

Cohomology of profinite groups of bounded rank

Peter Symonds

ABSTRACT

We generalise to profinite groups some of our previous results on the cohomology of pro- p groups of bounded sectional p -rank.

1. Introduction

The purpose of this note is to generalise to profinite groups our results of [6] for pro- p groups of bounded rank. The main one of these states that amongst the pro- p groups of rank bounded by a number r there are only finitely many mod- p cohomology rings up to isomorphism. Recall that rank here means the p -sectional rank, which is the maximum of the ranks of H/K where $H \leq G$ and $K \trianglelefteq H$ and H/K is an elementary abelian p -group. The generalisation is as follows.

THEOREM 1.1. *For given p and r , the profinite groups of p -sectional rank at most r have only finitely many graded isomorphism classes of mod- p cohomology rings between them and these are all Noetherian.*

Another result bounds the dimensions of the cohomology groups.

THEOREM 1.2. *If G is a profinite group of p -sectional rank at most r , then:*

- (1) $\dim_{\mathbb{F}_p} H^i(G; \mathbb{F}_p) \leq \binom{r(\lceil \log_2 r \rceil + 3 + e) + i - 1}{i}$, where $e = 0$ for p odd and $e = 1$ for $p = 2$;
- (2) there is a function $X(p, r)$ such that $\dim_{\mathbb{F}_p} H^i(G; \mathbb{F}_p) \leq X(p, r) \cdot i^{a-1}$, where a is the maximum rank of an elementary abelian p -subgroup.

In Section 3, we will generalise these results to pro-fusion systems.

The generalisation from pro- p to profinite is much harder than in the finite case, because a Sylow pro- p subgroup might have infinite index so we cannot use the transfer.

2. Proofs

We fix a prime p ; all cohomology groups will have coefficients in \mathbb{F}_p and rank will mean p -sectional rank. A homomorphism of cohomology rings means a homomorphism of graded rings. A subgroup will always mean a subgroup in the category of profinite groups.

First we prove part (1) of Theorem 1.2. This is proved for finite p -groups in [6, 1.2]. If G is a finite group with Sylow p -subgroup S , then a standard transfer argument shows that the restriction map embeds $H^*(G)$ in $H^*(S)$, so the bound hold for G .

Received 26 February 2021; published online 11 October 2021.

2020 *Mathematics Subject Classification* 20J06 (primary), 20E18 (secondary).

© 2021 The Authors. *Transactions of the London Mathematical Society* is copyright © London Mathematical Society. This is an open access article under the terms of the [Creative Commons Attribution](https://creativecommons.org/licenses/by/4.0/) License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

If G is profinite, say $G = \varprojlim G/N_i$ with the G/N_i finite, then $H^*(G) = \varinjlim H^*(G/N_i)$. The ranks of the G/N_i are also bounded by r , so the bound applies to each $H^*(G/N_i)$ and hence to their direct limit.

Now we prove Theorem 1.1. This is proved for finite p -groups in [6, 1.1] and for pro- p groups in [6, 1.4]. Let G be a finite group with Sylow p -subgroup S . Restriction embeds $H^*(G)$ in $H^*(S)$ and the image can be characterised as the subring of stable elements, see, for example, [2, XII 10.1].

The stable elements of $H^*(S)$ are the $x \in H^*(S)$ such that, for any inclusion of a subgroup $i_P: P \rightarrow S$ and any homomorphism $\varphi: P \rightarrow S$ induced by inclusion and then conjugation by an element of G , we have $(i_P^* - \varphi^*)(x) = 0$. The usual version only considers certain subgroups P , but all the conditions are certainly necessary on the image of $H^*(G)$, so this formulation is also valid.

By the validity of the result for finite p -groups, there are only finitely many possible isomorphism classes of rings $H^*(S)$ and $H^*(P)$. Since $H^*(S)$ is Noetherian, there are only finitely many graded ring homomorphisms $H^*(S) \rightarrow H^*(P)$. Thus there are only finitely many different conditions of the form $(i_P^* - \varphi^*)(x) = 0$ and hence only a finite number of possible subrings of stable elements. The cohomology of any finite group is Noetherian. This completes the case of a finite group.

