SPATIAL DISCRETIZATION FOR STOCHASTIC SEMILINEAR SUBDIFFUSION DRIVEN BY INTEGRATED MULTIPLICATIVE SPACE-TIME WHITE NOISE

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Abstract. Spatial discretization of the stochastic semilinear subdiffusion driven by integrated multiplicative space-time white noise is considered. The spatial discretization scheme discussed in Gyöngy [16] and Anton et al. [5] for stochastic quasi-linear parabolic partial differential equations driven by multiplicative space-time noise is extended to the stochastic subdiffusion. The nonlinear terms f and σ satisfy the global Lipschitz conditions and the linear growth conditions. The space derivative and the integrated multiplicative space-time white noise are discretized by using finite difference methods. Based on the approximations of the Green functions which are expressed with the Mittag-Leffler functions, the optimal spatial convergence rates of the proposed numerical method are proved uniformly in space under the suitable smoothness assumptions of the initial values.

Key words. Semilinear, space-time white noise, Caputo fractional derivative, fractionally integrated additive noise, error estimates.

AMS subject classifications. 65M12; 65M06; Secondary 65M70; 35S10

1. Introduction. In this paper, we will consider the spatial discretization of the following stochastic semilinear subdiffusion driven by integrated multiplicative space-time white noise, with $0<\alpha\leq 1$, $0\leq\gamma\leq 1$, [16], [30],

$$\begin{cases} {}_0^C D_t^{\alpha} u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = f(u(t,x)) + {}_0D_t^{-\gamma}\sigma(u(t,x))\frac{\partial^2 W(t,x)}{\partial t\partial x}, \quad 0<x<1,\ t>0, \\ u(t,0)=u(t,1)=0, \quad t\geq 0, \\ u(0,x)=u_0(x), \quad 0\leq x\leq 1, \end{cases} \tag{1.1}$$

where ${}_0^C D_t^{\alpha} v$ and ${}_0D_t^{-\gamma} v$ denote the Caputo fractional derivative and the Riemann-Liouville fractional integral of the function v, respectively [18], [27]. Here u_0 is the initial value which satisfies $u_0\in C[0,1]$ and $u_0(0)=u_0(1)=0$, where $C[0,1]$ denotes the continuous function space.

The main aim of this paper is to extend the spatial discretization scheme discussed in Gyöngy [16] and Anton et al. [5] for stochastic quasi-linear parabolic partial differential equations driven by multiplicative space-time white noise to the stochastic subdiffusion driven by integrated multiplicative space-time white noise. We obtain the error estimates uniformly in space for the proposed finite difference method. The error estimates are based on the bounds of the Green functions and the corresponding discrete Green functions as well as the error bounds between them under some suitable norms. Such Green functions are expressed in terms of the Mittag-Leffler functions involving the parameters $0<\alpha\leq 1$ and $0\leq\gamma\leq 1$. The exponential function $E(t)=e^{-t}, t>0$ in the mild solution of the stochastic parabolic equation discussed in [16], [5] has the exponential decay as $t\to\infty$. However the Mittag-Leffler function $E_{\alpha,\beta}(z), 0<\alpha\leq 1, \beta\in\mathbb{R}$ in the mild solution of the stochastic subdiffusion has no exponential decay property and instead satisfies the following asymptotic properties: [27, Theorem 1.6] [18, eq. (1.8.28)], with $\frac{\pi\alpha}{2}<\mu<\min(\pi,\alpha\pi)$,

$$\left|E_{\alpha,\beta}(z)\right|\leq C(1+|z|)^{-1}, \quad \mu\leq|\arg(z)|\leq\pi, \tag{1.2}$$

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and

$$|E_{\alpha,\alpha}(z)| \leq C(1+|z|)^{-2}, \quad \mu \leq |\arg(z)| \leq \pi, \tag{1.3}$$

which make the error estimates of the stochastic subdiffusion problem much more complicated than the stochastic parabolic equation. To the best of our knowledge, there are no error estimates uniformly in space for the stochastic subdiffusion driven by space-time white noise. In this study, we aim at filling this gap by providing the detailed error estimates based on the error bounds developed in this paper for the Green functions which are expressed by the Mittag-Leffler functions.

Let $(\Omega, \mathscr{F}, (\mathscr{F})_{t\geq 0}, \mathbf{P})$ be a stochastic basis carrying an $\mathscr{F}_t$-adapted Brownian sheet $W = \{W(t,x) : t \geq 0, x \in \mathbb{R}_+\}$. We recall that W is a zero mean Gaussian random field with covariance [16], [30],

$$\mathbb{E}(W(t,x)W(s,y)) = (t \wedge s)(x \wedge y),$$

where $\mathbb{E}$ denotes the expectation.

A random field $v = \{v(t,x) : (t,x) \in D\}$ is called $\mathscr{F}_t$-adapted if $v(t,x)$ is $\mathscr{F}_t$-measurable for every $(t,x) \in D$, where D is a subset of $[0,\infty) \times [0,1]$.

We assume that the nonlinear terms f and σ satisfy the following globally Lipschitz and the linear growth conditions [16], [5].

$$
\begin{aligned}
&(L) \qquad |f(r) - f(v)| + |\sigma(r) - \sigma(v)| \leq C|r - v|, \quad \text{for all } r,\ v \in \mathbb{R}, \\
&(LG) \qquad |f(r)| + |\sigma(r)| \leq C(1+|r|), \quad \text{for } r \in \mathbb{R}.
\end{aligned}
$$

Further we assume that α and γ satisfy the following condition [11], [17].

ASSUMPTION 1.1.

$$0 \leq \alpha \leq 1, \quad 0 \leq \gamma \leq 1, \quad \alpha + \gamma > \frac{1}{2}.$$

Under (L), (LG), and the Assumption 1.1, one may show that the model (1.1) has a unique solution [30], [11].

The model (1.1) is used to describe the random effects on transport of particles in medium with memory or particles subject to sticking and trapping [11]. The fractional integrated noise reflects the fact that the internal energy depends also on the past random effects. In recent years, the model (1.1) has been very actively researched [4], [9], [10], [11], [24]. Chen et al. [11] studied the L^2 theory of (1.1) in both divergence and non-divergence forms. Anh et al. [4] discussed sufficient conditions for a Gaussian solution (in the mean-square sense) and derived temporal, spatial and spatial-temporal Hölder continuity of the solution. Chen [9] analyzed moments, Hölder continuity and intermittency of the solution for 1D nonlinear stochastic subdiffusion. Liu et al. [24] analyzed the existence and uniqueness of the solution (1.1) with fairly general quasi-linear elliptic operators.

Let us review some numerical methods for solving (1.1). Jin et al. [17] considered a fully discrete scheme for approximating (1.1) with $f = 0$ and $\sigma(u) = 1$ and the space-time noise is the Hilbert space-valued Wiener process with covariance operator Q and the error estimates in the $L^p, p > 1$ norm in space is obtained. Wu et al. [31] introduced the L1 scheme to approximate (1.1) with $f = 0$ and $\sigma(u) = 1$ and the space-time noise is defined as in Jin et al. [17]. To the best of our knowledge, we did not find any numerical analysis for solving (1.1) in the multiplicative (i.e., $\sigma(u) \neq 1$) space-time white noise case in literature. In this paper, we will approximate the derivative $\frac{\partial^2 u(t,x)}{\partial x^2}$ and the space-time white noise $\frac{\partial^2 W(t,x)}{\partial t \partial x}$ with the finite difference methods as in Gyöngy [16] and Anton et al. [5] and obtain a spatial discretization scheme for approximating (1.1). The convergence rate in the mean-square sense is obtained, uniformly in $x \in [0,1]$.

There are many works for the numerical methods for solving the stochastic parabolic equations driven by additive or multiplicative noises, see, e.g, [1], [2], [3], [6], [7], [8], [12], [13], [14], [19], [20], [21], [22], [23], [25], [26], [28], [29], [32],[33] and the references therein. Most of these references are concerned with an interpretation of stochastic partial differential equations in Hilbert spaces and thus error estimates are provided in the $L^2([0,1])$ norm (or similar norms).

Let $0 = x_0 < x_1 < \cdots < x_{M-1} < x_M = 1$ be the partition of $[0,1]$ and $\Delta x = 1/M$ the space step size. At $x = x_k, k = 1,2,\dots,M-1$, we approximate the derivative $\frac{\partial^2 u(t,x)}{\partial x^2}$ and the space-time white noise $\frac{\partial^2 W(t,x)}{\partial t \partial x}$ by

$$\frac{\partial^2 u(t,x)}{\partial x^2}\Big|_{x=x_k} \approx \frac{u(t,x_{k+1}) - 2u(t,x_k) + u(t,x_{k-1})}{\Delta x^2},$$

and

$$\frac{\partial^2 W(t,x)}{\partial t \partial x}\Big|_{x=x_k} \approx \frac{d}{dt}\frac{W(t,x_{k+1}) - W(t,x_k)}{\Delta x}.$$

Denote $u^M(t,x_k) \approx u(t,x_k), k = 0,1,2,\dots,M$ the approximate solution of $u(t,x_k)$. We define the following finite difference method for solving (1.1).

$$\tag{1.4} \begin{cases} {}_0^C D_t^{\alpha} u^M(t,x_k) - \frac{u^M(t,x_{k+1}) - 2u^M(t,x_k) + u^M(t,x_{k-1})}{\Delta x^2} \\ \qquad = f(u^M(t,x_k)) + {}_0D_t^{-\gamma}\Big[\sigma(u^M(t,x_k))\frac{d}{dt}\Big(\frac{W(t,x_{k+1}) - W(t,x_k)}{\Delta x}\Big)\Big], \quad k = 1,2,\dots,M-1,\ t > 0, \\ u^M(t,0) = u^M(t,1) = 0, \quad t \geq 0, \\ u^M(0,x_k) = u_0(x_k), \quad k = 0,1,2,\dots,M. \end{cases}$$

When the initial value $u_0 \in C^1[0,1]$, $u_0(0) = u_0(1) = 0$, we obtain the following error estimates.

THEOREM 1.1. *Assume* $(L),(LG)$ *and Assumption 1.1 hold. Let* $u(t,x)$ *and* $u^M(t,x_k), k = 0,1,2,\dots,M$ *be the solutions of* (1.1) *and* (1.4)*, respectively. Further assume that* $u_0 \in C^1[0,1]$, $u_0(0) = u_0(1) = 0$.

(i). If $f = 0$*, then we have*

$$\tag{1.5} \begin{aligned} \mathbb{E}|u^M(t,x) - u(t,x)|^2 \leq & Ct^{-1+\varepsilon}\Delta x^{r_1} + C\Delta x^{r_3} \\ & + C\begin{cases} C\Delta x^{r_1}, & \text{if } 2(\alpha+\gamma-1) - \frac{\alpha}{2} + \varepsilon \geq 0, \\ C\Delta x^{2(\alpha+\gamma-1) - \frac{\alpha}{2} + \min\{1, r_1+\varepsilon\}}, & \text{if } 2(\alpha+\gamma-1) - \frac{\alpha}{2} + \varepsilon < 0. \end{cases} \end{aligned}$$

(ii). If $f \neq 0$*, then we have*

$$\tag{1.6} \begin{aligned} \mathbb{E}|u^M(t,x) - u(t,x)|^2 \leq & Ct^{-1+\varepsilon}\Delta x^{r_1} + C(\Delta x^{r_2} + \Delta x^{r_3}) \\ & + C\begin{cases} C\Delta x^{r_1}, & \text{if } 2(\alpha+\gamma-1) - \frac{\alpha}{2} + \varepsilon \geq 0, \\ C\Delta x^{2(\alpha+\gamma-1) - \frac{\alpha}{2} + \min\{1, r_1+\varepsilon\}}, & \text{if } 2(\alpha+\gamma-1) - \frac{\alpha}{2} + \varepsilon < 0. \end{cases} \end{aligned}$$

where, with small $\varepsilon > 0$*,*

$$\tag{1.7} r_1 = \begin{cases} 2, & \text{if } 0 \leq \alpha \leq \frac{2(1-\varepsilon)}{3}, \\ 4\big(\frac{1-\varepsilon}{2\alpha}\big) - 1, & \text{if } \frac{2(1-\varepsilon)}{3} \leq \alpha \leq 1, \end{cases}$$

and

$$r_2 = 3 - \frac{2}{\alpha}, \quad \text{if } 3 - 2/\alpha > 0, \tag{1.8}$$

and

$$r_3 = \begin{cases} 2, & \text{if } 2\gamma - 1 \geq 0, \\ 2, & \text{if } 2\gamma - 1 < 0,\ 0 \leq \frac{2(1-2\gamma)}{\alpha} \leq 1, \\ 3 - \frac{2(1-2\gamma)}{\alpha}, & \text{if } 2\gamma - 1 < 0,\ 1 \leq \frac{2(1-2\gamma)}{\alpha} \leq 3. \end{cases} \tag{1.9}$$

REMARK 1.2. *When $\alpha = 1$ and $\gamma = 0$, we obtain, from Theorem 1.1 for the initial data $u_0 \in C^1[0,1]$, $u_0(0) = u_0(1) = 0$,*

$$\sup_{t,k} \mathbb{E}|u^M(t,x_k) - u(t,x_k)|^2 \leq C\Delta x^{2(\alpha+\gamma-1)+(r_1+\varepsilon)} = C\Delta x^{\frac{1}{2}-2\varepsilon},$$

which is consistent with the spatial convergence rate obtained in [16, Theorem 3.1] for the stochastic parabolic equation driven by space-time white noise.

REMARK 1.3. *We may consider the error estimates with respect to the norm $\sup_{t,k} \mathbb{E}|u^M(t,x_k) - u(t,x_k)|^{2p}$ for any $p \geq 1$ as in [16, Theorem 3.1]. For simplicity of the notations of the proof, we only consider the case with $p = 1$ in Theorem 1.1.*

When the initial value u_0 is sufficiently smooth, that is, $u_0 \in C^3[0,1]$, $u_0(0) = u_0(1) = 0$, we may get higher convergence rates for some $2/3 \leq \alpha \leq 1$.

THEOREM 1.4. *Assume $(L), (LG)$ and Assumption 1.1 hold. Let $u(t,x)$ and $u^M(t,x_k), k = 0,1,2,\dots,M$ be the solutions of (1.1) and (1.4), respectively. Further assume that $u_0 \in C^3[0,1]$, $u_0(0) = u_0(1) = 0$.*

(i). If $f = 0$, then we have, for $\frac{2(1-2\gamma)}{\alpha} < 3$,

$$\sup_{t,k} \mathbb{E}|u^M(t,x_k) - u(t,x_k)|^2 \leq C\Delta x^{\min(r_2,r_3,2)},$$

where r_2 and r_3 are defined by (1.8) and (1.9), respectively.

(ii). If $f \neq 0$, then we have, for $\frac{2}{\alpha} < 3$, that is, $2/3 < \alpha \leq 1$,

$$\sup_{t,k} \mathbb{E}|u^M(t,x_k) - u(t,x_k)|^2 \leq C\Delta x^{\min(r_2,r_3,2)},$$

where r_3 is defined by (1.9).

REMARK 1.5. *Note that, by (1.8), $r_2 = 3 - 2/\alpha$, therefore the condition $2/3 < \alpha \leq 1$ is also necessary in case (i) in Theorem 1.4. In other words, we may only get the higher convergence rates for $2/3 < \alpha \leq 1$ when the initial value is sufficiently smooth and the convergence rates are $O(\Delta x^{\min(r_2,r_3,2)})$ in both cases for $f = 0$ and $f \neq 0$.*

REMARK 1.6. *When $\alpha = 1$ and $\gamma = 0$, we obtain, from Theorem 1.4 for the sufficiently smooth initial data,*

$$\sup_{t,k} \mathbb{E}|u^M(t,x_k) - u(t,x_k)|^2 \leq C\Delta x,$$

which is consistent with the spatial convergence rate obtained in [16, Theorem 3.1] for the stochastic parabolic equation driven by space-time white noise.

REMARK 1.7. *We may consider the error estimates with respect to the norm* $\sup_{t,k} \mathbb{E}|u^M(t,x_k) - u(t,x_k)|^{2p}$ *for any* $p \geq 1$ *as in [16, Theorem 3.1]. For simplicity of the notations of the proof, we only consider the case with* $p = 1$ *in Theorem 1.4.*

The paper is organized as follows. In Section 2, we consider the continuous problem, i.e., (1.1) and we obtain the mild solution of the problem and the spatial regularity of the solution of the model. Section 3 is devoted to the spatial discretization of the model (1.1) and the regularity of the solution of the spatial discretization problem is obtained. In Section 4, we consider the error estimates both in smooth and nonsmooth data cases. Finally, in Appendix, we give the error estimates of the Green functions expressed by using the different Mittag-Leffler functions.

Throughout this paper, we denote by C a generic constant depending on $u, u_0, t, \alpha, \gamma$, but independent of the space step size Δx, which could be different at different occurrences. Further, $\varepsilon > 0$ is always a small positive constant.

