OSCILLATORY AND STABILITY OF A MIXED TYPE DIFFERENCE EQUATION WITH VARIABLE COEFFICIENTS

SANDRA PINELAS[1], NEDJEM EDDINE RAMDANI[2], ALI FUAT YENIï¿ŒERIOGLU[3], YUBIN YAN[4]

[1]Academia Militar
Departamento de Ciencias Exatas e Naturais
2720-113 Amadora, Portugal
Email: sandra.pinelas@gmail.com
[2]Department of Civil Engineering
University of Saad Dahleb Blida, 1
Blida, Algeria
Email: nedjemeddine.ramdani@yahoo.com
[3]Kocaeli University
Faculty of Education
Kocaeli, 41380, Turkey
Email: fuatyenicerioglu@kocaeli.edu.tr
[4]Department of Mathematics
University of Chester, CH1 4BJ
Email: y.yan@chester.ac.uk

Abstract. The goal of this paper is to study the oscillatory and stability of the mixed type difference equation with variable coefficients

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n)x(\tau_i(n)) + \sum_{j=1}^{m} q_j(n)x(\sigma_i(n)), \quad n \geq n_0,$$

where $\tau_i(n)$ is the delay term and $\sigma_j(n)$ is the advance term and they are positive real sequences for $i = 1, \cdots, l$ and $j = 1, \cdots, m$, respectively, and $p_i(n)$ and $q_j(n)$ are real functions. This paper generalise some known results and the examples illustrate the results.

Keywords: Mixed type difference equation; Asymptotic behavior; Stability; Characteristic equation; Solution.

MSC: 39A10; 39A30

1. Introduction

This paper deals with the mixed type difference equation with variable coefficients

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n)x(\tau_i(n)) + \sum_{j=1}^{m} q_j(n)x(\sigma_i(n)), \quad n \geq n_0, \tag{1.1}$$

where $\tau_i(n)$ is the delay term and $\sigma_j(n)$ is the advance term and they are positive real sequences for $i = 1, \cdots, l$ and $j = 1, \cdots, m$, respectively, and $p_i(n)$ and $q_j(n)$ are real functions. We define:

$$P(n) = \sum_{i=1}^{\ell} p_i(n), \quad Q(n) = \sum_{j=1}^{m} q_j(n).$$

Assume that the sequences $(\tau_i(n))_{1\le i\le l}$ and $(\sigma_j(n))_{1\le j\le m}$ are increasing and the following conditions are hold:

$$\tau_i(n) \le n-1, \ \forall n \ge n_0, \text{ and } \lim_{n\tau\infty} \tau_i(n) = \infty, \quad 1 \le i \le \ell, \tag{1.2}$$

$$\sigma_j(n) \ge n+1, \ \forall n \ge n_0, \quad 1 \le j \le m, \tag{1.3}$$

$$\tau(n) = \min_{1\le i\le l} \tau_i(n) \text{ and } \sigma(n) = \min_{1\le j\le m} \sigma_i(n), \ \forall n \ge n_0.$$

The Section 2 is devoted to the study of oscillatory behavior of the solutions of the linear difference equation of mixed type. By a solution of equation (1.1), we mean a nontrivial real sequence $x(n)$ that is defined for all $n > n_0 - \tau(n_0)$ and satisfies equation (1.1) for all $n > n_0$. A solution $x(n)$ of equation (1.1) is said to be nonoscillatory if it is either eventually positive or eventually negative, and oscillatory otherwise. An equation is called oscillatory if all its solutions are oscillatory.

A particular case of the equation (1.1) is obtained when we consider $\tau_i(n) = n-\tau_i$ and $\sigma_j(n) = n+\sigma_j$ where $\tau_i, \sigma_j > 0$, $1 \le i \le \ell$ and $1 \le j \le m$. Then equation (1.1) becomes

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n)x(n-\tau_i) + \sum_{j=1}^{m} q_j(n)x(n+\sigma_j). \tag{1.4}$$

The Section 3 is devoted to the study of the behavior of the solutions of the linear difference equation of mixed type

$$\triangle x(n) = \sum_{i=1}^{\ell} p_i(n)x(n-\tau_i) + \sum_{j=1}^{m} q_j(n)x(n+\sigma_j), \quad n \ge 0. \tag{1.5}$$

where $p_i(n)$ for $i = 1, 2, \cdots, \ell$ and $q_j(n)$ for $j = 1, 2, \cdots, m$ are real functions, and ℓ, m are positive integers. Equation (1.5) can be obtained (as a special case) from (1.1) by taking $\tau_i(n) = n - \tau_i$ and $\sigma_(n) = n + \sigma_j$ where τ_i, σ_j are positive integers, $1 \le i \le \ell$ and $1 \le j \le m$.

Qualitative theory of difference equations has drawn considerable attention in the past two decades. For the general background of difference equations, one can refer to the books by Agarwal and Wong [1], Agarwal, Grace, and O'Regan [3], Agarwal [4], Elaydi [9], Gyï¿œri and Ladas [14], Kelley and Peterson [15], and Lakshmikantham and Trigiante [16].

The equation (1.5) has been adequately introduced in [5] and [21]. In [5], Berezansky and Pinelas have established the oscillatory criteria for the oscillatory linear mixed type difference equations of form (1.5). Later in [21], Pinelas obtained an asymptotic behavior of the linear difference equation of mixed equations of form (1.5). In this article, we have applied a different method for asymptotic behavior of (1.5).

When $q_j(n)$ in the equation (1.5) is the null function, one obtains a linear non-autonomous delay difference equation whose asymptotic and stability behavior are studied in [8, 17, 18, 19] and references therein.

A special case of the mixed type difference equation (1.5) is that of the linear autonomous mixed type difference equation

$$\triangle x(n) = \sum_{i=1}^{\ell} p_i x(n-i) + \sum_{j=1}^{m} q_j x(n+j),$$

where p_i for $i = 1, 2, \cdots, \ell$ and q_j for $j = 1, 2, \cdots, m$ are real numbers, and ℓ, m are positive integers (see [11] and [22]). In [11], Ferreira and Pinelas have established the oscillatory criteria for the oscillatory mixed difference systems. Also, the authors in [22] obtained asymptotic behavior and stability results of the linear autonomous mixed type difference equations. The similar equation is studied in [2] (see also, [3, Section 1.16]). Note that, the difference equations considered are the discrete versions of first order linear autonomous mixed differential equations with delays and advances (see, the book [10, pp. 355-364] and [12, 13, 20] and references therein).

We define

$$\rho = \min_{1 \le i \le \ell,\ 1 \le j \le m} \{\tau_i, \sigma_j\}.$$

2. Oscillation criteria

Theorem 1. *Assume that* (1.2), (1.3) *are hold and*

$$\limsup_{n\to\infty} \sum_{k=\tau(n)}^{\sigma(n)-1} Q(k) > 1, \tag{2.1}$$

with $(p_i(n))_{1\le i\le l}$ *and* $(q_j(n))_{1\le j\le m}$ *non negative sequences. Then the equation* (1.1) *is oscillatory.*

Proof. Assume that $(x(n))_{n\ge n_0}$ is a non oscillatory solution of Eq(1.1), then it is either eventually positive or eventually negative. As $(-x(n))_{n\ge n_0}$ is also a solution for Eq (1.1). Thus, we may consider only the case where $x(n) > 0$ for all large n. Let n_0 be an integer such that: $x(\tau(n)) > 0$, for all $n \ge n_0$. Then,

$$x(n) > 0,\ x(\tau_i(n)) > 0,\ x(\sigma_i(n)) > 0,\ \forall n \ge n_0,\ 1 \le i \le \ell,\ 1 \le j \le m.$$

Thus, from Eq(1.1), we have:

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n) x(\tau_i(n)) + \sum_{j=1}^{m} q_j(n) x(\sigma_j(n)) \ge 0, \quad \forall n \ge n_0,$$

which means the $(x(n))$ is eventually increasing.

From the definition of $(\tau(n))$, $(\sigma(n))$ and the fact that $(x(n))$ is increasing, summing up eq(1.1) from $\tau(n)$ to $\sigma(n)-1,\ n \geq n_0$, we get:

$$\sum_{k=\tau(n)}^{\sigma(n)-1}(x(n+1)-x(n)) = \sum_{k=\tau(n)}^{\sigma(n)-1}\sum_{i=1}^{\ell} p_i(k)x(\tau_i(k)) + \sum_{k=\tau(n)}^{\sigma(n)-1}\sum_{j=1}^{m} q_j(k)x(\sigma_j(k))$$

$$x(\sigma(n)) - x(\tau(n)) \geq \sum_{k=\tau(n)}^{\sigma(n)-1}\sum_{i=1}^{\ell} p_i(k)x(\tau(n)) + \sum_{k=\tau(n)}^{\sigma(n)-1}\sum_{j=1}^{m} q_j(k)x(\sigma(n))$$

$$0 \geq -x(\sigma(n)) + x(\tau(n)) + \sum_{k=\tau(n)}^{\sigma(n)-1}\sum_{i=1}^{\ell} p_i(k)x(\tau(n)) + \sum_{k=\tau(n)}^{\sigma(n)-1}\sum_{j=1}^{m} q_j(k)x(\sigma(n)),$$

or

$$\left(\sum_{k=\tau(n)}^{\sigma(n)-1} P(k) + 1\right) x(\tau(n)) + \left(\sum_{k=\tau(n)}^{\sigma(n)-1} Q(k) - 1\right) x(\sigma(n)) \leq 0.$$

which is contradiction with (2.1). That completes the proof. □

Corollary 2. *If*

$$\limsup_{n\to\infty} \sum_{j=1}^{m}\sum_{k=n-\rho}^{n+\rho-1} q_j(k) > 1,$$

where $(p_i(n))_{1\leq i\leq l}$ *and* $(q_j(n))_{1\leq j\leq m}$ *are non negative sequences, then all solution of Eq*(1.4) *oscillate.*

