any* $T>0$*, and some integer* $k=k(p)>0$*, and for any* $p\ge 1$ *and* $d=1,2,3$*, it holds that*

$$\mathbb{E}\Big(\sup_{0\le t\le T}\|u\|_{H^2}^{2p}\Big) + \varepsilon^p\mathbb{E}\Big(\Big[\int_0^T\|\Delta^2 u\|^2 ds\Big]^p\Big) \le \frac{c}{\varepsilon^{k(p)}}, \tag{2.63}$$

and therefore,

$$\mathbb{E}\Big(\sup_{0\le t\le T}\|u\|_\infty^{2p}\Big) \le c\mathbb{E}\Big(\sup_{0\le t\le T}\|u\|_{H^2}^{2p}\Big) \le \frac{c}{\varepsilon^{k(p)}}. \tag{2.64}$$

Proof. Observe that all the assumptions on the noise and on σ imposed in all the previous lemmas and theorems, are satisfied (see also the boundedness of σ that follows from (2.61)).

When $d=1$, L^∞ higher moments are derived through the H^1 estimate, and there the assumptions on the regularity of noise dW, and on σ can be weakened.

We have

$$d(\Delta u,\Delta u)=2(d\Delta u,\Delta u)+(d\Delta u,d\Delta u), \tag{2.65}$$

while the Laplacian operator on the stochastic Cahn-Hilliard equation (1.1) yields

$$d\Delta u=\Delta(L)dt+\varepsilon^\gamma\Delta(\sigma(u)dW). \tag{2.66}$$

Since

$$\begin{aligned}\Delta(\sigma(u)dW)=&\sigma''(u)|\nabla u|^2dW+\sigma'(u)\Delta u dW+2\sigma'(u)\nabla u\nabla dW+\sigma(u)\Delta dW\\=&\Delta(\sigma(u))dW+2\nabla(\sigma(u))\nabla dW+\sigma(u)\Delta dW=Bdt,\end{aligned} \tag{2.67}$$

and since

$$\|e_i\|_\infty+\|\nabla e_i\|_\infty+\|\Delta e_i\|_\infty\leq c\|e_i\|_{H^4},$$

and since from (2.62)

$$\sum_{i=1}^\infty a_i^2\|e_i\|_{H^4}^2\leq c,$$

we get by (2.66), (2.67) and (2.62)

$$(d\Delta u,d\Delta u)\leq c\varepsilon^{2\gamma}\sum a_i^2\|e_i\|^2_{H^4}\|\sigma(u)\|^2_{H^2}dt\leq c\varepsilon^{2\gamma}\|\sigma(u)\|^2_{H^2}dt. \tag{2.68}$$

Using now (2.65), and (2.68), we obtain

$$\begin{aligned}\frac{1}{2}d\|\Delta u\|^2=&(d\Delta u,\Delta u)+\frac{1}{2}(d\Delta u,d\Delta u)=(du,\Delta^2u)+\frac{1}{2}(d\Delta u,d\Delta u)\\\leq&(du,\Delta^2u)+c\varepsilon^{2\gamma}\|\sigma(u)\|^2_{H^2}dt\\=&-\varepsilon(\Delta^2u,\Delta^2u)dt+\frac{1}{\varepsilon}(\Delta(f(u)),\Delta^2u)dt+\varepsilon^\gamma(\Delta^2u,\sigma(u)dW)+c\varepsilon^{2\gamma}\|\sigma(u)\|^2_{H^2}dt\\=&-\varepsilon\|\Delta^2u\|^2dt+\frac{1}{\varepsilon}(6u|\nabla u|^2,\Delta^2u)dt+\frac{1}{\varepsilon}(3u^2\Delta u,\Delta^2u)dt\\&-\frac{1}{\varepsilon}(\Delta u,\Delta^2u)dt+\varepsilon^\gamma(\Delta^2u,\sigma(u)dW)+c\varepsilon^{2\gamma}\|\sigma(u)\|^2_{H^2}dt.\end{aligned} \tag{2.69}$$