2. Continuous problem. In this section, we shall consider the mild solution of (1.1) and study its spatial regularity.

Let $\{\lambda_j, \varphi_j\}_{j=1}^{\infty}$ be the eigenpairs of the Laplacian operator $A = -\frac{d^2}{dx^2}$ with $\mathscr{D}(A) = H_0^1(0,1) \cap H^2(0,1)$, that is,

$$\lambda_j = j^2\pi^2, \ \varphi_j(x) = \sqrt{2}\sin j\pi x, \ j = 1, 2, \ldots. \tag{2.1}$$

It is well known that $\{\varphi_j(x)\}_{j=1}^{\infty}$ forms an orthonormal basis in $H = L^2(0,1)$.

Let $E_{\alpha,\beta}(z), 0 < \alpha \leq 1, \beta \in \mathbb{R}$ denote the Mittag-Leffler function defined by [27],

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad 0 < \alpha \leq 1, \ \beta \in \mathbb{R}. \tag{2.2}$$

We have the following differentiation formulas of Mittag-Leffler functions which we shall use frequently in the error estimates of the Green functions.

LEMMA 2.1. *[27] Let* $0 < \alpha \leq 1$, $0 \leq \gamma \leq 1$. *We have*

$$\frac{d}{dt}E_{\alpha,1}(-t^{\alpha}\lambda) = \lambda t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}\lambda), \quad \lambda > 0, \tag{2.3}$$

$$\frac{d}{dt}\Big[t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-t^{\alpha}\lambda)\Big] = t^{\alpha+\gamma-2}E_{\alpha,\alpha+\gamma-1}(-t^{\alpha}\lambda), \quad \lambda > 0, \ \alpha+\gamma \neq 1. \tag{2.4}$$

2.1. The mild solution of (1.1)**.** In this subsection, we shall give the mild solution of (1.1).

LEMMA 2.2. *Assume* $(L), (LG)$ *and Assumption 1.1 hold. Let* $u(t,x)$ *be the solution of* (1.1)*. Further assume that* $u_0 \in C[0,1]$*. Then* (1.1) *has the following unique mild solution*

$$\begin{aligned} u(t,x) = &\int_0^1 G_1(t,x,y)u_0(y)\,\mathrm{d}y + \int_0^t\int_0^1 G_2(t-s,x,y)f(u(s,y))\,\mathrm{d}y\mathrm{d}s \\ &+ \int_0^t\int_0^1 G_3(t-s,x,y)\sigma(u(s,y))\,\mathrm{d}W(s,y), \end{aligned} \tag{2.5}$$

where

$$G_1(t,x,y) = \sum_{j=1}^{\infty} E_{\alpha,1}(-t^{\alpha}\lambda_j)\varphi_j(x)\varphi_j(y),$$
$$G_2(t,x,y) = \sum_{j=1}^{\infty} t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}\lambda_j)\varphi_j(x)\varphi_j(y),$$
$$G_3(t,x,y) = \sum_{j=1}^{\infty} t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-t^{\alpha}\lambda_j)\varphi_j(x)\varphi_j(y).$$

Here $E_{\alpha,\beta}(z)$ *denotes the Mittag-Leffler function defined in* (2.2) *and* $\{\lambda_j,\varphi_j\}_{j=1}^{\infty}$ *are eigenpairs defined in* (2.1)*. The integral*

$$\int_0^t\int_0^1 G_3(t-s,x,y)\sigma(u(s,y))\mathrm{d}W(s,y) = \int_0^t\int_0^1 G_3(t-s,x,y)\sigma(u(s,y))\frac{\partial^2 W(s,y)}{\partial s\partial y}\,\mathrm{d}y\mathrm{d}s$$

is understood in Itô's sense [16, page 3].

Proof. One may prove this lemma by the method of separation of variables. Assume that the solution $u(t,x)$ has the form

$$u(t,x) = \sum_{k=1}^{\infty} u_k(t)\varphi_k(x).$$

Substituting this form into (1.1), one may easily obtain the mild solution (2.5). We omit the details here.
□

2.2. The spatial regularity of the mild solution of (1.1)**.** In this subsection, we shall consider the spatial regularity of the mild solution (1.1). The mild solution $u(t,x)$ of (1.1) can be written into

$$u(t,x) = v(t,x) + w(t,x),$$

where $v(t,x)$ is the solution of the following homogeneous problem,

$$\begin{cases} {}_0^C D_t^{\alpha} v(t,x) - \frac{\partial^2 v(t,x)}{\partial x^2} = 0, \quad 0<x<1,\ t>0,\\ v(t,0) = v(t,1) = 0, \quad t\ge 0,\\ v(0,x) = u_0(x), \quad 0\le x\le 1, \end{cases} \tag{2.6}$$

and $w(t,x)$ is the solution of the following inhomogeneous problem,

$$\begin{cases} {}_0^C D_t^{\alpha} w(t,x) - \frac{\partial^2 w(t,x)}{\partial x^2} = f(u(t,x)) + {}_0D_t^{-\gamma}\sigma(u(t,x))\frac{\partial^2 W(t,x)}{\partial t\partial x}, \quad 0<x<1,\ t>0,\\ w(t,0) = w(t,1) = 0, \quad t\ge 0,\\ w(0,x) = u_0(x), \quad 0\le x\le 1. \end{cases} \tag{2.7}$$

Let $0 = y_0 < y_1 < \cdots < y_{M-1} < y_M = 1$ be a partition of $[0,1]$ and $\Delta x = 1/M$ be the step size. We define the piecewise constant function $k_M(y), 0\le y\le 1$ by

$$k_M(y) = \begin{cases} y_j, & y_j \le y < y_{j+1}, \quad j = 0,1,\ldots,M-1,\\ y_M, & y = y_M. \end{cases} \tag{2.8}$$

2.3. The case for the initial data $u_0 \in C[0,1], u_0(0) = u_0(1) = 0$. In this subsection, we shall consider the spatial regularity of the mild solution of (1.1) when the initial data $u_0 \in C[0,1], u_0(0) = u_0(1) = 0$.

LEMMA 2.3. *Let $v(t,x)$ be the solution of* (2.6). *Let $u_0 \in C[0,1]$, $u_0(0) = u_0(1) = 0$. Then we have*

$$\mathbb{E}|v(t,y) - v(t,K_M(y)|^2 \leq Ct^{-1+\varepsilon}\Delta x^{r_1},$$

where r_1 is denoted by (1.7).

Proof. Note that, by Cauchy-Schwarz inequality,

$$\begin{aligned}|v(t,y) - v(t,k_M(y)|^2 &= \Big|\int_0^1 \Big[G_1(t,y,z) - G_1(t,k_M(y),z)\Big]u_0(z)\,dz\Big|^2\\ &\leq \Big[\int_0^1 |G_1(t,y,z) - G(t,k_M(y),z)|^2\,\mathrm{d}z\Big]\Big[\int_0^1 |u_0(z)|^2\,\mathrm{d}z\Big].\end{aligned}\tag{2.9}$$

By Lemma 5.1, we get

$$|v(t,y) - v(t,k_M(y))|^2 \leq Ct^{-1+\varepsilon}\Delta x^{r_1},$$

where r_1 is defined by (1.7), which completes the proof of Lemma 2.3. □

We next consider the spatial regularity for the mild solution of the inhomogeneous problem (2.7). Recall that the solution of (2.7) has the following form

$$w(t,x) = \int_0^t\int_0^1 G_2(t-s,x,z)f(u(s,z))\,\mathrm{d}z\mathrm{d}s + \int_0^t\int_0^1 G_3(t-s,x,z)\sigma(u(s,z))\,\mathrm{d}W(s,z).$$

LEMMA 2.4. *Assume $(L),(LG)$ and Assumption 1.1 hold. Let $w(t,x)$ be the solution of the inhomogeneous problem* (2.7). *Then we have*

$$\mathbb{E}|w(t,y) - w(t,K_M(y)|^2 \leq C\big(\Delta x^{r_2} + \Delta x^{r_3}\big),$$

where r_2 and r_3 are defined by (1.8), (1.9)*, respectively.*

Proof. Denote $h(s,z) = f(u(s,z))$ or $\sigma(u(s,z))$. One may easily prove (we omit the proof here due to the length of the paper) that, under the assumptions (L) and (LG), $h(t,x), t \geq 0, 0 \leq x \leq 1$ satisfy

$$\sup_{t,x}\mathbb{E}|h(t,x)|^2 \leq C.$$

Denote

$$\begin{aligned}F(t,x) &= \int_0^t\int_0^1 G_2(t-s,x,z)h(s,z)\,\mathrm{d}z\mathrm{d}s,\\ H(t,x) &= \int_0^t\int_0^1 G_3(t-s,x,z)h(s,z)\,\mathrm{d}W(s,z).\end{aligned}$$

we will show that

$$\mathbb{E}|F(t,y) - F(t,k_M(y))|^2 \leq C\Delta x^{r_2},\tag{2.10}$$

$$\mathbb{E}|H(t,y) - H(t,k_M(y))|^2 \leq C\Delta x^{r_3},\tag{2.11}$$

where r_2 and r_3 are defined by (1.8) and (1.9), respectively.

We only prove (2.11) here since the proof of (2.10) is similar. By Burkholder's inequality [16] and the assumption for h, we have

$$\begin{aligned}\mathbb{E}|H(t,y)-H(t,k_M(y))|^2 &= \mathbb{E}\Big|\int_0^t\int_0^1\Big[G_3(t-s,y,z)h(s,z)-G_3(t-s,k_M(y),z)\Big]h(s,z)\,\mathrm{d}W(s,z)\Big|^2\\ &\le C\mathbb{E}\int_0^t\int_0^1|G_3(t-s,y,z)-G_3(t-s,k_M(y),z)|^2|h(s,z)|^2\,\mathrm{d}z\mathrm{d}s\\ &\le C\int_0^t\int_0^1|G_3(t-s,y,z)-G_3(t-s,k_M(y),z)|^2\,\mathrm{d}z\mathrm{d}s,\end{aligned}$$

which implies, by Lemma 5.4,

$$\mathbb{E}|H(t,y)-H(t,k_M(y)|^2 \le C\Delta x^{r_3},$$

where r_3 is defined by (1.9).

The proof of Lemma 2.4 is complete.

□

2.3.1. The case for sufficiently smooth initial data $u_0 \in C^2[0,1]$, $u_0(0)=u_0(1)=0$. In this subsection, we shall consider the spatial regularity of the mild solution of (1.1) with respect to the sufficiently smooth initial data $u_0 \in C^2[0,1]$, $u_0(0)=u_0(1)=0$.

LEMMA 2.5. *Let $v(t,x)$ be the solution of the homogeneous problem* (2.6). *Let $u_0 \in C^2[0,1]$, $u_0(0)=u_0(1)=0$. Then we have*

$$\mathbb{E}|v(t,y)-v(t,K_M(y)|^2 \le C\Delta x^{r_2},$$

where r_2 is defined by (1.8).

Proof. We have

$$v(t,y)=\int_0^1 G_1(t,y,z)u_0(z)\,\mathrm{d}z=\int_0^1\sum_{j=1}^{\infty}E_{\alpha,1}(-t^{\alpha}\lambda_j)\varphi_j(y)\varphi_j(z)u_0(z)\,\mathrm{d}z.$$

By Lemma 2.1, we have

$$\begin{aligned}v(t,y) &= \int_0^1\sum_{j=1}^{\infty}\Big[\int_0^t s^{\alpha-1}\lambda_j E_{\alpha,\alpha}(-s^{\alpha}\lambda_j)\,\mathrm{d}s+1\Big]\varphi_j(y)\varphi_j(z)u_0(z)\,\mathrm{d}z\\ &= \sum_{j=1}^{\infty}\int_0^1\varphi_j(y)\varphi_j(z)u_0(z)\,\mathrm{d}z+\int_0^t\int_0^1\sum_{j=1}^{\infty}s^{\alpha-1}E_{\alpha,\alpha}(-s^{\alpha}\lambda_j)\varphi_j(y)\big(\lambda_j\varphi_j(z)\big)u_0(z)\,\mathrm{d}s.\end{aligned}$$

Note that

$$\int_0^1\lambda_j\varphi_j(z)u_0(z)\,\mathrm{d}z=-\int_0^1\varphi_j''(z)u_0(z)\,\mathrm{d}z=-\int_0^1\varphi_j(z)u_0''(z)\,\mathrm{d}z.$$

We get

$$v(t,y)=u_0(y)-\int_0^t\int_0^1 G_2(s,y,z)u_0''(z)\,\mathrm{d}z\mathrm{d}s. \tag{2.12}$$

Thus, by using Cauchy-Schwarz inequality, with $y_k \le y \le y_{k+1}$, $k = 0, 1, \ldots, M-1$,

$$\begin{aligned}
|v(t,y) - v(t,k_M(y)|^2 &\le C|u_0(y) - u_0(k_M(y))|^2 \\
&\quad + C\Big| \int_0^t \int_0^1 \Big[G_2(s,y,z) - G_2(s,k_M(y),z) \Big] u_0''(z)\,\mathrm{d}z\mathrm{d}s \Big|^2 \\
&\le C|u_0(y) - u_0(k_M(y))|^2 \\
&\quad + C\Big[\int_0^t \int_0^1 \Big(G_2(s,y,z) - G_2(s,k_M(y),z) \Big)^2 \mathrm{d}z\mathrm{d}s \Big] \Big[\int_0^t \int_0^1 |u_0''(z)|^2\,\mathrm{d}z\mathrm{d}s \Big].
\end{aligned}$$

By Lemma 5.4 and using the error estimates of the liner interpolation function, we obtain, with $C = C(t)$,

$$|v(t,y) - v(t,k_M(y))|^2 \le C\Delta x^2 \|u_0\|^2_{C^1[0,1]} + C\Delta x^{r_2} \|u_0\|_{C^2[0,1]},$$

which completes the proof of Lemma 2.5. □

3. Spatial discretization. In this section, we shall consider the spatial discretization of (1.1).

3.1. The mild solution of the spatial discretization problem (1.4)**.** Let $\{\lambda_j^M, \vec{\varphi}_j^M\}_{j=1}^{M-1}$ be the eigenpairs of the following discrete Laplacian matrix $\vec{A}$ defined by

$$\vec{A} = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}_{(M-1)\times(M-1)}. \tag{3.1}$$

It is well known that [16]

$$\lambda_j^M = \frac{\sin^2\left(\frac{j\pi}{2M}\right)}{\left(\frac{1}{2M}\right)^2}, \quad \vec{\varphi}_j^M = \sqrt{\Delta x} \begin{pmatrix} \varphi_j(x_1) \\ \varphi_j(x_2) \\ \vdots \\ \varphi_j(x_{M-1}) \end{pmatrix}, \quad j = 1, 2, \ldots, M-1, \tag{3.2}$$

and $\vec{\varphi}_j^M$, $j = 1, 2, \ldots, M-1$ forms an orthonormal basis in $\mathbb{R}^{M-1}$.

LEMMA 3.1. *Assume* (L), (LG) *and Assumption 1.1 hold. Let* $u^M(t, x_k)$, $k = 0, 1, 2, \ldots, M$ *be the solution of spatial discretization problem* (1.4)*. Further assume that* $u_0 \in C[0,1]$*. Then* (1.4) *has the following unique mild solution*

$$\begin{aligned}
u^M(t,x) &= \int_0^1 G_1^M(t,x,y) u_0(k_M(y))\,\mathrm{d}y + \int_0^t \int_0^1 G_2^M(t-s,x,y) f(u^M(s,k_M(y)))\,\mathrm{d}y\mathrm{d}s \\
&\quad + \int_0^t \int_0^1 G_3^M(t-s,x,y) \sigma(u^M(s,k_M(y))) \frac{\partial^2 W_M(s,y)}{\partial s \partial y}\,\mathrm{d}y\mathrm{d}s,
\end{aligned} \tag{3.3}$$

where $u^M(t,x)$ *denotes the piecewise linear interpolation function of* $u^M(t,x_k), k=0,1,2,\dots,M$ *and*

$$G_1^M(t,x,y)=\sum_{j=1}^{M-1}E_{\alpha,1}(-t^\alpha\lambda_j^M)\varphi_j^M(x)\varphi_j(k_M(y)),$$
$$G_2^M(t,x,y)=\sum_{j=1}^{M-1}t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha\lambda_j^M)\varphi_j^M(x)\varphi_j(k_M(y)),$$
$$G_3^M(t,x,y)=\sum_{j=1}^{M-1}t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-t^\alpha\lambda_j^M)\varphi_j^M(x)\varphi_j(k_M(y)),$$

and

$$\frac{\partial^2 W_M(t,y)}{\partial t\partial y}=\frac{d}{dt}\Big(\frac{W(t,k_M(y)+\frac{1}{M})-W(t,k_M(y))}{\Delta x}\Big),\quad 0\le y\le 1. \tag{3.4}$$

Here $E_{\alpha,\beta}(z)$ *denote the Mittag-Leffler functions defined by* (2.2) *and* $\{\lambda_j^M,\vec{\varphi}_j^M\}_{j=1}^{M-1}$ *are the eigenpairs of the discrete Laplacian* $\vec{A}$ *defined in* (3.2)*. Here* $k_M(y), 0\le y\le 1$ *is defined by* (2.8) *and* $\varphi_j^M(x),\ j=1,2,\dots,M-1$ *are the piecewise linear interpolation functions of* $\varphi_j(x)$ *on the nodes* $x_j, j=0,1,\dots,M$.

