Weak convergence of the L1 scheme for a stochastic subdiffusion problem driven by fractionally integrated additive noise

Ye Hu[a], Changpin Li[b], Yubin Yan[*,c]

[a]*Department of Mathematics, Lvliang University, Lishi 033000, China*
[b]*Department of Mathematics, Shanghai University, Shanghai 200444, China*
[c]*Department of Physical, Mathematical and Engineering, University of Chester, CH1 4BJ, UK*

Abstract

The weak convergence of a fully discrete scheme for approximating a stochastic subdiffusion problem driven by fractionally integrated additive noise is studied. The Caputo fractional derivative is approximated by the L1 scheme and the Riemann-Liouville fractional integral is approximated with the first order convolution quadrature formula. The noise is discretized by using the Euler method and the spatial derivative is approximated with the linear finite element method. Based on the nonsmooth data error estimates of the corresponding deterministic problem, the weak convergence orders of the fully discrete schemes for approximating the stochastic subdiffusion problem driven by fractionally integrated additive noise are proved by using the Kolmogorov equation approach. Numerical experiments are given to show that the numerical results are consistent with the theoretical results.

Key words:
Stochastic subdiffusion; Fractional derivative; Finite element method; L1 scheme; Error estimates.

1. Introduction

We shall study the weak convergence of the L1 scheme for approximating the following stochastic subdiffusion problem driven by fractionally integrated additive noise, with $0 < \alpha < 1$, $0 \leq \gamma \leq 1$, see, e.g., [3, (1.3)] and [16, (1.1)],

$$\begin{cases} {}_C\mathrm{D}_{0,t}^{\alpha} u(t) + Au(t) = {}_0I_t^{\gamma} \dot{W}(t), \ 0 < t \leq T, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $A = -\Delta$, with $\mathcal{D}(A) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$ and $\mathcal{D} \subset \mathbb{R}^d, d = 1, 2, 3$ is a bounded domain with smooth boundary. Here $\dot{W}(t) = \frac{\mathrm{d}W(t)}{\mathrm{d}t}$ denotes the noise and $W(t)$ is a Hilbert space-valued Wiener process with covariance operator Q with respect to a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ on

*Corresponding author. Email: y.yan@chester.ac.uk
Email addresses: huye1015@163.com (Ye Hu), lcp@shu.edu.cn (Changpin Li),
y.yan@chester.ac.uk (Yubin Yan)

a probability space $(\Omega, \mathscr{F}, \mathbf{P})$. The initial value u_0 is an $\mathscr{F}_0$-measurable H-valued random variable, where $H = L_2(\mathcal{D})$ denotes the Hilbert space of square integrable functions defined on $\mathcal{D}$ with the inner product $(\cdot, \cdot)$ and the norm $\| \cdot \|$.

The Caputo fractional derivative ${}_C\mathrm{D}_{0,t}^{\alpha} v(t)$ in (1.1) is defined by, with $v'(s) = \frac{\mathrm{d}v(s)}{\mathrm{d}s}$ and $0 < \alpha < 1$, see, e.g., [23, 27],

$$ {}_C\mathrm{D}_{0,t}^{\alpha} v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} v'(s) \, \mathrm{d}s, $$

and the Riemann-Liouville fractional integral ${}_0I_t^{\gamma} v(t)$, $0 < \gamma \leq 1$ is defined by

$$ {}_0I_t^{\gamma} v(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} v(s) \, \mathrm{d}s, $$

with the convention ${}_0I_t^{0} v(t) = v(t)$. Here $\Gamma(\cdot)$ denotes the Gamma function.

The covariance operator Q of the Wiener process $W(t)$ is possibly an unbounded operator, see, e.g., [21, (1.1)]. In particular, if Q is in trace class (see the definition of the trace class operator in Subsection 2.1), i.e., $\mathrm{Tr}(Q) < \infty$, then W is an H-valued Wiener process. If $Q = I$ (I denotes the identity operator), then W is called the cylindrical Wiener process which is not H-valued, but it is H_1-valued Wiener process with covariance operator Q_1, where $H_1 \supset H$ is a Hilbert space. See [24, Example 10.13] for an example of the Hilbert space H_1 in one-dimensional case.

Although we assume $A = -\Delta$ with $\mathcal{D}(A) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$ in (1.1), our results in this paper are suitable also for a more general positive definite linear operator which satisfies the following resolvent estimate, for any $\pi/2 < \theta < \pi$ and with $C = C(\theta)$, see, e.g., [26, (1.6)],

$$ \|(zI + A)^{-1}\| \leq C|z|^{-1}, \quad \text{for } z \in \Sigma_\theta = \{ z \neq 0 : |\arg z| < \theta \}. \tag{1.2} $$

Note that $\arg z^\alpha = \alpha \arg z$, we see that (1.2) implies that, with $0 < \alpha < 1$,

$$ \| \left(z^\alpha I + A \right)^{-1} \| \leq C|z|^{-\alpha}, \quad \text{for } z \in \Sigma_\theta = \{ z \neq 0 : |\arg z| < \theta \}. \tag{1.3} $$

The model (1.1) can be used to describe the random effects of particles subject to the sticking and trapping in the medium with memory [7]. We briefly describe how to obtain the model (1.1) in [7] below. Consider the transport of particles in medium with memory (e.g, heat conduct). Let $u(t,x), e(t,x), \vec{F}(t,x)$ denote the temperature of materials, internal energy and flux density, respectively. Then we have, with $\beta, \lambda > 0$,

$$ \begin{aligned} &\frac{\partial e(t,x)}{\partial t} = -\mathrm{div}(\vec{F}), \\ &e(t,x) = \beta u(t,x), \\ &\vec{F}(t,x) = -\lambda \nabla u(t,x). \end{aligned} $$

The temperature of the materials satisfies the classical heat equation

$$\beta \frac{\partial u}{\partial t} = \lambda \Delta u.$$

However in the medium with memory, the internal energy satisfies

$$e(t,x) = \bar{\beta} u(t,x) + \int_0^t n(t-s) u(s,x)\, ds,$$

where $\bar{\beta} \geq 0$ and n denotes the kernel function. Here the convolution means that the internal energy $e(t,x)$ depends on the temperature $u(s,x)$ of the materials for the past time $0 < s < t$. In real problem, the internal energy $e(t,x)$ depends on the temperature in past time randomly and takes the following form

$$e(t,x) = \bar{\beta} u(t,x) + \int_0^t n(t-s) u(s,x)\, ds + \int_0^t l(t-s) h(s, u(s,x))\, dW(s), \tag{1.4}$$

where the Wiener process $W(t)$ is the same as introduced in (1.1).

Denote, with $\gamma_1, \gamma_2 \in (0,1)$,

$$u(0) = 0, \quad \bar{\beta} = 0,$$

and

$$n(t) = \frac{1}{\Gamma(1-\gamma_1)} t^{-\gamma_1}, \quad l(t) = \frac{t^{1-\gamma_2}}{\Gamma(2-\gamma_2)}, \quad 0 < \gamma_1, \gamma_2 < 1.$$

Taking the derivative on (1.4), we get

$$\begin{aligned}\lambda \Delta u = -\, div \vec{F} = \frac{\partial e(t,x)}{\partial t} &= \frac{1}{\Gamma(1-\gamma_1)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\gamma_1} u(s,x)\, ds \\ &+ \frac{1}{\Gamma(2-\gamma_2)} \frac{\partial}{\partial t} \int_0^t (t-s)^{1-\gamma_2} h(s, u(s,x))\, dW(s).\end{aligned} \tag{1.5}$$

Note that

$$\begin{aligned}{}_{RL}\mathrm{D}_{0,t}^{\gamma_2} \int_0^t h(s, u(s,x))\, dW(s) &= \frac{1}{\Gamma(1-\gamma_2)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\gamma_2} \int_0^s h(\tau, u(\tau,x))\, dW(\tau)\, ds \\ &= \frac{1}{\Gamma(2-\gamma_2)} \frac{\partial}{\partial t} \int_0^t (t-s)^{1-\gamma_2} h(s, u(s,x))\, dW(s),\end{aligned}$$

where ${}_{RL}\mathrm{D}_{0,t}^{\gamma_2}$ denotes the Riemann-Liouville fractional derivative. We therefore have, by (1.5),

$${}_{C}\mathrm{D}_{0,t}^{\gamma_1} u(t,x) = \lambda \Delta u(t,x) - {}_{RL}\mathrm{D}_{0,t}^{\gamma_2} \int_0^t h(s, u(s,x))\, dW(s).$$

Hence we obtain, with $\dot{W}(t,x) = \frac{dW(t,x)}{dt}$,

$${}_{C}\mathrm{D}_{0,t}^{\gamma_1} u(t,x) = \lambda \Delta u(t,x) - {}_{RL}\mathrm{D}_{0,t}^{\gamma_2 - 1} h(s, u(s,x))\, \dot{W}(t,x),$$

which is the model (1.1). The model of fractional ordinary differential equation may be referred to [15].

The existence, uniqueness and regularity of the solution of (1.1) are well studied, see, e.g., [3, 6, 7, 13, 25] and the references therein.

Recently the numerical method for solving (1.1) attracts increasing interests. Wu et al. [30] studied the strong convergence for a fully discrete scheme for approximating (1.1) where the Caputo time fractional derivative is approximated by using the L1 scheme and the Riemann-Liouville fractional integral is approximated by using the first order convolution quadrature formula. Jin et al. [16] introduced a fully discrete scheme for solving (1.1), where both the Caputo time fractional derivative and the Riemann Liouville fractional integral are approximated by the first order convolution quadrature formulas and the space is approximated by using the finite element method. The strong convergence error estimates are obtained based on the nonsmooth data error estimates of the corresponding deterministic problem. The weak convergence is also considered in [16] by using the Malliavin calculus. More references can be found for the weak convergences of the time discretization schemes for approximating the stochastic heat, wave and Volterra integral differential equations, see, e.g., [1], [2], [4], [5], [8], [9], [11], [12], [22], [29] and the references therein.

In this paper, we shall use the Kolmogorov equation approach to consider the weak convergence of a fully discrete scheme for solving stochastic subdiffusion problem driven by fractionally integrated additive noise. The Kolmogorov equation approach was used in [21] for studying the weak convergence of a fully discrete approximation of a linear stochastic evolution equation with a positive memory term.

Let us introduce our main theorem in this paper. By taking the Laplace transform on the first equation in (1.1), we obtain the following mild solution of (1.1) [16, (4.1)],

$$u(t) = E(t)u_0 + \int_0^t \widetilde{E}(t-s)\,\mathrm{d}W(s), \tag{1.6}$$

where

$$E(t) = E_{\alpha,1}(-t^\alpha A), \quad \widetilde{E}(t) = t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-t^\alpha A),$$

and $E_{\alpha,1}(z)$ and $E_{\alpha,\alpha+\gamma}(z)$ denote the Mittag-Leffler functions defined by, see Podlubny [27, (1.56)], with $z \in \mathbb{C}$,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0,\ \beta > 0. \tag{1.7}$$

Here $\int_0^t \widetilde{E}(t-s)\,\mathrm{d}W(s)$ is in the sense of the Itô stochastic integral. See [24, (10.3)] for the construction of the Itô stochastic integral for the operator-valued stochastic process with respect to the Wiener process $W(t)$.

In order to ensure the well posedness of problem (1.1), we assume that α, γ satisfy the following assumption, see [7, pp. 1473-1474] and [16, (1.2)],

Assumption 1.1.

$$0 < \alpha < 1, \quad 0 \le \gamma \le 1, \quad \alpha + \gamma > 1/2.$$

For the Wiener process $W(t)$, we assume that the covariance operator Q and the elliptic operator A satisfy the following regularity assumption:

Assumption 1.2. *There exists a $\beta \in [0, \kappa]$ with $\kappa > 0$ such that*

$$\|A^{\frac{\beta-\kappa}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty, \quad 0 \leq \beta \leq \kappa,$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt operator norm (see the definition of the Hilbert-Schmidt operator in Subsection 2.1) and κ is defined by, for $\epsilon \in (0, 2 - \frac{1-2\gamma}{\alpha})$ when $0 \leq \gamma \leq 1/2$,

$$\kappa = \begin{cases} 2, & \text{if } \ 1/2 < \gamma \leq 1, \\ 2 - \frac{1-2\gamma}{\alpha} - \epsilon, & \text{if } \ 0 \leq \gamma \leq 1/2. \end{cases} \tag{1.8}$$

Remark 1.1. *The similar assumption as the Assumption 1.2 has been used in [16, Theorem A.1] where κ is defined by*

$$\kappa = \begin{cases} 2, & \text{if } \ 1/2 < \gamma \leq 1, \\ 2 - \epsilon, & \text{if } \ \gamma = 1/2, \\ 2 - \frac{1-2\gamma}{\alpha}, & \text{if } \ 0 \leq \gamma < 1/2. \end{cases} \tag{1.9}$$

The definition for κ in (1.8) *implies that $\kappa \neq 0$ since we choose $\epsilon \in (0, 2 - \frac{1-2\gamma}{\alpha})$ which is necessary in Theorem 1.2 where we require $\kappa \neq 0$ in* (1.18)*.*

Remark 1.2. *When $\alpha = 1, \gamma = 0$, the Assumption 1.2 reduces to*

$$\|A^{\frac{\beta-(1-\epsilon)}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty, \quad 0 \leq \beta \leq 1 - \epsilon,$$

which is similar as the assumption proposed in [31, Theorem 2.1] for the stochastic heat equation.