So, we arrive at

$$\begin{aligned}\|\Delta u(t)\|^2+2\varepsilon\int_0^t\|\Delta^2u\|^2ds\leq&\|\Delta u(0)\|^2+\frac{2}{\varepsilon}\int_0^t(6u|\nabla u|^2,\Delta^2u)ds\\&+\frac{2}{\varepsilon}\int_0^t(3u^2\Delta u,\Delta^2u)ds-\frac{2}{\varepsilon}\int_0^t(\Delta u,\Delta^2u)ds\\&+2\varepsilon^\gamma\int_0^t(\Delta^2u,\sigma(u)dW)+c\varepsilon^{2\gamma}\int_0^t\|\sigma(u)\|^2_{H^2}ds.\end{aligned} \tag{2.70}$$

Due to Lemma 2.8, we obtain for

$$a_1:=\begin{cases}7/10 & d=2,\\5/6 & d=3,\end{cases}\quad a_2:=\begin{cases}13/30 & d=2,\\2/3 & d=3,\end{cases}$$

and for $\mu_1,\ \mu_2,\ \mu_3>0$ some rather large powers of $\|u\|_{H^1}$, and some $\ell_1>0$

$$\begin{aligned}\|\Delta u\|^2+\varepsilon\int_0^t\|\Delta^2u\|^2ds \leq& c\|\Delta u_0\|^2+\frac{c}{\varepsilon}\int_0^t\|\Delta^2u\|^{a_2+1}\|u\|_{H^1}^{\mu_2}ds\\ &+\frac{c}{\varepsilon}\int_0^t\|\Delta^2u\|^{a_2+1}ds+\frac{C}{\varepsilon}\int_0^t\|\Delta^2u\|ds\\ &+\frac{c}{\varepsilon}\int_0^t\|\Delta^2u\|^{a_1+1}\|u\|_{H^1}^{\mu_1}ds+\frac{C}{\varepsilon}\int_0^t\|\Delta^2u\|^{a_1+1}ds\\ &+\frac{c}{\varepsilon}\int_0^t\|u\|_{H^1}^{\mu_3}ds+c\varepsilon^{-\ell_1}\\ &+\frac{c}{\varepsilon^3}\int_0^t\|\Delta u\|^2ds\\ &+c\varepsilon^{2\gamma}\int_0^t\|\sigma(u)\|_{H^2}^2ds+c\varepsilon^{\gamma}\int_0^t(\Delta^2u,\sigma(u)dW),\end{aligned}$$

where we hidded $\frac{2}{\varepsilon}|\int_0^t(\Delta u,\Delta^2u)ds|\leq c_0\varepsilon\int_0^t\|\Delta^2u\|^2ds+c\varepsilon^{-3}\int_0^t\|\Delta u\|^2ds$, for $c_0>0$ as small, at the left.

Since $a_1+1,\ a_2+1<2$, using Young's inequality and hiding at the left the $\|\Delta^2u\|$ involving terms, we get for some $\mu_4,\mu_5,\ell_2>0$

$$\begin{aligned}\|\Delta u\|^2+\varepsilon\int_0^t\|\Delta^2u\|^2ds \leq& c\|\Delta u_0\|^2+\frac{c}{\varepsilon^{\mu_4}}\int_0^t\|u\|_{H^1}^{\mu_5}ds+c\varepsilon^{-\ell_2}\\ &+\frac{c}{\varepsilon^3}\int_0^t\|\Delta u\|^2ds\\ &+c\varepsilon^{2\gamma}\int_0^t\|\sigma(u)\|_{H^2}^2ds+c\varepsilon^{\gamma}\int_0^t(\Delta^2u,\sigma(u)dW).\end{aligned}\tag{2.71}$$

We use (2.34), i.e.

$$\|u\|^2+\varepsilon\int_0^t\|\Delta u\|^2ds\leq\frac{c}{\varepsilon^3}\int_0^t\|u\|^2ds+c\varepsilon^{\gamma}\int_0^t(\sigma(u)u,dW)+\|u_0\|^2+cT\varepsilon^{2\gamma},$$

which yields

$$\int_0^t\|\Delta u\|^2ds\leq c\varepsilon^{-4}\int_0^t\|u\|^2ds+c\varepsilon^{\gamma-1}\int_0^t(\sigma(u)u,dW)+\varepsilon^{-1}\|u_0\|^2+cT\varepsilon^{2\gamma-1}.\tag{2.72}$$