Proof. We write (1.4) into the following matrix form

$$\begin{cases} {}_0^CD_t^\alpha\vec{u}^M(t)+\vec{A}\vec{u}^M(t)=\vec{F}_1^M(t)+{}_0D_t^{-\gamma}\vec{F}_2^M(t), \quad t>0,\\ \vec{u}^M(0)=\begin{pmatrix} u_0(x_1)\\ u_0(x_2)\\ \vdots\\ u^0(x_{M-1})\end{pmatrix}, \end{cases} \tag{3.5}$$

where

$$\vec{u}^M(t)=\begin{pmatrix} u^M(t,x_1)\\ u^M(t,x_2)\\ \vdots\\ u^M(t,x_{M-1})\end{pmatrix},\quad \vec{F}_1^M(t)=\begin{pmatrix} f(u^M(t,x_1))\\ f(u^M(t,x_2))\\ \vdots\\ f(u^M(t,x_{M-1}))\end{pmatrix},$$

and

$$\vec{F}_2^M(t)=\begin{pmatrix} \sigma(u^M(t,x_1))\frac{d}{dt}\Big(\frac{W(t,x_2)-W(t,x_1)}{\Delta x}\Big)\\ \sigma(u^M(t,x_2))\frac{d}{dt}\Big(\frac{W(t,x_3)-W(t,x_2)}{\Delta x}\Big)\\ \vdots\\ \sigma(u^M(t,x_{M-1}))\frac{d}{dt}\Big(\frac{W(t,x_M)-W(t,x_{M-1})}{\Delta x}\Big)\end{pmatrix}.$$

The solution of (3.5) can be written into the following integration form

$$\vec{u}^M(t)=E_{\alpha,1}(-t^\alpha\vec{A})\vec{u}^M(0)+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha\vec{A})\vec{F}_1^M(s)\,\mathrm{d}s$$
$$+\int_0^t(t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-(t-s)^\alpha\vec{A})\vec{F}_2^M(s)\,\mathrm{d}s.$$

Thus we have, noting that $\{\lambda_j^M, \vec{\varphi}_j^M\}_{j=1}^{M-1}$ is an orthonormal basis in $\mathbb{R}^{M-1}$,

$$
\begin{aligned}
\vec{u}^M(t) = & \sum_{j=1}^{M-1} (\vec{u}^M(0), \vec{\varphi}_j^M) E_{\alpha,1}(-\lambda_j^M t^\alpha) \vec{\varphi}_j^M \\
& + \sum_{j=1}^{M-1} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j^M (t-s)^\alpha) \big(\vec{F}_1^M(s), \vec{\varphi}_j^M\big) \vec{\varphi}_j^M \, \mathrm{d}s \\
& + \sum_{j=1}^{M-1} \int_0^t (t-s)^{\alpha+\gamma-1} E_{\alpha,\alpha+\gamma}(-\lambda_j^M (t-s)^\alpha) \big(\vec{F}_2^M(s), \vec{\varphi}_j^M\big) \vec{\varphi}_j^M \, \mathrm{d}s,
\end{aligned}
$$

which implies that, with $k = 1, 2, \dots, M-1$,

$$
\begin{aligned}
u^M(t, x_k) = & \sum_{j=1}^{M-1} \Big[\Delta x \sum_{l=1}^{M-1} u_0(x_l) \varphi_j(x_l)\Big] E_{\alpha,1}(-\lambda_j^M t^\alpha) \varphi_j(x_k) \\
& + \sum_{j=1}^{M-1} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j^M (t-s)^\alpha) \Big[\Delta x \sum_{l=1}^{M-1} f(u^M(s, x_l)) \varphi_j(x_l)\Big] \varphi_j(x_k) \, \mathrm{d}s \\
& + \sum_{j=1}^{M-1} \int_0^t (t-s)^{\alpha+\gamma-1} E_{\alpha,\alpha+\gamma}(-\lambda_j^M (t-s)^\alpha) \cdot \\
& \cdot \Big[\Delta x \sum_{l=1}^{M-1} \sigma(u^M(s, x_l)) \frac{d}{ds} \Big(\frac{W(s, x_{l+1}) - W(s, x_l)}{\Delta x} \Big) \varphi_j(x_l)\Big] \varphi_j(x_k) \, \mathrm{d}s. \qquad (3.6)
\end{aligned}
$$

Replacing $\varphi_j(x_k)$ by the piecewise linear interpolation function $\varphi_j^M(x)$ in (3.6), where

$$
\varphi_j^M(x) = \varphi_j(x_k) + \frac{\varphi_j(x_{k+1}) - \varphi_j(x_k)}{\Delta x} (x - x_k), \quad x_k \le x \le x_{k+1}, \, k = 0, 1, 2, \dots, M-1,
$$

we get the following piecewise linear interpolation function of $u^M(t, x_k), k = 0, 1, 2, \dots, M$,

$$
\begin{aligned}
u^M(t, x) = & \int_0^1 \sum_{j=1}^{M-1} E_{\alpha,1}(-\lambda_j^M t^\alpha) \varphi_j^M(x) \varphi_j(k_M(y)) u_0(k_M(y)) \, \mathrm{d}y \\
& + \int_0^t \int_0^1 \sum_{j=1}^{M-1} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j^M (t-s)^\alpha) \varphi_j^M(x) \varphi_j(k_M(y)) f(u^M(s, k_M(y)) \, \mathrm{d}y \mathrm{d}s \\
& + \int_0^t \int_0^1 \sum_{j=1}^{M-1} (t-s)^{\alpha+\gamma-1} E_{\alpha,\alpha+\gamma}(-\lambda_j^M (t-s)^\alpha) \varphi_j^M(x) \varphi_j(k_M(y)) \sigma(u^M(s, k_M(y)) \frac{\partial^2 W_M(s, y)}{\partial s \partial y} \, \mathrm{d}y \mathrm{d}s,
\end{aligned}
$$

which shows (3.3), where $k_M(y)$ and $\frac{\partial^2 W_M(s,y)}{\partial s \partial y}$ are defined by (2.8) and (3.4), respectively. The proof of Lemma 3.1 is now complete. □

3.2. Spatial regularity for the space discretization problem. In this subsection, we shall consider the spatial regularity of the mild solution of the spatial discretization problem (3.3).

The solution $u^M(t, x)$ in (3.3) can be written into

$$
u^M(t, x) = v^M(t, x) + w^M(t, x),
$$

where

$$v^M(t,x) = \int_0^1 G_1^M(t,x,y)u_0(k_M(y))\,\mathrm{d}y, \tag{3.7}$$

and

$$\begin{aligned} w^M(t,x) = &\int_0^t \int_0^1 G_2^M(t-s,x,y) f(u_M(s,k_M(y)))\,\mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_0^1 G_3^M(t-s,x,y)\sigma(u_M(s,k_M(y)))\frac{\partial^2 W_M(s,y)}{\partial s \partial y}\,\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{3.8}$$

3.2.1. The case for the initial data $u_0 \in C[0,1], u_0(0) = u_0(1) = 0$. In this subsection, we consider the spatial regularity of the mild solution of (3.7) with respect to the nonsmooth data $u_0 \in C[0,1], u_0(0) = u_0(1) = 0$.

LEMMA 3.2. *Let $v^M(t,x)$ be the solution of* (3.7). *Let $u_0 \in C[0,1]$, $u_0(0) = u_0(1) = 0$. Then we have*

$$\mathbb{E}|v^M(t,y) - v^M(t,K_M(y)|^2 \le Ct^{-1+\varepsilon}\Delta x^{r_1},$$

where r_1 is defined by (1.7).

Proof. Note that, by Cauchy-Schwarz inequality,

$$\begin{aligned} |v^M(t,y) - v^M(t,k_M(y)|^2 &= \Big| \int_0^1 \Big[G_1^M(t,y,z) - G_1^M(t,k_M(y),z)\Big] u_0(k_M(z))\,\mathrm{d}z \Big|^2 \\ &\le \Big[\int_0^1 |G_1^M(t,y,z) - G_1^M(t,k_M(y),z)|^2\,\mathrm{d}z\Big]\Big[\int_0^1 |u_0(k_M(z))|^2\,\mathrm{d}z\Big]. \end{aligned} \tag{3.9}$$

By Lemma 5.2, we get

$$|v^M(t,y) - v^M(t,k_M(y))|^2 \le Ct^{-1+\varepsilon}\Delta x^{r_1},$$

which completes the proof of Lemma 3.2. □

We next consider the spatial regularity for the mild solution of the inhomogeneous problem (3.8). Recall that the solution of (3.8) has the following form

$$w^M(t,x) = \int_0^t \int_0^1 G_2^M(t-s,x,z) f(u^M(s,k_M(z)))\,\mathrm{d}z\mathrm{d}s + \int_0^t \int_0^1 G_3^M(t-s,x,z)\sigma(u^M(s,k_M(z)))\,\mathrm{d}W(s,z)$$

Following the same lines of the proof of Lemma 2.4, we may prove the following

LEMMA 3.3. *Assume $(L), (LG)$ and Assumption 1.1 hold. Let $w^M(t,x)$ be the solution of the inhomogeneous problem* (3.8). *Then we have*

$$\mathbb{E}|w^M(t,y) - w^M(t,K_M(y)|^2 \le C\big(\Delta x^{r_2} + \Delta x^{r_3}\big),$$

where r_2 and r_3 are defined by (1.8), (1.9)*, respectively.*

3.2.2. The case for the sufficiently smooth initial data $u_0 \in C^2[0,1]$, $u_0(0) = u_0(1) = 0$. In this subsection, we shall consider the spatial regularity of the mild solution of (3.3) when the initial value is sufficiently smooth, that is, $u_0 \in C^2[0,1]$, $u_0(0) = u_0(1) = 0$.

LEMMA 3.4. *Let $v^M(t,x)$ be the solution of* (3.7). *Let $u_0 \in C^2[0,1]$, $u_0(0) = u_0(1) = 0$. Then we have*

$$\mathbb{E}|v(t,y) - v(t,K_M(y)|^2 \le C\Delta x^{r_2},$$

where r_2 is defined by (1.8).

Proof. We have

$$v^M(t,y) = \int_0^1 G_1^M(t,y,z)u_0(k_M(z))\,\mathrm{d}z = \int_0^1 \sum_{j=1}^{M-1} E_{\alpha,1}(-t^\alpha \lambda_j^M)\varphi_j^M(y)\varphi_j(k_M(z))u_0(k-M(z))\,\mathrm{d}z.$$

By Lemma 2.1, we have

$$\begin{aligned} v^M(t,y) &= \int_0^1 \sum_{j=1}^{M-1} \Big[\int_0^t s^{\alpha-1}\lambda_j^M E_{\alpha,\alpha}(-s^\alpha \lambda_j^M)\,\mathrm{d}s + 1\Big]\varphi_j^M(y)\varphi_j(k_M(z))u_0(k_M(z))\,\mathrm{d}z \\ &= \sum_{j=1}^{M-1}\int_0^1 \varphi_j^M(y)\varphi_j(k_M(z))u_0(k_M(z))\,\mathrm{d}z \\ &\quad + \int_0^t \int_0^1 \sum_{j=1}^{M-1} s^{\alpha-1}E_{\alpha,\alpha}(-s^\alpha\lambda_j^M)\varphi_j^M(y)\big(\lambda_j^M\varphi_j(k_M(z))\big)u_0(k_M(z))\,\mathrm{d}s. \end{aligned} \tag{3.10}$$

For the first term of the last equality in (3.10), we have, with $k = 0,1,2,\dots,M$,

$$\begin{aligned} \sum_{j=1}^{M-1}\int_0^1 \varphi_j^M(y_k)\varphi_j(k_M(z))u_0(k_M(z))\,\mathrm{d}z &= \sum_{j=1}^{M-1}\varphi_j(y_k)\Big[\int_0^1 \varphi_j(k_M(z))u_0(k_M(z))\,\mathrm{d}z\Big] \\ &= \sum_{j=1}^{M-1}\big(\sqrt{\Delta x}\varphi_j(y_k)\big)\sum_{l=0}^{M-1}\big(\sqrt{\Delta x}\varphi_j(z_l)\big)u_0(z_l) \\ &= \sum_{j=1}^{M-1}\Big[\big(\vec{u}_0^M(0),\vec{\varphi}_j^M\big)\vec{\varphi}_j^M\Big](k) = \Big[\vec{u}_0^M(0)\Big](k) = u_0(y_k). \end{aligned}$$

Therefore $\sum_{j=1}^{M-1}\int_0^1 \varphi_j^M(y)\varphi_j(k_M(z))u_0(k_M(z))\,\mathrm{d}z$ is the piecewise linear interpolation function of $u_0(y_k), k = 0,1,2,\dots,M$ and we denote

$$I_h u_0(y) := \sum_{j=1}^{M-1}\int_0^1 \varphi_j^M(y)\varphi_j(k_M(z))u_0(k_M(z))\,\mathrm{d}z.$$

Further we assume that the following equality holds at the moment

$$\int_0^1 \lambda_j^M\varphi_j(k_M(z))u_0(k_M(z))\,\mathrm{d}z = -\int_0^1 \varphi_j(k_M(z))\frac{u_0(k_M(z)+\frac{1}{M}) - 2u_0(k_M(z)) + u_0(k_M(z)-\frac{1}{M})}{\Delta x^2}\,\mathrm{d}z, \tag{3.11}$$

which we shall prove later. We then get

$$v^M(t,y) = I_h u_0(y) - \int_0^t\int_0^1 G_2^M(s,y,z)\frac{u_0(k_M(z)+\frac{1}{M}) - 2u_0(k_M(z)) + u_0(k_M(z)-\frac{1}{M})}{\Delta x^2}\,\mathrm{d}z\mathrm{d}s. \tag{3.12}$$

Thus, by using Cauchy-Schwarz inequality,

$$
\begin{aligned}
&|v^M(t,y)-v^M(t,k_M(y)|^2 \le C|I_h u_0(y)-I_h u_0(k_M(y))|^2\\
&\quad + C\Big|\int_0^t\int_0^1 \Big[G_2^M(s,y,z)-G_2^M(s,k_M(y),z)\Big]\frac{u_0(k_M(z)+\frac{1}{M})-2u_0(k_M(z))+u_0(k_M(z)-\frac{1}{M})}{\Delta x^2}\,\mathrm{d}z\mathrm{d}s\Big|^2\\
&\le C|I_h u_0(y)-I_h u_0(k_M(y))|^2 + C\Big[\int_0^t\int_0^1\Big[G_2^M(s,y,z)-G_2^M(s,k_M(y),z)\Big]^2\mathrm{d}z\mathrm{d}s\Big]\cdot\\
&\qquad\qquad \cdot\Big[\int_0^t\int_0^1\Big|\frac{u_0(k_M(z)+\frac{1}{M})-2u_0(k_M(z))+u_0(k_M(z)-\frac{1}{M})}{\Delta x^2}\Big|^2\mathrm{d}z\mathrm{d}s\Big].
\end{aligned}
$$

By Lemma 5.4 and using the error estimates of the liner interpolation function and the mean-value theorem, we obtain

$$|v^M(t,y)-v^M(t,k_M(y))|^2 \le C\Delta x^2\|u_0\|^2_{C^1[0,1]} + C\Delta x^{r_2}\|u_0\|_{C^2[0,1]}.$$

It remains to prove (3.11). In fact, we have

$$
\begin{aligned}
&-\int_0^1 \lambda_j^M\varphi_j(k_M(y))u_0(k_M(y))\,\mathrm{d}y = \Big[\int_{y_1}^{y_2}+\cdots+\int_{y_{M-1}}^{y_M}\Big](-\lambda_j^M)\varphi_j(k_M(y))u_0(k_M(y))\,\mathrm{d}y\\
&= \sqrt{\Delta x}\big((-\lambda_j^M)\vec{\varphi}_j^M,\vec{u}_0^M\big)_{\mathbb{R}^{M-1}} = \sqrt{\Delta x}\big(\vec{\varphi}_j^M, A\vec{u}_0^M\big)_{\mathbb{R}^{M-1}}\\
&= \int_0^1 \varphi_j(k_M(y))\frac{u_0(k_M(z)+\frac{1}{M})-2u_0(k_M(z))+u_0(k_M(z)-\frac{1}{M})}{\Delta x^2}\,\mathrm{d}y, \qquad (3.13)
\end{aligned}
$$

where we use the fact $u_0(y_0)=u_0(y_M)=0$ in the last equality in (3.13). Hence (3.11) holds.