Since there are only finitely many possible cohomology rings for a finite group of rank at most r and they are all finitely generated, there is a number N , depending only on p and r , such that they are all generated in degrees at most N . The dimension of the sum of the cohomology groups in degrees 0 through N is bounded in terms of r and N , by Theorem 1.2(1), and this bounds the number of generators needed. The proof of [6, 1.3] now applies verbatim to prove part (2) of 1.2.

Let G be a profinite group of rank at most r , say $G = \varprojlim G/N_i$, so $H^*(G) = \varinjlim H^*(G/N_i)$. Thus $H^*(G)$ will also be generated in degrees at most N . Let S be the Sylow pro- p subgroup of G ; then the restriction map identifies $H^*(G)$ with its image under restriction to $H^*(S)$ (because it does so on each finite quotient). By the pro- p case of the theorem, there are only finitely many possible rings $H^*(S)$ and the part of $H^*(S)$ in degrees at most N is finite. It follows that there are only finitely many subrings generated in degrees at most N and these subrings are all finitely generated and so Noetherian. This completes the proofs of Theorems 1.1 and 1.2.

3. Inflation functors and pro-fusion systems

An inflation functor is defined in [7] (it is called a functor with Mackey structure in [5]). An inflation functor M is a contravariant functor M^* from the category of finite groups to the category of R -modules for some ring R that is also a covariant functor M_* on the subcategory of finite groups and injective group homomorphisms. The two structures are related, in particular the Mackey double coset formula holds. Examples are cohomology $H^*(G; R)$, where the covariant part is given by transfer and the representation or Green rings, where it is given by induction.

Such a functor M can be extended to profinite groups by setting $M(G) = \varinjlim M^*(G/N_i)$, when $G = \varprojlim G/N_i$. This is well defined as a contravariant functor on the category of profinite groups (and continuous homomorphisms) and the covariant structure is defined on injective homomorphisms with open image. The Mackey formula still holds. All of this is familiar in the case of cohomology, where the covariant structure is given by the transfer. We can also use the same construction on just a contravariant functor on finite groups or finite p -groups to obtain a functor on profinite groups or pro- p groups, respectively.

An inflation functor is said to be cohomological if $M_*(i_H) \circ M^*(i_H) = |G : H| \text{id}_{M(G)}$ for i_H the inclusion of a subgroup H in G . If this holds for all finite groups, then it holds whenever

G is profinite and H is an open subgroup. Cohomology $H^*(-; \mathbb{F}_p)$ is a cohomological inflation functor; the usual definition on a profinite group also satisfies the colimit property above.

Pro-fusion systems were introduced in [4] as a generalisation of fusion systems to pro- p groups. We will freely refer to that paper and use its notation without comment. Roughly speaking, a pro-fusion system \mathcal{F} on a pro- p group S is an inverse limit of fusion systems \mathcal{F}_i on certain finite quotients S_i of S .

For any pro-fusion system \mathcal{F} on a pro- p group S and any inflation functor M , we define the stable elements $M(S)^\mathcal{F}$ to be the submodule of elements $x \in M(S)$ such that, for any inclusion of a subgroup $i_P: P \rightarrow S$ and any homomorphism $\varphi \in \text{Hom}_\mathcal{F}(P, S)$, we have $(i_P^* - \varphi^*)(x) = 0$ (where we write f^* for $M^*(f)$). The method of stable elements, as mentioned above, is usually stated for cohomology, but the proof applies whenever G is a finite group with Sylow p -subgroup S , M is a cohomological inflation functor (or just a cohomological Mackey functor on G) and every rational prime except p is invertible in R , to show that restriction $M(G) \rightarrow M(S)$ is injective with image $M(S)^{\mathcal{F}_S(G)}$, where $\mathcal{F}_S(G)$ is the fusion system on S induced by G . Our aim is to extend this to profinite groups.

If G is a profinite group with Sylow p -subgroup S , say $G = \varprojlim G/N_i$, then $M(G) = \varinjlim M^*(G/N_i) \cong \varinjlim M(S_i)^{\mathcal{F}_{S_i}(G/N_i)}$, where $S_i = SN_i/N_i$. There is also a pro-fusion system $\mathcal{F}_S(G) = \varprojlim \mathcal{F}_{S_i}(G/N_i)$ on S , which is pro-saturated. Thus what we need is the next result, which has also been considered in [3]. I am grateful to the authors of [3] for pointing out an error in my original proof.