Theorem 3. *Assume that* (1.2), (1.3) *hold. If*

$$\limsup_{n\to\infty} \sum_{k=\tau(n)}^{\sigma(n)-1} P(k) < -1, \tag{2.2}$$

with $(p_i(n))_{1\leq i\leq \ell}$ *and* $\tau(q_j(n))_{1\leq j\leq m}$ *non positive sequences, then the equation* (1.1) *is oscillatory.*

Proof. Proceed similarly as Theorem 1, we assume the contrary, that means $(x(n))_{n\geq n_0}$ is a non oscillatory solution of Eq(1.1), then it is either eventually positive or eventually negative. As $(-x(n))_{n\geq n_0}$ is also a solution for Eq (1.1) we will restrict our study to the case where $x(n) > 0$ for all large n. Let n_0 be an integer such that: $x(\tau(n)) > 0$, for all $n \geq n_0$. Then,

$$x(n) > 0,\ x(\tau_i(n)) > 0,\ x(\sigma_i(n)) > 0,\ \forall n \geq n_0,\ 1 \leq i \leq \ell,\ 1 \leq j \leq m.$$

Thus, from Eq(1.1), we have:

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n)x(\tau_i(n)) + \sum_{j=1}^{m} q_j(n)x(\sigma_j(n)) \leq 0, \quad \forall n \geq n_0,$$

which means the $(x(n))$ is eventually decreasing.

From the definition of $(\tau(n))$, $(\sigma(n))$ and the fact that $(x(n))$ is decreasing, summing up eq(1.1) from $\tau(n)$ to $\sigma(n)-1$, $n \geq n_0$, we get:

$$\sum_{k=\tau(n)}^{\sigma(n)-1}(x(n+1)-x(n)) = \sum_{k=\tau(n)}^{\sigma(n)-1}\sum_{i=1}^{\ell} p_i(k)x(\tau_i(k)) + \sum_{k=\tau(n)}^{\sigma(n)-1}\sum_{j=1}^{m} q_j(k)x(\sigma_j(k))$$

$$x(\sigma(n)) - x(\tau(n)) \leq \sum_{k=\tau(n)}^{\sigma(n)-1}\sum_{i=1}^{\ell} p_i(k)x(\tau(n)) + \sum_{k=\tau(n)}^{\sigma(n)-1}\sum_{j=1}^{m} q_j(k)x(\sigma(n))$$

$$0 \leq -x(\sigma(n)) + x(\tau(n)) + \sum_{k=\tau(n)}^{\sigma(n)-1}\sum_{i=1}^{\ell} p_i(k)x(\tau(n)) + \sum_{k=\tau(n)}^{\sigma(n)-1}\sum_{j=1}^{m} q_j(k)x(\sigma(n)),$$

or

$$\left(\sum_{k=\tau(n)}^{\sigma(n)-1} P(k) + 1\right) x(\tau(n)) + \left(\sum_{k=\tau(n)}^{\sigma(n)-1} Q(k) - 1\right) x(\sigma(n)) \geq 0.$$

Which is contradiction with (2.2). That completes the proof. □

Corollary 4. *If*

$$\limsup_{n\to\infty} \sum_{i=1}^{\ell} \sum_{k=n-\rho}^{n+\rho-1} p_i(k) < -1,$$

with $(p_i(n))_{1\leq i\leq \ell}$ and $\tau(q_j(n))_{1\leq j\leq m}$ non positive sequences, then the equation(1.4) *is oscillatory.*

Theorem 5. *Assume that the sequences $(\tau_i(n))_{1\leq i\leq l}$, $(\sigma_j(n))_{1\leq j\leq m}$ are such that* (1.2), (1.3), $-1 < P(k) < 0$ *and*

$$\liminf_{n\to\infty} \sum_{k=\tau_i(n)}^{n-1} P(k) < -\frac{1}{e}, \tag{2.3}$$

with $(p_i(n))_{1\leq i\leq l}$ and $(q_j(n))_{1\leq j\leq m}$ non-positive sequences. Then, all solutions of equation (1.1) *oscillate.*

Proof. Proceed the same way as Theorem 3, i.e. $(x(n)) > 0$, $\forall n \geq n_0$ and is decreasing. Set

$$b_i(n) = \left(\frac{n-\tau_i(n)}{n-\tau_i(n)+1}\right)^{n-\tau_i(n)+1}, \tag{2.4}$$

with $n \geq n_0$, $1 \leq i \leq l$, $1 \leq j \leq m$. Clearly, by the exponential inequality

$$\frac{1}{4} \leq b_i(n) \leq \frac{1}{e}. \tag{2.5}$$

By (2.3) there exist an integer $n > n_0 \geq n_2$ and a small positive number ϵ_0 such that

$$\sum_{k_0=\tau_i(n)}^{n-1} P(k_0) < -\frac{1}{e} - \epsilon_0, \tag{2.6}$$

Let

$$d = e(\frac{1}{e} + \epsilon_0). \tag{2.7}$$

Combining (2.5), (2.6) and (2.7), we get

$$-\sum_{k_0=\tau_i(n)}^{n-1}\frac{P(k_0)}{b_i(n)} > -\sum_{k_0=\tau_i(n)}^{n-1}\frac{P(k_0)}{1/e}$$
$$= -e\sum_{k_0=\tau_i(n)}^{n-1}P(k_0) > -e(-\frac{1}{e}-\epsilon_0) = e(\frac{1}{e}+\epsilon_0) = d > 1,$$

or

$$-\sum_{k_0=\tau_i(n)}^{n-1}\frac{P(k_0)}{b_i(n)} > d > 1. \tag{2.8}$$

Since $(x(n))$ is decreasing, clearly

$$\frac{x(\tau_i(n))}{x(n)} \geq 1, \quad \frac{x(\sigma_j(n))}{x(n)} \leq 1, \text{ for all } 1 \leq i \leq \ell,\ 1 \leq j \leq m. \tag{2.9}$$

By Eq(1.1), we have

$$\frac{x(n+1)}{x(n)} = 1 + \sum_{i=1}^{\ell} p_i(n)\frac{x(\tau_i(n))}{x(n)} + \sum_{j=1}^{m} q_j(n)\frac{x(\sigma_j(n))}{x(n)}, \forall n \geq n_0. \tag{2.10}$$

On the other hand

$$\frac{x(n)}{x(\tau_i(n))} = \prod_{k=\tau_i(n)}^{n-1}\frac{x(k+1)}{x(k)}$$
$$= \prod_{k=\tau_i(n)}^{n-1}\left[1+\sum_{i=1}^{\ell}p_i(k)\frac{x(\tau_i(k))}{x(k)} + \sum_{j=1}^{m}q_j(k)\frac{x(\sigma_j(k))}{x(k)}\right]$$
$$\leq \prod_{k=\tau_i(n)}^{n-1}\left(1+\sum_{i=1}^{\ell}p_i(k)\right),$$

because $\sum_{j=1}^{m} q_j(n) < 0$, Thus

$$\frac{x(n)}{x(\tau_i(n))} \leq \prod_{k=\tau_i(n)}^{n-1}\left(1+\sum_{i=1}^{\ell}p_i(k)\right). \tag{2.11}$$

By using (2.11) and the well-known inequality between the arithmetic and geometric means, we find that

$$\frac{x(n)}{x(\tau_i(n))} \leq \prod_{k=\tau_i(n)}^{n-1}\left(1+\sum_{i=1}^{\ell}p_i(k)\right)$$
$$\leq \left[1+\frac{1}{n-\tau_i(n)}\sum_{i=1}^{\ell}\sum_{k=\tau_i(n)}^{n-1}p_i(k)\right]^{n-\tau_i(n)},$$

or

$$\frac{x(\tau_i(n))}{x(n)} \geq \left[1+\frac{1}{n-\tau_i(n)}\sum_{i=1}^{\ell}\sum_{k=\tau_i(n)}^{n-1}p_i(k)\right]^{-(n-\tau_i(n))}, \tag{2.12}$$

In view of

$$y(1+y)^{\rho} \geq -\frac{\rho^{\rho}}{(1+\rho)^{1+\rho}}, \ \forall y \in (-1,0), \ \rho \in \mathbb{N}$$

Inequality (2.12) becomes

$$\frac{x(\tau_i(n))}{x(n)} \geq (1+y)^{-\rho} \geq -\sum_{k=\tau_i(n)}^{n-1} P(k) \left(\frac{n-\tau_i(n)+1}{n-\tau_i(n)}\right)^{n-\tau_i(n)+1} \tag{2.13}$$

Combining (2.13), (2.4) and (2.8), we get

$$\frac{x(\tau_i(n))}{x(n)} \geq -\sum_{k=\tau_i(n)}^{n-1} \frac{P(k)}{b_i(n)} > d, \ \text{for all}\, n \geq n_0. \tag{2.14}$$