The proof of Lemma 3.4 is now complete. □

4. Error estimates. In this section, we will prove the error bounds between $u(t,x)$ and $u^M(t,x)$ under the suitable smoothness assumptions of the initial values. We need the following Grönwall Lemma [16, Lemma 3.4].

LEMMA 4.1. *Let $z:\mathbb{R}_+\to\mathbb{R}_+$ be a Borel function satisfying for all $t\in[0,T]$ the inequality*

$$0\le z(t)\le a+K\int_0^t (t-s)^\sigma z(s)\,\mathrm{d}s,$$

with some constants $a\ge 0, K$ and $\sigma>-1$. Then there exists a constant $C=C(\sigma,K,T)$ such that $z(t)\le aC$ for all $t\in[0,T]$.

4.1. Proof of Theorem 1.4. In this subsection, we shall prove Theorem 1.4 when the initial data is sufficiently smooth, that is, $u_0\in C^3[0,1]$, $u_0(0)=u_0(1)=0$.

We first consider the case (i), that is, $f=0$. We divide the proof into two steps.

Step 1. We consider the approximation of the homogeneous problem of (2.2). The solution of the homogeneous problem of (2.2) has the form

$$v(t,x)=\int_0^1 G_1(t,x,y)u_0(y)\,\mathrm{d}y.$$

The approximate solution of the homogeneous problem of (2.2) has the form

$$v^M(t,x)=\int_0^1 G_1^M(t,x,y)u_0(k_M(y))\,\mathrm{d}y.$$

By (3.12), we have

$$v(t,x) = u_0(x) + \int_0^t \int_0^1 G_2(s,x,y) u_0''(y)\, \mathrm{d}y \mathrm{d}s.$$

By (3.8), we have

$$v^M(t,x) = I_h u_0(x) + \int_0^t \int_0^1 G_2^M(s,x,y) \frac{u_0(k_M(y) + \frac{1}{M}) - 2u_0(k_M(y)) + u_0(k_M(y) - \frac{1}{M})}{\Delta x^2}\, \mathrm{d}y \mathrm{d}s,$$

where $I_h u_0(x)$ is the piecewise linear interpolation function of $u_0(x_k), k = 0, 1, 2, \dots, M$.

Hence we have

$$\begin{aligned}
|v(t,x) - v^M(t,x)|^2 &\leq C|I_h u_0(x) - u_0(x)|^2 + C\Big| \int_0^t \int_0^1 \Big[G_2^M(s,x,y) - G_2(s,x,y) \Big] u_0''(y)\, \mathrm{d}y \mathrm{d}s \Big|^2 \\
&\quad + C\Big| \int_0^t \int_0^1 G_2^M(s,x,y) \Big[u_0''(y) - \frac{u_0(k_M(y) + \frac{1}{M}) - 2u_0(k_M(y)) + u_0(k_M(y) - \frac{1}{M})}{\Delta x^2} \Big] \mathrm{d}y \mathrm{d}s \Big|^2 \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

For I_1, we have, using the error estimates of the linear interpolation function,

$$I_1 = |I_h u_0(x) - u_0(x)| \leq C\|u_0\|^2_{C^2[0,1]} \Delta x^2.$$

For I_2, we have, by Cauchy-Schwartz inequality,

$$\begin{aligned}
I_2 &= \Big| \int_0^t \int_0^1 \big(G_2^M(s,x,y) - G_2(s,x,y) \big) u_0''(y)\, \mathrm{d}y \mathrm{d}s \Big|^2 \\
&\leq \Big[\int_0^t \int_0^1 \Big| G_2^M(s,x,y) - G_2(s,x,y) \Big|^2 \mathrm{d}y \mathrm{d}s \Big] \Big[\int_0^t \int_0^1 |u_0''(y)|^2\, \mathrm{d}y \mathrm{d}s \Big] \\
&\leq C\Big[\int_0^t \int_0^1 \Big| G_2^M(s,x,y) - G_2(s,x,y) \Big|^2 \mathrm{d}y \mathrm{d}s \Big] \|u_0\|^2_{C^2[0,1]}.
\end{aligned}$$

Hence we have, by Lemma 5.9,

$$I_2 \leq C\Delta x^{r_2} \|u_0\|^2_{C^2[0,1]},$$

where r_2 is defined by (1.8).

For I_3, we have

$$\begin{aligned}
I_3 &= \Big| \int_0^t \int_0^1 G_2^M(s,x,y) \Big[u_0''(y) - \frac{u_0(k_M(y) + \frac{1}{M}) - 2u_0(k_M(y)) + u_0(k_M(y) - \frac{1}{M})}{\Delta x^2} \Big] \mathrm{d}y \mathrm{d}s \Big|^2 \\
&\leq \Big[\int_0^t \int_0^1 |G_2^M(s,x,y)|^2\, \mathrm{d}y \mathrm{d}s \Big] \Big[\int_0^t \int_0^1 \Big| u_0''(y) - \frac{u_0(k_M(y) + \frac{1}{M}) - 2u_0(k_M(y)) + u_0(k_M(y) - \frac{1}{M})}{\Delta x^2} \Big|^2 \mathrm{d}y \mathrm{d}s.
\end{aligned}$$

Note that, with $y_l \leq y < y_{l+1}$, $l = 0, 1, 2, \dots, M-1$,

$$\begin{aligned}
&\Big| u_0''(y) - \frac{u_0(y_{l+1}) - 2u_0(y_l) + u_0(y_{l+1})}{\Delta x^2} \Big| \\
&\leq |u_0''(y) - u_0''(y_l)| + \Big| u_0''(y_l) - \frac{u_0(y_{l+1}) - 2u_0(y_l) + u_0(y_{l+1})}{\Delta x^2} \Big| \\
&\leq C|u_0'''(c)\Delta x| \leq C\Delta x \|u_0\|_{C^3[0,1]}.
\end{aligned}$$

Hence we get, by Lemma 5.8,

$$I_3 \le C\Delta x^2 \|u_0\|^2_{C^3[0,1]}.$$

Therefore we obtain

$$\begin{aligned}\mathbb{E}|v(t,x)-v^M(t,x)|^2 &\le C\Delta x^2\|u_0\|^2_{C^2[0,1]} + C\Delta x^{r_2}\|u_0\|^2_{C^2[0,1]} + C\Delta x^2\|u_0\|^2_{C^3[0,1]} \\ &\le C(\Delta x^2 + \Delta x^{r_2}), \end{aligned} \tag{4.1}$$

where r_2 is defined by (1.8).

Step 2. We now consider the approximation of the inhomogeneous problem of (2.2). The solution of the inhomogeneous problem of (2.2) has the form, since $f=0$,

$$w(t,x) = \int_0^t \int_0^1 G_3(t-s,x,y)\sigma(u(s,y))\frac{\partial^2 W(s,y)}{\partial s \partial y}\,\mathrm{d}y\mathrm{d}s, \tag{4.2}$$

and the approximate solution of the inhomogeneous problem of (2.2) has the form

$$w^M(t,x) = \int_0^t \int_0^1 G_3^M(t-s,x,y)\sigma(u^M(s,k_M(y)))\frac{\partial^2 W_M(s,y)}{\partial s \partial y}\,\mathrm{d}y\mathrm{d}s. \tag{4.3}$$

Thus we have

$$\begin{aligned} &\mathbb{E}|w^M(t,x)-w(t,x)|^2 \\ =\ &\mathbb{E}\Big|\int_0^t\int_0^1 G_3^M(t-s,x,y)\sigma(u^M(s,k_M(y)))\frac{\partial^2 W_M(s,y)}{\partial s\partial y}\,\mathrm{d}y\mathrm{d}s \\ &-\int_0^t\int_0^1 G_3(t-s,x,y)\sigma(u(s,y))\frac{\partial^2 W(s,y)}{\partial s\partial y}\,\mathrm{d}y\mathrm{d}s\Big|^2 \\ \le\ & C\mathbb{E}\Big|\int_0^t\int_0^1 \Big[G_3^M(t-s,x,y)\sigma(u^M(s,k_M(y))) - G_3(t-s,x,y)\sigma(u(s,y))\Big]\frac{\partial^2 W(s,y)}{\partial s\partial y}\,\mathrm{d}y\mathrm{d}s\Big|^2 \\ &+C\mathbb{E}\Big|\int_0^t\int_0^1 G_3^M(t-s,x,y)\sigma(u^M(s,k_M(y)))\Big[\frac{\partial^2 W(s,y)}{\partial s\partial y} - \frac{\partial^2 W_M(s,y)}{\partial s\partial y}\Big]\,\mathrm{d}y\mathrm{d}s\Big|^2 \\ =\ & I_1 + I_2. \end{aligned}$$

For I_1, we have

$$\begin{aligned} I_1 &= \mathbb{E}\Big|\int_0^t\int_0^1 \Big[G_3^M(t-s,x,y)\sigma(u^M(s,k_M(y))) - G_3(t-s,x,y)\sigma(u(s,y))\Big]\mathrm{d}W(s,y)\Big|^2 \\ &\le C\mathbb{E}\Big|\int_0^t\int_0^1 \Big[G_3^M(t-s,x,y) - G_3(t-s,x,y)\Big]\sigma(u^M(s,k_M(y)))\mathrm{d}W(s,y)\Big|^2 \\ &\quad + C\mathbb{E}\Big|\int_0^t\int_0^1 G_3(t-s,x,y)\Big[\sigma(u^M(s,k_M(y))) - \sigma(u(s,y))\Big]\mathrm{d}W(s,y)\Big|^2. \end{aligned}$$

By Burkholder inequality [16], we have

$$\begin{aligned} I_1 &\le C\int_0^t\int_0^1 \Big[G_3^M(t-s,x,y) - G_3(t-s,x,y)\Big]^2 \sup_{s,y}\mathbb{E}\|\sigma(u^M(s,k_M(y)))\|^2\,\mathrm{d}y\mathrm{d}s \\ &\quad + C\int_0^t\int_0^1 [G_3(t-s,x,y]^2 \sup_y \mathbb{E}\|\sigma(u^M(s,k_M(y))) - \sigma(u(s,y))\|^2\,\mathrm{d}y\mathrm{d}s. \end{aligned}$$

By the Assumptions (L) and (LG), we have, using the boundedness of the solution u^M,

$$I_1 \le C\int_0^t\int_0^1 \Big[G_3^M(t-s,x,y)-G_3(t-s,x,y)\Big]^2 \,\mathrm{d}y\mathrm{d}s + C\int_0^t\int_0^1 [G_3(t-s,x,y]^2 \sup_y \mathbb{E}\|u^M(s,k_M(y))-u(s,y)\|^2 \,\mathrm{d}y\mathrm{d}s.$$

Hence we have, by Lemmas 5.4 and 5.6,

$$I_1 \le C\Delta x^{r_3} + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \sup_y \mathbb{E}|u^M(s,k_M(y))-u(s,y)|^2 \,\mathrm{d}s,$$

where r_3 is defined by (1.9).

For I_2, we have

$$\begin{aligned} I_2 &= \mathbb{E}\Big|\int_0^t\int_0^1 G_3^M(t-s,x,y)\sigma(u^M(s,k_M(y)))\frac{\partial^2 W(s,y)}{\partial s\partial y}\,\mathrm{d}y\mathrm{d}s \\ &\quad - \int_0^t\int_0^1 G_3^M(t-s,x,y)\sigma(u^M(s,k_M(y)))\frac{\partial^2 W_M(s,y)}{\partial s\partial y}\,\mathrm{d}y\mathrm{d}s\Big|^2 \\ &= \mathbb{E}\Big|\sum_{k=0}^{M-1}\int_0^t\int_{y_k}^{y_{k+1}} G_3^M(t-s,x,y)\sigma(u^M(s,k_M(y)))\frac{\partial^2 W(s,y)}{\partial s\partial y}\,\mathrm{d}y\mathrm{d}s \\ &\quad - \sum_{k=0}^{M-1}\int_0^t\int_{y_k}^{y_{k+1}} G_3^M(t-s,x,\bar{y})\sigma(u^M(s,k_M(\bar{y})))\frac{\partial^2 W_M(s,\bar{y})}{\partial s\partial \bar{y}}\,\mathrm{d}\bar{y}\mathrm{d}s\Big|^2. \end{aligned}$$

By (3.4), we have, for $y_k \le \bar{y} \le y_{k+1},\ k=0,1,2,\dots,M-1$,

$$\frac{\partial^2 W_M(s,\bar{y})}{\partial s\partial \bar{y}} = \frac{d}{ds}\Big[\frac{W(s,y_{k+1})-W(s,y_k)}{\Delta x}\Big] = \frac{1}{\Delta x}\int_{y_k}^{y_{k+1}} \frac{\partial^2 W(s,y)}{\partial s\partial y}\,\mathrm{d}y.$$

Thus we get

$$\begin{aligned} I_2 &= \mathbb{E}\Big|\sum_{k=0}^{M-1}\int_0^t\int_{y_k}^{y_{k+1}} G_3^M(t-s,x,y)\sigma(u^M(s,k_M(y)))\frac{\partial^2 W(s,y)}{\partial s\partial y}\,\mathrm{d}y\mathrm{d}s \\ &\quad - \sum_{k=0}^{M-1}\int_0^t\int_{y_k}^{y_{k+1}} G_3^M(t-s,x,\bar{y})\sigma(u^M(s,k_M(\bar{y})))\Big[\frac{1}{\Delta x}\int_{y_k}^{y_{k+1}} \frac{\partial^2 W(s,y)}{\partial s\partial y}\,\mathrm{d}y\Big]\,\mathrm{d}\bar{y}\mathrm{d}s\Big|^2 \\ &= \mathbb{E}\Big|\sum_{k=0}^{M-1}\int_0^t\int_{y_k}^{y_{k+1}} \Big[\frac{1}{\Delta x}\int_{y_k}^{y_{k+1}} G_3^M(t-s,x,y)\sigma(u^M(s,k_M(y)))\,\mathrm{d}\bar{y} \\ &\quad - \frac{1}{\Delta x}\int_{y_k}^{y_{k+1}} G_3^M(t-s,x,\bar{y})\sigma(u^M(s,k_M(\bar{y})))\,\mathrm{d}\bar{y}\Big]\,\mathrm{d}W(s,y)\Big|^2. \end{aligned}$$

Note that, for $y_k \le y,\bar{y} \le y_{k+1}, k=0,1,2,\dots,M-1$,

$$\begin{aligned} G_3^M(t-s,x,y) &= \sum_{j=1}^{M-1}(t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_j^M(t-s)^\alpha)\varphi_j^M(x)\varphi_j(k_M(y)) \\ &= \sum_{j=1}^{M-1}(t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_j^M(t-s)^\alpha)\varphi_j^M(x)\varphi_j(k_M(\bar{y})) \\ &= G_3^M(t-s,x,\bar{y}), \end{aligned}$$

which implies that

$$I_2 = \mathbb{E}\Big|\sum_{k=0}^{M-1}\int_0^t\int_{y_k}^{y_{k+1}} 0\,\mathrm{d}W(s,y)\Big|^2 = 0.$$

Thus we get

$$\mathbb{E}|w^M(t,x)-w(t,x)|^2 \le C\Delta x^{r_3} + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \sup_y \mathbb{E}|u^M(s,k_M(y))-u(s,y)|^2\,\mathrm{d}s. \tag{4.4}$$

By the spatial regularity Lemmas 3.4 and 3.3 and the error estimate (4.1) for $\mathbb{E}|v^M(s,y)-v(s,y)|^2$, we obtain

$$\begin{aligned}
&\mathbb{E}|u^M(s,k_M(y))-u(s,y)|^2\\
&\le \mathbb{E}|w^M(s,k_M(y))-w^M(s,y)|^2 + \mathbb{E}|w^M(s,y)-w(s,y)|^2\\
&\quad + \mathbb{E}|v^M(s,k_M(y))-v^M(s,y)|^2 + \mathbb{E}|v^M(s,y)-v(s,y)|^2\\
&\le C(\Delta x^2+\Delta x^{r_2}+\Delta x^{r_3}) + C\mathbb{E}|w^M(s,y)-w(s,y)|^2.
\end{aligned}$$

Hence we get, if $2(\alpha+\gamma-1)-\frac{\alpha}{2} > -1$, i.e., $\frac{2(1-2\gamma)}{\alpha} < 3$,

$$\begin{aligned}
&\mathbb{E}|w^M(t,x)-w(t,x)|^2\\
&\le C\Delta x^{r_3} + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}}\Big[(\Delta x^2+\Delta x^{r_2}+\Delta x^{r_3}) + \sup_y \mathbb{E}|w^M(s,y)-w(s,y)|^2\Big]\,\mathrm{d}s\\
&\le C(\Delta x^2+\Delta x^{r_2}+\Delta x^{r_3}) + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}}\Big[\sup_y \mathbb{E}|w^M(s,y)-w(s,y)|^2\Big]\,\mathrm{d}s,
\end{aligned}$$

which implies that

$$\begin{aligned}
\sup_x \mathbb{E}|w^M(t,x)-w(t,x)|^2 \le\,& C(\Delta x^2+\Delta x^{r_2}+\Delta x^{r_3})\\
&+ C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}}\Big[\sup_y \mathbb{E}|w^M(s,y)-w(s,y)|^2\Big]\,\mathrm{d}s.
\end{aligned}$$