For any $s \in \mathbb{R}$, as in [28, page 38], we denote $\dot{H}^s = \dot{H}^s(\mathcal{D}) = \mathcal{D}(A^{s/2})$ with the norm $|\cdot|_s = \|A^{s/2} \cdot\|$. Jin et al. [16, Theorem A.1] proved the following regularity results for the solution of (1.6).

Lemma 1.1. *Assume that the Assumptions 1.1 and 1.2 hold. Let $r, q \in \mathbb{R}$ with $0 \leq r - q \leq 2$. Let $0 \leq r \leq \kappa$ with κ defined in the Assumption 1.2. Assume that $u_0 \in L^p(\Omega; \dot{H}^q)$, $p \geq 1$. For all $T > 0$ there exists a constant C such that for all $t \in [0, T]$ it holds that*

$$\|u(t)\|_{L^2(\Omega; \dot{H}^r(\mathcal{D}))} \leq C t^{-\alpha \frac{r-q}{2}} |u_0|_q + C t^{(1-\frac{\kappa}{2})\alpha + \gamma - \frac{1}{2}} \|A^{\frac{r-\kappa}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}. \tag{1.10}$$

We remark that in Jin et al. [16, Theorem A.1], they did not require $0 \leq r \leq \kappa$ in the term $\|A^{\frac{r-\kappa}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}$ in (1.10). However, in our paper, the Assumption 1.2 requires that $0 \leq r \leq \kappa$ in the term $\|A^{\frac{r-\kappa}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}$ in (1.10).

We next consider the semidiscrete scheme for approximating (1.1). Let $\mathcal{T}_h$ be a shape regular quasi-uniform triangulation of the domain $\mathcal{D} \subset \mathbb{R}^2$ (One may consider the partition of the domain $\mathcal{D} \subset \mathbb{R}^3$ similarly). Let $S_h \subset H_0^1(\mathcal{D})$ be the space of continuous piecewise linear functions on $\mathcal{T}_h$, where h denotes the maximal length of the sides of the triangulation $\mathcal{T}_h$.

For each $0 < t \leq T$, the finite element method for solving (1.1) is to find $u_h(t) \in S_h$ such that, with $0 < \alpha < 1$ and $0 \leq \gamma \leq 1$,

$$\begin{cases} {}_C\mathrm{D}_{0,t}^{\alpha} u_h(t) + A_h u_h(t) = P_h\big({}_0I_t^{\gamma} \dot{W}(t)\big), \\ u_h(0) = P_h u_0, \end{cases} \tag{1.11}$$

where $A_h : S_h \to S_h$ denotes the discrete Laplacian [28, (1.33)],

$$(A_h \psi, \chi) = (\nabla \psi, \nabla \chi), \quad \forall\, \chi \in S_h,$$

and $P_h : H \to S_h$ denotes the L_2 projection. Similar to (1.6), the mild solution of (1.11) takes the form

$$u_h(t) = E_h(t) P_h u_0 + \int_0^t \widetilde{E}_h(t-s) P_h \, \mathrm{d}W(s), \tag{1.12}$$

where $E_h(t) = E_{\alpha,1}(-t^{\alpha} A_h)$ and $\widetilde{E}_h(t) = t^{\alpha+\gamma-1} E_{\alpha,\alpha+\gamma}(-t^{\alpha} A_h)$. Here $E_{\alpha,1}(z)$ and $E_{\alpha,\alpha+\gamma}(z)$ are the Mittag-Leffler functions defined by (1.7).

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be the time partition of $[0, T]$ with $t_n = n\tau, n = 0, 1, \ldots, N$, where $\tau = T/N$ is the time step size. Let $U^n \approx u_h(t_n)$, $n = 0, 1, \ldots, N$ be the approximate solution of $u_h(t_n)$. We may define the following fully discrete scheme for approximating (1.11): Find $U^n \approx u_h(t_n)$, $n = 1, 2, \ldots, N$, such that, with $\Delta W^k = W(t_k) - W(t_{k-1})$, $k = 1, 2, \ldots, n$, and $\Delta W^0 = 0$,

$$\begin{cases} \tau^{-\alpha} \sum_{k=0}^{n} w_{n-k}^{(\alpha)} U^k + A_h U^n = \tau^{\gamma} P_h \sum_{k=0}^{n} w_{n-k}^{(-\gamma)} \big(\tau^{-1} \Delta W^k\big), \\ U^0 = P_h u_0, \end{cases} \tag{1.13}$$

where the weights $w_k^{(\alpha)}$, $k = 0, 1, \ldots, n$ are generated by the L1 scheme [17, (1.3)],

$$\Gamma(2-\alpha) w_k^{(\alpha)} = \begin{cases} 1, \quad \text{for } k = 0, \\ -2k^{1-\alpha} + (k-1)^{1-\alpha} + (k+1)^{1-\alpha}, \quad \text{for } k = 1, 2, \ldots, \end{cases} \tag{1.14}$$

and the weights $w_k^{(-\gamma)} = (-1)^k \binom{-\gamma}{k}$, $k = 0, 1, 2, \ldots$ are generated by the first order convolution quadrature formula [26, (1.15)],

$$(1-\zeta)^{-\gamma} = \sum_{k=0}^{\infty} w_k^{(-\gamma)} \zeta^k. \tag{1.15}$$

The convergence and stability of the fully discretized scheme (1.13) have been studied in [30]. See also [16] where the time fractional derivative is approximated by using the first order convolution quadrature formula.

We remark that the solution of (1.13) is unique. To see this, let us assume that U_1^n and $U_2^n, n = 0, 1, \ldots, N$ are two solutions of (1.13) with the initial values $U_1^0 = U_2^0 = P_h u_0$. Then $\epsilon^n := U_1^n - U_2^n$ satisfies the following equation

$$\begin{cases} \tau^{-\alpha} \sum_{k=0}^{n} w_{n-k}^{(\alpha)} \epsilon^k + A_h \epsilon^n = 0, \\ \epsilon^0 = 0, \end{cases} \tag{1.16}$$

By [17, (2.9)], the solution ϵ^n can be represented by the contour integral involving the resolvent of A_h and the initial value ϵ^0, which implies that $\epsilon^n = 0$ for all $n = 1, 2, \ldots, N$ since $\epsilon^0 = 0$.

Let $\phi : H \to \mathbb{R}$ be a functional satisfying [21, (4.1)],

$$\phi \in C(H, \mathbb{R}), \quad D\phi \in C(H, H), \quad D^2\phi \in C_b(H, \mathscr{L}(H)), \tag{1.17}$$

where $C(X, Y)$ and $C_b(X, Y)$ denote the space of continuous resp. continuous and bounded functions from X to Y and D denotes the Fréchet derivative. Here $\mathscr{L}(H)$ denotes the Banach space of the bounded operator from H to H.

We then have the following main theorem:

Theorem 1.2. *Let u and $(U^n)_{n=1}^N$ be the solutions of* (1.6) *and* (1.13)*, respectively. Assume that the Assumptions 1.1 and 1.2 hold and the functional $\phi : H \to \mathbb{R}$ satisfies* (1.17)*. Further assume that $\mathbf{E}|u_0|_q < \infty$ with $0 \le q \le 2$. Let $\kappa > 0$ be defined by* (1.8)*.*

1. *If $\beta \in [0, \kappa]$ satisfies*

$$(\kappa - \beta)\alpha + \frac{\beta}{\kappa} < 2(\alpha + \gamma) - 1, \tag{1.18}$$

then there exists a constant C which is independent of the time and space step sizes τ and h such that the following weak convergence order error estimate holds:

$$\big|\mathbf{E}\phi(U^n) - \mathbf{E}\phi(u(t_n))\big| \le C\big(\tau t_n^{-1+\frac{q}{2}\alpha} + h^2 t_n^{-\alpha\frac{2-q}{2}}\big)(1 + \mathbf{E}|u_0|_q^2) + C\big(\tau^{\frac{\beta}{\kappa}} + h^{\frac{2\beta}{\kappa}}\big)$$

where $\mathbf{E}$ denotes the expectation.

2. *If $\beta \in [0, \kappa]$ satisfies*

$$(\kappa - \beta)\alpha + \frac{\beta}{\kappa} = 2(\alpha + \gamma) - 1, \tag{1.19}$$

then there exists a constant C which is independent of the time and space step sizes τ and h such that the following weak convergence order error estimate holds:

$$\begin{aligned} \big|\mathbf{E}\phi(U^n) - \mathbf{E}\phi(u(t_n))\big| \le & C\big(\tau t_n^{-1+\frac{q}{2}\alpha} + h^2 t_n^{-\alpha\frac{2-q}{2}}\big)(1 + \mathbf{E}|u_0|_q^2) \\ & + C \ln\big(T/(\tau + h^2)\big)\big(\tau^{\frac{\beta}{\kappa}} + h^{\frac{2\beta}{\kappa}}\big). \end{aligned}$$

Remark 1.3. *The condition* (1.18) *is necessary since we need to make the integrals in* (6.6) *and* (6.7) *to be finite in the proof of Theorem 1.2.*

Let us discuss the following important cases of the weak convergence orders in Theorem 1.2.

Case 1. Consider the trace class case, i.e., $\mathrm{Tr}(Q) < \infty$. In this case, we may choose $\beta = \kappa$ in (1.2), which requires, by (1.18), $\alpha + \gamma \geq 1$. The weak convergence orders in Theorem 1.2 are $O(\tau + h^2)$ which are the same as the orders obtained in [16, Theorem 5.2]. In numerical section 7, we give the numerical simulations in Tables 3 and 5 for this case.

Case 2. Consider the one-dimensional case with $Q = I$, $\alpha = 1$ and $\gamma = 0$, that is, the stochastic heat equation case. In this case, we have, by (1.8), $\kappa = 1 - \epsilon$ for sufficiently small $\epsilon > 0$. Choose $\beta = 1/2 - 2\epsilon$ which implies that the following noise regularity assumption condition (1.2) holds,

$$\|A^{\frac{\beta-\kappa}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 = \|A^{-\frac{1}{4}-\frac{\epsilon}{2}}\|_{\mathrm{HS}}^2 = \sum_{j=1}^{\infty} \lambda_j^{-1/2-\epsilon} \approx \sum_{j=1}^{\infty} j^{-1-2\epsilon} < \infty,$$

where $\lambda_j \approx j^2$ is the eigenvalue of the operator $A = -\Delta$ with $\mathcal{D}(A) = H^2(0,1) \cap H_0^1(0,1)$. It is also easy to see that the condition (1.18) holds which follows from

$$\begin{aligned}
&(\kappa - \beta)\alpha + \frac{\beta}{\kappa} - \big(2(\alpha+\gamma) - 1\big) \\
&= \Big((1-\epsilon) - (\frac{1}{2} - 2\epsilon)\Big) \cdot 1 + \frac{\frac{1}{2} - 2\epsilon}{1-\epsilon} - \big(2(1+0) - 1\big) \\
&= \frac{-\epsilon^2 - \frac{1}{2}\epsilon}{1-\epsilon} < 0, \quad \text{as } \epsilon \to 0.
\end{aligned}$$

By Theorem 1.2, the weak convergence orders are nearly $O(\tau^{\frac{\beta}{\kappa}} + h^{\frac{2\beta}{\kappa}}) \approx O(\tau^{\frac{1}{2}} + h)$, which are the same orders as obtained in Jin et al. [16, Theorem 5.2] where the weak convergence order in time in this case is, with $s = \frac{1}{2}, \alpha = 1, \gamma = 0$ and $p = \frac{2}{2-2(\alpha+\gamma)+s\alpha}$,

$$O(\tau^{\min(1,(1-\frac{s}{2})\alpha+\gamma-\frac{1}{p})}) \approx O(\tau^{\min(1,2(\alpha+\gamma)-s\alpha-1)}) = O(\tau^{\frac{1}{2}}).$$

Case 3. Consider the one-dimensional case with $Q = I$, $0 < \alpha < 1$ and $1/2 < \gamma \leq 1$. In this case, we have, by (1.8), $\kappa = 2$. The following noise regularity assumption condition (1.2) holds,

$$\|A^{\frac{\beta-\kappa}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 = \|A^{\frac{\beta-\kappa}{2}}\|_{\mathrm{HS}}^2 = \sum_{j=1}^{\infty} \lambda_j^{\beta-\kappa} \approx \sum_{j=1}^{\infty} j^{2(\beta-\kappa)} < \infty,$$

if $2(\beta - \kappa) < -1$, that is, $\beta < \kappa - \frac{1}{2} = 2 - \frac{1}{2} = \frac{3}{2}$. By Theorem 1.2, in this case the weak convergence orders are nearly $O(\tau^{\frac{\beta}{\kappa}} + h^{\frac{2\beta}{\kappa}}) \approx O(\tau^{\frac{3}{4}} + h^{\frac{3}{2}})$.

Case 4. Consider the one-dimensional case with $Q = I$, $\frac{1}{2} < \alpha < 1$ and $0 \leq \gamma \leq \frac{1}{2}$. In this case, we have, by (1.8), $\kappa = 2 - \frac{1-2\gamma}{\alpha} - \epsilon$. The following noise regularity assumption condition (1.2) holds,

$$\|A^{\frac{\beta-\kappa}{2}} Q^{\frac{1}{2}}\|_{\text{HS}}^2 = \|A^{\frac{\beta-\kappa}{2}}\|_{\text{HS}}^2 = \sum_{j=1}^{\infty} \lambda_j^{\beta-\kappa} \approx \sum_{j=1}^{\infty} j^{2(\beta-\kappa)} < \infty,$$

if $2(\beta - \kappa) < -1$, that is, $\beta < \kappa - \frac{1}{2} = \left(2 - \frac{1-2\gamma}{\alpha} - \epsilon\right) - \frac{1}{2} = \frac{3}{2} - \frac{1-2\gamma}{\alpha} - \epsilon$. By Theorem 1.2, in this case the weak convergence orders are nearly $O(\tau^{\frac{\beta}{\kappa}} + h^{\frac{2\beta}{\kappa}})$ with $\frac{\beta}{\kappa} \approx \frac{\frac{3}{2} - \frac{1-2\gamma}{\alpha}}{2 - \frac{1-2\gamma}{\alpha}}$. In numerical section 7, we give the numerical simulations in Tables 4 and 6 for this case.