Similarly,

$$\begin{aligned}
\frac{x(n)}{x(\tau_i(n))} &= \prod_{k=\tau_i(n)}^{n-1} \frac{x(k+1)}{x(k)} \\
&= \prod_{k=\tau_i(n)}^{n-1} \left[1+\sum_{i=1}^{\ell} p_i(k)\frac{x(\tau_i(k))}{x(k)} + \sum_{j=1}^{m} q_j(k)\frac{x(\sigma_j(k))}{x(k)}\right] \\
&\leq \prod_{k=\tau_i(n)}^{n-1} \left(1+\sum_{i=1}^{\ell} p_i(k)\right) \\
&\leq \left[1+\frac{d}{n-\tau_i(n)}\sum_{i=1}^{l}\sum_{k=\tau_i(n)}^{n-1} p_i(k)\right]^{n-\tau_i(n)}.
\end{aligned}$$

Therefore

$$\frac{x(\tau_i(n))}{x(n)} \geq d\sum_{i=1}^{\ell}\sum_{k=\tau_i(n)}^{n-1} \frac{p_i(k)}{b_i(n)} > d^2, \ \text{for all}\, n \geq n_3 \geq n_0.$$

Applying this procedure t-times, we obtain

$$\frac{x(\tau_i(n))}{x(n)} > d^t, \ \text{for all}\, n. \tag{2.15}$$

On the other hand, since $(p_i(n))_{1\leq i \leq l}$ is non-positive. Then, $\limsup_{n\to\infty} P(n) < 0$, which implies that there exists a subsequence of integers $\tau(n)$ such that

$$\limsup_{n\to\infty} P(\tau(n)) \leq D < 0.$$

By (2.10), we have

$$0 < \frac{x(n+1)}{x(n)} = 1+\sum_{i=1}^{\ell} p_i(n)\frac{x(\tau_i(n))}{x(n)} + \sum_{j=1}^{m} q_j(n)\frac{x(\sigma_j(n))}{x(n)}$$

or

$$0 < 1+\sum_{i=1}^{\ell} p_i(n)\frac{x(\tau(n))}{x(n)},$$

then

$$0 \geq \frac{x(\tau(n))}{x(n)}P(n) > -1.$$

Thus, there exist a sequence $(\theta(n))_{n\in\mathbb{N}}$ such that $\lim_{n\to n}\theta(n)=+\infty$ and

$$-1<\frac{x(\tau(\theta(n)))}{x(\theta(n))}P(\theta(n))<0,$$

or

$$0<\frac{x(\tau(\tau(n)))}{x(\tau(n))}<\frac{-1}{P(\tau(n))}\leq\frac{-1}{D}.$$

Then, $\lim_{n\to\infty}\frac{x(\tau(n))}{x(n)}$ exists. This is contradiction with (2.15). The proof is complete. □

Corollary 6. *If*

$$\liminf_{n\to\infty}\sum_{i=1}^{\ell}\sum_{k=n-\tau_i}^{n-1}p_i(k)<-\frac{1}{e},$$

with $(p_i(n))_{1\leq i\leq\ell}$ *and* $\tau(q_j(n))_{1\leq j\leq m}$ *non positive sequences, then the equation* (1.4) *is oscillatory.*

Theorem 7. *Assume that the sequences* $(\tau_i(n))_{1\leq i\leq\ell}$ *and* $(\sigma_j(n))_{1\leq j\leq m}$ *are increasing,* (1.2), (1.3) *hold and for all* $1\leq i\leq\ell,\ 1\leq j\leq m$

$$\limsup_{n\to\infty}\sum_{k=n}^{\sigma_j(n)-1}Q(k)>\frac{1}{e},\tag{2.16}$$

with $(p_i(n))_{1\leq i\leq\ell}$, $(q_j(n))_{1\leq j\leq m}$ *are non-negative sequences. Then the equation* (1.1) *is oscillatory.*

Proof. Proceed the same way as Theorem 1, i.e. $(x(n))>0,\ \forall n\geq n_0$ and is increasing. Set

$$c_j(n)=\left(\frac{\sigma_j(n)-n}{1+\sigma_j(n)-n}\right)^{1+\sigma_j(n)-n},\tag{2.17}$$

with $n\geq n_0,\ 1\leq i\leq l,\ 1\leq j\leq m$. Clearly,

$$\frac{1}{4}\leq c_j(n)\leq\frac{1}{e},\quad n\geq n_0,\ 1\leq i\leq l,\ 1\leq j\leq m.\tag{2.18}$$

By (2.3) there exist an integer $n_0\geq n_2$ and a small positive number ϵ_0 such that

$$\sum_{k=n}^{\sigma_j(n)-1}Q(k)>\frac{1}{e}+\epsilon_0,\ \text{for } n\geq n_0.\tag{2.19}$$

Let

$$d=e(\frac{1}{e}+\epsilon_0).\tag{2.20}$$

Combining (2.18), (2.19) and (2.20), we get

$$\begin{aligned}\sum_{k=n}^{\sigma_j(n)-1}\frac{Q(k)}{c_j(n)}&>\sum_{k=n}^{\sigma_j(n)-1}\frac{Q(k)}{1/e}\\&=e\sum_{k=n}^{\sigma_j(n)-1}Q(k)>e(\frac{1}{e}+\epsilon_0)=d>1,\ \text{for } n\geq n_0,\end{aligned}$$

or

$$\sum_{k=n}^{\sigma_j(n)-1} \frac{Q(k)}{c_j(n)} > d > 1. \tag{2.21}$$

Since $(x(n))$ is increasing, clearly

$$\frac{x(\tau_i(n))}{x(n+1)} \leq 1, \quad \frac{x(\sigma_j(n))}{x(n+1)} \geq 1, \text{ for all } 1 \leq i \leq l, \ 1 \leq j \leq m. \tag{2.22}$$

By Eq(1.1), we have

$$\frac{x(n)}{x(n+1)} = 1 - \sum_{i=1}^{l} p_i(n) \frac{x(\tau_i(n))}{x(n+1)} - \sum_{j=1}^{m} q_j(n) \frac{x(\sigma_j(n))}{x(n+1)}, \forall n \geq n_0. \tag{2.23}$$

On the other hand

$$\begin{aligned}
\frac{x(n)}{x(\sigma_j(n))} &= \prod_{k=n}^{\sigma_j(n)-1} \frac{x(k)}{x(k+1)} \\
&= \prod_{k=n}^{\sigma_j(n)-1} \left[1 - \sum_{i=1}^{\ell} p_i(k) \frac{x(\tau_i(k))}{x(k+1)} - \sum_{j=1}^{m} q_j(k) \frac{x(\sigma_j(k))}{x(k+1)}\right] \\
&\leq \prod_{k=n}^{\sigma_j(n)-1} \left(1 - \sum_{j=1}^{m} q_j(k) \frac{x(\sigma_j(k))}{x(k+1)}\right) \\
&\leq \prod_{k=n}^{\sigma_j(n)-1} \left(1 - \sum_{j=1}^{m} q_j(k)\right),
\end{aligned}$$

because $-\sum_{i=1}^{\ell} p_i(n) < 0$. Thus

$$\frac{x(n)}{x(\sigma_j(n))} \leq \prod_{k=n}^{\sigma_j(n)-1} (1 - Q(k)). \tag{2.24}$$

By using (2.24) and the well-known inequality between the arithmetic and geometric means, we find that

$$\begin{aligned}
\frac{x(n)}{x(\sigma_j(n))} &\leq \prod_{k=n}^{\sigma_j(n)-1} (1 - Q(k)) \\
&\leq \left[1 - \frac{1}{\sigma_j(n) - n} \sum_{k=n}^{\sigma_j(n)-1} Q(k)\right]^{\sigma_j(n)-n},
\end{aligned}$$

or

$$\frac{x(\sigma_j(n))}{x(n)} \geq \left[1 - \frac{1}{\sigma_j(n) - n} \sum_{k=n}^{\sigma_j(n)-1} Q(k)\right]^{-(\sigma_j(n)-n)}. \tag{2.25}$$

In view of

$$y(1-y)^{\rho} \leq \frac{\rho^{\rho}}{(1+\rho)^{1+\rho}}, \ \forall y \in (0,1), \ \rho \in \mathbb{N}.$$

Inequality (2.25) becomes

$$\frac{x(\sigma_j(n))}{x(n)} \geq \sum_{k=n}^{\sigma_j(n)-1} Q(k) \left(\frac{1+\sigma_j(n)-n}{\sigma_j - n} \right)^{1+\sigma_j(n)-n}. \tag{2.26}$$

Combining (2.26), (2.17) and (2.21), we obtain

$$\frac{x(\sigma_j(n))}{x(n)} \geq \sum_{k=n}^{\sigma_j(n)-1} \frac{Q(k)}{c_j(n)} \geq d, \quad \forall n \geq n_0. \tag{2.27}$$

Similarly, we can prove that

$$\frac{x(n)}{x(\sigma_j(n))} \leq \left[1 - \frac{d}{\sigma_j(n)-n} \sum_{k=n}^{\sigma_j(n)-1} Q(k) \right]$$

Therefore

$$\frac{x(\sigma_j(n))}{x(n)} > d^2, \ \forall n \geq n_3 \geq n_0.$$

Applying this procedure t-times, we obtain

$$\frac{x(\sigma_j(n))}{x(n)} > d^t, \ \text{for all large } n. \tag{2.28}$$

On the other hand, since (2.16) holds, there exists a subsequence of integers $\tau(n)$ such that

$$\lim_{n \tau \infty} Q(\tau(n)) \geq C > 0.$$

By (2.23), we have

$$0 < \frac{x(n)}{x(n+1)} = 1 - \sum_{i=1}^{l} p_i(n) \frac{x(\tau_i(n))}{x(n+1)} - \sum_{j=1}^{m} q_j(n) \frac{x(\sigma_j(n))}{x(n+1)}$$

or

$$0 < 1 - \sum_{j=1}^{m} q_j(n) \frac{x(\sigma(n))}{x(n)},$$

then

$$\frac{x(\sigma(n))}{x(n)} Q(n) < 1.$$

Thus, there exist a sequence $(\theta(n))_{n \in \mathbb{N}}$ such that $\lim_{n \to n} \theta(n) = +\infty$ and

$$\frac{x(\sigma(\theta(n)))}{x(\theta(n))} Q(\theta(n)) < 1,$$

or

$$\frac{x(\sigma(\theta(n)))}{x(\theta(n))} < \frac{1}{Q(\theta(n))} \leq \frac{1}{C} < +\infty, \ \text{for all large } n,$$

i.e., $\lim_{n \to \infty} \frac{x(\sigma(n))}{x(n)}$ exists. This contradicts with (2.28).