Using (2.72) in (2.71), we obtain for some integer $m\geq 2$

(2.73)

$$\begin{aligned}\|\Delta u\|^2+\varepsilon\int_0^t\|\Delta^2 u\|^2ds\leq& c\|\Delta u_0\|^2+\frac{c}{\varepsilon^{\mu_4}}\int_0^t\|u\|_{H^1}^{\mu_3}ds+c\varepsilon^{-\ell_2}\\&+\frac{c}{\varepsilon^3}\int_0^t\|\Delta u\|^2ds+c\varepsilon^{2\gamma}\int_0^t\|\sigma(u)\|_{H^2}^2ds+c\varepsilon^{\gamma}\int_0^t(\Delta^2u,\sigma(u)dW)\\\leq& c\|\Delta u_0\|^2+\frac{c}{\varepsilon^{\mu_4}}\int_0^t\|u\|_{H^1}^{\mu_3}ds+c\varepsilon^{-\ell_2}\\&+c\varepsilon^{-7}\int_0^t\|u\|^2ds+c\varepsilon^{\gamma-4}\int_0^t(\sigma(u)u,dW)+\varepsilon^{-4}\|u_0\|^2+cT\varepsilon^{2\gamma-4}\\&+c\varepsilon^{2\gamma}\int_0^t\|\sigma(u)\|_{H^2}^2ds+c\varepsilon^{\gamma}\int_0^t(\Delta^2u,\sigma(u)dW)\\\leq& c\|\Delta u_0\|^2+c\varepsilon^{-m}\int_0^t\|u\|_{H^1}^{m}ds+c\varepsilon^{-m}\Big|\int_0^t(\sigma(u)u,dW)\Big|+\varepsilon^{-m}\|u_0\|^2\\&+c\varepsilon^{2\gamma}\int_0^t\|\sigma(u)\|_{H^2}^2ds+c\varepsilon^{\gamma}\Big|\int_0^t(\Delta^2u,\sigma(u)dW)\Big|+c\varepsilon^{-m}.\end{aligned}$$

Thus, taking p powers and then supremum in $[0,T]$, we arrive at

$$\begin{aligned}\sup_{0\leq t\leq T}\|\Delta u\|^{2p}+\varepsilon^p\Big[\int_0^T\|\Delta^2u\|^2ds\Big]^p\leq& c\|\Delta u_0\|^{2p}+c\varepsilon^{-mp}\sup_{0\leq t\leq T}\|u\|_{H^1}^{mp}\\&+c\varepsilon^{-mp}\sup_{0\leq t\leq T}\Big|\int_0^t(\sigma(u)u,dW)\Big|^p+\varepsilon^{-mp}\|u_0\|^{2p}\\&+c\varepsilon^{\gamma p}\sup_{0\leq t\leq T}\Big|\int_0^t(\sigma(u)dW,\Delta^2u)\Big|^p\\&+c\varepsilon^{2\gamma p}\Big(\int_0^T\|\sigma(u)\|_{H^2}^2ds\Big)^p+c\varepsilon^{-mp}.\end{aligned}\tag{2.74}$$

Hence, we get

$$\begin{aligned}&\mathbb{E}\Big(\sup_{0\leq t\leq T}\|\Delta u\|^{2p}\Big)+\varepsilon^p\mathbb{E}\Big(\Big[\int_0^T\|\Delta^2u\|^2ds\Big]^p\Big)\\&\leq c\mathbb{E}(\|\Delta u_0\|^{2p})+c\varepsilon^{-mp}\mathbb{E}(\sup_{0\leq t\leq T}\|u\|_{H^1}^{mp})\\&\quad+c\varepsilon^{-mp}\mathbb{E}\Big(\sup_{0\leq t\leq T}\Big|\int_0^t(\sigma(u)u,dW)\Big|^p\Big)+\varepsilon^{-mp}\mathbb{E}(\|u_0\|^{2p})\\&\quad+c\varepsilon^{\gamma p}\mathbb{E}\Big(\sup_{0\leq t\leq T}\Big|\int_0^t(\sigma(u)dW,\Delta^2u)\Big|^p\Big)\\&\quad+c\varepsilon^{2\gamma p}\mathbb{E}\Big(\int_0^T\|\sigma(u)\|_{H^2}^2ds\Big)^p+c\varepsilon^{-mp}.\end{aligned}\tag{2.75}$$

The $\|\sigma(u)\|_{H^2}$ term in the above inequality motivated (2.61).