In Tables 1 and 2, we show the weak convergence orders obtained in Theorem 1.2 in the cases 1 and 2 and compare them with the orders available in [16, Theorem 5.2].

Table 1: Weak convergence orders for the trace class $\text{Tr}(Q) < \infty$

(α, γ)	weak order in [16]	Weak order in Theorem 1.2
$\alpha + \gamma \geq 1$	$O(\tau)$	$O(\tau)$

Table 2: Weak convergence orders for the case with $Q = I$

(α, γ)	weak order in [16]	Weak orders in Theorem 1.2
$(\alpha, \gamma) = (1, 0)$	$O(\tau^{1/2})$	$O(\tau^{1/2})$

The main contributions of the paper are as follows:

1. The weak convergences of the numerical method for solving the stochastic subdiffusion problem driven by fractionally integrated additive noise are considered where the time fractional derivative is approximated by using the L1 scheme.

2. The proof of the weak convergence of the proposed numerical method is based on the Kolmogorov equation approach.

3. The weak convergence error estimates are applicable for both trace class and cylindrical wiener processes.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and lemmas. In Section 3, we consider the error estimates of the spatial discretization scheme. In Section 4, we derive the error estimates of the time discretization scheme. In Section 5, we obtain the error representation formula for the weak error by using Kolmogorov equation approach. In Section 6, we prove the weak convergence error estimates based on the error representation formula. In Section 7, we provide some numerical simulations to show that the numerical results are consistent with the theoretical results.

For the sake of convenience, the generic constant C may vary in the different places but are independent of the time and space step sizes τ and h.

2. Preliminaries

In this section, we introduce some notations and lemmas which will be used later on.

2.1. Trace class operator and Hilbert-Schmidt operator

In this subsection, we shall give the definitions and some properties of the trace class operator and the Hilbert-Schmidt operator which we will use in the proof of Theorem 1.2, see [10, Appendix C] for more details.

An operator $T \in \mathscr{L}(H)$ is called a trace class operator if there are sequences $\{a_j\}, \{b_j\} \subset H$ such that

$$\sum_{j=1}^{\infty} \|a_j\|\|b_j\| < \infty,$$

and T has the representation

$$Tx = \sum_{j=1}^{\infty}(x, b_j)a_j, \quad x \in H.$$

We define the space

$$\mathscr{L}_1(H) = \{T \in \mathscr{L}(H) : \ T \text{ is a trace class operator}\}.$$

The space $\mathscr{L}_1(H)$ is a Banach space under the norm

$$\|T\|_{\mathrm{Tr}} = \inf\Big\{\sum_{j=1}^{\infty} \|a_j\|\|b_j\| : \ Tx = \sum_{j=1}^{\infty}(x, b_j)a_j\Big\}.$$

If $T \in \mathscr{L}_1(H)$, then the trace of T, defined by

$$\mathrm{Tr}(T) = \sum_{k=1}^{\infty}(T\varphi_k, \varphi_k),$$

is finite, where $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis in H. The number $\mathrm{Tr}(T)$ is independent of the choice of the orthonormal basis, and satisfies

$$|\mathrm{Tr}(T)| \le \|T\|_{\mathrm{Tr}}, \quad \forall\, T \in \mathscr{L}_1(H).$$

If $T \in \mathscr{L}_1(H)$, then $T^* \in \mathscr{L}_1(H)$ and

$$\mathrm{Tr}(T) = \mathrm{Tr}(T^*), \tag{2.1}$$

where T^* is the adjoint operator of T.

If $T \in \mathscr{L}_1(H)$ and $S \in \mathscr{L}(H)$, then $TS \in \mathscr{L}_1(H)$ and $ST \in \mathscr{L}_1(H)$ and

$$\mathrm{Tr}(TS) = \mathrm{Tr}(ST). \tag{2.2}$$

We next introduce the definition of the Hilbert-Schmidt operator. An operator $T \in \mathscr{L}(H)$ is called a Hilbert-Schmidt operator if for some orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$, the sum

$$\|T\|_{\mathrm{HS}}^2 := \sum_{k=1}^{\infty} \|T\varphi_k\|^2 < \infty.$$

The number $\|T\|_{\mathrm{HS}}^2$ is independent of the choice of the orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$.

We define the space

$$\mathscr{L}_2(H) = \{T \in \mathscr{L}(H) :\ T \text{ is a Hilbert-Schmidt operator}\}.$$

The space $\mathscr{L}_2(H)$ is a separable Hilbert space with the norm

$$\|T\|_{\mathrm{HS}} = \Big(\sum_{k=1}^{\infty} \|T\varphi_k\|^2\Big)^{\frac{1}{2}}, \quad \forall\, T \in \mathscr{L}_2(H),$$

and the corresponding inner product is defined by

$$\langle S, T\rangle_{\mathrm{HS}} = \sum_{k=1}^{\infty} (S\varphi_k, T\varphi_k), \quad \forall\, T, S \in \mathscr{L}_2(H).$$

If $T \in \mathscr{L}_2(H)$, then $T^* \in \mathscr{L}_2(H)$ and

$$\|T\|_{\mathrm{HS}} = \|T^*\|_{\mathrm{HS}}. \tag{2.3}$$

If $T \in \mathscr{L}_2(H)$ and $S \in \mathscr{L}(H)$, then $TS \in \mathscr{L}_2(H)$ and $ST \in \mathscr{L}_2(H)$ and

$$\|TS\|_{\mathrm{HS}} \le \|T\|_{\mathrm{HS}}\|S\|, \quad \|ST\|_{\mathrm{HS}} \le \|T\|_{\mathrm{HS}}\|S\|. \tag{2.4}$$

Furthermore, if $T \in \mathscr{L}_2(H)$, $S \in \mathscr{L}_2(H)$, then $TS \in \mathscr{L}_1(H)$ and

$$\|TS\|_{\mathrm{Tr}} \le \|T\|_{\mathrm{HS}}\|S\|_{\mathrm{HS}}, \quad \forall\, T, S \in \mathscr{L}_2(H),$$

which implies that

$$\mathrm{Tr}(TS) \le \|T\|_{\mathrm{HS}}\|S\|_{\mathrm{HS}}, \quad \forall\, T, S \in \mathscr{L}_2(H). \tag{2.5}$$

2.2. Mittag-Leffler functions

The two-parameter function of the Mittag-Leffler type defined by (1.7) plays a very important role in fractional calculus.

Lemma 2.1. *[27, Theorem 1.6] [18, (1.8.28)] Let $0 < \alpha < 1$ and $\beta \in \mathbb{R}$ and assume that $\pi\alpha/2 < \mu < \alpha\pi$. Then there exists a constant $C = C(\alpha, \beta, \mu)$ such that*

$$|E_{\alpha,\beta}(z)| \le C(1 + |z|)^{-1}, \quad \mu \le |\arg(z)| \le \pi, \tag{2.6}$$

and

$$|E_{\alpha,\alpha}(z)| \le C(1 + |z|)^{-2}, \quad \mu \le |\arg(z)| \le \pi. \tag{2.7}$$

We also need the following differentiation formulas for the Mittag-Leffler functions.

Lemma 2.2. *[27, (1.83)] For $\lambda > 0$, $\alpha \in (0,1)$, $\gamma \in [0,1]$, and $t > 0$, there hold*

$$\frac{d}{dt}E_{\alpha,1}(-\lambda t^{\alpha}) = -\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha}), \tag{2.8}$$

and

$$\frac{d}{dt}\big(t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda t^{\alpha})\big) = t^{\alpha+\gamma-2}E_{\alpha,\alpha+\gamma-1}(-\lambda t^{\alpha}),\ \alpha+\gamma \neq 1. \tag{2.9}$$

2.3. Limited smoothing properties of the solution operators

Recall that the mild solution of (1.1) has the following form, with $0 \le t \le T$,

$$u(t) = E(t)u_0 + \int_0^t \widetilde{E}(t-s)\,\mathrm{d}W(s), \tag{2.10}$$

where

$$E(t) = E_{\alpha,1}(-t^{\alpha}A), \quad \widetilde{E}(t) = t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-t^{\alpha}A), \tag{2.11}$$

respectively. Here $E_{\alpha,1}(z)$ and $E_{\alpha,\alpha+\gamma}(z)$ are the Mittag-Leffler functions defined by (1.7).

Further, we denote

$$\overline{E}(t) = t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}A). \tag{2.12}$$

We have the following smoothing properties for the Mittag-Leffler solution operators.

Lemma 2.3. *For $s \in [0,1]$ and $t > 0$, there hold, with $v \in H$,*

$$\|A^s\widetilde{E}(t)v\| \le Ct^{(1-s)\alpha+\gamma-1}\|v\|, \tag{2.13}$$

$$\|A^s\overline{E}(t)v\| \le Ct^{(1-s)\alpha-1}\|v\|, \tag{2.14}$$

$$\Big\|A^s\dot{\widetilde{E}}(t)v\Big\| \le Ct^{(1-s)\alpha+\gamma-2}\|v\|, \tag{2.15}$$

where $\dot{\widetilde{E}}(t) = \frac{\mathrm{d}\widetilde{E}(t)}{\mathrm{d}t}$ denotes the derivative of $\widetilde{E}(t)$.

With κ defined by (1.8), we further have

$$\int_0^t \|A^{\kappa/2}\widetilde{E}(s)v\|^2\,\mathrm{d}s \le Ct^{(2-\kappa)\alpha+2\gamma-1}\|v\|^2. \tag{2.16}$$

Proof. The proofs of (2.13) and (2.14) are similar to the proof of (2.15). We only prove (2.15) here.

Case 1. If $\alpha+\gamma \neq 1$, then we have, by (2.6) and (2.9),

$$\begin{aligned}
\|A^s \dot{\widetilde{E}}(t)v\|^2 &= |\dot{\widetilde{E}}v|_{2s}^2 = |t^{\alpha+\gamma-2}E_{\alpha,\alpha+\gamma-1}(-t^\alpha A)v|_{2s}^2 \\
&= \sum_{j=1}^{\infty} \lambda_j^{2s} t^{2(\alpha+\gamma-2)} (E_{\alpha,\alpha+\gamma-1}(-t^\alpha \lambda_j))^2 (v,\varphi_j)^2 \\
&\leq Ct^{2(\alpha+\gamma-2)} \sum_{j=1}^{\infty} \frac{(\lambda_j t^\alpha)^{2s}}{t^{2s\alpha}} \frac{1}{(1+\lambda_j t^\alpha)^2} (v,\varphi_j)^2 \\
&= Ct^{2(\alpha+\gamma-2)-2s\alpha} \sum_{j=1}^{\infty} \frac{(\lambda_j t^\alpha)^{2s}}{(1+\lambda_j t^\alpha)^2} (v,\varphi_j)^2 \\
&\leq Ct^{2(\alpha+\gamma-2-s\alpha)} \|v\|^2,
\end{aligned}$$

where in the last inequality we use $\sup_{x\in[0,\infty)} \frac{x^{2s}}{(1+x)^2} < \infty$ for any fixed $s \in [0,1]$.

Case 2. If $\alpha+\gamma = 1$, then we have, by (2.7) and (2.8),

$$\begin{aligned}
\|A^s \dot{\widetilde{E}}(t)v\|^2 &= |\dot{\widetilde{E}}v|_{2s}^2 = |At^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha A)v|_{2s}^2 \\
&= \sum_{j=1}^{\infty} \lambda_j^{2s+2} t^{2(\alpha-1)} (E_{\alpha,\alpha}(-t^\alpha \lambda_j))^2 (v,\varphi_j)^2 \\
&\leq Ct^{2(\alpha-1)} \sum_{j=1}^{\infty} \frac{(\lambda_j t^\alpha)^{2s+2}}{t^{(2s+2)\alpha}} \frac{1}{(1+\lambda_j t^\alpha)^4} (v,\varphi_j)^2 \\
&= Ct^{2(\alpha-1)-(2s+2)\alpha} \sum_{j=1}^{\infty} \frac{(\lambda_j t^\alpha)^{2s+2}}{(1+\lambda_j t^\alpha)^4} (v,\varphi_j)^2 \\
&\leq Ct^{2(\alpha+\gamma-2-s\alpha)} \|v\|^2,
\end{aligned}$$

where in the last inequality we use $\sup_{x\in[0,\infty)} \frac{x^{2s+2}}{(1+x)^4} < \infty$ for any fixed $s \in [0,1]$. Combining these two cases complete the proof of (2.15).