The proof is complete. □

Corollary 8. *If*

$$\liminf_{n\to\infty}\sum_{j=1}^{m}\sum_{k=n-\tau_i}^{n+\sigma_i-1} q_j(k) > \frac{1}{e},$$

with $(p_i(n))_{1\le i\le \ell}$ *and* $\tau(q_j(n))_{1\le j\le m}$ *non negative sequences, then the equation* (1.4) *is oscillatory.*

In order to illustrate our results for all the previous theorems, we give the following examples.

Example 9. Consider the mixed type difference equation

$$\Delta x(n) = x(n-1)+x(n-2)+\frac{c_1}{3\ln(n+2)^{n+2}}x(n^2+1)+\frac{2c_1}{3\ln(n+2)^{n+2}}x(n^2+2),\ n\ge 1, \tag{2.29}$$

with $c_1 = \frac{e}{\ln 4}$.

Here $\tau_1(n) = n-1,\ \tau_2(n) = n-2\ \sigma_1(n) = n^2+1 \ge n+1,\ \sigma_2(n) = n^2+2 \ge n+1$,. Easily we can see that that (1.2) and (1.3) hold and

$$\tau(n) = \min_{1\le i\le 2} \tau_i(n) = n-2.$$

$$\sigma(n) = \min_{1\le j\le 2} \sigma_j(n) = n^2+1.$$

We claim

$$\limsup_{n\to\infty}\sum_{k=\tau(n)}^{\sigma(n)-1} Q(k) > 1.$$

In fact,

$$\sum_{k=n-2}^{n^2} Q(k) = \sum_{k=n-2}^{n^2}\left(\frac{c_1}{3\ln(k+2)^{k+2}}+\frac{2c_1}{3\ln(k+2)^{k+2}}\right) = \sum_{k=n-2}^{n^2}\left(\frac{c_1}{\ln(k+2)^{k+2}}\right).$$

We have the following inequality, for a positive decreasing function $f(x)$, we get

$$\int_{b-1}^{b} f(x)dx \ge f(b) \ge \int_{b}^{b+1} f(x)dx.$$

Therefore,

$$\begin{aligned}\sum_{k=n-2}^{n^2}\left(\frac{c_1}{\ln(k+2)^{k+2}}\right) &\ge c_1\sum_{k=n-2}^{n^2}\int_{k}^{k+1}\frac{ds}{\ln(s+2)^{(s+2)}}\\ &= c_1\int_{n-2}^{n^2+1}\frac{ds}{(s+2)\ln(s+2)} = c_1\ln\left(\frac{\ln(n^2+3)}{\ln(n)}\right).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\sum_{k=n-2}^{n^2}\left(\frac{c_1}{\ln(k+2)^{k+2}}\right) &\le c_1\sum_{k=n-2}^{n^2}\int_{k-1}^{k}\frac{ds}{\ln(s+2)^{(s+2)}} =\\ &= c_1\int_{n-3}^{n^2}\frac{ds}{(s+2)\ln(s+2)} = c_1\ln\left(\frac{\ln(n^2+2)}{\ln(n-1)}\right).\end{aligned}$$

Thus,

$$\lim_{n\to\infty} c_1 \ln\left(\frac{\ln(n^2+2)}{\ln(n-1)}\right) = \lim_{n\to\infty} c_1 \ln\left(\frac{\ln(n^2+3)}{\ln(n)}\right) = c_1\ln(2) = \frac{e}{2} > 1.$$

Therefore

$$\limsup_{n\to\infty} \sum_{k=n-2}^{n^2} Q(k) = \frac{e}{2} > 1.$$

Thus, all the conditions of Theorem 1 are satisfied and therefore all solutions of equation (2.29) oscillate.

Example 10. Consider the mixed type difference equation

$$\Delta x(n) = -\frac{c_2}{3n}x([0.5n]) - \frac{2c_2}{3n}x([0.75n]) - x(n+1) - 2x(n^2+2), \tag{2.30}$$

with $c_2 = \frac{3}{2\ln 2}$. Here $\tau_1(n) = [0.5n]$, $\tau_2(n) = [0.75n]$, $\sigma_1(n) = n+1$, $\sigma_2(n) = n^2+2$. Easily we can see that (1.2) and (1.3) hold.

Furthermore,

$$\tau(n) = [0.5n]$$

and

$$\sigma(n) = n+1.$$

We will prove that

$$\liminf_{n\to\infty} \sum_{k=\tau(n)}^{\sigma(n)-1} P(k) < -1.$$

In fact, by the similar procedure as in Example 9, we get

$$\sum_{k=[0.5n]}^{n} P(k) = \sum_{k=[0.5n]}^{n} \left(\frac{-c_2}{3k} + \frac{-2c_2}{3k}\right) = \sum_{k=[0.5n]}^{n} \left(\frac{-c_2}{k}\right).$$

Furthermore,

$$-c_2 \sum_{k=[0.5n]}^{n} \int_{k-1}^{k} \frac{ds}{s} \le -\sum_{k=[0.5n]}^{n} \frac{c_2}{k} \le -c_2 \sum_{k=[0.5n]}^{n} \int_{k}^{k+1} \frac{ds}{s}$$

i.e.,

$$\lim_{n\to\infty}\left[-\sum_{k=[0.5n]}^{n} \frac{c_2}{k}\right] = -c_2\ln 2 = -\frac{3}{2} < -1.$$

Hence, all the conditions of Theorem 3 are satisfied and therefore all the solutions of equation (2.30) oscillate.

Example 11. Consider the mixed type difference equation

$$\Delta x(n) = -\frac{1}{3e}x(n-1) - \frac{5}{6e}x(n-2) - \frac{1}{2}x(n+1) - \frac{1}{2}x(n+5). \tag{2.31}$$

We proceed the same way as in the Example 9. Since $\tau_1(n) = n-1,\ \tau_2(n) = n-2$, we observe that

$$\begin{aligned}\liminf_{n\to\infty} \sum_{k=n-1}^{n-1} P(k) &= p_1(n-1) + p_2(n-1)\\ &= \left[\frac{-1}{3e} + \frac{-5}{6e}\right] = -\frac{7}{6e} < -\frac{1}{e},\\ \liminf_{n\to\infty} \sum_{k=n-2}^{n-1} P(k) &= p_1(n-2) + p_1(n-1) + p_2(n-2) + p_2(n-1)\\ &= -\frac{7}{3e} < -\frac{1}{e}.\end{aligned}$$

Hence, all the conditions of Theorem 5 are satisfied and therefore all the solutions of equation (2.31) oscillate.

Example 12. Consider the mixed type difference equation

$$\Delta x(n) = \frac{1}{3}x(n-1) + \frac{2}{3}x(n-4) + \frac{1}{3e}x(n+1) - \frac{4}{5e}x(n+3). \tag{2.32}$$

We proceed the same way as in the Example 9. Here $\sigma_1(n) = n+1,\ \sigma_2(n) = n+3$. Since

$$\begin{aligned}\liminf_{n\to\infty} \sum_{k=n}^{n} Q(k) &= q_1(n) + q_2(n)\\ &= \left[\frac{1}{3e} + \frac{4}{5e}\right] = \frac{17}{15e} > \frac{1}{e},\\ \liminf_{n\to\infty} \sum_{k=n}^{n+2} Q(k) &= q_1(n) + q_1(n+1) + q_1(n+2) + q_2(n) + q_2(n+1) + q_2(n+2)\\ &= \frac{17}{5e} > \frac{1}{e}.\end{aligned}$$

Hence, all the conditions of Theorem 7 are satisfied and therefore all the solutions of equation (2.31) oscillate.

Next we will exemplify the previous theorems using a numerical simulation.

Consider the mixed type difference equation

$$\Delta x(n) = C_1x(n-1) + C_2x(n-2) + C_3x(n+1) + C_4x(n+2),\ n \geq 1, \tag{2.33}$$

where C_1, C_2, C_3, C_4 are given constants.
Since, $\tau_1(n) = n-1,\ \tau_2(n) = n-2,\ \sigma_1(n) = n+1$, and $\sigma_2(n) = n+2$, we get

$$\tau(n) = \min_{1\leq i\leq 2} \tau_i(n) = n-2,$$

$$\sigma(n) = \min_{1\leq j\leq 2} \sigma_j(n) = n+1.$$

Example 13. Let the Equation (2.33) with $C_1 = C_2 = 1,\ C_3 = 1/2,\ C_4 = 3/2$. We claim

$$\limsup_{n\to\infty} \sum_{k=\tau(n)}^{\sigma(n)-1} Q(k) > 1.$$

In fact,

$$\sum_{k=n-2}^{n} Q(k) = \sum_{k=n-2}^{n} (C_3 + C_4) = 6 \geq 1$$

and consequently

$$\limsup_{n\to\infty} \sum_{k=n-2}^{n} Q(k) > 1.$$

Thus, all the conditions of Theorem 1 are satisfied and therefore all solutions of equation (2.33) oscillate.