THEOREM 3.1. *Let M be a contravariant functor from p -groups to R -modules, extended to pro- p groups as above. Let S be a pro- p group and let $\mathcal{F} = \varprojlim \mathcal{F}_i$ be a pro-saturated pro-fusion system on S . Then inflation induces an isomorphism*

$$M(S)^\mathcal{F} \cong \varinjlim M(S_i)^{\mathcal{F}_i}.$$

Proof. It is straightforward to show that the inflation maps induce an embedding of $\varinjlim M(S_i)^{\mathcal{F}_i}$ into $M(S)^\mathcal{F}$, so we concentrate on proving that this map is surjective. We use notation from [4].

Because \mathcal{F} is pro-saturated, we know from [4, 4.5] that we can take the \mathcal{F}_i to be the saturated fusion systems \mathcal{F}/N as N runs through the open strongly \mathcal{F} -closed subgroups of S . It follows that for any $P \leq S_i$, $\varphi \in \text{Hom}_{\mathcal{F}_i}(P, S_i)$ and $j \geq i$, there is a $\tilde{\varphi}_j \in \text{Hom}_{\mathcal{F}_j}(\tilde{P}_j, S_j)$ such that $f_{i,j}\tilde{\varphi}_j = \varphi f_{i,j}|_{\tilde{P}_j}$, where $f_{j,k}: S_k \rightarrow S_j$ is the quotient map and $\tilde{P}_j = f_{i,j}^{-1}(P)$. Similarly, φ can also be lifted to $\tilde{\varphi} \in \text{Hom}_\mathcal{F}(\tilde{P}, S)$.

Let $x \in M(S)^\mathcal{F}$; then $x = f_j^*(x_j)$ for some j and some $x_j \in M(S_j)$, where $f_j: S \rightarrow S_j$ is the quotient map. The element x satisfies the conditions $(i_P - \varphi)^*(x) = 0$ for all $P \leq S$ and $\varphi \in \text{Hom}_\mathcal{F}(P, S)$.

Let $Q \leq S_j$ and $\theta \in \text{Hom}_{\mathcal{F}_j}(Q, S_j)$. Then

$$\begin{aligned} f_j|_Q^* (i_Q^* - \theta^*)(x_j) &= (i_Q f_j|_Q - \theta f_j|_Q)^*(x_j) = (f_j i_Q - f_j \tilde{\theta})^*(x_j) \\ &= (i_Q - \tilde{\theta})^* f_j^*(x_j) = (i_Q - \tilde{\theta})^*(x) = 0. \end{aligned}$$

Thus there is an $\ell = \ell(Q, \theta) \geq j$ such that $f_{j,\ell}|_{Q_\ell}^* (i_Q^* - \theta^*)(x_j) = 0$. There are only finitely many possible different Q and θ so there is a k such that $k \geq \ell(Q, \theta)$ for all of them and hence $(i_{Q_k} - \tilde{\theta}_k^*) f_{j,k}^*(x_j) = f_{j,k}|_{Q_k}^* (i_Q^* - \theta^*)(x_j) = 0$ for all Q and θ .

Set $x_k = f_{j,k}^*(x_j) \in M(S_k)$; we need to show that $x_k \in M(S_k)^{\mathcal{F}_k}$. Suppose that $P \leq S_k$ and $\varphi \in \text{Hom}_{\mathcal{F}_k}(P, S_k)$. Let $Q = f_{j,k}(P)$, then φ induces a $\theta \in \text{Hom}_{\mathcal{F}_j}(Q, S_j)$ such that $f_{j,k}\varphi = \theta f_{j,k}|_P$. As mentioned at the beginning of the proof, there is a $\hat{\theta} \in \text{Hom}_{\mathcal{F}_k}(\hat{Q}_k, S_k)$, where

$\hat{Q}_k = f_{j,k}^{-1}(Q)$, such that $f_{j,k}\hat{\theta}_k = \theta f_{j,k}|_{\hat{Q}_k}$. Thus $f_{j,k}\hat{\theta}_k i_P^{\hat{Q}_k} = f_{j,k}\varphi$, where $i_P^{\hat{Q}_k}$ is the inclusion of P in \hat{Q}_k . Now

$$\begin{aligned} (i_P - \varphi)^*(x_k) &= (i_P - \varphi)^* f_{j,k}^*(x_j) = (f_{j,k}i_P - f_{j,k}\varphi)^*(x_j) \\ &= (f_{j,k}i_{\hat{Q}_k} i_P^{\hat{Q}_k} - f_{j,k}\hat{\theta}_k i_P^{\hat{Q}_k})^*(x_j) = i_P^{\hat{Q}_k} (i_{\hat{Q}_k} - \hat{\theta}_k)^* f_{j,k}^*(x_j) = 0, \end{aligned}$$

as required. □

As was pointed out above, we can now deduce the next result.