By using Grönwall Lemma 4.1, we get, if $2(\alpha+\gamma-1)-\frac{\alpha}{2} > -1$, i.e., $\frac{2(1-2\gamma)}{\alpha} < 3$,

$$\sup_x \mathbb{E}|w^M(t,x)-w(t,x)|^2 \le C(\Delta x^2+\Delta x^{r_2}+\Delta x^{r_3}).$$

We now consider the case (ii), that is, $f \neq 0$. In this case, the approximation of the solution for the homogeneous problem of (2.2) is the same as in the case (i). For the inhomogeneous problem of (2.2), the solution has the form

$$\begin{aligned}
w(t,x) =& \int_0^t\int_0^1 G_2(t-s,x,y) f(u(s,y))\,\mathrm{d}y\mathrm{d}s\\
&+ \int_0^t\int_0^1 G_3(t-s,x,y)\sigma(u(s,y))\frac{\partial^2 W(s,y)}{\partial s\partial y}\,\mathrm{d}y\mathrm{d}s.
\end{aligned} \tag{4.5}$$

The approximate solution of the inhomogeneous problem of (2.2) has the form

$$
\begin{aligned}
w^M(t,x) = &\int_0^t \int_0^1 G_2^M(t-s,x,y) f(u^M(s,k_M(y)))\,\mathrm{d}y\mathrm{d}s \\
&+ \int_0^t \int_0^1 G_3^M(t-s,x,y)\sigma(u^M(s,k_M(y)))\frac{\partial^2 W_M(s,y)}{\partial s \partial y}\,\mathrm{d}y\mathrm{d}s.
\end{aligned} \tag{4.6}
$$

Following the same arguments as in Step 2, we may get, if $2(\alpha-1)-\frac{\alpha}{2} > -1$, i.e., $\frac{2}{\alpha} < 3$,

$$
\begin{aligned}
&\mathbb{E}|w^M(t,x)-w(t,x)|^2 \\
&\leq C(\Delta x^{r_2}+\Delta x^{r_3}) \\
&\quad + C\int_0^t (t-s)^{2(\alpha-1)-\frac{\alpha}{2}} \Big[(\Delta x^2+\Delta x^{r_2}+\Delta x^{r_3}) + \sup_y \mathbb{E}|w^M(s,y)-w(s,y)|^2\Big]\,\mathrm{d}s \\
&\quad + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Big[(\Delta x^2+\Delta x^{r_2}+\Delta x^{r_3}) + \sup_y \mathbb{E}|w^M(s,y)-w(s,y)|^2\Big]\,\mathrm{d}s \\
&\leq C(\Delta x^{r_3}+\Delta x^{r_2}+\Delta x^2) \\
&\quad + C\int_0^t (t-s)^{2(\alpha-1)-\frac{\alpha}{2}} \Big[\sup_y \mathbb{E}|w^M(s,y)-w(s,y)|^2\Big]\,\mathrm{d}s, \\
&\quad + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Big[\sup_y \mathbb{E}|w^M(s,y)-w(s,y)|^2\Big]\,\mathrm{d}s,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\sup_x \mathbb{E}|w^M(t,x)-w(t,x)|^2 \leq & C(\Delta x^{r_2}+\Delta x^{r_3}+\Delta x^2) \\
&+ C\int_0^t (t-s)^{2(\alpha-1)-\frac{\alpha}{2}} \Big[\sup_y \mathbb{E}|w^M(s,y)-w(s,y)|^2\Big]\,\mathrm{d}s \\
&+ C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Big[\sup_y \mathbb{E}|w^M(s,y)-w(s,y)|^2\Big]\,\mathrm{d}s.
\end{aligned}
$$

By using Grönwall Lemma 4.1, we get, if $2(\alpha-1)-\frac{\alpha}{2} > -1$, i.e., $\frac{2}{\alpha} < 3$,

$$
\sup_x \mathbb{E}|w^M(t,x)-w(t,x)|^2 \leq C(\Delta x^{r_2}+\Delta x^{r_3}+\Delta x^2).
$$

Together this with (4.1) shows (ii).

The proof of Theorem 1.4 is now complete.

4.2. Proof of Theorem 1.1. In this subsection, we shall prove Theorem 1.1 when the initial data $u_0 \in C^1[0,1]$, $u_0(0) = u_0(1) = 0$. The proof is similar as the proof of Theorem 1.4.

We first consider the case (i), that is, $f = 0$. We divide the proof into two steps.

Step 1. We consider the approximation of the homogeneous problem of (2.2). The solution of the homogeneous problem of (2.2) has the form

$$
v(t,x) = \int_0^1 G_1(t,x,y)u_0(y)\,\mathrm{d}y.
$$

The approximate solution of the homogeneous problem of (3.1) has the form

$$v^M(t,x) = \int_0^1 G_1^M(t,x,y) u_0(k_M(y))\, \mathrm{d}y.$$

Hence we have, by Cauchy-Schwarz inequality,

$$\begin{aligned}
|v(t,x) - v^M(t,x)|^2 &\le C\Big| \int_0^1 \big(G_1^M(t,x,y) - G_1(t,x,y) \big) u_0(k_M(y))\, \mathrm{d}y \Big|^2 \\
&\quad + C\Big| \int_0^1 G_1(t,x,y) \big(u_0(k_M(y)) - u_0(y) \big)\, \mathrm{d}y \Big|^2 \\
&\le C\Big[\int_0^1 |G_1^M(t,x,y) - G_1(t,x,y)|^2\, \mathrm{d}y \Big] \Big[\int_0^1 |u_0(k_M(y))|^2\, \mathrm{d}y \Big] \\
&\quad + C\Big[\int_0^1 |G_1(t,x,y)|^2\, \mathrm{d}y \Big] \Big[\int_0^1 |u_0(k_M(y)) - u_0(y)|^2\, \mathrm{d}y \Big].
\end{aligned}$$

Thus we have, by mean-value theorem,

$$\begin{aligned}
|v(t,x) - v^M(t,x)|^2 &\le C\Big[\int_0^1 |G_1^M(t,x,y) - G_1(t,x,y)|^2\, \mathrm{d}y \Big] \|u_0\|^2_{C[0,1]} \\
&\quad + C\Big[\int_0^1 |G_1(t,x,y)|^2\, \mathrm{d}y \Big] \Delta x^2 \|u_0\|^2_{C^1[0,1]}.
\end{aligned}$$

By Lemmas 5.1 and 5.3, we get

$$|v(t,x) - v^M(t,x)|^2 \le Ct^{-1+\varepsilon} \Delta x^{r_1} \|u_0\|^2_{C[0,1]} + Ct^{-\frac{\alpha}{2}} \Delta x^2 \|u_0\|^2_{C^1[0,1]} \le Ct^{-1+\varepsilon} \Delta x^{r_1} \|u_0\|^2_{C^1[0,1]}, \tag{4.7}$$

where r_1 is defined by (1.7).

Step 2. We now consider the approximation of the inhomogeneous problem of (2.2). Following the proof of (4.4), we get

$$\mathbb{E}|w^M(t,x) - w(t,x)|^2 \le C\Delta x^{r_3} + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \sup_y \mathbb{E}|u^M(s,k_M(y)) - u(s,y)|^2\, \mathrm{d}s,$$

where r_3 is defined by (1.9).

Noting that

$$\begin{aligned}
\mathbb{E}|u^M(s,k_M(y)) - u(s,y)|^2 &\le \mathbb{E}|w^M(s,k_M(y)) - w^M(s,y)|^2 + \mathbb{E}|w^M(s,y) - w(s,y)|^2 \\
&\quad + \mathbb{E}|v^M(s,k_M(y)) - v^M(s,y)|^2 + \mathbb{E}|v^M(s,y) - v(s,y)|^2,
\end{aligned}$$

we therefore obtain

$$
\begin{aligned}
&\mathbb{E}|w^M(t,x)-w(t,x)|^2\\
&\le C\Delta x^{r_3} + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Big[\sup_y \mathbb{E}\big|w^M(s,k_M(y))-w^M(s,y)\big|^2\Big]\,\mathrm{d}s\\
&\quad + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Big[\sup_y \mathbb{E}\big|w^M(s,k_M(y))-w(s,y)\big|^2\Big]\,\mathrm{d}s\\
&\quad + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Big[\sup_y \mathbb{E}\big|v^M(s,k_M(y))-v^M(s,y)\big|^2\Big]\,\mathrm{d}s\\
&\quad + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Big[\sup_y \mathbb{E}\big|v^M(s,y)-v(s,y)\big|^2\Big]\,\mathrm{d}s\\
&= C\Delta x^{r_3} + J_1(t)+J_2(t)+J_3(t)+J_4(t). \qquad (4.8)
\end{aligned}
$$

For $J_1(t)$, if $2(\alpha+\gamma-1)-\frac{\alpha}{2} > -1$, we then have, applying Lemma 3.3 for the case $f=0$,

$$
J_1(t) \le C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Delta x^{r_3}\,\mathrm{d}s \le C\Delta x^{r_3},
$$

where r_3 is defined by (1.9).

For $J_3(t)$, we consider the following two cases.

Case 1. If $2(\alpha+\gamma-1)-\frac{\alpha}{2}+\varepsilon \ge 0$, then we have, by Lemma 2.3,

$$
J_3(t) \le \mathbb{E}\int_0^t (t-s)^{2(\alpha+\gamma)-1-\frac{\alpha}{2}} \Delta^{r_1} s^{-1+\varepsilon}\,\mathrm{d}s \le C\Delta x^{r_1} t^{2(\alpha+\gamma-1)-\frac{\alpha}{2}+\varepsilon} \le C\Delta x^{r_1},
$$

where r_1 is defined by (1.7).

Case 2. If $2(\alpha+\gamma-1)-\frac{\alpha}{2}+\varepsilon < 0$, then we have, for $t > \Delta x$, (the case $t < \Delta x$ is easy to estimate and we omit the detail here)

$$
\begin{aligned}
J_3(t) &= \int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Big[\sup_y \mathbb{E}\big|v^M(s,k_M(y))-v^M(s,y)\big|^2\Big]\,\mathrm{d}s\\
&\le C\int_0^{\Delta x} (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Big[\sup_y \mathbb{E}\big|v^M(s,k_M(y))-v^M(s,y)\big|^2\Big]\,\mathrm{d}s\\
&\quad + C\int_{\Delta x}^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Big[\sup_y \mathbb{E}\big|v^M(s,k_M(y))-v^M(s,y)\big|^2\Big]\,\mathrm{d}s\\
&= J_{31}(t)+J_{32}(t).
\end{aligned}
$$

For $J_{31}(t)$, if $2(\alpha+\gamma-1)-\frac{\alpha}{2} > -1$, then we have, by using the boundedness of $v^M(s,y)$,

$$
J_{31}(t) \le C\int_0^{\Delta x} (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}}\,\mathrm{d}s = \Delta x^{2(\alpha+\gamma-1)-\frac{\alpha}{2}+1}.
$$

For $J_{32}(t)$, we have, by Lemma 2.3,

$$
J_{32}(t) \le C\int_{\Delta x}^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Delta x^{r_1} s^{-1+\varepsilon}\,\mathrm{d}s \le C\Delta x^{r_1} t^{2(\alpha+\gamma-1)-\frac{\alpha}{2}+\varepsilon} \le C\Delta x^{r_1+2(\alpha+\gamma-1)-\frac{\alpha}{2}+\varepsilon},
$$

where r_1 is defined in (1.7).

Thus we get

$$J_3(t) \le \begin{cases} C\Delta x^{r_1}, & \text{if } 2(\alpha+\gamma-1)-\frac{\alpha}{2}+\varepsilon \ge 0, \\ C\Delta x^{2(\alpha+\gamma-1)-\frac{\alpha}{2}+\min\{1,r_1+\varepsilon\}}, & \text{if } 2(\alpha+\gamma-1)-\frac{\alpha}{2}+\varepsilon < 0. \end{cases}$$

Following the same arguments as the estimate of $J_3(t)$, we may obtain

$$J_4(t) \le \begin{cases} C\Delta x^{r_1}, & \text{if } 2(\alpha+\gamma-1)-\frac{\alpha}{2}+\varepsilon \ge 0, \\ C\Delta x^{2(\alpha+\gamma-1)-\frac{\alpha}{2}+\min\{1,r_1+\varepsilon\}}, & \text{if } 2(\alpha+\gamma-1)-\frac{\alpha}{2}+\varepsilon < 0. \end{cases}$$

Thus we have the following two cases.

Case 1. If $2(\alpha+\gamma-1)-\frac{\alpha}{2}+\varepsilon \ge 0$, then we have

$$\mathbb{E}|w^M(t,x)-w(t,x)|^2 \le C\Delta x^{r_3} + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \left[\sup_y \mathbb{E}|w^M(s,y)-w(s,y)|^2 + \Delta x^{r_1}.\right.$$

By Grönwall Lemma 4.1, we get

$$\mathbb{E}|w^M(t,x)-w(t,x)|^2 \le C(\Delta x^{r_1} + \Delta x^{r_3}),$$

where r_1 and r_3 are defined by (1.7) and (1.9), respectively.

Case 2. If $2(\alpha+\gamma-1)-\frac{\alpha}{2}+\varepsilon < 0$, then we have

$$\begin{aligned}&\mathbb{E}|w^M(t,x)-w(t,x)|^2 \\ &\le C\Delta x^{r_3} + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \left[\sup_y \mathbb{E}|w^M(s,y)-w(s,y)|^2\right] + \Delta x^{2(\alpha+\gamma-1)-\frac{\alpha}{2}+\min\{1,r_1+\varepsilon\}}.\end{aligned}$$

By Grönwall Lemma 4.1, we get

$$\mathbb{E}|w^M(t,x)-w(t,x)|^2 \le C(\Delta x^{2(\alpha+\gamma-1)-\frac{\alpha}{2}+\min\{1,r_1+\varepsilon\}} + \Delta x^{r_3}).$$

Thus we get

$$\begin{aligned}\mathbb{E}|u^M(t,x)-u(t,x)|^2 \le& Ct^{-1+\varepsilon}\Delta x^{r_1} + C\Delta x^{r_3} \\ &+ C\begin{cases} C\Delta x^{r_1}, & \text{if } 2(\alpha+\gamma-1)-\frac{\alpha}{2}+\varepsilon \ge 0, \\ C\Delta x^{2(\alpha+\gamma-1)-\frac{\alpha}{2}+\min\{1,r_1+\varepsilon\}}, & \text{if } 2(\alpha+\gamma-1)-\frac{\alpha}{2}+\varepsilon < 0, \end{cases}\end{aligned} \tag{4.9}$$

where r_1 and r_3 are defined by (1.7) and (1.9), respectively.

We now consider the case (ii), that is, $f \neq 0$. In this case, the approximation of the solution for the homogeneous problem of (2.2) is the same as in the case (i). For the inhomogeneous problem of (2.2), the solution has the form

$$\begin{aligned}w(t,x) =& \int_0^t \int_0^1 G_2(t-s,x,y) f(u(s,y))\,\mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_0^1 G_3(t-s,x,y)\sigma(u(s,y)) \frac{\partial^2 W(s,y)}{\partial s \partial y}\,\mathrm{d}y\mathrm{d}s.\end{aligned} \tag{4.10}$$

The approximate solution of the inhomogeneous problem of (2.2) has the form

$$\begin{aligned} w^M(t,x) &= \int_0^t \int_0^1 G_2^M(t-s,x,y) f(u^M(s,k_M(y)))\,\mathrm{d}y\mathrm{d}s \\ &\quad + \int_0^t \int_0^1 G_3^M(t-s,x,y)\sigma(u^M(s,k_M(y)))\frac{\partial^2 W_M(s,y)}{\partial s \partial y}\,\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{4.11}$$

Following the same arguments as in Step 2, we may get,

$$\begin{aligned} &\mathbb{E}|w^M(t,x)-w(t,x)|^2 \\ &\le C\Delta x^{r_3} + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Big[\sup_y \mathbb{E}\big|w^M(s,k_M(y))-w^M(s,y)\big|^2\Big]\,\mathrm{d}s \\ &\quad + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Big[\sup_y \mathbb{E}\big|w^M(s,k_M(y))-w(s,y)\big|^2\Big]\,\mathrm{d}s \\ &\quad + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Big[\sup_y \mathbb{E}\big|v^M(s,k_M(y))-v^M(s,y)\big|^2\Big]\,\mathrm{d}s \\ &\quad + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Big[\sup_y \mathbb{E}\big|v^M(s,y)-v(s,y)\big|^2\Big]\,\mathrm{d}s \\ &= C\Delta x^{r_3} + J_1'(t) + J_2'(t) + J_3(t) + J_4(t), \end{aligned}$$

where $J_3(t)$ and $J_4(t)$ are defined as in (4.8) since $v(s,y)$ and $v^M(s,y)$ are the same as in the case (i).