We choose the starting values $x(-2) = 1, = x(-1) = -1, x(0) = 0, x(1) = 0$. In Figure **??**, we see that the solutions of equation (2.33) oscillate.

Example 14. Let the Equation (2.33) with $C_1 = -3/4,\ C_2 = -1/2,\ C_3 = C_4 = -1$.
By the assumptions of Theorem 3 we must have

$$\limsup_{n\to\infty} \sum_{k=\tau(n)}^{\sigma(n)-1} P(k) \leq -1.$$

In fact,

$$\sum_{k=n-2}^{n} P(k) = \sum_{k=n-2}^{n} (C_1 + C_2) = -15/4 \leq -1,$$

and

$$\limsup_{n\to\infty} \sum_{k=n-2}^{n} P(k) < -1.$$

Hence, all the conditions of Theorem 3 are satisfied and therefore all the solutions of equation (2.33) oscillate.

We choose the starting values $x(-2) = 1, = x(-1) = -1, x(0) = 0, x(1) = 0$. In Figure **??**, we see that the solutions of equation (2.33) oscillate.

Example 15. Let the Equation (2.33) with $C_1 = \frac{-ln2}{e}$, $C_2 = \frac{-ln4}{e}$, $C_3 = C_4 = -1$. We claim

$$\limsup_{n\to\infty} \sum_{k=\tau(n)}^{\sigma(n)-1} P(k) < \frac{-1}{e}.$$

In fact,

$$\sum_{k=n-2}^{n} P(k) = \sum_{k=n-2}^{n} (C_1 + C_2) = \frac{-9ln2}{e} \leq \frac{-1}{e}$$

so,

$$\limsup_{n\to\infty} \sum_{k=n-2}^{n} P(k) < \frac{-1}{e}.$$

Hence, all the conditions of Theorem 5 are satisfied and therefore all the solutions of equation (2.33) oscillate.

We choose the starting values $x(-2)=1, = x(-1)=-1, x(0)=0, x(1)=0$. In Figure **??**, we see that the solutions of equation (2.33) oscillate.

Example 16. Let the Equation (2.33) with $C_1=C_2=1,\ C_3=\frac{4}{e},\ C_4=\frac{1}{3e}$. By the assumptions of Theorem 7,

$$\limsup_{n\to\infty}\sum_{k=\tau(n)}^{\sigma(n)-1} Q(k) > \frac{1}{e}.$$

In fact,

$$\sum_{k=n-2}^{n} Q(k) = \sum_{k=n-2}^{n}(C_3+C_4) = \frac{13}{e} \geq \frac{1}{e}$$

and consequently

$$\limsup_{n\to\infty}\sum_{k=n-2}^{n} Q(k) > \frac{1}{e}.$$

Hence, all the conditions of Theorem 7 are satisfied and therefore all the solutions of equation (2.33) oscillate.

We choose the starting values $x(-2)=1, = x(-1)=-1, x(0)=0, x(1)=0$. In Figure **??**, we see that the solutions of equation (2.33) oscillate.

In the following section, we will focus on the special case of the non-autonomous linear mixed-type difference equation. Namely, we'll consider the equation (1.4).

3. Asymptotic Behavior and Stability Criteria

Our goal in this section is to give new results on asymptotic behavior and stability for a class of (1.5). Our results will be obtained via an appropriate solution of the corresponding generalized characteristic equation. The techniques applied in obtaining our results are originated in a combination of the methods used in [17, 18, 19].

Throughout the paper, τ represents the positive integer determined by

$$\tau = \max_{i=1,\cdots,\ell} \tau_i \quad \text{and} \quad \sigma = \max_{j=1,\cdots,m} \sigma_j$$

and also, let's show the set of all $\phi = (\phi(n))_{n=-\tau}^{\sigma}$ by Φ with $\phi(n) \in \mathbf{R}$ for $n=-\tau,\cdots,0,\cdots,\sigma$; this set is a finite dimensional space with the usual sup-norm $\|\cdot\|$ denoted by

$$\|\phi\| = \sup_{n=-\tau,\cdots,0,\cdots,\sigma} |\phi(n)| \quad \text{for any} \quad \phi=(\phi(n))_{n=-\tau}^{\sigma} \quad \text{in} \quad \Phi.$$

Along with the mixed type difference equation (1.5), we specify an initial condition

$$x(n)=\phi(n) \quad \text{for} \quad n=-\tau,\cdots,0,\cdots,\sigma, \tag{3.1}$$

where the initial function ϕ is a given real-valued function for $n=-\tau,\cdots,0,\cdots,\sigma$ satisfying the "consistency condition"

$$\phi(1)-\phi(0)=\sum_{i=1}^{\ell} p_i(0)\phi(-\tau_i)+\sum_{j=1}^{m} q_j(0)\phi(\sigma_j).$$

As is known, by a *solution* of the mixed type difference equation (1.5) is meant a sequence of real numbers $(x(n))_{n\geq -\tau}$ that provide (1.5) for all integers $n \geq 0$. In order to guarantee its existence and uniqueness for given initial values (3.1), we will assume throughout this paper that the numbers q_j for $j = 1, 2, \cdots, m$ and for all $n \geq 0$ are such that

$$\begin{array}{lll} q_1 \neq 1 & \text{if} & m = 1 \text{ and } \sigma_1 = 1 \\ q_1 \neq 0 & \text{if} & m = 1 \text{ and } \sigma_1 > 1 \\ q_m \neq 0 & \text{if} & m \geq 2 \end{array}$$

with no restrictions in other cases (see [14], Chapter 7]).

Note the usual rule of $\prod_{s=0}^{-1} = 1$ throughout the paper.

The following equation is called the *generalized characteristic equation* of (1.5):

$$\lambda_n - 1 = \sum_{i=1}^{\ell} p_i(n) \left(\prod_{s=n-\tau_i}^{n-1} \lambda_s \right)^{-1} + \sum_{j=1}^{m} q_j(n) \left(\prod_{s=n}^{n+\sigma_j-1} \lambda_s \right). \tag{3.2}$$

The last equation was obtained from the equation (1.5) using the solutions in the form

$$x(n) = \begin{cases} \prod_{s=0}^{n-1} \lambda_s & \text{for} \quad n \geq 0 \\ \left(\prod_{s=n}^{-1} \lambda_s \right)^{-1} & \text{for} \quad n = -\tau, \cdots, 0. \end{cases} \tag{3.3}$$

Here, $(\lambda_n)_{n\geq -\tau}$ is a sequence of positive real numbers.

The first result is the following theorem, which creates a basic asymptotic property for solutions of the mixed-type difference equation (1.5).

Theorem 17. *We assume that* $\lambda_0 = \left(\lambda_n^0\right)_{n\geq -\tau}$ *be a solution of* (3.2) *with the property*

$$\mu(\lambda_0) = \sup_{n\geq 0} \left\{ \frac{1}{\lambda_n^0} \left[\sum_{i=1}^{\ell} \tau_i \left| p_i(n) \right| \left(\prod_{s=n-\tau_i}^{n-1} \lambda_s \right)^{-1} + \sum_{j=1}^{m} \sigma_j \left| q_j(n) \right| \left(\prod_{s=n}^{n+\sigma_j-1} \lambda_s \right) \right] \right\} < 1. \tag{3.4}$$

Set

$$\Psi_{\lambda_0}(n) = \begin{cases} \prod_{s=0}^{n-1} \lambda_s^0 & \textit{for} \quad n \geq 0 \\ \left(\prod_{s=n}^{-1} \lambda_s^0 \right)^{-1} & \textit{for} \quad n = -\tau, \cdots, 0. \end{cases} \tag{3.5}$$

Then, for $\phi = (\phi(n))_{n=-\tau}^{\sigma}$ *in* Φ*, it holds*

$$\lim_{n\to\infty} \frac{x(n)}{\Psi_{\lambda_0}(n)} = L(\lambda_0; \phi). \tag{3.6}$$

Here, $L(\lambda_0; \phi)$ *is real number depending to* λ_0 *and denoted by* ϕ*.*

Proof. First of all, we have

$$\Psi_{\lambda_0}(n - \tau_i) = \Psi_{\lambda_0}(n) \left(\prod_{s=n-\tau_i}^{n-1} \lambda_s^0 \right)^{-1} \quad \text{for all } n \geq 0, \quad (i = 1, \cdots, \ell) \tag{3.7}$$

and

$$\Psi_{\lambda_0}(n+\sigma_j) = \Psi_{\lambda_0}(n)\left(\prod_{s=n}^{n+\sigma_j-1} \lambda_s^0\right) \quad \text{for all } n \geq 0, \quad (j=1,\cdots,m). \tag{3.8}$$

In fact, if i is an any index at $1,...,\ell$, then we get

$$\frac{\Psi_{\lambda_0}(n-\tau_i)}{\Psi_{\lambda_0}(n)} = \begin{cases} \dfrac{\prod_{s=0}^{n-\tau_i-1}\lambda_s^0}{\prod_{s=0}^{n-1}\lambda_s^0} & \text{if } \quad n \geq \tau_i \\ \dfrac{\left(\prod_{s=n-\tau_i}^{-1}\lambda_s^0\right)^{-1}}{\prod_{s=0}^{n-1}\lambda_s^0} & \text{if } \quad 0 \leq n \leq \tau_i \end{cases}$$

$$= \begin{cases} \dfrac{1}{\left(\prod_{s=0}^{n-1}\lambda_s^0\right)\left(\prod_{s=0}^{n-\tau_i-1}\lambda_s^0\right)^{-1}} & \text{if } \quad n \geq \tau_i \\ \dfrac{1}{\left(\prod_{s=0}^{n-1}\lambda_s^0\right)\left(\prod_{s=n-\tau_i}^{-1}\lambda_s^0\right)} & \text{if } \quad 0 \leq n \leq \tau_i \end{cases} = \left(\prod_{s=n-\tau_i}^{n-1}\lambda_s^0\right)^{-1}$$

i.e., (3.7) holds true. If j is an any index at $1,...,m$, then we get

$$\Psi_{\lambda_0}(n+\sigma_j) = \prod_{s=0}^{n+\sigma_j-1}\lambda_s^0 = \left(\prod_{s=0}^{n-1}\lambda_s^0\right)\left(\prod_{s=n}^{n-\sigma_j-1}\lambda_s^0\right) = \Psi_{\lambda_0}(n)\prod_{s=n}^{n-\sigma_j-1}\lambda_s^0$$

i.e., (3.8) is satisfied.