We now turn to the proof of (2.16). Obviously,

$$\begin{aligned}
\int_0^t \|A^{\frac{\kappa}{2}} \widetilde{E}(s)v\|^2 \,\mathrm{d}s &= \sum_{j=1}^{\infty} \int_0^t \lambda_j^\kappa s^{2(\alpha+\gamma-1)} \left(E_{\alpha,\alpha+\gamma}\left(-\lambda_j s^\alpha\right)\right)^2 \left(v,\varphi_j\right)^2 \,\mathrm{d}s \\
&\leq C \sum_{j=1}^{\infty} \int_0^t \frac{\lambda_j^\kappa s^{2(\alpha+\gamma-1)}}{\left(1+\lambda_j s^\alpha\right)^2} \left(v,\varphi_j\right)^2 \,\mathrm{d}s.
\end{aligned}$$

Noting that $\sup_{x\in[0,\infty)} \frac{x^\kappa}{(1+x)^2} < \infty$ for any fixed $\kappa \in [0,2]$, one gets

$$\int_0^t \|A^{\frac{\kappa}{2}} \widetilde{E}(s)v\|^2 \,\mathrm{d}s \leq C \sum_{j=1}^{\infty} \left(v,\varphi_j\right)^2 \int_0^t s^{(2-\kappa)\alpha+2\gamma-2} \,\mathrm{d}s. \tag{2.17}$$

Now we estimate the integral of the right hand of (2.17). Recalling the definition of κ from equation (1.8), we proceed as follows for each of the following cases.

Case 1. If $\frac{1}{2} < \gamma \leq 1$, then we have, with $\kappa = 2$,

$$\int_0^t \|A^{\frac{\kappa}{2}}\widetilde{E}(s)v\|^2 \, \mathrm{d}s \leq Ct^{2\gamma-1}\|v\|^2 = Ct^{(2-\kappa)\alpha+2\gamma-1}\|v\|^2.$$

Case 2. If $\gamma = \frac{1}{2}$, then we have, with $\kappa = 2 - \epsilon$ and $\epsilon > 0$,

$$\int_0^t \|A^{\frac{\kappa}{2}}\widetilde{E}(s)v\|^2 \, \mathrm{d}s \leq Ct^{\epsilon\alpha}\|v\|^2 = Ct^{(2-\kappa)\alpha+2\gamma-1}\|v\|^2.$$

Case 3. If $0 \leq \gamma < \frac{1}{2}$, then we have, with $\kappa = 2 - \frac{1-2\gamma}{\alpha} - \epsilon$ and $\epsilon > 0$,

$$\int_0^t \|A^{\frac{\kappa}{2}}\widetilde{E}(s)v\|^2 \, \mathrm{d}s \leq Ct^{\epsilon\alpha}\|v\|^2 = Ct^{(2-\kappa)\alpha+2\gamma-1}\|v\|^2.$$

All this completes the proof. □

3. Spatial discretization

In the following section, we shall consider the spatial discretization of (1.1). Recall that the mild solution of (1.1) has the form

$$u(t) = E(t)u_0 + \int_0^t \widetilde{E}(t-s) \, \mathrm{d}W(s), \tag{3.1}$$

where

$$E(t) = E_{\alpha,1}(-t^\alpha A), \quad \widetilde{E}(t) = t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-t^\alpha A),$$

and the mild solution of the finite element approximation of (3.1) has the form

$$u_h(t) = E_h(t)P_h u_0 + \int_0^t \widetilde{E}_h(t-s)P_h \, \mathrm{d}W(s), \tag{3.2}$$

where

$$E_h(t) = E_{\alpha,1}(-t^\alpha A_h), \quad \widetilde{E}_h(t) = t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-t^\alpha A_h).$$

We have the following lemma for the error estimates of the spatial approximations of $E(t)$ and $\widetilde{E}_h(t)$.

Lemma 3.1. *Let $v \in \dot{H}^q$ with $0 \leq q \leq 2$. Let $E(t), E_h(t), \widetilde{E}(t)$ and $\widetilde{E}_h(t)$ be defined in* (3.1) *and* (3.2)*, respectively. Then one has*

$$\|(E(t) - E_h(t)P_h)v\| \leq Ch^2 t^{-\alpha\frac{2-q}{2}}|v|_q, \tag{3.3}$$

$$\|(\widetilde{E}(t) - \widetilde{E}_h(t)P_h)v\| \leq Ch^2 t^{\gamma-1}\|v\|. \tag{3.4}$$

Proof. The estimate (3.3) can be found in [17, Theorem 2.1] and the estimate of (3.4) is the case with $s = 0$ and $r = 0$ in [16, Lemma 4.4]. For completeness, we slightly give the sketch of the proof of (3.4) here. By the inverse Laplace transform, for any given $v \in H$, we have

$$\widetilde{E}(t)v = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} e^{zt} \left(z^{\alpha} + A\right)^{-1} z^{-\gamma} v \, \mathrm{d}z, \tag{3.5}$$

and

$$\widetilde{E}_h(t)P_h v = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} e^{zt} \left(z^{\alpha} + A_h\right)^{-1} z^{-\gamma} P_h v \, \mathrm{d}z, \tag{3.6}$$

where the contour Γ is a line in the complex plane $\mathbb{C}$ with $\Re z = a > 0$ for some $a > 0$. One can deform Γ into Γ_θ where, with $\theta \in (\pi/2, \pi)$,

$$\Gamma_\theta = \{z \in \mathbb{C} : |\arg z| = \theta\} \text{ (with } \Im z \text{ running from } -\infty \text{ to } \infty). \tag{3.7}$$

When the integral is singular at $z = 0$ and not integrable along Γ_θ, the curve will be interpreted as, with some $\delta > 0$, see [26, (2.1)],

$$\Gamma_{\theta,\delta} = \{z \in \mathbb{C} : z = re^{\pm i\theta}, r \geq \delta\} \cup \{z \in \mathbb{C} : z = \delta e^{i\phi}, \, |\phi| \leq \theta\}. \tag{3.8}$$

Now let us show (3.4). By (3.5) and (3.6), one gets

$$\big\|\big(\widetilde{E}(t) - \widetilde{E}_h(t)P_h\big)v\big\| \leq C \int_{\Gamma_{\theta,\delta}} e^{\Re(z)t} \big\|\big(\big(z^{\alpha} + A\big)^{-1} - \big(z^{\alpha} + A_h\big)^{-1} P_h\big)v\big\| |z|^{-\gamma} |\,\mathrm{d}z|.$$

Using the resolvent estimate (1.3), one may show, see the proof of [16, Lemma 4.4],

$$\big\|\big((z^{\alpha} + A)^{-1} - (z^{\alpha} + A_h)^{-1} P_h\big)v\big\| \leq Ch^2 \|v\|, \ \forall z \in \Sigma_\theta.$$

Hence we have

$$\begin{aligned}
&\|(\widetilde{E}(t) - \widetilde{E}_h(t)P_h)v\| \leq Ch^2 \|v\| \int_{\Gamma_{\theta,\delta}} e^{\Re(z)t} |z|^{-\gamma} |\mathrm{d}z| \\
&= Ch^2 \|v\| \Big(\int_{\{z \in \mathbb{C} : z = \delta e^{i\phi}, |\phi| \leq \theta\}} + \int_{\{z \in \mathbb{C} : z = re^{\pm i\theta}, r \geq \delta\}} \Big) e^{\Re(z)t} |z|^{-\gamma} |\mathrm{d}z| \\
&= I + II.
\end{aligned}$$

For I, one has, with $z = \delta e^{i\phi}$, $\delta = t^{-1}$, where $t^{-1} < \frac{\pi}{\tau}$ for sufficiently small τ,

$$I \leq Ch^2 \|v\| \int_{-\theta}^{\theta} e^{t\delta \cos\phi} \delta^{1-\gamma} \, \mathrm{d}\phi \leq Ch^2 \|v\| \delta^{1-\gamma} \int_{-\theta}^{\theta} e^{\cos\phi} \, \mathrm{d}\phi \leq Ch^2 t^{\gamma-1} \|v\|.$$

For II, one gets, with $z = re^{\pm i\theta}$, $r \geq \delta$, $\delta = t^{-1}$,

$$\begin{aligned}
II &\leq Ch^2 \|v\| \int_{\delta}^{\infty} e^{tr\cos\theta} r^{-\gamma} \, \mathrm{d}r \leq Ch^2 \|v\| \int_{t^{-1}}^{\infty} e^{-ctr} r^{-\gamma} \, \mathrm{d}r \\
&\leq Ch^2 \|v\| t^{\gamma-1} \int_{c}^{\infty} e^{-x} x^{-\gamma} \, \mathrm{d}x \leq Ch^2 t^{\gamma-1} \|v\|,
\end{aligned}$$

where c is a suitable positive constant. Hence (3.4) is shown. □

4. Temporal discretization

Now we will consider the temporal discretization of (3.2). To understand the approximation of the solution operator $\widetilde{E}_h(t)$, we write (3.2) into the following form, with $f_h(s) = P_h \frac{\mathrm{d}W(s)}{\mathrm{d}s}$,

$$u_h(t) = E_h(t)P_h u_0 + \int_0^t \widetilde{E}_h(t-s) f_h(s)\,\mathrm{d}s, \tag{4.1}$$

where

$$E_h(t) = E_{\alpha,1}(-t^\alpha A_h), \quad \widetilde{E}_h(t) = t^{\alpha+\gamma-1} E_{\alpha,\alpha+\gamma}(-t^\alpha A_h).$$

Here $E_{\alpha,1}(z)$ and $E_{\alpha,\alpha+\gamma}(z)$ are the Mittag-Leffler functions defined by (1.7).

By (1.13), the time discretization scheme for approximating (4.1) is defined by, with $U^n \approx u_h(t_n), n = 0, 1, \dots, N$ and $U^0 = P_h u_0$, where $t_n = n\tau, \tau = T/N$, $n = 0, 1, \dots, N$,

$$\tau^{-\alpha} \sum_{k=0}^{n} w_{n-k}^{(\alpha)} U^k + A_h U^n = \tau^\gamma \sum_{k=0}^{n} w_{n-k}^{(-\gamma)} f_h^k, \ n = 1, 2, \dots, N, \tag{4.2}$$

where $f_h^0 = 0$ and $f_h^k = \tau^{-1} P_h \Delta W^k$ with $\Delta W^k = W(t_k) - W(t_{k-1}),\ k = 1, 2, \dots, n$. The weights $w_k^{(\alpha)}$ and $w_k^{(-\gamma)}$ are defined by (1.14), (1.15), respectively.

Further we write the solution U^n of (4.2) into

$$U^n = G^n + V^n, \ n = 1, 2, \dots, N, \tag{4.3}$$

where G^n is the solution of the following homogeneous problem, with $G^0 = P_h u_0$,

$$\tau^{-\alpha} \sum_{k=0}^{n} w_{n-k}^{(\alpha)} G^k + A_h G^n = 0, \ n = 1, 2, \dots, N, \tag{4.4}$$

and where V^n is the solution of the following inhomogeneous problem, with $V^0 = 0$,

$$\tau^{-\alpha} \sum_{k=0}^{n} w_{n-k}^{(\alpha)} V^k + A_h V^n = \tau^\gamma \sum_{k=0}^{n} w_{n-k}^{(-\gamma)} f_h^k, \ n = 1, 2, \dots, N, \tag{4.5}$$

in which $f_h^0 = 0$ and $f_h^k = \tau^{-1} P_h \Delta W^k$ with $\Delta W^k = W(t_k) - W(t_{k-1}),\ k = 1, 2, \dots, n$.

For the approximate solution G^n of the homogeneous problem (4.4), following the proofs of [17, Theorem 3.10, 3.12] with the operator A replaced by the discrete analogue A_h, one may prove the following lemma.

Lemma 4.1. *Let $0 \le q \le 2$, $u_0 \in \dot{H}^q$. And let G^n, $n = 0, 1, \dots, N$ be the solution of (4.4). Then there exists a constant C which is independent of the time and space step sizes τ and h such that*

$$\|G^n - E_h(t_n) P_h u_0\| \le C \tau t_n^{-1+\frac{q}{2}\alpha} |u_0|_q, \ n = 1, 2, \dots, N. \tag{4.6}$$

We now turn to the approximate solutions $V^n, n = 1, 2, \dots, N$ of the inhomogeneous problem (4.5). Let us first find the expression of V^n in (4.5). We shall use the discrete Laplace transform method as in Kovács and Printems [20, (5.2)] for finding the expressions of the solutions of the fully time discretization schemes for a linear stochastic Volterra type evolution equation. Multiplying by ζ^n in both sides of (4.5) and summing over n from 1 to ∞, one has, noting that $V^0 = 0$ and $f_h^0 = 0$,

$$\tau^{-\alpha} \sum_{n=1}^{\infty} \Big(\sum_{k=0}^{n} w_{n-k}^{(\alpha)} V^k \Big) \zeta^n + \sum_{n=1}^{\infty} (A_h V^n) \zeta^n = \tau^{\gamma} \sum_{n=1}^{\infty} \Big(\sum_{k=0}^{n} w_{n-k}^{(-\gamma)} f_h^k \Big) \zeta^n. \tag{4.7}$$

We shall use the notational convention $\tilde{w}(\zeta) = \sum_{n=0}^{\infty} w_n \zeta^n$ for the discrete Laplace transform or generating function of a sequence $(w_n)_{n=0}^{\infty}$. Similarly we denote $\widetilde{V}(\zeta) = \sum_{n=0}^{\infty} V^n \zeta^n$ and $\widetilde{f}_h(\zeta) = \sum_{n=0}^{\infty} f_h^n \zeta^n$.