For $J_1'(t)$, if $2(\alpha+\gamma-1)-\frac{\alpha}{2} > -1$, then we have, by Lemma 3.3,

$$J_1(t) \le C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \big(\Delta x^{r_2}+\Delta x^{r_3}\big)\,\mathrm{d}s \le C\big(\Delta x^{r_2}+\Delta x^{r_3}\big).$$

Thus we have the following two cases.

Case 1. If $2(\alpha+\gamma-1)-\frac{\alpha}{2}+\varepsilon \ge 0$, then we have

$$\begin{aligned} &\mathbb{E}|w^M(t,x)-w(t,x)|^2 \\ &\le C\big(\Delta x^{r_2}+\Delta x^{r_3}\big) + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Big[\sup_y \mathbb{E}|w^M(s,y)-w(s,y)|^2 + \Delta x^{r_1}. \end{aligned}$$

By Grönwall Lemma 4.1, we get

$$\mathbb{E}|w^M(t,x)-w(t,x)|^2 \le C(\Delta x^{r_1}+\Delta x^{r_2}+\Delta x^{r_3}),$$

where r_1, r_2 and r_3 are defined by (1.7), (1.8) and (1.9), respectively.

Case 2. If $2(\alpha+\gamma-1)-\frac{\alpha}{2}+\varepsilon < 0$, then we have

$$\begin{aligned} &\mathbb{E}|w^M(t,x)-w(t,x)|^2 \\ &\le C\big(\Delta x^{r_2}+\Delta x^{r_3}\big) + C\int_0^t (t-s)^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \Big[\sup_y \mathbb{E}|w^M(s,y)-w(s,y)|^2\Big] + \Delta x^{2(\alpha+\gamma-1)-\frac{\alpha}{2}+\min\{1,r_1+\varepsilon\}}. \end{aligned}$$

By Grönwall Lemma 4.1, we have

$$\mathbb{E}|w^M(t,x)-w(t,x)|^2 \le C(\Delta x^{2(\alpha+\gamma-1)-\frac{\alpha}{2}+\min\{1,r_1+\varepsilon\}}+\Delta x^{r_2}+\Delta x^{r_3}).$$

Thus we obtain

$$\mathbb{E}|u^M(t,x)-u(t,x)|^2 \leq Ct^{-1+\varepsilon}\Delta x^{r_1}+C(\Delta x^{r_2}+\Delta x^{r_3}) \\ +C\begin{cases} C\Delta x^{r_1}, & \text{if } 2(\alpha+\gamma-1)-\frac{\alpha}{2}+\varepsilon \geq 0,\\ C\Delta x^{2(\alpha+\gamma-1)-\frac{\alpha}{2}+\min\{1,r_1+\varepsilon\}}, & \text{if } 2(\alpha+\gamma-1)-\frac{\alpha}{2}+\varepsilon<0,\end{cases} \tag{4.12}$$

where r_1, r_2 and r_3 are defined by (1.7), (1.8) and (1.9), respectively.

The proof of Theorem 1.4 is now complete.

5. Appendix. In this Appendix, we shall consider the approximations of the Green functions $G_i(t,x,y)$ by $G_i^M(t,x,y), i=1,2,3$, where $G_i(t,x,y)$ and $G_i^M(t,x,y), i=1,2,3$ are defined by

$$G_1(t,x,y)=\sum_{j=1}^{\infty}E_{\alpha,1}(-t^{\alpha}\lambda_j)\varphi_j(x)\varphi_j(y),$$
$$G_2(t,x,y)=\sum_{j=1}^{\infty}t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}\lambda_j)\varphi_j(x)\varphi_j(y),$$
$$G_3(t,x,y)=\sum_{j=1}^{\infty}t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-t^{\alpha}\lambda_j)\varphi_j(x)\varphi_j(y),$$

and

$$G_1^M(t,x,y)=\sum_{j=1}^{M-1}E_{\alpha,1}(-t^{\alpha}\lambda_j^M)\varphi_j^M(x)\varphi_j(k_M(y)),$$
$$G_2^M(t,x,y)=\sum_{j=1}^{M-1}t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}\lambda_j^M)\varphi_j^M(x)\varphi_j(k_M(y)),$$
$$G_3^M(t,x,y)=\sum_{j=1}^{M-1}t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-t^{\alpha}\lambda_j^M)\varphi_j^M(x)\varphi_j(k_M(y)),$$

where $E_{\alpha,\beta}(z), \alpha>0, \beta\in\mathbb{C}$ denote the Mittag-Leffler functions defined in (2.2) and where $\{\lambda_j,\varphi_j\}_{j=1}^{\infty}$ and $\{\lambda_j^M\}_{j=1}^{M-1}$ are defined by (2.1) and (3.2), respectively. Here $\varphi_j^M(x), j=1,2\ldots,$ denote the piecewise linear interpolation functions of $\varphi_j(x)$ on the grids $0=x_0<x_1<\ldots x_M=1$ and the piecewise constant function $k_M(y), 0\leq y\leq 1$ is defined by (2.8).

5.1. Green functions $G_1(t,x,y)$ and its approximation $G_1^M(t,x,y)$. In this subsection, we will consider the bounds of $G_1(t,x,y)$ and its approximation $G_1^M(t,x,y)$ and the error bounds of $G_1(t,x,y)-G_1^M(t,x,y)$

in some suitable norms.

LEMMA 5.1. *Let $0<\alpha\le 1$. There exist some positive constant $\delta>0$ and sufficiently small $\varepsilon>0$ such that, with $t>0$,*

$$\int_0^1 |G_1(t,x,y)|^2\,\mathrm{d}y \le Ct^{-\frac{\alpha}{2}}, \quad 0\le x\le 1, \tag{5.1}$$

$$\int_0^t\int_0^1 |G_1(s,x,y)|^2\,\mathrm{d}y\mathrm{d}s \le Ct^{\delta}, \quad ,0\le x\le 1, \tag{5.2}$$

$$\int_0^1 |G_1(t,y,z)-G_1(t,k_M(y),z)|^2\,\mathrm{d}z \le Ct^{-1+\varepsilon}\Delta x^{r_1}, \quad 0\le y\le 1, \tag{5.3}$$

$$\int_0^t\int_0^1 |G_1(s,y,z)-G_1(s,k_M(y),z)|^2\,\mathrm{d}z\mathrm{d}s \le Ct^{\delta}\Delta x^{r_1}, \quad 0\le y\le 1, \tag{5.4}$$

where r_1 is defined by (1.7).

Proof. For (5.1), we have

$$\int_0^1 |G_1(t,x,y)|^2\,\mathrm{d}y = \int \Big[\sum_{j=1}^{\infty} E_{\alpha,1}(-t^{\alpha}\lambda_j)\varphi_j(x)\varphi_j(y)\Big]^2\,\mathrm{d}y.$$

Since $\{\varphi_j(y)\}_{j=1}^{\infty}$ is an orthonormal basis in $H=L^2(0,1)$ and $\varphi_j(x), j=1,2,\dots$ are bounded, we get

$$\int_0^1 |G_1(t,x,y)|^2\,\mathrm{d}y = \sum_{j=1}^{\infty} E^2_{\alpha,1}(-t^{\alpha}\lambda_j). \tag{5.5}$$

By boundedness of the Mittag-Leffler function (1.2), we have, with $0\le\gamma_1\le 1$,

$$\begin{aligned}\int_0^1 |G_1(t,x,y)|^2\,\mathrm{d}y &\le C\sum_{j=1}^{\infty}\Big(\frac{1}{1+t^{\alpha}\lambda_j}\Big)^{2\gamma_1} \le C\sum_{j=1}^{\infty}\Big(\frac{1}{1+t^{\alpha}j^2}\Big)^{2\gamma_1}\\ &\le C\int_0^{\infty}\Big(\frac{1}{1+t^{\alpha}x^2}\Big)^{2\gamma_1}\mathrm{d}x \le Ct^{-\frac{\alpha}{2}}\int_0^{\infty}\Big(\frac{1}{1+y^2}\Big)^{2\gamma_1}\mathrm{d}y.\end{aligned} \tag{5.6}$$

Then (5.1) follows by choosing some $\gamma_1\in(1/4,1]$ in (5.6).

For (5.2), we have, by (5.1),

$$\int_0^t\int_0^1 |G_1(s,x,y)|^2\,\mathrm{d}y\mathrm{d}s \le C\int_0^t s^{-\frac{\alpha}{2}}\,\mathrm{d}s \le C.$$

For (5.3), we have

$$\begin{aligned}&\int_0^1 |G_1(t,y,z)-G_1(t,k_M(y),z)|^2\,\mathrm{d}z\\ &=\int_0^1\Big|\sum_{j=1}^{\infty} E_{\alpha,1}(-t^{\alpha}\lambda_j)\Big[\varphi_j(y)-\varphi_j(k_M(y))\Big]\varphi_j(z)\Big|^2\,\mathrm{d}z.\end{aligned}$$

Since $\{\varphi_j(z)\}_{j=1}^{\infty}$ is an orthonormal basis in $H = L^2(0,1)$, we get

$$\begin{aligned}
&\int_0^1 |G_1(t,y,z) - G_1(t,k_M(y),z)|^2 \,\mathrm{d}z \\
&= \sum_{j=M}^{\infty} E_{\alpha,1}^2 \Big[\varphi_j(y) - \varphi_j(k_M(y))\Big]^2 + \sum_{j=1}^{M-1} E_{\alpha,1}^2 \Big[\varphi_j(y) - \varphi_j(k_M(y))\Big]^2 \\
&= I_1 + I_2.
\end{aligned}$$

For I_1, we have, using the boundedness of $\varphi_j(y)$ and (1.2), with $0 \leq \gamma_1 \leq 1$,

$$I_1 \leq C \sum_{j=M}^{\infty} E_{\alpha,1}^2(-t^{\alpha}\lambda_j) \leq C \sum_{j=M}^{\infty} \Big(\frac{1}{1+t^{\alpha} j^2}\Big)^{2\gamma_1} \leq C t^{-2\alpha\gamma_1} \sum_{j=M}^{\infty} \frac{1}{j^{4\gamma_1}}. \tag{5.7}$$

Case 1. If $1/2 \leq \alpha \leq 1$, then we choose $-2\alpha\gamma_1 = -1+\varepsilon$, $\varepsilon > 0$, i.e., $\gamma_1 = \frac{1-\varepsilon}{2\alpha}$, we get

$$I_1 \leq C t^{-1+\varepsilon} \sum_{j=M}^{\infty} \frac{1}{j^{4\left(\frac{1-\varepsilon}{2\alpha}\right)}} \leq C t^{-1+\varepsilon} \Delta x^{4\left(\frac{1-\varepsilon}{2\alpha}\right)-1}.$$

Case 2. If $0 \leq \alpha < 1/2$, then we choose $\gamma_1 = 1$ and obtain

$$I_1 \leq C t^{-2\alpha} \sum_{j=M}^{\infty} \frac{1}{j^4} \leq C t^{-2\alpha} \Delta x^3.$$

Thus we get

$$I_1 \leq \begin{cases} C t^{-1+\varepsilon} \Delta x^{4\left(\frac{1-\varepsilon}{2\alpha}\right)-1} & 1/2 \leq \alpha \leq 1, \\ C t^{-2\alpha} \Delta x^3, \quad 0 \leq \alpha < 1/2. \end{cases}$$

For I_2, we get

$$I_2 = \sum_{j=1}^{M-1} E_{\alpha,1}^2(-t^{\alpha}\lambda_i) \Big[\varphi_j(y) - \varphi_j(k_M(y))\Big]^2.$$

Note that

$$|\varphi_j(y) - \varphi_j(k_M(y))| = |\varphi_j'(c)(y - k_M(y))| \leq C(j\pi)\frac{1}{M} \leq C\frac{j}{M}.$$

We get

$$I_2 \leq \sum_{j=1}^{M-1} E_{\alpha,1}^2(-t^{\alpha}\lambda_j)\big(\frac{j}{M}\big)^2. \tag{5.8}$$

By (1.2), we get, with $0 \leq \gamma_1 \leq 1$,

$$I_2 \leq C \sum_{j=1}^{M-1} \Big(\frac{1}{1+t^{\alpha}\lambda_j}\Big)^{2\gamma_1} \frac{j^2}{M^2} \leq C t^{-2\alpha\gamma_1} \frac{1}{M^2} \sum_{j=1}^{M-1} \frac{1}{j^{4\gamma_1 - 2}}.$$

Case 1. If $0 \leq \alpha \leq \frac{2(1-\varepsilon)}{3}$, then we have

$$I_2 \leq Ct^{-1+\varepsilon}\frac{1}{M^2}\sum_{j=1}^{M-1}\frac{1}{j^{4\left(\frac{1-\varepsilon}{2\alpha}\right)-2}} \leq Ct^{-1+\varepsilon}\frac{1}{M^2}\int_1^M x^{-4\left(\frac{1-\varepsilon}{2\alpha}\right)+2}\,\mathrm{d}x \leq Ct^{-1=\varepsilon}\Delta x^2.$$

Case 2. If $\frac{2(1-\varepsilon)}{3} \leq \alpha \leq 1$, then we have

$$I_2 \leq Ct^{-1+\varepsilon}\frac{1}{M^2}\int_1^M x^{-4\left(\frac{1-\varepsilon}{2\alpha}\right)+2}\,\mathrm{d}x \leq Ct^{-1+\varepsilon}\Delta x^{4\left(\frac{1-\varepsilon}{2\alpha}\right)-1}.$$

Thus we get

$$I_2 \leq \begin{cases} Ct^{-1+\varepsilon}\Delta x^2, & 0 \leq \alpha \leq \frac{2(1-\varepsilon)}{3} \\ Ct^{-1+\varepsilon}\Delta x^{4\left(\frac{1-\varepsilon}{2\alpha}\right)-1}, & \frac{2(1-\varepsilon)}{3} \leq \alpha \leq 1. \end{cases}$$

Note that the convergence order in I_2 is higher than the convergence order in I_1, we therefore obtain

$$\int_0^1 |G_1(t,y,z) - G_1(t,k_M(y),z)|^2\,\mathrm{d}z \leq Ct^{-1+\varepsilon}\Delta x^{r_1},$$

where r_1 is defined by (1.7), which shows (5.3).

For (5.4), we get, by (5.3),

$$\int_0^t\int_0^1 |G_1(s,y,z) - G_1(s,k_M(y),z)|^2\,\mathrm{d}z\mathrm{d}s \leq C\int_0^t s^{-1+\varepsilon}\Delta x^{r_1}\,\mathrm{d}s \leq Ct^{\delta}\Delta x^{r_1}.$$

Together these estimates complete the proof of Lemma 5.1. □

LEMMA 5.2. *Let $0 < \alpha \leq 1$. There exist some positive constant $\delta > 0$ and sufficiently small $\varepsilon > 0$ such that, with $t > 0$,*

$$\int_0^1 |G_1^M(t,x,y)|^2\,\mathrm{d}y \leq Ct^{-\frac{\alpha}{2}}, \quad 0 \leq x \leq 1, \tag{5.9}$$

$$\int_0^t\int_0^1 |G_1^M(s,x,y)|^2\,\mathrm{d}y\mathrm{d}s \leq Ct^{\delta}, \quad ,0 \leq x \leq 1, \tag{5.10}$$

$$\int_0^1 |G_1^M(t,y,z) - G_1^M(t,k_M(y),z)|^2\,\mathrm{d}z \leq Ct^{-1+\varepsilon}\Delta x^{r_1}, \quad 0 \leq y \leq 1, \tag{5.11}$$

$$\int_0^t\int_0^1 |G_1^M(s,y,z) - G_1^M(s,k_M(y),z)|^2\,\mathrm{d}z\mathrm{d}s \leq Ct^{\delta}\Delta x^{r_1}, \quad 0 \leq y \leq 1, \tag{5.12}$$

where r_1 is defined by (1.7).

Proof. For (5.2), we have

$$\int_0^1 |G_1^M(t,x,y)|^2\,\mathrm{d}y = \int\Big[\sum_{j=1}^{M-1}E_{\alpha,1}(-t^{\alpha}\lambda_j^M)\varphi_j(x)\varphi_j(k_M(y))\Big]^2\,\mathrm{d}y.$$

Note that

$$\int_0^1 \varphi_j(k_M(y))\varphi_l(k_M(y))\,\mathrm{d}y = \Delta x\sum_{k=1}^{M-1}\varphi_j(y_k)\varphi_l(y_k) = \begin{cases} 1, & j = l, \\ 0, & j \neq l, \end{cases} \tag{5.13}$$

and $\varphi_j(x), j = 1, 2, \dots$ are bounded, we get

$$\int_0^1 |G_1^M(t,x,y)|^2 \,\mathrm{d}y = \sum_{j=1}^{M-1} E_{\alpha,1}^2(-t^\alpha \lambda_j^M). \tag{5.14}$$

Note that $\lambda_j^M \approx \lambda_j,\ j = 1, 2, \dots, M-1$, we have, by (5.14),

$$\int_0^1 |G_1^M(t,x,y)|^2 \,\mathrm{d}y \le C \sum_{j=1}^{M-1} E_{\alpha,1}^2(-t^\alpha \lambda_j) \le C \sum_{j=1}^{\infty} E_{\alpha,1}^2(-t^\alpha \lambda_j) \le C t^{-\frac{\alpha}{2}}.$$

which shows (5.2).