Let us define

$$y(n) = \frac{x(n)}{\Psi_{\lambda_0}(n)} \quad \text{for} \quad n \geq -\tau. \tag{3.9}$$

Then, by taking into account (3.7) and (3.8), for $n \geq 0$, we have

$$\begin{aligned}
&\triangle x(n) - \sum_{i=1}^{\ell} p_i(n)x(n-\tau_i) - \sum_{j=1}^{m} q_j(n)x(n+\sigma_j) \\
&= \triangle\left[y(n)\Psi_{\lambda_0}(n)\right] - \sum_{i=1}^{\ell} p_i(n)y(n-\tau_i)\Psi_{\lambda_0}(n-\tau_i) - \sum_{j=1}^{m} q_j(n)y(n+\sigma_j)\Psi_{\lambda_0}(n+\sigma_j) \\
&= \left[(\triangle y(n))\,\Psi_{\lambda_0}(n+1) + y(n)\triangle\left(\Psi_{\lambda_0}(n)\right)\right] \\
&\qquad - \sum_{i=1}^{\ell} p_i(n)y(n-\tau_i)\Psi_{\lambda_0}(n-\tau_i) - \sum_{j=1}^{m} q_j(n)y(n+\sigma_j)\Psi_{\lambda_0}(n+\sigma_j) \\
&= (\triangle y(n))\,\lambda_n^0\Psi_{\lambda_0}(n) + y(n)\left(\lambda_n^0 - 1\right)\Psi_{\lambda_0}(n) \\
&- \sum_{i=1}^{\ell} p_i(n)y(n-\tau_i)\Psi_{\lambda_0}(n)\left(\prod_{s=n-\tau_i}^{n-1}\lambda_s^0\right)^{-1} - \sum_{j=1}^{m} q_j(n)y(n+\sigma_j)\Psi_{\lambda_0}(n)\left(\prod_{s=n}^{n+\sigma_j-1}\lambda_s^0\right) \\
&= \Big\{(\triangle y(n))\,\lambda_n^0 + y(n)\left(\lambda_n^0 - 1\right) \\
&\qquad - \sum_{i=1}^{\ell} p_i(n)y(n-\tau_i)\left(\prod_{s=n-\tau_i}^{n-1}\lambda_s^0\right)^{-1} - \sum_{j=1}^{m} q_j(n)y(n+\sigma_j)\left(\prod_{s=n}^{n+\sigma_j-1}\lambda_s^0\right)\Big\}\Psi_{\lambda_0}(n).
\end{aligned}$$

Thus, by using the hypothesis that λ_n^0 is a solution of (3.2), we get for $n \geq 0$

$$\begin{aligned}
&\triangle x(n) - \sum_{i=1}^{\ell} p_i(n) x(n-\tau_i) - \sum_{j=1}^{m} q_j(n) x(n+\sigma_j) \\
&= \Bigg\{ (\triangle y(n))\,\lambda_n^0 + y(n) \left[\sum_{i=1}^{\ell} p_i(n) \left(\prod_{s=n-\tau_i}^{n-1} \lambda_s^0 \right)^{-1} + \sum_{j=1}^{m} q_j(n) \left(\prod_{s=n}^{n+\sigma_j-1} \lambda_s^0 \right) \right] \\
&- \sum_{i=1}^{\ell} p_i(n) y(n-\tau_i) \left(\prod_{s=n-\tau_i}^{n-1} \lambda_s^0 \right)^{-1} - \sum_{j=1}^{m} q_j(n) y(n+\sigma_j) \left(\prod_{s=n}^{n+\sigma_j-1} \lambda_s^0 \right) \Bigg\} \Psi_{\lambda_0}(n) \\
&= \Bigg\{ (\triangle y(n))\,\lambda_n^0 + \sum_{i=1}^{\ell} p_i(n) \left(y(n) - y(n-\tau_i) \right) \left(\prod_{s=n-\tau_i}^{n-1} \lambda_s^0 \right)^{-1} \\
&\quad + \sum_{j=1}^{m} q_j(n) \left(y(n) - y(n+\sigma_j) \right) \left(\prod_{s=n}^{n+\sigma_j-1} \lambda_s^0 \right) \Bigg\} \Psi_{\lambda_0}(n).
\end{aligned}$$

Hence, $(x(n))_{n\geq -\tau}$ satisfies (1.5), followed by the condition that $(y(n))_{n\geq -\tau}$ satisfies

$$\begin{aligned}
\triangle y(n) = -\frac{1}{\lambda_n^0} \Bigg\{ &\sum_{i=1}^{\ell} p_i(n) \left(y(n) - y(n-\tau_i) \right) \left(\prod_{s=n-\tau_i}^{n-1} \lambda_s^0 \right)^{-1} \\
&+ \sum_{j=1}^{m} q_j(n) \left(y(n) - y(n+\sigma_j) \right) \left(\prod_{s=n}^{n+\sigma_j-1} \lambda_s^0 \right) \Bigg\} \quad \text{for} \quad n \geq 0.
\end{aligned}$$

We can easily see that last equation can equivalently be written in the form

$$\begin{aligned}
\triangle y(n) = -\frac{1}{\lambda_n^0} \Bigg\{ &\sum_{i=1}^{\ell} p_i(n) \left(\sum_{s=n-\tau_i}^{n-1} \triangle y(s) \right) \left(\prod_{s=n-\tau_i}^{n-1} \lambda_s^0 \right)^{-1} \\
&- \sum_{j=1}^{m} q_j(n) \left(\sum_{s=n}^{n+\sigma_j-1} \triangle y(s) \right) \left(\prod_{s=n}^{n+\sigma_j-1} \lambda_s^0 \right) \Bigg\} \quad \text{for} \quad n \geq 0. \qquad (3.10)
\end{aligned}$$

This equation is equivalent to equation (1.5). The initial condition (3.1) can be written in the following equivalent form:

$$y(n) = \frac{\phi(n)}{\Psi_{\lambda_0}(n)} \quad \text{for} \quad n = -\tau, \cdots, 0, \cdots, \sigma. \qquad (3.11)$$

Now, let us define following expression:

$$H_{\lambda_0}(\phi) = \max_{n=-\tau,\cdots,-1} \left| \triangle \left(\frac{\phi(n)}{\Psi_{\lambda_0}(n)} \right) \right|. \qquad (3.12)$$

It is clear, from (3.11), we obtain

$$|\triangle y(n)| \leq H_{\lambda_0}(\phi) \quad \text{for} \quad n = -\tau, \cdots, -1. \qquad (3.13)$$

Now, let us show that the following inequality is satisfied

$$|\triangle y(n)| \leq H_{\lambda_0}(\phi) \quad \text{for} \quad n \geq -\tau. \qquad (3.14)$$

In fact, let ϵ be an any positive number. We claim that

$$|\triangle y(n)| < H_{\lambda_0}(\phi) + \epsilon \qquad \text{for every} \quad n \geq -\tau. \tag{3.15}$$