We remark that $\widetilde{f}_h(\zeta)$ is almost surely finite. For example, in the trace class case, that is, $\mathrm{Tr}(Q) < \infty$, we have, by Isometry property, see e.g., [31, (1.2)],

$$\begin{aligned} \mathbf{E}\|\widetilde{f}_h(\zeta)\|^2 &= \sum_{n=1}^{\infty} \mathbf{E}\|\tau^{-1} P_h \Delta W^n \zeta^n\|^2 = \tau^{-2} \sum_{n=1}^{\infty} \mathbf{E}\Big\| \int_{t_{n-1}}^{t_n} P_h \, \mathrm{d}W(t) \Big\|^2 |\zeta|^{2n} \\ &= \tau^{-2} \sum_{n=1}^{\infty} \int_{t_{n-1}}^{t_n} \|P_h Q^{\frac{1}{2}}\|_{HS}^2 \, \mathrm{d}t |\zeta|^{2n} \le \tau^{-1} \|Q^{\frac{1}{2}}\|_{HS}^2 \sum_{n=1}^{\infty} |\zeta|^{2n} \\ &= \tau^{-1} \mathrm{Tr}(Q) \sum_{n=1}^{\infty} |\zeta|^{2n}, \end{aligned}$$

which is convergent for $|\zeta| < \rho$ with some sufficiently small $\rho \in (0, 1)$. Thus $\widetilde{f}_h(\zeta)$ is almost surely finite for $|\zeta| < \rho$ with some sufficiently small $\rho \in (0, 1)$.

Denote

$$\delta_1(\zeta) := \Big(\sum_{j=0}^{\infty} w_j^{(\alpha)} \zeta^j \Big)^{1/\alpha}, \tag{4.8}$$

and

$$\delta_2(\zeta) = \Big(\sum_{j=0}^{\infty} w_j^{(-\gamma)} \zeta^j \Big)^{-1/\gamma}. \tag{4.9}$$

Applying the following equalities

$$\begin{aligned} \sum_{n=1}^{\infty} \Big(\sum_{k=0}^{n} w_{n-k}^{(\alpha)} V^k \Big) \zeta^n &= \Big(\sum_{j=0}^{\infty} w_j^{(\alpha)} \zeta^j \Big) \widetilde{V}(\zeta), \\ \sum_{n=1}^{\infty} \Big(\sum_{k=0}^{n} w_{n-k}^{(-\gamma)} f_h^k \Big) \zeta^n &= \Big(\sum_{j=0}^{\infty} w_j^{(-\gamma)} \zeta^j \Big) \widetilde{f}_h(\zeta), \end{aligned}$$

we obtain from (4.7) that

$$\widetilde{V}(\zeta) = \big(\tau^{-\alpha}\delta_1(\zeta)^\alpha + A_h\big)^{-1}\tau^{\gamma}\delta_2(\zeta)^{-\gamma}\,\widetilde{f}_h(\zeta). \tag{4.10}$$

By the definitions of $\delta_1(\zeta)$ and $\delta_2(\zeta)$ in (4.8) and (4.9), we see that

$$\big(\tau^{-\alpha}\delta_1(\zeta)^\alpha + A_h\big)^{-1}\tau^{\gamma}\delta_2(\zeta)^{-\gamma}$$

is analytic at $\zeta = 0$. Thus we may find the coefficients $(B_k)_{k=0}^{\infty}$ such that, with $B_0 = I$, (I is the identity operator)

$$\widetilde{B}(\zeta) := \sum_{k=0}^{\infty} B_k \zeta^k = I + \zeta\big(\tau^{-\alpha}\delta_1(\zeta)^\alpha + A_h\big)^{-1}\tau^{\gamma-1}\delta_2(\zeta)^{-\gamma}, \tag{4.11}$$

which implies that, by (4.10),

$$\begin{aligned}\widetilde{V}(\zeta) &= \tau\frac{\tilde{B}(\zeta) - I}{\zeta}\widetilde{f}_h(\zeta) = \tau\big(B_1 + B_2\zeta + B_3\zeta^2 + \dots\big)\big(f_h^1\zeta + f_h^2\zeta^2 + \dots\big)\\ &= \tau\sum_{n=1}^{\infty}\Big(\sum_{k=0}^{n-1} B_{n-k} f_h^{k+1}\Big)\zeta^n = \tau\sum_{n=1}^{\infty}\Big(\sum_{k=0}^{n} B_{n-(k-1)} f_h^{k}\Big)\zeta^n.\end{aligned}$$

Hence we obtain, noting $\widetilde{V}(\zeta) = \sum\limits_{n=0}^{\infty} V^n\zeta^n$ and $V^0 = 0$,

$$V^n = \tau\sum_{k=0}^{n} B_{n-(k-1)} f_h^k, \quad n = 1, 2, \dots, N. \tag{4.12}$$

We remark that $\tau\sum_{k=0}^{n} B_{n-(k-1)} f_h^k$ is the discrete convolution quadrature for approximating the convolution integral $\int_0^t \widetilde{E}_h(t-s) f_h(s)\,\mathrm{d}s$ in (4.1).

We close this section by showing the following nonsmooth data error estimate of $\|(\widetilde{E}_h(t_n) - B_n)P_h g\|$ with $g \in H$.

Theorem 4.2. *Let $\widetilde{E}_h(t_n)$ and B_n be defined by* (4.1) *and* (4.11)*, respectively. Let $g \in H$. Then there exists a constant C which is independent of the time and space step sizes τ and h, such that, with $n = 1, 2, \dots, N$,*

$$\|(\widetilde{E}_h(t_n) - B_n)P_h g\| \le C\tau t_n^{\alpha+\gamma-2}\|g\|. \tag{4.13}$$

To prove this Theorem, we need the following lemma:

Lemma 4.3. *Let $0 < \alpha < 1$, $0 \le \gamma \le 1$ and $\tau \in (0, \infty)$. Denote*

$$\Gamma_{\theta,\delta}^{\tau} = \{z \in \Gamma_{\theta,\delta} : |\Im z| \le \frac{\pi}{\tau}\},$$

where $\Gamma_{\theta,\delta}$ is defined by (3.8). *Set*

$$z_1(z) := \frac{\delta_1(e^{-z\tau})}{\tau}, \quad z_2(z) := \frac{\delta_2(e^{-z\tau})}{\tau}, \quad \textit{for } z \in \Gamma^\tau_{\theta,\delta}, \tag{4.14}$$

where $\delta_1(\zeta)$ and $\delta_2(\zeta)$ are defined by (4.8) *and* (4.9)*, respectively. Then there exist some positive constants c and C which are independent of the time step size τ, such that,*

$$c|z| \le |z_1(z)| \le C|z|, \quad z \in \Gamma^\tau_{\theta,\delta} \tag{4.15}$$

$$c|z| \le |z_2(z)| \le C|z|, \quad z \in \Gamma^\tau_{\theta,\delta} \tag{4.16}$$

$$|z_1(z) - z| \le C\tau^{2-\alpha}|z|^{3-\alpha}, \quad z \in \Gamma^\tau_{\theta,\delta} \tag{4.17}$$

$$|z_2(z) - z| \le C\tau|z|^2, \quad z \in \Gamma^\tau_{\theta,\delta} \tag{4.18}$$

and, with $z \in \Gamma^\tau_{\theta,\delta}$,

$$\|(z^\alpha + A_h)^{-1}z^{-\gamma} - (z_1(z) + A_h)^{-1}z_2(z)^{-\gamma}e^{-z\tau}\| \le C\tau|z|^{-\alpha-\gamma+1}\big(1 + \tau^{1-\alpha}|z|^{1-\alpha}\big). \tag{4.19}$$

Proof. The proof of (4.15) is given in [32, equations (24)], and the proofs of (4.16) and (4.18) can be found in [14, (3.10)-(3.11)], respectively.

Let us first prove (4.17). By the estimates of $z_1(z) - z$ in [32, Page 217], we have

$$z_1(z) - z = O(\tau^{2-\alpha}z^{3-\alpha}), \quad \text{as } z\tau \to 0,$$

which implies that there exists $0 < \delta_0 < \pi$ such that

$$|z_1(z) - z| \le C\tau^{2-\alpha}|z|^{3-\alpha} \quad \text{for } 0 < |z\tau| \le \delta_0,\ z \in \Gamma^\tau_{\theta,\delta}.$$

For large $|z\tau|$ with $\delta_0 \le |z\tau| \le \pi$, $z \in \Gamma^\tau_{\theta,\delta}$, we have, by (4.15),

$$\begin{aligned} |z_1(z) - z| &\le |z_1(z)| + |z| \le C|z| \le C\tau^{2-\alpha}|z|^{3-\alpha}\frac{1}{|z\tau|^{2-\alpha}} \\ &\le C\tau^{2-\alpha}|z|^{3-\alpha}\frac{1}{\delta_0^{2-\alpha}} \le C\tau^{2-\alpha}|z|^{3-\alpha}, \quad z \in \Gamma^\tau_{\theta,\delta}, \end{aligned} \tag{4.20}$$

which shows (4.17).

We now turn to (4.19). The proof is based on the resolvent estimate (1.3) which also holds with A replaced by A_h, see [26, page 6] for the explanation. Note that

$$\begin{aligned} &\big\|(z^\alpha + A_h)^{-1}z^{-\gamma} - \big((z_1(z))^\alpha + A_h\big)^{-1}(z_2(z))^{-\gamma}e^{-z\tau}\big\| \\ \le &\big\|(z^\alpha + A_h)^{-1}z^{-\gamma} - \big((z_1(z))^\alpha + A_h\big)^{-1}(z_1(z))^{-\gamma}\big\| \\ &+ \big\|\big((z_1(z))^\alpha + A_h\big)^{-1}(z_1(z))^{-\gamma} - \big((z_1(z))^\alpha + A_h\big)^{-1}(z_2(z))^{-\gamma}\big\| \\ &+ \big\|\big((z_1(z))^\alpha + A_h\big)^{-1}(z_2(z))^{-\gamma} - \big((z_1(z)^\alpha + A_h\big)^{-1}(z_2(z))^{-\gamma}e^{-z\tau}\big\| \\ = &I + II + III. \end{aligned}$$

For I, following the idea of the proof of Theorem 4.2 in [26], we have, with $\bar{z}$ lying between z and $z_1(z)$ and by the mean-value theorem,

$$
\begin{aligned}
&(z^\alpha + A_h)^{-1} z^{-\gamma} - \big((z_1(z))^\alpha + A_h\big)^{-1}(z_1(z))^{-\gamma} \\
&= \big((-1)(\bar{z}^\alpha + A_h)^{-2}\alpha \bar{z}^{\alpha-1}\bar{z}^{-\gamma} + (\bar{z}^\alpha + A_h)^{-1}(-\gamma)\bar{z}^{-\gamma-1}\big)(z - z_1(z)).
\end{aligned}
$$

Thus we get, by (1.3), (4.17), and noting that $\bar{z}$ and z are equivalent for $z \in \Gamma^\tau_{\theta,\delta}$,

$$
\begin{aligned}
I &= \|(z^\alpha + A_h)^{-1} z^{-\gamma} - \big((z_1(z))^\alpha + A_h\big)^{-1}(z_1(z))^{-\gamma}\| \\
&\le \|(-1)(\bar{z}^\alpha + A_h)^{-2}\alpha \bar{z}^{\alpha-1}\bar{z}^{-\gamma} + (\bar{z}^\alpha + A_h)^{-1}(-\gamma)\bar{z}^{-\gamma-1}\||z - z_1(z)| \\
&\le C\|(z^\alpha + A_h)^{-1}\||z|^{-\gamma-1}|z - z_1(z)| \le C|z|^{-\alpha-\gamma-1}|z - z_1(z)| \\
&\le C|z|^{-\alpha-\gamma-1}\big(\tau^{2-\alpha}|z|^{3-\alpha}\big) = C|z|^{-\alpha-\gamma+1}\tau^{2-\alpha}|z|^{1-\alpha} \\
&\le C\tau|z|^{-\alpha-\gamma+1}(\tau|z|)^{1-\alpha}. \qquad (4.21)
\end{aligned}
$$

For II, we have, with $\bar{z}$ lying between $z_1(z)$ and $z_2(z)$,

$$
(z_1(z))^{-\gamma} - (z_2(z))^{-\gamma} = -\gamma \bar{z}^{-\gamma-1}(z_1(z) - z_2(z)).
$$

Thus we obtain, by (4.18) and (4.17),

$$
\begin{aligned}
II &= \big\|\big((z_1(z))^\alpha + A_h\big)^{-1}(z_1(z))^{-\gamma} - \big((z_1(z))^\alpha + A_h\big)^{-1}(z_2(z))^{-\gamma}\big\| \\
&\le \big\|\big((z_1(z))^\alpha + A_h\big)^{-1}\big\|\big|(z_1(z))^{-\gamma} - (z_2(z))^{-\gamma}\big| \\
&\le C\big\|\big((z_1(z))^\alpha + A_h\big)^{-1}\big\||z|^{-\gamma-1}|z_1(z) - z_2(z)| \\
&\le C\big\|\big((z_1(z))^\alpha + A_h\big)^{-1}\big\||z|^{-\gamma-1}\big(|z_1(z) - z| + |z_2(z) - z|\big) \\
&\le C\tau|z|^{-\alpha-\gamma+1}\big(1 + (\tau|z|)^{1-\alpha}\big). \qquad (4.22)
\end{aligned}
$$

Finally we have, by (4.16) and (4.15), and noting that $|1 - e^{-z\tau}| \le C|\tau z|$ for $z \in \Gamma^\tau_{\theta,\delta}$,

$$
\begin{aligned}
III &= \big\|\big((z_1(z))^\alpha + A_h\big)^{-1}(z_2(z))^{-\gamma} - \big((z_1(z))^\alpha + A_h\big)^{-1}(z_2(z))^{-\gamma}e^{-z\tau}\big\| \\
&= \big\|(z_1(z)^\alpha + A_h)^{-1}(z_2(z))^{-\gamma}\big(1 - e^{-z\tau}\big)\big\| \\
&\le C\|(z_1(z)^\alpha + A_h)^{-1}\||z_2(z)|^{-\gamma}|\tau z| \le C\tau|z|^{-\alpha-\gamma+1}.
\end{aligned}
$$