COROLLARY 3.2. *Let M be a cohomological inflation functor over a ring R in which every rational prime except p is invertible, extended to profinite groups as above. Let G be a profinite group; then $M(G) \cong M(S)^{\mathcal{F}_S(G)}$.*

This motivates the study of $M(S)^{\mathcal{F}}$ for an arbitrary pro-fusion system. There is also a dual version of this theory that applies to homology.

THEOREM 3.3. *For given p and r , consider the rings $H^*(S)^{\mathcal{F}}$, where S is a pro- p group with p -sectional rank at most r and \mathcal{F} is a pro-saturated pro-fusion system on S . Then there are only finitely many such rings up to isomorphism and they are all Noetherian.*

Proof. By hypothesis, \mathcal{F} can be expressed as an inverse limit of fusion systems \mathcal{F}_i on finite p -groups S_i . By Theorem 3.1, $H^*(S)^{\mathcal{F}}$ is the direct limit of the $H^*(S_i)^{\mathcal{F}_i}$. The finiteness of the number of rings is now proved in the same way as in the case of finite groups in Theorem 1.1.

For the Noetherian property, we use the fact from [1, 5.2] that each $H^*(S_i)^{\mathcal{F}_i}$ is Noetherian. We have just seen that there are only finitely many isomorphism classes of such rings, thus there is a number N such that they are all generated in degrees at most N . This property passes to the direct limit, which is $H^*(S)^{\mathcal{F}}$.

We are therefore considering subrings of $H^*(S)$ that are generated in degrees at most N . But $H^*(S)$ is finite in this range of degrees, so the subrings must be Noetherian. □

Of course, Theorem 1.1 can be seen as a corollary of these last two results.

4. The Steenrod algebra

In any situation where we have only finitely many non-isomorphic cohomology rings and they are all Noetherian, we also have only finitely many non-isomorphic algebras over the Steenrod algebra. This is because, on any particular ring, the action of a Steenrod power is determined by the images of the generators of the ring, by the Cartan formula, and there are only finitely many possibilities. Also, sufficiently high powers act trivially, because the Noetherian assumption implies that there is a bound on the degrees of the generators, so these are sent to 0, by the condition for being unstable.

If we take cohomology with coefficients in the p -adic integers instead of \mathbb{F}_p , we get infinitely many different cohomology rings. This happens even in the case of sectional p -rank 1, which includes the cyclic groups of p -power order.

References

1. C. BROTO, R. LEVI and B. OLIVER, ‘The homotopy theory of fusion systems’, *J. Amer. Math. Soc.* 16 (2003) 779–856.
2. H. CARTAN and H. EILENBERG, *Homological algebra* (Princeton University Press, Princeton, NJ, 1956).

3. A. DÍAZ, O. GARAIALDE, N. MAZZA and S. PARK, ‘Cohomology of pro-fusion systems’, Preprint, 2020.
4. R. STANCU and P. SYMONDS, ‘Fusion Systems for profinite groups’, *J. Lond. Math. Soc.* 89 (2014) 461–481.
5. P. SYMONDS, ‘A splitting principle for group representations’, *Comment. Math. Helv.* 66 (1991) 169–184.
6. P. SYMONDS, ‘Rank, coclass and cohomology’, *Int. Math. Res. Notices* (2019). <https://doi.org/10.1093/imrn/rnz287>
7. P. J. WEBB, ‘Two classifications of simple Mackey functors with applications to group cohomology and the decomposition of classifying spaces’, *J. Pure App. Algebra* 88 (1993) 265–304.

Peter Symonds

Department of Mathematics

University of Manchester

Manchester M13 9PL

United Kingdom

Peter.Symonds@manchester.ac.uk