Since $|\sigma(u)| \leq \|\sigma(u)\|_{H^2} \leq c$, then we obtain

$$
\begin{aligned}
c\varepsilon^{\gamma p}\mathbb{E}\Big(\sup_{0\leq t\leq T}\Big|\int_0^t(\Delta^2 u,\sigma(u)dW)\Big|^p\Big) \leq & c\varepsilon^{\gamma p}\mathbb{E}\Big(\Big(\int_0^T\|\sigma(u)\Delta^2 u\|^2 ds\Big)^{p/2}\Big)\\
\leq & c\varepsilon^{\gamma p}\mathbb{E}\Big(\Big(\int_0^T\|\Delta^2 u\|^2 ds\Big)^{p/2}\Big)\\
\leq & c\varepsilon^{2\gamma p-p}+\frac{\varepsilon^p}{2}\mathbb{E}\Big(\Big(\int_0^T\|\Delta^2 u\|^2 ds\Big)^{p}\Big),
\end{aligned}
\tag{2.76}
$$

and

$$
\mathbb{E}\Big(\sup_{0\leq t\leq T}\Big|\int_0^t(\sigma(u)u,dW)\Big|^p\Big) \leq c\mathbb{E}\Big(\Big(\int_0^T\|\sigma(u)u\|^2 ds\Big)^{p/2}\Big) \leq c\mathbb{E}\Big(\Big(\int_0^T\|u\|^2 ds\Big)^{p/2}\Big).
\tag{2.77}
$$

Using in (2.75) the relations (2.61), (2.76) and (2.77), and the H^1 p-moments estimate (2.41) (which is of negative polynomial order in ε), we obtain for some integer $k_1 = k_1(p) > 0$ and $m_1, m_2 > 0$

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leq t\leq T}\|\Delta u\|^{2p}\Big)+\frac{\varepsilon^p}{2}\mathbb{E}\Big(\Big[\int_0^T\|\Delta^2 u\|^2 ds\Big]^p\Big)\\
\leq & c\mathbb{E}(\|\Delta u_0\|^{2p})+c\varepsilon^{-mp}\mathbb{E}(\sup_{t\leq T}\|u\|_{H^1}^{mp})+c\varepsilon^{-mp}\mathbb{E}\Big(\Big(\int_0^T\|u\|^2 ds\Big)^{p/2}\Big)\\
&+\varepsilon^{-mp}\mathbb{E}(\|u_0\|^{2p})+c\varepsilon^{2\gamma p-p}+c\varepsilon^{2\gamma p}+c\varepsilon^{-mp}\\
\leq & C\mathbb{E}(\|\Delta u_0\|^{2p})+c\varepsilon^{-m_1 p}\mathbb{E}(\sup_{0\leq t\leq T}\|u\|_{H^1}^{m_1 p})+\varepsilon^{-mp}\mathbb{E}(\|u_0\|^{2p})+c\varepsilon^{2\gamma p-p}+c\varepsilon^{2\gamma p}+c\varepsilon^{-m_2 p}\\
\leq & c\varepsilon^{-k_1}.
\end{aligned}
$$

So, using again (2.41), and since $\|u\|_{H^2}^2 \leq c\|u\|^2 + c\|\Delta u\|^2$ we arrive at the final estimate for some integer $k = k(p) > 0$

$$
\mathbb{E}\Big(\sup_{0\leq t\leq T}\|u\|_{\infty}^{2p}\Big) \leq c\mathbb{E}\Big(\sup_{0\leq t\leq T}\|u\|_{H^2}^{2p}\Big)+\varepsilon^p\mathbb{E}\Big(\Big[\int_0^T\|\Delta^2 u\|^2 ds\Big]^p\Big) \leq c\varepsilon^{-k(p)}.
\tag{2.78}
$$

□

Remark 2.10. *Under the assumptions of the above Theorem 2.3, by* (2.75)*, we also derive the next inequality for any* $p \geq 1$

$$
\mathbb{E}\Big(\sup_{0\leq t\leq T}\|\Delta u\|^{2p}\Big)+\varepsilon^p\mathbb{E}\Big(\Big[\int_0^T\|\Delta^2 u\|^2 ds\Big]^p\Big) \leq c\varepsilon^{-mp},
$$

for some integer $m > 0$.