For (5.10), we have, by (5.9),

$$\int_0^t \int_0^1 |G_1^M(s,x,y)|^2 \,\mathrm{d}y\mathrm{d}s \le C \int_0^t s^{-\frac{\alpha}{2}} \,\mathrm{d}s \le C.$$

For (5.11), we have, by (5.13),

$$\begin{aligned}
&\int_0^1 |G_1^M(t,y,z) - G_1^M(t,k_M(y),z)|^2 \,\mathrm{d}z \\
&= \int_0^1 \Big| \sum_{j=1}^{M-1} E_{\alpha,1}(-t^\alpha \lambda_j^M) \Big[\varphi_j^M(y) - \varphi_j^M(k_M(y))\Big] \varphi_j(k_M(z)) \Big|^2 \,\mathrm{d}z \\
&= \sum_{j=1}^{M-1} E_{\alpha,1}^2(-t^\alpha \lambda_j^M) \Big[\varphi_j^M(y) - \varphi_j^M(k_M(y))\Big]^2.
\end{aligned}$$

Note that, for $y \in [y_k, y_{k+1}]$, $k = 0, 1, 2, \dots, M$,

$$\begin{aligned}
|\varphi_j^M(y) - \varphi_j^M(k_M(y))| &= \Big| \frac{\varphi_j^M(y_{k+1}) - \varphi_j^M(y_k)}{y_{k+1} - y_k} (y - y_k) \Big| \\
&= \Big| \frac{\varphi_j(y_{k+1}) - \varphi_j(y_k)}{y_{k+1} - y_k} (y - y_k) \Big| = |\varphi_j'(c)(y - y_k)| \le C \frac{j}{M}.
\end{aligned}$$

Thus we have, by (5.8),

$$\int_0^1 |G_1^M(t,y,z) - G_1^M(t,k_M(y),z)|^2 \,\mathrm{d}z \le C \sum_{j=1}^{M-1} E_{\alpha,1}^2(-t^\alpha \lambda_j) \big(\frac{j}{M}\big)^2 \le C t^{-1+\varepsilon} \Delta x^{r_1}, \tag{5.15}$$

where r_1 is defined by (1.7).

For (5.12), we get, by (5.11),

$$\int_0^t \int_0^1 |G_1^M(s,y,z) - G_1^M(s,k_M(y),z)|^2 \,\mathrm{d}z\mathrm{d}s \le C \int_0^t s^{-1+\varepsilon} \Delta x^{r_1} \,\mathrm{d}s \le C t^\delta \Delta x^{r_1}.$$

Together these estimates complete the proof of Lemma 5.2. □

LEMMA 5.3. *Let $0 < \alpha \le 1$. There exist some positive constant $\delta > 0$ and sufficiently small $\varepsilon > 0$ such that, with $t > 0$,*

$$\int_0^1 |G_1(t,x,y) - G_1^M(t,x,y)|^2 \,\mathrm{d}y \le C t^{-1+\varepsilon} \Delta x^{r_1}, \quad 0 \le x \le 1, \tag{5.16}$$

$$\int_0^t \int_0^1 |G_1(s,x,y) - G_1^M(s,x,y)|^2 \,\mathrm{d}y\mathrm{d}s \le C t^\delta \Delta x^{r_1}, \quad 0 \le x \le 1, \tag{5.17}$$

where r_1 is defined by (1.7).

Proof. For (5.16), we have

$$\begin{aligned}
&\int_0^1 |G_1(t,x,y)-G_1^M(t,x,y)|^2\,\mathrm{d}y\\
&=\int_0^1 \Big|\sum_{j=1}^{\infty} E_{\alpha,1}(-\lambda_j t^{\alpha})\varphi_j(x)\varphi_j(y)-\sum_{j=1}^{M-1} E_{\alpha,1}(-\lambda_j^M t^{\alpha})\varphi_j^M(x)\varphi_j(k_M(y))\Big|^2\,\mathrm{d}y\\
&\le C\int_0^1 \Big|\sum_{j=M}^{\infty} E_{\alpha,1}(-\lambda_j t^{\alpha})\varphi_j(x)\varphi_j(y)\Big|^2\,\mathrm{d}y\\
&\quad +C\int_0^1 \Big|\sum_{j=1}^{M-1} E_{\alpha,1}(-\lambda_j t^{\alpha})\varphi_j(x)\Big[\varphi_j(y)-\varphi_j(k_M(y))\Big]\Big|^2\,\mathrm{d}y\\
&\quad +C\int_0^1 \Big|\sum_{j=1}^{M-1} E_{\alpha,1}(-\lambda_j t^{\alpha})\Big[\varphi_j(x)-\varphi_j^M(x)\Big]\varphi_j(k_M(y))\Big|^2\,\mathrm{d}y\\
&\quad +C\int_0^1 \Big|\sum_{j=1}^{M-1} \Big[E_{\alpha,1}(-\lambda_j t^{\alpha})-E_{\alpha,1}(-\lambda_j^M t^{\alpha})\Big]\varphi_j^M(x)\varphi_j(k_M(y))\Big|^2\,\mathrm{d}y\\
&=I_1(t)+I_2(t)+I_3(t)+I_4(t).
\end{aligned}$$

For $I_1(t)$, we have, by (5.7),

$$I_1(t)=\int_0^1 \Big|\sum_{j=M}^{\infty} E_{\alpha,1}(-\lambda_j t^{\alpha})\varphi_j(x)\varphi_j(y)\Big|^2\,\mathrm{d}y=\sum_{j=M}^{\infty} E_{\alpha,1}^2(-\lambda_j t^{\alpha})\le Ct^{-1+\varepsilon}\Delta x^{r_1},$$

where r_1 is defined by (1.7).

For $I_2(t)$, we have

$$\begin{aligned}
I_2(t)&=\int_0^1 \Big|\sum_{j=1}^{M-1} E_{\alpha,1}(-\lambda_j t^{\alpha})\varphi_j(x)\Big[\varphi_j(y)-\varphi_j(k_M(y))\Big]\Big|^2\,\mathrm{d}y\\
&=\sum_{k=0}^{M-1}\int_{y_k}^{y_{k+1}} \Big|\sum_{j=1}^{M-1} E_{\alpha,1}(-\lambda_j t^{\alpha})\varphi_j(x)\int_{y_k}^{y}\varphi_j'(z)\,\mathrm{d}z\Big|^2\,\mathrm{d}y\\
&=\sum_{k=0}^{M-1}\int_{y_k}^{y_{k+1}} \Big|\int_{y_k}^{y}\sum_{j=1}^{M-1} E_{\alpha,1}(-\lambda_j t^{\alpha})\varphi_j(x)\varphi_j'(z)\,\mathrm{d}z\Big|^2\,\mathrm{d}y\\
&\le\sum_{k=0}^{M-1}\int_{y_k}^{y_{k+1}} \frac{1}{M}\int_{y_k}^{y_{k+1}}\Big|\sum_{j=1}^{M-1} E_{\alpha,1}(-\lambda_j t^{\alpha})\varphi_j(x)\varphi_j'(z)\Big|^2\,\mathrm{d}z\mathrm{d}y\\
&=\frac{1}{M^2}\int_0^1 \Big|\sum_{j=1}^{M-1} E_{\alpha,1}(-\lambda_j t^{\alpha})\varphi_j(x)\varphi_j'(z)\Big|^2\,\mathrm{d}z.
\end{aligned}$$

Note that

$$\int_0^1 \varphi_j'(z)\varphi_l'(z)=\begin{cases} j^2\pi^2, & j=l,\\ 0, & j\ne l,\end{cases}$$

and $\varphi_j(x), j = 1,2,\dots$ are bounded, we get

$$I_2(t) \le \big(\frac{1}{M}\big)^2 \sum_{j=1}^{M-1} E_{\alpha,1}^2(-\lambda_j t^\alpha) j^2 = \sum_{j=1}^{M-1} E_{\alpha,1}^2(-\lambda_j t^\alpha)\big(\frac{j}{M}\big)^2.$$

By (5.8), we get

$$I_2(t) \le Ct^{-1+\varepsilon}\Delta x^{r_1},$$

where r_1 is defined by (1.7).

For $I_3(t)$, we have

$$\begin{aligned} I_3(t) &= \int_0^1 \Big| \sum_{j=1}^{M-1} E_{\alpha,1}(-\lambda_j t^\alpha)\Big[\varphi_j(x) - \varphi_j^M(x)\Big]\varphi_j(k_M(y))\Big|^2 \mathrm{d}y \\ &= \sum_{j=1}^{M-1} E_{\alpha,1}^2(-\lambda_j t^\alpha)\Big[\varphi_j(x) - \varphi_j^M(x).\Big]^2. \end{aligned}$$

Note that, for $y_k \le y \le y_{k+1}, k = 0,1,2,\dots,M-1$,

$$|\varphi_j(y) - \varphi_j^M(y)| = |\varphi_j''(c)(y-y_k)(y-y_{k+1})| \le C(j^2\pi^2)\big(\frac{1}{M}\big)^2 \le C\big(\frac{j}{M}\big)^2.$$

Hence we get, by (5.8),

$$I_3(t) \le C\sum_{j=1}^{M-1} E_{\alpha,1}^2(-\lambda_j t^\alpha)\big(\frac{j}{M}\big)^4 \le C\sum_{j=1}^{M-1} E_{\alpha,1}^2(-\lambda_j t^\alpha)\big(\frac{j}{M}\big)^2 \le Ct^{-1+\varepsilon}\Delta x^{r_1},$$

where r_1 is defined by (1.7).

Finally we consider $I_4(t)$. We have

$$\begin{aligned} I_4(t) &= \int_0^t \Big| \sum_{j=1}^{M-1} \Big[E_{\alpha,1}(-t^\alpha\lambda_j) - E_{\alpha,1}(-t^\alpha\lambda_j^M)\Big]\varphi_j^M(x)\varphi_j(k_M(y))\Big|^2 \mathrm{d}y \\ &\le C\sum_{j=1}^{M-1} \Big|E_{\alpha,1}(-t^\alpha\lambda_j) - E_{\alpha,1}(-t^\alpha\lambda_j^M)\Big|^2. \end{aligned}$$

Note that

$$\frac{d}{dt}E_{\alpha,1}(-t^\alpha\lambda_j) = E'_{\alpha,1}(-t^\alpha\lambda_j)\frac{d}{dt}(-t^\alpha\lambda_j),$$

we have, by Lemma 2.1,

$$E'_{\alpha,1}(-t^\alpha\lambda_j) = \frac{\frac{d}{dt}E_{\alpha,1}(-t^\alpha\lambda_j)}{\frac{d}{dt}(-t^\alpha\lambda_j)} = \frac{t^{\alpha-1}\lambda_j E_{\alpha,\alpha}(-t^\alpha\lambda_j)}{-\alpha t^{\alpha-1}\lambda_j} = -\frac{1}{\alpha}E_{\alpha,\alpha}(-t^\alpha\lambda_j).$$

Thus, we get, by using the mean-value theorem

$$\begin{aligned} I_4(t) &\le \sum_{j=1}^{M-1} |E_{\alpha,1}(-t^\alpha\lambda_j) - E_{\alpha,1}(-t^\alpha\lambda_j^M)|^2 = \sum_{j=1}^{M-1} \Big|E'_{\alpha,1}(c)\big(t^\alpha(\lambda_j - \lambda_j^M)\big)\Big|^2 \\ &\le C\sum_{j=1}^{M-1} \Big|E'_{\alpha,1}(-t^\alpha\lambda_j)\big(t^\alpha(\lambda_j - \lambda_j^M)\big)\Big|^2 \le C\sum_{j=1}^{M-1} E_{\alpha,\alpha}^2(-t^\alpha\lambda_j)|t^\alpha(\lambda_j - \lambda_j^M)|^2. \end{aligned}$$

By (1.3), we get, with $0 \leq \gamma_1 \leq 2$,

$$I_4(t) \leq C \sum_{j=1}^{M-1} \frac{1}{(t^{\alpha}\lambda_j)^{2\gamma_1}} |t^{\alpha}(\lambda_j - \lambda_j^M)|^2.$$

Note that [16, line -4, page 7]

$$|\lambda_j - \lambda_j^M| \leq C \frac{j^4}{M^2}, \tag{5.18}$$

we have

$$I_4(t) \leq C \sum_{j=1}^{M-1} \frac{1}{(t^{\alpha}\lambda_j)^{2\gamma_1}} t^{2\alpha} \cdot \frac{j^8}{M^4} \leq C t^{2\alpha - 2\alpha\gamma_1} \sum_{j=1}^{M-1} \frac{j^{8-4\gamma_1}}{M^4}.$$

Case 1. For $0 < \alpha < 1/2$, we have, with $\gamma_1 = 2$,

$$I_4(t) \leq C t^{2\alpha - 4\alpha} \sum_{j=1}^{M-1} \frac{1}{j^8} \cdot \frac{j^8}{M^4} \leq C t^{-2\alpha} \sum_{j=1}^{M-1} \frac{1}{M^4} \leq C t^{-2\alpha} \Delta x^3.$$

Case 2. For $1/2 \leq \alpha \leq 1$, we have, with $2\alpha - 2\alpha\gamma_1 = -1 + \varepsilon$, $\varepsilon > 0$,

$$\begin{aligned} I_4(t) &\leq C t^{-1+\varepsilon} \frac{1}{M^4} \sum_{j=1}^{M-1} \frac{1}{j^{4\frac{2\alpha+1-\varepsilon}{2\alpha} - 8}} \leq C t^{-1+\varepsilon} \frac{1}{M^4} \int_1^M x^{-4\frac{2\alpha+1-\varepsilon}{2\alpha} + 8} \, \mathrm{d}x \\ &\leq C t^{-1+\varepsilon} \frac{1}{M^4} M^{9 - 4\frac{2\alpha+1-\varepsilon}{2\alpha}} \leq C t^{-1+\varepsilon} \Delta x^{4\left(\frac{1-\varepsilon}{2\alpha}\right) - 1}. \end{aligned}$$

Note that the convergence order of $I_4(t)$ is higher than the order Δx^{r_1}, we therefore have

$$I_4(t) \leq C t^{-1+\varepsilon} \Delta x^{r_1},$$

where r_1 is defined by (1.7).

Thus we get

$$\int_0^1 |G_1(t,x,y) - G_1^M(t,x,y)|^2 \, \mathrm{d}y \leq C t^{-1+\varepsilon} \Delta x^{r_1}.$$

For (5.17), we have

$$\int_0^t \int_0^1 |G_1(s,x,y) - G_1^M(s,x,y)|^2 \, \mathrm{d}y \mathrm{d}s \leq C t^{\delta} \Delta x^{r_1}.$$

Together these estimates complete the proof of Lemma 5.3.
□

5.2. Green functions $G_3(t,x,y)$ and its approximation $G_3^M(t,x,y)$. In this subsection, we will consider the bounds of $G_3(t,x,y)$ and its approximation $G_3^M(t,x,y)$ and the error bounds of $G_3(t,x,y)-G_3^M(t,x,y)$ in some suitable norms.

LEMMA 5.4. *Assume that the Assumption 1.1 holds. There exist some positive constant $\delta > 0$ and sufficiently small $\varepsilon > 0$ such that, with $t > 0$,*

$$\int_0^1 |G_3(t,x,y)|^2 \,\mathrm{d}y \le Ct^{-\frac{\alpha}{2}}, \quad 0 \le x \le 1, \tag{5.19}$$

$$\int_0^t \int_0^1 |G_3(s,x,y)|^2 \,\mathrm{d}y\mathrm{d}s \le Ct^{\delta}, \quad ,0 \le x \le 1, \quad \textit{if } \frac{2(1-2\gamma)}{\alpha} < 3, \tag{5.20}$$

$$\int_0^1 |G_3(t,y,z) - G_3(t,k_M(y),z)|^2 \,\mathrm{d}z \le Ct^{-1+\varepsilon}\Delta x^{r_3}, \quad 0 \le y \le 1, \tag{5.21}$$

$$\int_0^t \int_0^1 |G_3(s,y,z) - G_1(s,k_M(y),z)|^2 \,\mathrm{d}z\mathrm{d}s \le Ct^{\delta}\Delta x^{r_3}, \quad 0 \le y \le 1, \tag{5.22}$$

where r_3 is defined by (1.9).