Otherwise, in view of (3.13), there is an integer $n_0 \geq 0$ with

$$|\triangle y(n)| < H_{\lambda_0}(\phi) + \epsilon \quad \text{for } n = -\tau, \cdots, n_0 - 1, n_0 + 1, \cdots, n_0 + \sigma$$

and

$$|\triangle y(n_0)| = H_{\lambda_0}(\phi) + \epsilon.$$

Then, using (3.4), from (3.10) we obtain

$$\begin{aligned}
H_{\lambda_0}(\phi) + \epsilon &= |\triangle y(n_0)| \\
&\leq \frac{1}{\lambda^0_{n_0}} \left\{ \sum_{i=1}^{\ell} |p_i(n_0)| \left(\sum_{s=n_0-\tau_i}^{n_0-1} |\triangle y(s)| \right) \left(\prod_{s=n_0-\tau_i}^{n_0-1} \lambda^0_s \right)^{-1} \right. \\
&\qquad \left. + \sum_{j=1}^{m} |q_j(n_0)| \left(\sum_{s=n_0}^{n_0+\sigma_j-1} |\triangle y(s)| \right) \left(\prod_{s=n_0}^{n_0+\sigma_j-1} \lambda^0_s \right) \right\} \\
&\leq \frac{1}{\lambda^0_{n_0}} \left\{ \sum_{i=1}^{\ell} |p_i(n_0)| \tau_i \left(\prod_{s=n_0-\tau_i}^{n_0-1} \lambda^0_s \right)^{-1} \right. \\
&\qquad \left. + \sum_{j=1}^{m} |q_j(n_0)| \sigma_j \left(\prod_{s=n_0}^{n_0+\sigma_j-1} \lambda^0_s \right) \right\} [H_{\lambda_0}(\phi) + \epsilon] \\
&\leq \mu(\lambda_0)[H_{\lambda_0}(\phi) + \epsilon] \\
&< H_{\lambda_0}(\phi) + \epsilon,
\end{aligned}$$

which in view of (3.4), leads to a contradiction. So, our claim is true, i.e. (3.15) holds. We have thus proved that (3.15) is fulfilled for all numbers $\epsilon > 0$. Hence, (3.14) is satisfied. Now, by virtue of (3.14), from (3.10) we derive for $n \geq 0$

$$\begin{aligned}
|\triangle y(n)| &\leq \frac{1}{\lambda^0_n} \left\{ \sum_{i=1}^{\ell} |p_i(n)| \left(\sum_{s=n-\tau_i}^{n-1} |\triangle y(s)| \right) \left(\prod_{s=n-\tau_i}^{n-1} \lambda^0_s \right)^{-1} \right. \\
&\qquad \left. + \sum_{j=1}^{m} |q_j(n)| \left(\sum_{s=n}^{n+\sigma_j-1} |\triangle y(s)| \right) \left(\prod_{s=n}^{n+\sigma_j-1} \lambda^0_s \right) \right\} \\
&\leq \frac{1}{\lambda^0_n} \left\{ \sum_{i=1}^{\ell} |p_i(n)| \tau_i \left(\prod_{s=n-\tau_i}^{n-1} \lambda^0_s \right)^{-1} + \sum_{j=1}^{m} |q_j(n)| \sigma_j \left(\prod_{s=n}^{n+\sigma_j-1} \lambda^0_s \right) \right\} H_{\lambda_0}(\phi) \\
&\leq \mu(\lambda_0) H_{\lambda_0}(\phi).
\end{aligned}$$

Consequently, by the definition of $\mu(\lambda_0)$, we have

$$|\triangle y(n)| \leq \mu(\lambda_0) H_{\lambda_0}(\phi), \qquad \text{for} \quad n \geq 0. \tag{3.16}$$

By (3.14) and (3.16), an easy induction leads to the conclusion that $(y(n))_{n \geq -\tau}$ satisfies

$$|\triangle y(n)| \leq (\mu(\lambda_0))^v H_{\lambda_0}(\phi), \qquad \text{for} \quad n \geq v\tau - \tau, \tag{3.17}$$

where $v = 0, 1, \cdots$.

Now, by following the same procedure (taking into account (3.4) and using inequality (3.17)) in the article of previous authors [18] , we can obtain

$$|y(n)-y(N)| \leq \frac{H_{\lambda_0}(\phi)}{1-(\mu(\lambda_0))^{1/\tau}} (\mu(\lambda_0))^{N/\tau}, \qquad \text{for all} \quad n \geq N \geq -\tau. \tag{3.18}$$

Note that, property (3.4) implies $1-(\mu(\lambda_0))^{1/\tau} > 0$. Due to the condition (3.4), we get

$$\lim_{N\to\infty} (\mu(\lambda_0))^{N/\tau} = 0.$$

So, by the Cauchy convergence criterion, it follows from (3.18) that

$$\lim_{n\to\infty} y(n) \quad \text{exists (in the real line).}$$

Obviously, this limit depends to λ_0 and is denoted by solution $(x(n))_{n\geq-\tau}$ (and thus denoted by ϕ). Therefore, we call this limit $L(\lambda_0;\phi)$. This shows the limit in Theorem 17 and completes the proof. □

The second main result in this section is Theorem 18 below. This theorem makes an estimate for the solutions of the mixed-type difference equation (1.5), which leads in a stability result for the trivial solution of the equation (1.5).

Theorem 18. *We assume that* $\lambda_0 = \left(\lambda_n^0\right)_{n\geq-\tau}$ *be a solution of the generalized characteristic equation* (3.2) *with the property* (3.4) *and let* $\mu(\lambda_0)$ *and* $\Psi_{\lambda_0}(n)$ *be defined by* (3.4) *and* (3.5)*, respectively. Then, for* $\phi = (\phi(n))_{n=-\tau}^{\sigma}$ *in* Φ*, the solution* $(x(n))_{n\geq-\tau}$ *of* (1.5)-(3.1) *satisfies*

$$|x(n)| \leq N(\lambda_0) \parallel \phi \parallel \Psi_{\lambda_0}(n), \qquad \textit{for all} \quad n \geq 0, \tag{3.19}$$

where

$$N(\lambda_0) = 1 + \frac{1}{1-(\mu(\lambda_0))^{1/\tau}} \max_{n=-\tau,\cdots,-1} \left[\left(\frac{1}{\lambda_n^0} + 1 \right) \left(\prod_{s=n}^{-1} \lambda_s^0 \right) \right]. \tag{3.20}$$

Furthermore, the trivial solution of (1.5) *is asymptotically stable if and only if*

$$\limsup_{n\to\infty} \Psi_{\lambda_0}(n) = 0. \tag{3.21}$$

and the trivial solution of (1.5) *stable if and only if*

$$\limsup_{n\to\infty} \Psi_{\lambda_0}(n) < \infty \tag{3.22}$$

Proof. Define $(y(n))_{n\geq-\tau}$ and $H_{\lambda_0}(\phi)$ by (3.9) and (3.12), respectively. Later, as in the proof of Theorem 17, $(y(n))_{n\geq-\tau}$ proves that it satisfies

$$|y(n)-y(N)| \leq \frac{H_{\lambda_0}(\phi)}{1-(\mu(\lambda_0))^{1/\tau}} (\mu(\lambda_0))^{N/\tau}, \qquad \text{for all} \quad n \geq N \geq -\tau.$$

For $N = 0$, from the last inequality we obtain

$$|y(n)-y(0)| \leq \frac{H_{\lambda_0}(\phi)}{1-(\mu(\lambda_0))^{1/\tau}}, \qquad \text{for} \quad n \geq 0.$$

Thus, since $y(0) = \frac{x(0)}{\Psi_{\lambda_0}(0)} = \phi(0)$, we get

$$|y(n)| \leq |\phi(0)| + \frac{H_{\lambda_0}(\phi)}{1 - (\mu(\lambda_0))^{1/\tau}}, \qquad \text{for} \quad n \geq 0. \tag{3.23}$$

Furthermore, from $H_{\lambda_0}(\phi)$, by following the same steps as the proof of the second theorem in the article of the authors [18], we obtain

$$\begin{aligned}
H_{\lambda_0}(\phi) &\equiv \max_{n=-\tau,\cdots,-1} \left| \triangle\Big(\frac{\phi(n)}{\Psi_{\lambda_0}(n)}\Big) \right| \\
&= \max_{n=-\tau,\cdots,-1} \left| \triangle \left(\phi(n) \left(\prod_{s=n}^{-1} \lambda_s^0 \right) \right) \right| \\
&= \max_{n=-\tau,\cdots,-1} \left| \phi(n+1) \left(\prod_{s=n+1}^{-1} \lambda_s^0 \right) - \phi(n) \left(\prod_{s=n}^{-1} \lambda_s^0 \right) \right| \\
&= \max_{n=-\tau,\cdots,-1} \left| \left(\phi(n+1) \frac{1}{\lambda_n^0} - \phi(n) \right) \left(\prod_{s=n}^{-1} \lambda_s^0 \right) \right| \\
&\leq \max_{n=-\tau,\cdots,-1} \left[\left(|\phi(n+1)| \frac{1}{\lambda_n^0} + |\phi(n)| \right) \left(\prod_{s=n}^{-1} \lambda_s^0 \right) \right] \\
&\leq \| \phi \| \max_{n=-\tau,\cdots,-1} \left[\left(\frac{1}{\lambda_n^0} + 1 \right) \left(\prod_{s=n}^{-1} \lambda_s^0 \right) \right].
\end{aligned}$$

Thus, for $n \geq 0$, (3.23) gives

$$|y(n)| \leq \left\{ 1 + \frac{1}{1 - (\mu(\lambda_0))^{1/\tau}} \max_{n=-\tau,\cdots,-1} \left[\left(\frac{1}{\lambda_n^0} + 1 \right) \left(\prod_{s=n}^{-1} \lambda_s^0 \right) \right] \right\} \| \phi \| . \tag{3.24}$$

So, using (3.20), we obtain

$$|y(n)| \leq N(\lambda_0) \| \phi \|, \qquad \text{for} \quad n \geq 0.$$

Finally, due to the definition of $(y(n))_{n \geq -\tau}$, we get

$$|x(n)| \leq N(\lambda_0) \| \phi \| \Psi_{\lambda_0}(n), \qquad \text{for all} \quad n \geq 0.$$

This completes the proof of the first part of the theorem.

We can now prove the stability results (i) and (ii), the same procedure as in the proof of the second theorem in [18]. Therefore, evidence of (i) and (ii) will be ignored.