Moreover, see for example in [17]*, pg. 270, if* $\partial\mathcal{D}$ *is* C^1*, using the Sobolev embedding for Hölder norms and since* $2 > \frac{d}{2}$ *when* $d = 1, 2, 3$

$$
\|u\|_{C^{2-[\frac{d}{2}]-1,\gamma}(\overline{\mathcal{D}})} \leq c\|u\|_{H^2(\mathcal{D})},
$$

for any $0<\gamma<1$ *if* $\frac{d}{2}=\left[\frac{d}{2}\right]$*, i.e. when* $\frac{d}{2}$ *is integer, and for* $\gamma=\left[\frac{d}{2}\right]+1-\frac{d}{2}$ *if* $\frac{d}{2}$ *is not integer (where* $[\cdot]$ *denotes the integer part), we obtain by* (2.78)*: for*

$$\|u(\cdot,t)\|_{C^{0,\gamma}(\overline{\mathcal{D}})}:=\sup_{x\neq y\in\overline{\mathcal{D}}}\frac{|u(x,t)-u(y,t)|}{|x-y|^{\gamma}},$$

for $|x-y|$ *for example the euclidean metric in* $\mathbb{R}^d$

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}\|u(\cdot,t)\|^{2p}_{C^{0,\gamma}(\overline{\mathcal{D}})}\Big)\leq c\mathbb{E}\Big(\sup_{0\leq t\leq T}\|u(\cdot,t)\|^{2p}_{H^2(\mathcal{D})}\Big)\leq c\varepsilon^{-k(p)}$$

for some integer $k(p)>0$*, for any* $0<\gamma<1$ *for* $d=2$*, and for* $\gamma=\frac{1}{2}$ *when* $d=3$ *(and analogously for* $d=1$*, however there lower regularity than* H^2 *is also sufficient for Hölder estimates, not the case in* $d=2,3$*). Therefore, the stochastic solution has a.s. continuous paths in space, while the Hölder norm bound path-wisely depends on the realization.*

3. Special cases

The problem (1.1) is the generalized statement of the ε-dependent stochastic Cahn-Hilliard equation with multiplicative Fourier noise, smooth in space. We have provided sufficient conditions for the noise regularity and the noise diffusion σ so that the stochastic solution is regular in space.

In this section we shall consider layered initial conditions of bounded energy as $\varepsilon\to 0$ as a special case. Applying our H^1 estimate, we will prove that on the sharp interface limit $\varepsilon\to 0$, the stochastic solution u converges to ± 1 a.s. Moreover, we will present some cases of a mass-conservative noise definition for which our results are applicable.

3.1. **Layered initial data.** Under the assumptions of Theorem 2.1, we consider the solution u of the stochastic Cahn-Hilliard equation (1.1). There, we have assumed that σ satisfies

$$|\sigma^2(v)f'(v)|\leq c\int_{\mathcal{D}}F(v)dx+c,\quad \|\sigma(v)\|^2+\|\nabla\sigma(v)\|^2\leq c\int_{\mathcal{D}}F(v)dx+c\|\nabla v\|^2+c,$$

uniformly for any v, and a noise sufficiently regular in space, in H^2 for $d=1$, or in H^3 for $d=2,3$.

In the scaling of the problem, the energy is defined by

$$\mathcal{E}(u):=\int_{\mathcal{D}}\Big(\frac{\varepsilon}{2}|\nabla u|^2+\frac{F(u)}{\varepsilon}\Big)dx, \tag{3.1}$$

see [11].

So, (2.22) yields, when u_0 is deterministic

$$\begin{aligned}\mathbb{E}\Big(\int_{\mathcal{D}}F(u)dx\Big)+\frac{\varepsilon^2}{2}\mathbb{E}(\|\nabla u(t)\|^2)=&\varepsilon\mathbb{E}\Big(\mathcal{E}(u)\Big)\\ \leq& c\varepsilon^{2\gamma}+c\varepsilon\mathcal{E}(u_0).\end{aligned} \tag{3.2}$$