Hence we prove (4.19). All this ends the proof. □

Proof of Theorem 4.2. First we show that B_n can be written into the following integration form,

$$
B_n P_h g = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma^\tau_{\theta,\delta}} e^{zt_n}(z_1(z)^\alpha + A_h)^{-1} z_2(z)^{-\gamma} e^{-z\tau} P_h g \, \mathrm{d}z, \qquad (4.23)
$$

where $\Gamma_{\theta,\delta}^{\tau} = \{z \in \Gamma_{\theta,\delta} : |\Im z| \leq \frac{\pi}{\tau}\}$. In fact, noting (4.11) and using the Cauchy integral formula, we have, for some $\rho > 0$,

$$B_n = \frac{1}{2\pi i} \int_{|\zeta|=\rho} \zeta^{-n-1} \widetilde{B}(\zeta) \, \mathrm{d}\zeta.$$

Furthermore we have, by (4.14),

$$\begin{aligned} B_n &= \frac{1}{2\pi i} \int_{|\zeta|=\rho} \zeta^{-n-1} \Big(1 + \zeta(\tau^{-\alpha}\delta_1(\zeta)^{\alpha} + A_h)^{-1}\tau^{\gamma-1}\delta_2(\zeta)^{-\gamma}\Big) \, \mathrm{d}\zeta \\ &= \frac{1}{2\pi i} \int_{|\zeta|=\rho} \zeta^{-n-1} \Big(\zeta(\tau^{-\alpha}\delta_1(\zeta)^{\alpha} + A_h)^{-1}\tau^{\gamma-1}\delta_2(\zeta)^{-\gamma}\Big) \, \mathrm{d}\zeta \\ &= \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}^{\tau}} e^{zt_n} (z_1(z)^{\alpha} + A_h)^{-1} z_2(z)^{-\gamma} e^{-z\tau} \, \mathrm{d}z, \quad n = 1, 2, \ldots, \end{aligned} \tag{4.24}$$

where in the last equality, we use the variable change $\zeta = e^{-z\tau}$ and deform $|\zeta| = \rho$ into $\Gamma_{\theta,\delta}^{\tau}$, see [17, (2.9)]. Thus we have, by (3.6) and (4.23),

$$\begin{aligned} & \|(\widetilde{E}_h(t_n) - B_n)P_h\| \\ \leq & \left\| \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}^{\tau}} e^{zt_n} \Big[(z^{\alpha} + A_h)^{-1} z^{-\gamma} - \big(z_1(z)^{\alpha} + A_h\big)^{-1} z_2(z)^{-\gamma} e^{-z\tau}\Big] P_h g \, \mathrm{d}z \right\| \\ & + \left\| \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta} \setminus \Gamma_{\theta,\delta}^{\tau}} e^{zt_n} (z^{\alpha} + A_h)^{-1} z^{-\gamma} P_h g \, \mathrm{d}z \right\| \\ = & \, I_1 + I_2. \end{aligned}$$

For I_1, noting that $1/2 < \alpha + \gamma < 2$ by Assumption 1.1 and $|z\tau| \leq C$ for $z \in \Gamma_{\theta,\delta}^{\tau}$, one has, by (4.19),

$$\begin{aligned} \|I_1\| &= \left\| \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}^{\tau}} e^{zt_n} \Big[(z^{\alpha} + A_h)^{-1} z^{-\gamma} - \big(z_1(z)^{\alpha} + A_h\big)^{-1} z_2(z)^{-\gamma} e^{-z\tau}\Big] P_h g \, \mathrm{d}z \right\| \\ &\leq C\tau \int_{\Gamma_{\theta,\delta}^{\tau}} |e^{zt_n}| |z|^{-\alpha-\gamma+1} \big(1 + (\tau|z|)^{1-\alpha}\big) |\, \mathrm{d}z| \, \|P_h g\| \\ &\leq C\tau \int_0^{\infty} e^{-crt_n} (t_n r)^{-\alpha-\gamma+1} t_n^{\alpha+\gamma-1} t_n^{-1} \, \mathrm{d}(t_n r) \, \|P_h g\| \\ &\leq C\tau t_n^{\alpha+\gamma-2} \big(\tau/t_n\big)^{1-\alpha} \|g\| \leq C\tau t_n^{\alpha+\gamma-2} \|g\|. \end{aligned}$$

For I_2, one gets, noting again that $1/2 < \alpha + \gamma < 2$ by Assumption 1.1

$$
\begin{aligned}
\|I_2\| &= \Big\|\frac{1}{2\pi i}\int_{\Gamma_{\theta,\delta}\backslash\Gamma^\tau_{\theta,\delta}} e^{zt_n}(z^\alpha + A_h)^{-1}z^{-\gamma}P_h g\,\mathrm{d}z\Big\| \\
&\le C\int_{\frac{\pi}{\tau}}^\infty e^{-crt_n}(r^{-1}r)r^{-\alpha}r^{-\gamma}\,\mathrm{d}r\,\|P_h g\| \le C\tau\int_{\frac{\pi}{\tau}}^\infty e^{-crt_n}r^{1-\alpha-\gamma}\,\mathrm{d}r\|P_h g\| \\
&= C\tau\int_0^\infty e^{-crt_n}(t_n r)^{1-\alpha-\gamma}t_n^{\alpha+\gamma-1}t_n^{-1}\,\mathrm{d}(t_n r)\,\|P_h g\| \le C\tau t_n^{\alpha+\gamma-2}\|g\|.
\end{aligned}
$$

Hence we prove that (4.13) is true. The proof is thus finished. □

5. Error representation formula

In the present section, we will give the error representation formula for, with $T = t_N, N \ge 1$,

$$
e(T) = \mathbf{E}\phi(U^N) - \mathbf{E}\phi(u(T)),
$$

in which the functional $\phi : \mathbb{R} \to H$ satisfies (1.17). Here $u(T)$ and $U^N \in S_h$ are defined by (1.6) and (1.13), respectively, that is,

$$
u(T) = E(T)u_0 + \int_0^T \widetilde{E}(T-s)\,\mathrm{d}W(s), \tag{5.1}
$$

and, with $\Delta W^k = W(t_k) - W(t_{k-1}), k = 1, 2, \dots, N$ and $\Delta W^0 = 0$,

$$
\begin{cases}
\tau^{-\alpha}\sum_{k=0}^N w_{N-k}^{(\alpha)}U^k + A_h U^N = \tau^\gamma P_h \sum_{k=0}^N w_{N-k}^{(-\gamma)}\big(\tau^{-1}\Delta W^k\big), \\
U^0 = P_h u_0.
\end{cases} \tag{5.2}
$$

By (4.3) and (4.12), we rewrite $U^N \in S_h, N = 1, 2, \dots,$ into the following integration form

$$
\begin{aligned}
U^N &= G^N + V^N = G^N + \tau\sum_{k=0}^N B_{N-(k-1)}P_h f_h^k \\
&= G^N + \tau\sum_{k=0}^N B_{N-(k-1)}P_h\big(\tau^{-1}\Delta W^k\big) = G^N + \int_0^T \overline{B}_N(T-s)\,\mathrm{d}W(s),
\end{aligned} \tag{5.3}
$$

where

$$
\overline{B}_N(t) = \begin{cases}
B_1 P_h, & t \in (t_0, t_1], \\
B_2 P_h, & t \in (t_1, t_2], \\
\vdots & \\
B_N P_h, & t \in (t_{N-1}, t_N].
\end{cases} \tag{5.4}
$$

Lemma 5.1. *Let $\widetilde{E}(t)$ and $\overline{B}_N(t)$ be defined by* (2.11) *and* (5.4)*, respectively. Then one has, with $r \in [0,2]$, and $0 < s \le t \le T$,*

$$\|\widetilde{E}(t) - \widetilde{E}(s)\| \le C s^{\alpha+\gamma-2} |t-s|, \tag{5.5}$$

$$\|A^{\frac{r}{2}} \overline{B}_N(t)\| \le C t^{(1-\frac{r}{2})\alpha+\gamma-1}, \quad \textit{for } (1-\frac{r}{2})\alpha+\gamma-1<0, \tag{5.6}$$

and, with κ defined by (1.8)*,*

$$\big\|A^{\frac{\kappa-\beta}{2}}\big(\widetilde{E}(t) - \overline{B}_N(t)\big)\big\| \le C t^{(1-\frac{\kappa-\beta}{2})\alpha+\gamma-1-\frac{\beta}{\kappa}} (\tau+h^2)^{\frac{\beta}{\kappa}}, \quad \textit{for } \ 0 \le \beta \le \kappa. \tag{5.7}$$

Proof. The estimate (5.5) follows from, by (2.15),

$$\begin{aligned}\|\widetilde{E}(t)v - \widetilde{E}(s)v\| &= \Big\| \int_s^t \dot{\widetilde{E}}(\tau) v \, \mathrm{d}\tau \Big\| \le \int_s^t \|\dot{\widetilde{E}}(\tau)v\| \, \mathrm{d}\tau \\ &\le C\|v\| \int_s^t \tau^{\alpha+\gamma-2} \, \mathrm{d}\tau \le C s^{\alpha+\gamma-2}|t-s| \|v\|.\end{aligned}$$

The estimate (5.6) follows from, for $t \in (t_{n-1}, t_n]$, with $r \in [0,2]$,

$$\big\|A^{\frac{r}{2}} \overline{B}_N(t)\big\| = \big\|A^{\frac{r}{2}} B_n P_h\big\| \le C t_n^{(1-\frac{r}{2})\alpha+\gamma-1} \le C t^{(1-\frac{r}{2})\alpha+\gamma-1}, \text{ for } (1-\frac{r}{2})\alpha+\gamma-1<0. \tag{5.8}$$

The first inequality in (5.8) can be proved as in [16, Lemma 4.6]. To prove this inequality, we need to use the resolvent estimates $\|(z_1(z)^\alpha + A_h)^{-1}\| \le C|z|^{-\alpha}$ and $\|A_h(z_1(z)^\alpha + A_h)^{-1}\| \le C$ in (4.23) and such estimates hold for both $L1$ scheme and the first order convolution quadrature formula.

For (5.7), we have, by (3.4) and (4.13) and noting that $0<\alpha<1$,

$$\begin{aligned}&\|\widetilde{E}(t_n) - B_n P_h\| \le \|\widetilde{E}(t_n) - \widetilde{E}_h(t_n)P_h\| + \|\widetilde{E}_h(t_n)P_h - B_nP_h\| \\ &\le Ch^2 t_n^{\gamma-1} + C\tau t_n^{\alpha+\gamma-2} \le C t_n^{\alpha+\gamma-2}(\tau + t_n^{1-\alpha}h^2) \le C t_n^{\alpha+\gamma-2}(\tau+h^2).\end{aligned} \tag{5.9}$$

For $t \in (t_{n-1}, t_n]$, one has, noting that $\alpha+\gamma-2<0$ and using (5.5),

$$\begin{aligned}&\|\widetilde{E}(t) - \overline{B}_N(t)\| = \|\widetilde{E}(t) - \widetilde{E}(t_n) + \widetilde{E}(t_n) - \overline{B}_N(t)\| \\ &\le \|\widetilde{E}(t) - \widetilde{E}(t_n)\| + \|\widetilde{E}(t_n) - B_nP_h\| \le C t^{\alpha+\gamma-2}|t_n - t| + C t_n^{\alpha+\gamma-2}(\tau+h^2) \\ &\le C t^{\alpha+\gamma-2}(\tau+h^2).\end{aligned} \tag{5.10}$$

Applying the following interpolation result, with $0 \le \theta \le 1$, $t>0, s>0$, one gets

$$\|A^{t\theta + s(1-\theta)} v\| \le \|A^t v\|^\theta \cdot \|A^s v\|^{1-\theta}. \tag{5.11}$$

Choosing $t=0, s=\frac{\kappa}{2}$, $\theta=\frac{\beta}{\kappa}$ in (5.11), and using (2.13), (5.6) and (5.10) yield

$$\begin{aligned}\left\|A^{\frac{\kappa-\beta}{2}}\left(\widetilde{E}(t)-\overline{B}_N(t)\right)\right\| &\leq \left\|\widetilde{E}(t)-\overline{B}_N(t)\right\|^{\frac{\beta}{\kappa}}\left\|A^{\frac{\kappa}{2}}\left(\widetilde{E}(t)-\overline{B}_N(t)\right)\right\|^{1-\frac{\beta}{\kappa}} \\ &\leq C\left\|\widetilde{E}(t)-\overline{B}_N(t)\right\|^{\frac{\beta}{\kappa}}\left(\left\|A^{\frac{\kappa}{2}}\widetilde{E}(t)\right\|^{1-\frac{\beta}{\kappa}}+\left\|A^{\frac{\kappa}{2}}\overline{B}_N(t)\right\|^{1-\frac{\beta}{\kappa}}\right) \\ &\leq Ct^{(1-\frac{\kappa-\beta}{2})\alpha+\gamma-1-\frac{\beta}{\kappa}}(\tau+h^2)^{\frac{\beta}{\kappa}}.\end{aligned}$$

The proof is thus ended. □

We now introduce the error representation formula based on the Kolmogorov equation approach. This approach has been used in [19] for the weak convergence of finite element approximations of a linear stochastic evolution equation with additive noise and has also been used in [21] for the weak convergence of a fully discrete scheme for solving a linear stochastic evolution equation with a positive-type memory term.