□

An application of Theorem 17 and Theorem 18 with $\lambda_n^0 = 1,\ n \geq -\tau$ leads to the following particular result:

Remark 19. Suppose that

$$\sum_{i=1}^{\ell} p_i(n) + \sum_{j=1}^{m} q_j(n) = 0, \quad n \geq 0.$$

and

$$\mu(1) = \sup_{n\geq 0}\left[\sum_{i=1}^{\ell} \tau_i \left|p_i(n)\right| + \sum_{j=1}^{m} \sigma_j \left|q_j(n)\right|\right] < 1. \tag{3.25}$$

Then, the solution $(x(n))_{n\geq -\tau}$ of (1.5)- (3.1) satisfies

$$\lim_{n\to\infty} x(n) = l(\phi)$$

and

$$|x(n)| \leq \left[1 + \frac{2}{1-(\mu(1))^{1/\tau}}\right] \parallel \phi \parallel, \qquad \text{for all} \quad n \geq 0.$$

Here, $l(\phi)$ is a real number denoted by ϕ.

Moreover, the trivial solution of (1.5) is stable if and only if $\lambda_n^0 = 1$.

In the following examples, we will apply the stability criteria of the Theorem 18.

Example 20. Consider

$$\begin{aligned}\triangle x(n) =& \frac{1}{2(n+2)}x(n-1) - \frac{1}{4(n^2+3)}x(n-2)\\ &- \frac{1}{2(n+2)}x(n+1) + \frac{1}{4(n^2+3)}x(n+2), \qquad n \geq 0\end{aligned} \tag{3.26}$$

$$x(n) = \phi(n) \quad \text{for} \quad n = -2, -1, 0, 1, 2,$$

where $\phi(n) \in \mathbf{R}$ such that ϕ provides a consistency condition.

It is easy to verify that $\lambda_n^0 = 1$ is a solution of (3.2) and satisfies (3.4) (or, equivalently, (3.25)). Indeed, we can easily check that

$$\begin{aligned}\mu(1) &= \sup_{n\geq 0}\left[\frac{1}{2(n+2)} + \frac{2}{4(n^2+3)} + \frac{1}{2(n+2)} + \frac{2}{4(n^2+3)}\right]\\ &= \sup_{n\geq 0}\left[\frac{1}{n+2} + \frac{1}{n^2+3}\right]\\ &= \frac{5}{6} < 1\end{aligned}$$

So, the trivial solution of (3.26) is stable by Remark 19.

We choose the starting values $x(-2) = x(-1) = x(0) = x(1) = 1$. In Figure **??**, we observe that the trivial solution is indeed asymptotically stable.

Example 21. Consider

$$\triangle x(n) = -\frac{1}{9(n+1)}x(n-1) - \frac{2n+1}{4(n+1)}x(n+1), \qquad n \geq 0 \tag{3.27}$$

$$x(n) = \phi(n) \quad \text{for} \quad n = -1, 0, 1,$$

where $\phi(n) \in \mathbf{R}$ such that ϕ provides a consistency condition.

From the characteristic equation (3.2), it follows that

$$\lambda_n - 1 = -\frac{1}{9(n+1)}\left(\lambda_{n-1}\right)^{-1} - \frac{2n+1}{4(n+1)}\lambda_n. \tag{3.28}$$

A solution of the equation (3.28) is $\lambda_n^0 = \frac{2}{3}$. Then, for $\lambda_0 = \frac{2}{3}$, from (3.4) we get

$$\mu(\frac{2}{3}) = \sup_{n\geq 0}\left\{\frac{3}{2}\left[\frac{1}{9(n+1)}\,\frac{3}{2} + \frac{2n+1}{4(n+1)}\,\frac{2}{3}\right]\right\}$$

$$= \sup_{n\geq 0}\left[\frac{1}{4(n+1)} + \frac{2n+1}{4(n+1)}\right] = \frac{1}{2} < 1.$$

Thus, since

$$\limsup_{n\to\infty} \Psi_{\lambda_0}(n) = \limsup_{n\to\infty}\left(\frac{2}{3}\right)^n = 0,$$

the trivial solution of (3.27) is asymptotically stable by Theorem 18.

We choose the starting values $x(-1) = (2/3)^{-1}, x(0) = 1$. In Figure **??**, we observe that the trivial solution is indeed asymptotically stable.

4. CONCLUSIONS

In this article, not only generalise the study introduced in [6] and [7] as well we have proved that there is a fundamental asymptotic criterion for the solutions of the initial value problem (1.5)-(3.1). The proof of theorem 17 is obtained according to $\triangle y$. Also, we obtained a useful boundary for the solutions of the initial value problem (1.5)-(3.1) and the stability of the trivial solution of (1.5) was shown. These results were obtained using an appropriate solution for (3.2). We also gave two examples for stability.

The statement $L(\lambda_0;\phi)$ in Theorem 17, has been given explicitly in two special cases: For linear autonomous mixed type difference equations (see [22]) and for linear mixed type difference equations with periodic coefficient shaving a common period and constant delays and advances that are multiples of this period. The current results will be the subject of a future study for linear mixed type difference equations with periodic coefficients.

For the generalized characteristic equation (3.2), it is an obvious problem to find the sufficient coefficients of the equation (1.5) to have a solution λ_0 such that $\mu(\lambda_0)$ holds. It is also a more obvious problem to extend the results of the present article to the more general state of mixed-type difference equations with variable coefficients and variable delays and advances.

Finally, we notice that it is an open question if the results of this paper can be generalized for equation (1.1).

References

[1] R. P. Agarwal and P.J.Y. Wong; *Advanced Topics in Difference equations*, Springer-Science+Business Media, B.V. 1997. 1

[2] R. P. Agarwal and S. R. Grace, The oscillation of certain difference equations, Mathematical and Computer Modelling 30 (1999), no. 1-2, 53-66. 1

[3] R. P. Agarwal, S. R. Grace, and D. O'Regan, Oscillation Theory for Difference and Functional Differential Equations, Kluwer Academic, Dordrecht, 2000. 1

[4] R. P. Agarwal; *Difference equations and Inequalities, Theory, Methods and Applications*, Second Edition, Marcel Dekker, New York, 2000. 1

[5] L. Berezansky and S. Pinelas, Oscillation Properties for a Scalar Linear Difference Equation of Mixed Type, *Mathematica Bohemica*, (2016); 141(2): 169-182. 1

[6] G. E. Chatzarakis, S. Pinelas and I. P. Stavroulakis, Oscillations of Difference Equations with Several Deviated Arguments, Aequationes Mathematicea (2013) 88:105. 4

[7] G. E. Chatzarakis, S. Pinelas and I. P. Stavroulakis, Correction: Oscillations of Difference Equations with Several Deviated Arguments, Aequationes Mathematicea (2018) 92:1195. 4

[8] R.D. Driver, G. Ladas and P.N. Vlahos, Asymptotic Behavior of a Linear Delay Difference Equation, *Proceedings of the American Mathematical Society*, Vol. 115, No. 1, (1992), 105-112. 1

[9] S. Elaydi; *An Introduction to Difference Equations*, Third Edition, Springer, New York, 2005. 1

[10] S. Elaydi, H. Oliveira, J.M. Ferreira and J.F. Alves; *Discrete Dynamics and Difference Equations*, Proceedings of the Twelfth International Conference on Difference Equations and Applications, World Scientific Publishing, Lisbon, 2007. 1

[11] J.M. Ferreira and S. Pinelas, Oscillatory mixed difference systems. *Advances in Difference Equations*, (2006) 1-18. 1

[12] J.M. Ferreira and S. Pinelas, Oscillatory mixed differential systems, *Funkcialaj Ekvacioj*, 53 (2010) 1-120. 1

[13] J.M. Ferreira, N.J. Ford, Md.A. Malique, S. Pinelas and Y. Yan, Analytical and numerical treatment of oscillatory mixed differential equations with differentiable delays and advances, Journal of Computational and Applied Mathematics, 235 (2011) 5112-5130. 1

[14] I. Gyï¿œri and G. Ladas, Oscillation Theory of Delay Differential Equations, Oxford Mathematical Monographs, Oxford University Press, New York, 1991. 1, 3

[15] W. G . Kelly, A. C. Peterson; *Difference Equations, An Introduction with Applications*, Academic Press, New York 1991. 1

[16] V. Lakshmikantham and D. Trigiante, *Theory of Difference Equations: Numerical Methods and Applications*, Academic Press Inc., New York, 1988. 1

[17] M. Pituk, The limits of the solutions of a nonautonomous linear delay difference equation, Comput. Math. Appl. 42 (2001), 543-550. 1, 3

[18] Ch. G. Philos and I. K. Purnaras, An asymptotic result and a stability criterion for linear nonautonomous delay difference equations, Arch. Math. (Basel) 83 (2004), 243-255. 1, 3, 3, 3, 3

[19] Ch. G. Philos and I. K. Purnaras, Asymptotic behavior and stability to linear nonautonomous neutral delay difference equations, J. Differ. Equations Appl. 11 (2005), 503-513. 1, 3

[20] S. Pinelas, Oscillations and nonoscillations in mixed differential equations with monotonic delays and advances, Advances in Dynamical Systems and Applications, Volume 4, Number 1, pp. 107-121 (2009). 1

[21] S. Pinelas, Asymptotic Behavior of a Scalar Linear Difference Equation of Mixed Equations, *UPI Journal of Mathematics and Biostatistics*, (2018); 1(1): 13-21. 1

[22] A.F. Yeniï¿œerioglu, S. Pinelas, and Y. Yan, On the behavior of the solutions for linear autonomous mixed type difference equation, *Rendiconti del Circolo Matematico di Palermo Series 2*, (2019); In press. 1, 4