3.1.1. *Bounded initial energy.* Let us consider that the initial data satisfy the condition proposed by X.F. Chen in [11] for the deterministic version of (1.1), i.e., for $\sigma=0$. More precisely, there the initial condition

$$u_0(\cdot,0):=u_0(\cdot,0;\varepsilon),$$

was assumed depending on ε so that the initial energy of the problem is uniformly bounded by a constant $\mathcal{E}_0$ for any $\varepsilon \in (0,1]$. This means

$$\mathcal{E}(u_0) = \frac{1}{\varepsilon}\widetilde{F}(u_0) = \int_{\mathcal{D}} \Big(\frac{\varepsilon}{2}|\nabla u_0|^2 + \frac{1}{\varepsilon}F(u_0)\Big)dx \leq \mathcal{E}_0 < \infty. \tag{3.3}$$

Remark 3.1. *Let us observe that the above is generally not true for ε-independent u_0. Due to the ε^{-1} term, and since F is non negative, we have*

$$\mathcal{E}(u_0) = \frac{\varepsilon}{2}\|\nabla u_0\|^2_{L^2(\mathcal{D})} + \frac{1}{\varepsilon}\int_{\mathcal{D}} F(u_0)dx \to \infty \quad \textit{for } \varepsilon \to 0,$$

unless $F(u_0) \equiv 0$.

From (3.2) we see that the bound (3.3) on $\mathcal{E}(u_0)$ yields

$$\frac{1}{4}\mathbb{E}\Big(\int_{\mathcal{D}}(u^2-1)^2dx\Big) = \mathbb{E}\Big(\int_{\mathcal{D}} F(u)dx\Big) \leq c(\varepsilon + \varepsilon^{2\gamma}). \tag{3.4}$$

So, if $\gamma > 0$, we obtain

$$u \to \pm 1 \ \text{ as } \ \varepsilon \to 0 \ \text{ a.s.}$$

In particular, the next Theorem holds.

Theorem 3.1. *Under the assumptions of Theorem 2.1, if u is the solution of the stochastic Cahn-Hilliard equation* (1.1) *with initial condition u_0 bounded in energy by* (3.3)*, then there exists a constant $c > 0$ such that*

$$\mathbb{E}(\|u^2-1\|^2) \leq c(\varepsilon + \varepsilon^{2\gamma}),$$

and

$$\mathbb{E}(\||u|-1\|^2) \leq c(\varepsilon + \varepsilon^{2\gamma}). \tag{3.5}$$

Moreover, for $\gamma > 0$, and any $0 < \alpha < \min\{\gamma, 1/2\}$ it holds that

$$\lim_{\varepsilon \to 0^+} P(\||u|-1\| \geq \varepsilon^\alpha) = 0. \tag{3.6}$$

Proof. The first statement of the Theorem follows by rewriting of (3.4) using the $L^2(\mathcal{D})$-norm. The second one, follows by observing that for all $u \in \mathbb{R}$

$$(|u|-1)^2 = \frac{(u^2-1)^2}{(|u|+1)^2} \leq (u^2-1)^2 = 4F(u).$$

Using Markov's inequality and (3.5), we derive (3.6) since

$$P(\||u|-1\| \geq \varepsilon^\alpha) = P(\||u|-1\|^2 \geq \varepsilon^{2\alpha}) \leq \frac{\mathbb{E}(\||u|-1\|^2)}{\varepsilon^{2\alpha}} \leq \frac{c(\varepsilon+\varepsilon^{2\gamma})}{\varepsilon^{2\alpha}} \to 0 \ \text{ as } \ \varepsilon \to 0^+.$$

□

Note that in case of a bounded noise diffusion, the above convergence for $d = 1, 2, 3$, even only in L^2 in expectation, under the assumptions of Theorem 2.3 (or under weaker assumptions on the diffusion when $d = 1$) corresponds to almost surely continuous paths in space.

3.2. **Noise diffusion and mass conservation.** Let us state some useful observations on the noise diffusion $\sigma(u)$ and mass conservation.

Integration of (1.1) in space, due to the Neumann b.c., gives

$$\partial_t \int_{\mathcal{D}} u(x,t)dx = \int_{\mathcal{D}} \varepsilon^{\gamma}\sigma(u)\dot{W}(x,t)dx.$$

An assumption on the noise of the next form

$$\int_{\mathcal{D}} \sigma(u)\dot{W}(x,t)dx = 0, \tag{3.7}$$

would yield that

$$\int_{\mathcal{D}} u(x,t)dx = \int_{\mathcal{D}} u_0(x)dx = \text{const}, \tag{3.8}$$

and therefore, a mass-conservative solution.