Define the auxiliary problem as follows,

$$\begin{aligned}&dZ(t)=\widetilde{E}(T-t)\mathrm{dW(t)}, \quad \tau<\mathrm{t}\leq \mathrm{T},\\ &Z(\tau)=\xi,\end{aligned}$$

which has the solution

$$Z(t;\tau,\xi)=\xi+\int_{\tau}^{t}\widetilde{E}(T-s)\,\mathrm{d}W(s),$$

where ξ is the $\mathscr{F}_t$-measurable random variable.

For the functional $\phi: H\to\mathbb{R}$ satisfying (1.17), let

$$w(x,t)=\mathbf{E}\big(\phi\big(Z(T;t,x)\big)\big),\ x\in H,\ t\in[0,T]. \tag{5.12}$$

Then $w(x,t)$ is a solution of the following Kolmogorov equation,

$$\begin{cases} w_t(x,t)+\frac{1}{2}\operatorname{Tr}\left(w_{xx}(x,t)\widetilde{E}(T-t)Q\widetilde{E}(T-t)^*\right)=0,\ (x,t)\in H\times[0,T), \\ w(x,T)=\phi(x),\ x\in H. \end{cases} \tag{5.13}$$

Furthermore, for any $\mathscr{F}_t$-measurable random variable ξ, one has

$$w(\xi,t)=\mathbf{E}\big(\phi\big(Z(T;t,\xi)\big)\big|\mathscr{F}_t\big),$$

which implies, by the law of double expectation,

$$\mathbf{E}\big(w(\xi,t)\big)=\mathbf{E}\big(\mathbf{E}\big(\phi\big(Z(T;t,\xi)\big)\big|\mathscr{F}_t\big)\big)=\mathbf{E}\phi(Z(T;t,\xi)).$$

Choosing $\xi=E(T)u_0$ gives

$$\mathbf{E}\big(w(E(T)u_0,0)\big)=\mathbf{E}\big(\phi(Z(T;0,E(T)u_0))\big)=\mathbf{E}\phi(u(T)).$$

If $\xi = U^N$ is chosen, then one has

$$\mathbf{E}\big(w(U^N,T)\big) = \mathbf{E}\big(\phi\big(Z(T;T,U^N)\big)\big) = \mathbf{E}\phi(U^N).$$

Thus one gets

$$\begin{aligned}\mathbf{E}\phi(U^N) - \mathbf{E}\phi(u(T)) &= \mathbf{E}w(U^N,T) - \mathbf{E}w(E(T)u_0,0)\\ = \Big(\mathbf{E}w(G^N,0) - \mathbf{E}w(E(T)u_0,0)\Big) &+ \Big(\mathbf{E}w(U^N,T) - \mathbf{E}w(G^N,0)\Big).\end{aligned}$$

Next, it is needed to use the Itô formula to estimate $\mathbf{E}w(U^N,T) - \mathbf{E}w(G^N,0)$. To see this, consider the following problem

$$\begin{aligned}&d\widetilde{Y}(t) = \overline{B}_N(T-t)\,\mathrm{d}W(t),\\ &\widetilde{Y}(0) = G^N,\end{aligned}$$

which has the solution

$$\begin{aligned}&\widetilde{Y}(t) = G^N + \int_0^t \overline{B}_N(T-t)\,\mathrm{d}W(t),\ 0 < t \le T,\\ &\widetilde{Y}(0) = G^N.\end{aligned}$$

Note that $\widetilde{Y}(T) = U^N$ and $\widetilde{Y}(0) = G^N$. By the Itô formula, see [21, Proposition 4.1], and the Kolmogorov equation (5.13), we get

$$\begin{aligned}&\mathbf{E}w(U^N,T) - \mathbf{E}w(G^N,0)\\ &= \mathbf{E}w(\widetilde{Y}(T),T) - \mathbf{E}w(\widetilde{Y}(0),0)\\ &= \mathbf{E}\int_0^T w_t(\widetilde{Y}(t),t)\,\mathrm{d}t + \mathbf{E}\int_0^T \frac{1}{2}\mathrm{Tr}\Big(w_{xx}(\widetilde{Y}(t),t)\big[\overline{B}_N(T-t)Q\overline{B}_N(T-t)^*\big]\Big)\,\mathrm{d}t\\ &= \frac{1}{2}\int_0^T \mathrm{Tr}\Big(w_{xx}(\widetilde{Y}(t),t)\Big\{\big[\overline{B}_N(T-t)Q\overline{B}_N(T-t)^*\big] - \big[E(T-t)QE(T-t)^*\big]\Big\}\Big)\,\mathrm{d}t.\end{aligned}$$

By using the same arguments as that in [21, Theorem 4.3], we obtain

$$\begin{aligned}&\mathbf{E}w(U^N,T) - \mathbf{E}w(G^N,0)\\ &= \frac{1}{2}\int_0^T \mathrm{Tr}\big(w_{xx}(\widetilde{Y}(t),t)\big[\overline{B}_N(T-t) + E(T-t)\big]Q\big[\overline{B}_N(T-t) - E(T-t)\big]^*\,\mathrm{d}t.\end{aligned}$$

Hence we get the following error representation formula,

$$\begin{aligned}
&\mathbf{E}\phi(U^N) - \mathbf{E}\phi(u(T)) \\
&= \mathbf{E}\big(w(G^N, 0) - w(E(T)u_0, 0)\big) + \frac{1}{2}\mathbf{E}\int_0^T \operatorname{Tr}\left(w_{xx}(\widetilde{Y}(t), t)O(t)\right) \mathrm{d}t,
\end{aligned} \tag{5.14}$$

where

$$O(t) = \big(\overline{B}_N(T-t) + \widetilde{E}(T-t)\big)Q\big(\overline{B}_N(T-t) - \widetilde{E}(T-t)\big)^*.$$

Now we may turn to the proof of the main result in this paper.

6. Proof of Theorem 1.2

Proof. By (5.14), we have

$$\begin{aligned}
\mathbf{E}\phi(U^N) - \mathbf{E}\phi(u(T)) &= \mathbf{E}\big(w(G^N, 0) - w(E(T)u_0, 0)\big) \\
&\quad + \frac{1}{2}\mathbf{E}\int_0^T \operatorname{Tr}\left(w_{xx}(\widetilde{Y}(t), t)O(t)\right) \mathrm{d}t \\
&=: I + \frac{1}{2} II,
\end{aligned} \tag{6.1}$$

in which

$$O(t) = \big(\overline{B}_N(T-t) + \widetilde{E}(T-t)\big)Q\big(\overline{B}_N(T-t) - \widetilde{E}(T-t)\big)^*.$$

For I, we have, by Taylor's series expansion and using the assumption (1.17) for ϕ,

$$|\phi(x) - \phi(y)| \le \|D\phi(y)\| \cdot \|x-y\| + C\|x-y\|^2,$$

where $C = \sup_{x\in H} \|D^2\phi(x)\|_{\mathscr{L}(H)}$ and that $\|D\phi(x)\| \le K(1+\|x\|)$ with $K = \max\{C, \|D\phi(0)\|\}$, which implies that

$$|\phi(x) - \phi(y)| \le C(1 + \|y\|) \cdot \|x-y\| + C\|x-y\|^2. \tag{6.2}$$

Using the law of double expectations and noting that G^n, $n = 0, 1, \ldots, N$ are the solutions of the homogeneous problem (4.4), we have, by (5.12) and Lemmas 3.1 and 4.1,

$$\begin{aligned}
|I| &= \left|\mathbf{E}(w(G^N, 0) - w(E(T)u_0, 0))\right| \\
&= \left|\mathbf{E}\Big(\mathbf{E}\Big[\phi\big(G^N + \int_0^T \widetilde{E}(T-s)\,\mathrm{d}W(s)\big) - \phi\big(E(T)u_0 + \int_0^T \widetilde{E}(T-s)\,\mathrm{d}W(s)\big)\Big]\Big|\mathcal{F}_0\Big)\right| \\
&\le C\mathbf{E}\big(\|G^N - E(T)u_0\| \cdot (1 + \|u(T)\|)\big) + C\mathbf{E}(\|G^N - E(T)u_0\|^2) \\
&\le C(\tau T^{-1+\frac{q}{2}\alpha} + h^2 T^{-\alpha\frac{2-q}{2}})\mathbf{E}\big(|u_0|_q(1 + \|u(T)\|)\big) + \mathbf{E}\left\|G^N - E(T)u_0\right\|^2 \\
&\le C(\tau T^{-1+\frac{q}{2}\alpha} + h^2 T^{-\alpha\frac{2-q}{2}})\left(1 + \mathbf{E}\big(|u_0|_q^2 + \|u(T)\|^2\big)\right) \\
&\quad + (\tau T^{-1+\frac{q}{2}\alpha} + h^2 T^{-\alpha\frac{2-q}{2}})^2\mathbf{E}|u_0|_q^2 \\
&\le C(\tau T^{-1+\frac{q}{2}\alpha} + h^2 T^{-\alpha\frac{2-q}{2}})\big(1 + \mathbf{E}|u_0|_q^2\big).
\end{aligned} \tag{6.3}$$

Next, we estimate II. By (2.1) and (2.2), one gets, since $w_{xx}(\widetilde{Y}(t),t)$ is selfadjoint, and Q is selfadjoint, positive semidefinite,

$$\begin{aligned} II :&= \mathbf{E}\int_0^T \mathrm{Tr}\left(w_{xx}(\widetilde{Y}(t),t)\big[\overline{B}_N(T-t)+\widetilde{E}(T-t)\big]Q\big[\overline{B}_N(T-t)-\widetilde{E}(T-t)\big]^*\right)\mathrm{d}t \\ &= \mathbf{E}\int_0^T \mathrm{Tr}\left(w_{xx}(\widetilde{Y}(t),t)\big[\overline{B}_N(T-t)+\widetilde{E}(T-t)\big]^*Q\big[\overline{B}_N(T-t)-\widetilde{E}(T-t)\big]\right)\mathrm{d}t \\ &= \mathbf{E}\int_0^T \mathrm{Tr}\Big(w_{xx}(\widetilde{Y}(t),t)\big[\overline{B}_N(T-t)+\widetilde{E}(T-t)\big]^*A^{\frac{\kappa-\beta}{2}}A^{\frac{\beta-\kappa}{2}}Q^{\frac{1}{2}} \\ &\qquad\qquad \cdot Q^{\frac{1}{2}}A^{\frac{\kappa-\beta}{2}}A^{\frac{\beta-\kappa}{2}}\big[\overline{B}_N(T-t)-\widetilde{E}(T-t)\big]\Big)\,\mathrm{d}t. \end{aligned} \tag{6.4}$$

Noting that $A^{\frac{\kappa-\beta}{2}}v_h, 0\le\beta\le\kappa$ is well defined for $v_h\in S_h$ when $\frac{\kappa-\beta}{2}\in[0,1/2]$, we then have

$$\Big(A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(T-t)+\widetilde{E}(T-t)\big]\Big)^* = \big[\overline{B}_N(T-t)+\widetilde{E}(T-t)\big]^*A^{\frac{\kappa-\beta}{2}}.$$

Thus we get

$$\begin{aligned} II = \mathbf{E}\int_0^T \mathrm{Tr}\Big(&w_{xx}(\widetilde{Y}(t),t)\Big(A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(T-t)+\widetilde{E}(T-t)\big]\Big)^*\big(A^{\frac{\beta-\kappa}{2}}Q^{\frac{1}{2}}\big) \\ &\cdot\big(Q^{\frac{1}{2}}A^{\frac{\beta-\kappa}{2}}\big)A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(T-t)-\widetilde{E}(T-t)\big]\Big)\,\mathrm{d}t. \end{aligned}$$

By using (2.3), (2.4) and (2.5), we have

$$
\begin{aligned}
\|II\| \leq & \, \mathbf{E}\int_0^T \Big\| w_{xx}(\widetilde{Y}(t),t)\Big(A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(T-t)+\widetilde{E}(T-t)\big]\Big)^*\big(A^{\frac{\beta-\kappa}{2}}Q^{\frac{1}{2}}\big)\Big\|_{\mathrm{HS}} \\
& \qquad \cdot \Big\|\big(Q^{\frac{1}{2}}A^{\frac{\beta-\kappa}{2}}\big)\Big(A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(T-t)-\widetilde{E}(T-t)\big]\Big)\Big\|_{\mathrm{HS}}\,\mathrm{d}t \\
\leq & \, \mathbf{E}\int_0^T \|w_{xx}(\widetilde{Y}(t),t)\|\Big\|\big(A^{\frac{\beta-\kappa}{2}}Q^{\frac{1}{2}}\big)^*\Big(A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(T-t)+\widetilde{E}(T-t)\big]\Big)\Big\|_{\mathrm{HS}} \\
& \qquad \cdot \Big\|\big(Q^{\frac{1}{2}}A^{\frac{\beta-\kappa}{2}}\big)\Big(A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(T-t)-\widetilde{E}(T-t)\big]\Big)\Big\|_{\mathrm{HS}}\,\mathrm{d}t \\
\leq & \, \mathbf{E}\int_0^T \|w_{xx}(\widetilde{Y}(t),t)\|\|\big(A^{\frac{\beta-\kappa}{2}}Q^{\frac{1}{2}}\big)^*\|_{\mathrm{HS}}\Big\|A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(T-t)+\widetilde{E}(T-t)\big]\Big\| \\
& \qquad \cdot \|Q^{\frac{1}{2}}A^{\frac{\beta-\kappa}{2}}\|_{\mathrm{HS}}\Big\|A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(T-t)-\widetilde{E}(T-t)\big]\Big\|\,\mathrm{d}t \\
\leq & \, \mathbf{E}\int_0^T \|w_{xx}(\widetilde{Y}(t),t)\|\|A^{\frac{\beta-\kappa}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2\Big\|A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(T-t)+\widetilde{E}(T-t)\big]\Big\| \\
& \qquad \cdot \Big\|A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(T-t)-\widetilde{E}(T-t)\big]\Big\|\,\mathrm{d}t \\
\leq & \sup_{(x,t)\in H\times[0,T]} \big\|w_{xx}(x,t)\big\| \cdot \|A^{\frac{\beta-\kappa}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \\
& \qquad \cdot \int_0^T \|A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(t)+\widetilde{E}(t)\big]\| \cdot \|A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(t)-\widetilde{E}(t)\big]\|\,\mathrm{d}t.
\end{aligned}
$$

Note that, see [21, page 950]

$$
\sup_{(x,t)\in H\times[0,T]} \|w_{xx}(x,t)\| \leq \sup_{x\in H}\|D^2\phi(x)\|.
$$

Combining this with the assumption (1.17) for ϕ and the Assumption 1.2, we obtain

$$
\|II\| \leq C\int_0^T \|A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(t)+\widetilde{E}(t)\big]\| \cdot \|A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(t)-\widetilde{E}(t)\big]\|\,\mathrm{d}t. \tag{6.5}
$$

If $(\kappa-\beta)\alpha+\frac{\beta}{\kappa}<2(\alpha+\gamma)-1$, $\beta\in[0,\kappa], \kappa>0$, then we get, using (5.6), (2.13), (5.7) and noting $\beta\in[0,\kappa]$,

$$
\begin{aligned}
& \int_0^T \|A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(t)+\widetilde{E}(t)\big]\| \cdot \|A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(t)-\widetilde{E}(t)\big]\|\,\mathrm{d}t \\
& \leq C\int_0^T \big(\|A^{\frac{\kappa-\beta}{2}}\overline{B}_N(t)\| + \|A^{\frac{\kappa-\beta}{2}}\widetilde{E}(t)\|\big)\|A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(t)-\widetilde{E}(t)\big]\|\,\mathrm{d}t \\
& \leq C(\tau+h^2)^{\frac{\beta}{\kappa}}\int_0^T t^{2(1-\frac{\kappa-\beta}{2})\alpha+2\gamma-2-\frac{\beta}{\kappa}}\,\mathrm{d}t \leq C(\tau+h^2)^{\frac{\beta}{\kappa}} \leq C(\tau^{\frac{\beta}{\kappa}}+h^{\frac{2\beta}{\kappa}}).
\end{aligned} \tag{6.6}
$$

Together (6.6) with (6.5) and (6.3) show (1.18).

If $2(\alpha+\gamma)-1=(\kappa-\beta)\alpha+\frac{\beta}{\kappa}$, $\beta\in[0,\kappa], \kappa>0$, we need to split the integral on $[0,T]$ into two sub-integrals $[0,\tau+h^2]$ and $[\tau+h^2,T]$. For the first sub-integral $[0,\tau+h^2]$, we have, by using the basic inequality and (2.13) and (5.6),

$$
\begin{aligned}
&\int_0^{\tau+h^2} \big\|A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(t)+\widetilde{E}(t)\big]\big\|\cdot\big\|A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(t)-\widetilde{E}(t)\big]\big\|\,\mathrm{d}t\\
&\le C\int_0^{\tau+h^2}\Big(\big\|A^{\frac{\kappa-\beta}{2}}\overline{B}_N(t)\big\|^2+\big\|A^{\frac{\kappa-\beta}{2}}\widetilde{E}(t)\big\|^2\Big)\,\mathrm{d}t \qquad (6.7)\\
&\le C\int_0^{\tau+h^2} t^{2(1-\frac{\kappa-\beta}{2})\alpha+2\gamma-2}\,\mathrm{d}t\le C(\tau+h^2)^{2(1-\frac{\kappa-\beta}{2})\alpha+2\gamma-1}\\
&=C(\tau+h^2)^{\frac{\beta}{\kappa}}\le C(\tau^{\frac{\beta}{\kappa}}+h^{\frac{2\beta}{\kappa}}).
\end{aligned}
$$

For the second sub-integral, we have, by using (2.13), (5.6) and (5.7),

$$
\begin{aligned}
&\int_{\tau+h^2}^{T} \big\|A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(t)+\widetilde{E}(t)\big]\big\|\cdot\big\|A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(t)-\widetilde{E}(t)\big]\big\|\,\mathrm{d}t\\
&\le C\int_{\tau+h^2}^{T}\Big(\big\|A^{\frac{\kappa-\beta}{2}}\overline{B}_N(t)\big\|+\big\|A^{\frac{\kappa-\beta}{2}}\widetilde{E}(t)\big\|\Big)\big\|A^{\frac{\kappa-\beta}{2}}\big[\overline{B}_N(t)-\widetilde{E}(t)\big]\big\|\,\mathrm{d}t \qquad (6.8)\\
&\le C(\tau+h^2)^{\frac{\beta}{\kappa}}\int_{\tau+h^2}^{T} t^{2(1-\frac{\kappa-\beta}{2})\alpha+2\gamma-2-\frac{\beta}{\kappa}}\,\mathrm{d}t\le C\ln\Big(\frac{T}{\tau+h^2}\Big)(\tau+h^2)^{\frac{\beta}{\kappa}}\\
&\le C\ln\Big(\frac{T}{\tau+h^2}\Big)(\tau^{\frac{\beta}{\kappa}}+h^{\frac{2\beta}{\kappa}}).
\end{aligned}
$$

Together (6.7), (6.8) with (6.5) and (6.3) show (1.19). The proof of Theorem 1.2 is now complete.

□

7. Numerical Simulations

Now we present some numerical results for the stochastic subdiffusion problem (1.1) on the unit interval $D=(0,1)$, in order to illustrate the error estimates derived in Theorem 1.2.

First, we describe the implementation of the noise term $W(t)$. We assume that $W(t)$ has the following Fourier series expansion [24, Definition 10.6],

$$
W(t)=\sum_{l=1}^{\infty}\gamma_l^{1/2}e_l\beta_l(t), \qquad (7.1)
$$

where $\{\beta_l(t)\}$ is a sequence of the real-valued independent and identically distributed Brownian motions and $\{(\gamma_l,e_l)\}$ are the eigenpairs of the covariance operator $Q\in\mathscr{L}(H)$ with

$Q \geq 0$ (selfadjoint and positive semidefinite). If $\mathrm{Tr}(Q) = \sum_{l=1}^{\infty} \gamma_l < \infty$, then Q is in trace class. We also assume that the covariance operator Q and A has a common basis of eigenvectors. Then the L_2 projection $P_h W(t) \in S_h$ of the noise $W(t)$ satisfies [31],

$$(P_h W(t), \chi) = \sum_{l=1}^{\infty} \gamma_l^{\frac{1}{2}} \beta_l(t) \, (e_l, \chi) \approx \sum_{l=1}^{M} \gamma_l^{\frac{1}{2}} \beta_l(t) \, (e_l, \chi) \,, \quad \forall \chi \in S_h,$$

where the eigenfunctions $e_l(x)$ are given by $\sqrt{2}\sin(l\pi x)$ and $\gamma_l = l^{-m}$, $m \geq 0$ with M truncation terms. In the fully discretization scheme (1.13), we have

$$W(t_k) - W(t_{k-1}) = \sum_{l=1}^{\infty} \gamma_l^{1/2} e_l(\beta_l(t_k) - \beta_l(t_{k-1})), k = 1, 2, \ldots,$$

where $\beta_l(t_k) - \beta_l(t_{k-1}) = \mathcal{N}(0, t_k - t_{k-1})$ denotes the normally distributed random variable with mean 0 and variance $t_k - t_{k-1}$.

To check the weak convergence order, we consider the $L_2(\Omega; H)$ norm and choose $\phi(u(t)) = \int_D u(t)^2 \, \mathrm{d}x$ for the weak convergence.

7.1. Temporal convergence

We illustrate the theoretical findings in Theorem 1.2 with one-dimensional examples on the unit interval $D = (0, 1)$. We fix the initial data $u_0 = 0$ to focus our discussions on the effect of the noise $W(t)$. In the following computations, we divide the unit interval $D = (0, 1)$ into M equally spaced subintervals, with a mesh size $h = 1/M$, and fix the time step size τ at $\tau = t/N$, where t is the time of interest. All the expected values are computed with 100 trajectories. We examine separately the spatial and temporal weak convergence orders.

In the numerical experiments, we fix the final time t at $t = 0.1$ and the number M of spatial subintervals at $M = 100$. The reference solution is computed with a much finer temporal mesh with $N = 6400$. The numerical results for various combinations of the fractional orders α and γ with $\alpha + \gamma \geq 1$ and trace class noise (with $m = 2$) are given in Tables 3. In the table, the numbers in the bracket in the last column denotes the theoretical order predicted by Theorem 1.2, which is of order (τ) in the trace class noise case when $\beta = \kappa$ and $\gamma + \alpha \geq 1$ (see Case 1 after Theorem 1.2). The empirical orders are in good agreement with the theoretical predictions, fully confirming the error analysis, despite the relatively small number of trajectories for computing expectation.

In Table 4, we give the numerical results for $Q = I$ in one dimensional case. In particular, when $\alpha = 1, \gamma = 0$, the expected theoretical weak convergence order in Theorem 1.2 is $O(\tau^{1/2})$ (see Cases 2, 4 after Theorem 1.2) which is indeed confirmed by the numerical results.

7.2. Spatial convergence

Next we examine the spatial convergence. Here, we fix the number M of space steps at $M = 200$ and the final time t at $t = 1$, and compute the reference solution at $N = 480$. The numerical results are given in Table 5 for trace class noise (with $m = 2$) with various

Table 3: The $L^2(\Omega; H)$-error for trace class noise ($m = 2$) at $t = 0.1$.

γ	$\alpha\backslash N$	40	80	160	320	640	order
0.5	0.5	8.19e-5	4.15e-5	2.10e-5	1.06e-5	5.40e-6	0.98 (1.00)
	0.7	6.25e-6	3.14e-6	1.55e-6	7.83e-7	3.92e-7	1.00 (1.00)
	0.9	8.46e-7	4.22e-7	2.12e-7	1.05e-7	5.29e-8	1.00 (1.00)
0.8	0.5	8.23e-8	4.11e-8	2.05e-8	1.04e-8	5.28e-9	0.99 (1.00)
	0.7	2.01e-8	1.02e-8	5.11e-9	2.51e-9	1.28e-9	1.00 (1.00)
	0.9	2.06e-8	1.18e-8	5.42e-9	2.87e-9	1.48e-9	0.96 (1.00)

Table 4: The $L^2(\Omega; H)$-error for $Q = I$ ($m = 0$) at $t = 0.1$.

γ	$\alpha\backslash N$	40	80	160	320	640	order
0.0	0.7	1.40e-1	1.26e-1	1.09e-1	8.92e-1	7.26e-2	0.10 (0.12)
	0.8	3.54e-2	2.84e-2	2.28e-2	1.72e-2	1.26e-2	0.31 (0.33)
	0.9	8.90e-3	6.54e-3	4.73e-3	3.43e-3	2.34e-3	0.42 (0.43)
	1.0	4.21e-3	2.86e-3	1.89e-3	1.18e-3	7.31e-4	0.49 (0.50)

fractional orders α and γ. An $O(h^2)$ convergence order is observed for all combinations, in excellent agreement with the theoretical predictions from Theorem 1.2 (see Case 1 after Theorem 1.2).

Table 5: The $L^2(\Omega; H)$-error with trace class noise ($m = 2$) at $t = 1$.

γ	$\alpha\backslash M$	10	20	40	80	160	order
0.5	0.5	3.17e-3	8.39e-4	2.15e-4	5.35e-5	1.23e-05	2.00 (2.00)
	0.7	2.87e-3	7.63e-4	1.96e-4	4.89e-5	1.13e-05	1.99 (2.00)
	0.9	2.74e-3	7.28e-4	1.87e-4	4.67e-5	1.08e-05	1.99 (2.00)
0.8	0.5	2.01e-3	5.24e-4	1.33e-4	3.29e-5	7.60e-06	2.01 (2.00)
	0.7	2.01e-3	5.25e-4	1.33e-4	3.30e-5	7.62e-06	2.01 (2.00)
	0.9	2.04e-3	5.32e-4	1.35e-4	3.34e-5	7.71e-06	2.01 (2.00)

In Table 6, we consider the spatial convergence orders for the white noise case, i.e., $m = 0$. It is observed that the spatial convergence orders deteriorate when the noise regularity is lowered to the white noise case. The empirical rates are slightly higher than the theoretical ones when $m = 0$ (see Case 4 after Theorem 1.2).

Table 6: The $L^2(\Omega; H)$-error for $Q = I$ ($m = 0$) at $t = 1$.

γ	$\alpha\backslash N$	40	80	160	320	640	order
0.4	0.3	9.58e-3	3.48e-3	1.25e-3	4.36e-3	1.38e-4	1.52 (1.25)
	0.5	9.20e-3	3.40e-3	1.23e-3	4.33e-4	1.37e-4	1.51 (1.37)
	0.7	8.75e-3	3.30e-3	1.21e-3	4.29e-4	1.36e-4	1.49 (1.41)
	0.9	8.27e-3	3.17e-3	1.19e-3	4.24e-4	1.36e-4	1.48 (1.43)

8. Conclusions

In this work, we study the weak convergence of the numerical method for solving stochastic subdiffusion problem. To our knowledge, this is the first work to consider the weak approximation of stochastic subdiffusion problems by using the Kolmogorov equation approach. The error estimates are proved based on the nonsmooth data error estimates of the corresponding deterministic problems. The numerical experiments are given to show that the numerical results are consistent with the theoretical results.

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