However, in the general case, since the noise is multiplicative, as $\sigma(u)$ depends on u, mass conservation (or equivalently (3.7)) is not holding true.

In what follows, we present some special cases for σ and the Fourier noise, where our results will be still applicable and the solution will keep the mass conservation property.

When

$$\sigma(u) := \text{const}, \tag{3.9}$$

if we assume that the process W satisfies

$$\int_{\mathcal{D}} \dot{W}(x,t)\; dx = 0 \quad \text{for any } \; t \geq 0, \tag{3.10}$$

then mass conservation (i.e., (3.8)) holds true.

When only one Brownian motion β is involved in the noise definition, and for

$$\sigma(u(x,t)) := \sigma(x), \tag{3.11}$$

(1.3) takes the form

$$W(x,t) = \sum_{i=1}^{\infty} a_i\beta_i(t)e_i(x) = \sigma(x)\beta(t), \tag{3.12}$$

where $\beta_1 := \beta$, $a_1e_1(x) := \sigma(x)$ and $a_i = 0$ for any $i \geq 2$. Obviously, the mass conservation condition (3.8) for this noise is then valid, if $\sigma(x)$ satisfies

$$\int_{\mathcal{D}} \sigma(x)\; dx = 0. \tag{3.13}$$

4. Appendix

We present first a useful version of Burkholder-Davis-Gundy inequality we used throughout our manuscript.

Let $g = g(x,r)$ be a Hilbert-space valued process defined for any $x \in \mathcal{D}$, and $r > 0$. By the Burkholder-Davis-Gundy inequality for local martingales, for any $\ell > 1$, and for any $0 \leq s \leq t$, it holds that

$$\begin{aligned}\mathbb{E}\Big[\sup_{\tau\in[s,t]}\Big|\int_s^\tau(g(\cdot,r),dW(r))\Big|^\ell\Big] \leq & c\mathbb{E}\Big[\Big|\int_s^t(g(\cdot,r),\mathcal{Q}g(\cdot,r))dr\Big|^{\ell/2}\Big]\\ \leq & c\mathbb{E}\Big[\Big(\int_s^t\|g(\cdot,r)\|^2dr\Big)^{\ell/2}\Big]\\ = & c\mathbb{E}\Big[\Big(\int_s^t\int_{\mathcal{D}}|g(x,r)|^2dxdr\Big)^{\ell/2}\Big],\end{aligned}$$

if $\sum_{i=1}^{\infty}a_i^2<\infty$, and thus $\|\mathcal{Q}\|\leq\sum_{i=1}^{\infty}a_i^2<\infty$. This inequality was used in [2] for $s:=0$ and for the general case of t stochastic (stopping time), cf. also in [23] where for g smooth enough so that $\|g(\cdot,r)\|$ is continuous, $\ell>0$ can take values below 1.

So, if the Fourier noise satisfies $\sum_{i=1}^{\infty}a_i^2<\infty$, then for any $0\leq s\leq t$, we obtain

$$\begin{aligned}\mathbb{E}\Big[\Big|\int_s^t\int_{\mathcal{D}}g(x,r)dW(x,r)dx\Big|^\ell\Big] = & \mathbb{E}\Big[\Big|\int_s^t(g(\cdot,r),dW(r))\Big|^\ell\Big]\\ \leq & \mathbb{E}\Big[\sup_{\tau\in[s,t]}\Big|\int_s^\tau(g(\cdot,r),dW(r))\Big|^\ell\Big]\\ \leq & c\mathbb{E}\Big[\Big(\int_s^t\int_{\mathcal{D}}|g(\cdot,r)|^2dxdr\Big)^{\ell/2}\Big].\end{aligned}$$

Moreover, we present a convenient stochastic version of Gronwall's Lemma, cf. [2] for a more general statement.

Let X, F be real valued processes, $g=g(x,s)$ a Hilbert-space valued process on $L^2(\mathcal{D})$, and $c\in\mathbb{R}$. If the next inequality is satisfied

$$dX\leq cXdt+Fdt+(g,dW),$$

then

$$X(t)\leq e^{ct}X(0)+\int_0^t e^{ct-cs}F(s)ds+\int_0^t e^{ct-cs}(g(\cdot,s),dW(s)).$$

Note that X, F may dependent on the space variables, but are smooth as functions of $x\in\mathcal{D}$.

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