Proof. For (5.19), we have

$$\int_0^1 |G_3(t,x,y)|^2 \,\mathrm{d}y = \int \Big[\sum_{j=1}^{\infty} t^{\alpha+\gamma-1} E_{\alpha,\alpha+\gamma}(-t^{\alpha}\lambda_j)\varphi_j(x)\varphi_j(y)\Big]^2 \,\mathrm{d}y.$$

Since $\{\varphi_j(y)\}_{j=1}^{\infty}$ is an orthonormal basis in $H = L^2(0,1)$ and $\varphi_j(x), j = 1,2,\dots$ are bounded, we get

$$\int_0^1 |G_3(t,x,y)|^2 \,\mathrm{d}y = \sum_{j=1}^{\infty} t^{2(\alpha+\gamma-1)} E^2_{\alpha,\alpha+\gamma}(-t^{\alpha}\lambda_j). \tag{5.23}$$

By boundedness of the Mittag-Leffler function (1.2), we have, with $0 \le \gamma_1 \le 1$,

$$\begin{aligned}\int_0^1 |G_3(t,x,y)|^2 \,\mathrm{d}y &\le Ct^{2(\alpha+\gamma-1)} \sum_{j=1}^{\infty} \Big(\frac{1}{1+t^{\alpha}\lambda_j}\Big)^{2\gamma_1} \\ &\le Ct^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \int_0^{\infty} \Big(\frac{1}{1+y^2}\Big)^{2\gamma_1} \mathrm{d}y. \end{aligned}\tag{5.24}$$

Then (5.19) follows by choosing some $\gamma_1 \in (1/4,1]$ in (5.24).

For (5.20), we have, by (5.19),

$$\int_0^t \int_0^1 |G_3(s,x,y)|^2 \,\mathrm{d}y\mathrm{d}s \le C \int_0^t s^{2(\alpha+\gamma-1)-\frac{\alpha}{2}} \,\mathrm{d}s \le C,$$

if $2(\alpha+\gamma-1) - \frac{\alpha}{2} > -1$, i.e., $\frac{2(1-2\gamma)}{\alpha} < 3$.

For (5.21), we have

$$\begin{aligned}&\int_0^1 |G_3(t,y,z) - G_3(t,k_M(y),z)|^2 \,\mathrm{d}z \\ &= \sum_{j=M}^{\infty} t^{2(\alpha+\gamma-1)} E^2_{\alpha,\alpha+\gamma}(-t^{\alpha}\lambda_j)\Big[\varphi_j(y) - \varphi_j(k_M(y))\Big]^2 \\ &\quad + \sum_{j=1}^{M-1} t^{2(\alpha+\gamma-1)} E^2_{\alpha,\alpha+\gamma}(-t^{\alpha}\lambda_j)\Big[\varphi_j(y) - \varphi_j(k_M(y))\Big]^2 \\ &= I_1 + I_2.\end{aligned}$$

For I_1, we have, using the boundedness of $\varphi_j(y)$ and (1.2), with $0 \leq \gamma_1 \leq 1$,

$$I_1 \leq C \sum_{j=M}^{\infty} t^{2(\alpha+\gamma-1)} \Big(\frac{1}{1+t^\alpha j^2} \Big)^{2\gamma_1} \leq C t^{2(\alpha+\gamma-1)-2\alpha\gamma_1} \sum_{j=M}^{\infty} \frac{1}{j^{4\gamma_1}}. \tag{5.25}$$

Case 1. If $2\gamma-1<0$ and noting that, by Assumption 1.1, $\gamma+\alpha>1/2$, then we choose $\gamma_1 = 1+\frac{2\gamma-1}{2\alpha} - \frac{\varepsilon}{2\alpha}$ and we obtain

$$\begin{aligned} I_1 &\leq C t^{-1+\varepsilon} \sum_{j=M}^{\infty} \frac{1}{j^{4\left(1+\frac{2\gamma-1}{2\alpha}-\frac{\varepsilon}{2\alpha}\right)}} \leq C t^{-1+\varepsilon} \int_M^\infty \frac{1}{x^{4\left(1+\frac{2\gamma-1}{2\alpha}-\frac{\varepsilon}{2\alpha}\right)}} \, \mathrm{d}x \\ &\leq C t^{-1+\varepsilon} \Delta x^{3-\frac{2(1-2\gamma)}{\alpha}-\frac{2\varepsilon}{\alpha}}, \end{aligned}$$

if $0 < \frac{2(1-2\gamma)}{\alpha} < 3 - \frac{2\varepsilon}{\alpha}$.

Case 2. If $2\gamma-1 \geq 0$, then we choose $\gamma_1 = 1-\frac{\varepsilon}{2\alpha}$ and we have

$$\begin{aligned} I_1 &\leq C t^{2(\alpha+\gamma-1)-2\alpha\gamma_1} \sum_{j=M}^{\infty} \frac{1}{j^{4\gamma_1}} = C t^{2(\alpha+\gamma-1)-2\alpha\left(1-\frac{\varepsilon}{2\alpha}\right)} \sum_{j=M}^{\infty} \frac{1}{j^{4\left(1-\frac{\varepsilon}{2\alpha}\right)}} \\ &\leq C t^{(2\gamma-1)-1+\varepsilon} \int_M^\infty \frac{1}{x^{4\left(1-\frac{\varepsilon}{2\alpha}\right)}} \, \mathrm{d}x \leq C t^{(2\gamma-1)-1+\varepsilon} \Delta x^{3-\frac{2\varepsilon}{\alpha}}. \end{aligned}$$

Thus we get

$$I_1 \leq \begin{cases} C t^{-1+\varepsilon} \Delta x^{3-\frac{2(1-2\gamma)}{\alpha}-\frac{2\varepsilon}{\alpha}}, & 0 < \frac{2(1-2\gamma)}{\alpha} < 3-\frac{2\varepsilon}{\alpha}, \quad 2\gamma-1<0, \\ C t^{(2\gamma-1)-1+\varepsilon} \Delta x^{3-\frac{2\varepsilon}{\alpha}}, & 2\gamma-1 \geq 0. \end{cases}$$

For I_2, we get

$$\begin{aligned} I_2 &= \sum_{j=1}^{M-1} t^{2(\alpha+\gamma-1)} E^2_{\alpha,\alpha+\gamma}(-t^\alpha \lambda_j) \Big[\varphi_j(y) - \varphi_j(k_M(y)) \Big]^2 \\ &\leq C \sum_{j=1}^{M-1} t^{2(\alpha+\gamma-1)} E^2_{\alpha,\alpha+\gamma}(-t^\alpha \lambda_j) \big(\frac{j}{M} \big)^2. \end{aligned} \tag{5.26}$$

By (1.2), we get, with $0 \leq \gamma_1 \leq 1$,

$$I_2 \leq C t^{2(\alpha+\gamma-1)-2\alpha\gamma_1} \frac{1}{M^2} \sum_{j=1}^{M-1} \frac{1}{j^{4\gamma_1-2}}.$$

Case 1. If $2\gamma-1<0$ and noting that, by Assumption 1.1, $\alpha+\gamma>1/2$, then we choose $2(\alpha+\gamma-1) - 2\alpha\gamma_1 = -1+\varepsilon$, that is, $\gamma_1 = 1+\frac{2\gamma-1}{2\alpha} - \frac{\varepsilon}{2\alpha}$ and we get

$$I_2 \leq C t^{-1+\varepsilon} \frac{1}{M^2} \sum_{j=1}^{M-1} \frac{1}{j^{4\left(1+\frac{2\gamma-1}{2\alpha}-\frac{\varepsilon}{2\alpha}\right)-2}} \leq C t^{-1+\varepsilon} \frac{1}{M^2} \int_1^M x^{-4\left(1+\frac{2\gamma-1}{2\alpha}-\frac{\varepsilon}{2\alpha}\right)+2} \, \mathrm{d}x,$$

which implies that

$$I_2 = \begin{cases} Ct^{-1+\varepsilon}\Delta x^2, & \frac{2(1-2\gamma)}{\alpha} \leq 1 - \frac{2\varepsilon}{\alpha} \\ Ct^{-1+\varepsilon}\Delta x^{3-\frac{2(1-2\gamma)}{\alpha}-\frac{2\varepsilon}{\alpha}}, & 1-\frac{2\varepsilon}{\alpha} \leq \frac{2(1-2\gamma)}{\alpha} < 3 - \frac{2\varepsilon}{\alpha}. \end{cases}$$

Case 2. If $2\gamma - 1 \geq 0$, then we choose $\gamma_1 = 1 - \frac{\varepsilon}{2\alpha}$ and we have

$$\begin{aligned} I_2 &\leq Ct^{2(\alpha+\gamma-1)-2\alpha\gamma_1} \sum_{j=1}^{M-1} \frac{1}{j^{4\gamma_1}} \frac{j^2}{M^2} \leq Ct^{2(\alpha+\gamma-1)-2\alpha\left(1-\frac{\varepsilon}{2\alpha}\right)} \frac{1}{M^2} \sum_{j=1}^{M-1} \frac{1}{j^{4\left(1-\frac{\varepsilon}{2\alpha}\right)-2}} \\ &\leq Ct^{(2\gamma-1)-1+\varepsilon} \frac{1}{M^2} \int_1^M x^{-4\left(1-\frac{\varepsilon}{2\alpha}\right)+2}\, \mathrm{d}x \leq Ct^{(2\gamma-1)-1+\varepsilon}\Delta x^2. \end{aligned}$$

Thus we get

$$I_2 \leq \begin{cases} Ct^{-1+\varepsilon}\Delta x^2, & \frac{2(1-2\gamma)}{\alpha} \leq 1 - \frac{2\varepsilon}{\alpha}, \quad 2\gamma - 1 < 0, \\ Ct^{-1+\varepsilon}\Delta x^{3-\frac{2(1-2\gamma)}{\alpha}-\frac{2\varepsilon}{\alpha}}, & 1-\frac{2\varepsilon}{\alpha} \leq \frac{2(1-2\gamma)}{\alpha} < 3 - \frac{2\varepsilon}{\alpha}, \quad 2\gamma-1<0, \\ Ct^{(2\gamma-1)-1+\varepsilon}\Delta x^2, & 2\gamma - 1 \geq 0. \end{cases}$$

Note that the convergence order in I_2 is higher than the convergence order in I_1, we therefore obtain

$$\int_0^1 |G_3(t,y,z) - G_3(t,k_M(y),z)|^2\, \mathrm{d}z \leq Ct^{-1+\varepsilon}\Delta x^{r_3},$$

where

$$r_3 = \begin{cases} 2, & \frac{2(1-2\gamma)}{\alpha} \leq 1 - \frac{2\varepsilon}{\alpha}, \quad 2\gamma - 1 < 0, \\ 3 - \frac{2(1-2\gamma)}{\alpha} - \frac{2\varepsilon}{\alpha}, & 1-\frac{2\varepsilon}{\alpha} \leq \frac{2(1-2\gamma)}{\alpha} < 3 - \frac{2\varepsilon}{\alpha}, \quad 2\gamma-1<0, \\ 2, & 2\gamma - 1 \geq 0. \end{cases} \tag{5.27}$$

For (5.22), we get, by (5.21),

$$\int_0^t \int_0^1 |G_3(s,y,z) - G_3(s,k_M(y),z)|^2\, \mathrm{d}z\mathrm{d}s \leq C\int_0^t s^{-1+\varepsilon}\Delta x^{r_3}\, \mathrm{d}s \leq Ct^{\delta}\Delta x^{r_3}.$$

Together these estimates complete the proof of Lemma 5.4. □

LEMMA 5.5. *Assume that the Assumption 1.1 holds. There exist some positive constant $\delta > 0$ and sufficiently small $\varepsilon > 0$ such that, with $t > 0$,*

$$\int_0^1 |G_3^M(t,x,y)|^2\, \mathrm{d}y \leq Ct^{-\frac{\alpha}{2}}, \quad 0 \leq x \leq 1, \tag{5.28}$$

$$\int_0^t \int_0^1 |G_3^M(s,x,y)|^2\, \mathrm{d}y\mathrm{d}s \leq Ct^{\delta}, \quad ,0 \leq x \leq 1, \quad \text{if } \frac{2(1-2\gamma)}{\alpha} < 3, \tag{5.29}$$

$$\int_0^1 |G_3^M(t,y,z) - G_3^M(t,k_M(y),z)|^2\, \mathrm{d}z \leq Ct^{-1+\varepsilon}\Delta x^{r_3}, \quad 0 \leq y \leq 1, \tag{5.30}$$

$$\int_0^t \int_0^1 |G_3^M(s,y,z) - G_1^M(s,k_M(y),z)|^2\, \mathrm{d}z\mathrm{d}s \leq Ct^{\delta}\Delta x^{r_3}, \quad 0 \leq y \leq 1, \tag{5.31}$$

where r_3 is defined by (1.9).

Proof. The proof is of Lemma 5.5 is similar to the proof of Lemma 5.2. We omit the proof here. □

LEMMA 5.6. *Assume that the Assumption 1.1 holds. There exist some positive constant $\delta > 0$ and sufficiently small $\varepsilon > 0$ such that, with $t > 0$,*

$$\int_0^1 |G_3(t,x,y) - G_3^M(t,x,y)|^2 \,\mathrm{d}y \leq Ct^{-1+\varepsilon}\Delta x^{r_3}, \quad 0 \leq x \leq 1, \tag{5.32}$$

$$\int_0^t \int_0^1 |G_3(s,x,y) - G_3^M(s,x,y)|^2 \,\mathrm{d}y\mathrm{d}s \leq Ct^{\delta}\Delta x^{r_3}, \quad 0 \leq x \leq 1, \tag{5.33}$$

where r_3 is defined by (1.9).

Proof. For (5.32), we have

$$\begin{aligned}
&\int_0^1 |G_3(t,x,y) - G_3^M(t,x,y)|^2 \,\mathrm{d}y \\
&\leq C\int_0^1 \Big| \sum_{j=M}^{\infty} t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_j t^{\alpha})\varphi_j(x)\varphi_j(y)\Big|^2 \mathrm{d}y \\
&\quad + C\int_0^1 \Big| \sum_{j=1}^{M-1} t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_j t^{\alpha})\varphi_j(x)\Big[\varphi_j(y) - \varphi_j(k_M(y))\Big]\Big|^2 \mathrm{d}y \\
&\quad + C\int_0^1 \Big| \sum_{j=1}^{M-1} t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_j t^{\alpha})\Big[\varphi_j(x) - \varphi_j^M(x)\Big]\varphi_j(k_M(y))\Big|^2 \mathrm{d}y \\
&\quad + C\int_0^1 \Big| \sum_{j=1}^{M-1} \Big[t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_j t^{\alpha}) - t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_j^M t^{\alpha})\Big]\varphi_j^M(x)\varphi_j(k_M(y))\Big|^2 \mathrm{d}y \\
&= I_1(t) + I_2(t) + I_3(t) + I_4(t).
\end{aligned}$$

For $I_1(t)$, we have, by (5.25),

$$\begin{aligned}
I_1(t) &= \int_0^1 \Big| \sum_{j=M}^{\infty} t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_j t^{\alpha})\varphi_j(x)\varphi_j(y)\Big|^2 \mathrm{d}y \\
&= \sum_{j=M}^{\infty} t^{2(\alpha+\gamma-1)}E^2_{\alpha,\alpha+\gamma}(-\lambda_j t^{\alpha}) \leq Ct^{-1+\varepsilon}\Delta x^{r_3},
\end{aligned}$$

where r_3 is defined by (1.9).

For $I_2(t)$, we have

$$I_2(t) \leq C\sum_{j=1}^{M-1} t^{2(\alpha+\gamma-1)}E^2_{\alpha,\alpha+\gamma}(-\lambda_j t^{\alpha})\big(\frac{j}{M}\big)^2.$$

By (5.26), we get

$$I_2(t) \leq Ct^{-1+\varepsilon}\Delta x^{r_3},$$

where r_3 is defined by (1.9).

For $I_3(t)$, we have, by (5.26),

$$I_3(t) \le C \sum_{j=1}^{M-1} t^{2(\alpha+\gamma-1)} E^2_{\alpha,\alpha+\gamma}(-\lambda_j t^\alpha)\big(\frac{j}{M}\big)^2 \le C t^{-1+\varepsilon} \Delta x^{r_3},$$

where r_3 is defined by (1.9).

Finally we consider $I_4(t)$. We have

$$\begin{aligned} I_4(t) &= \int_0^t \Big| \sum_{j=1}^{M-1} t^{\alpha+\gamma-1} \Big[E_{\alpha,\alpha+\gamma}(-t^\alpha \lambda_j) - E_{\alpha,\alpha+\gamma}(-t^\alpha \lambda_j^M) \Big] \varphi_j^M(x) \varphi_j(k_M(y)) \Big|^2 \mathrm{d}y \\ &\le C \sum_{j=1}^{M-1} t^{2(\alpha+\gamma-1)} \Big| E_{\alpha,1}(-t^\alpha \lambda_j) - E_{\alpha,1}(-t^\alpha \lambda_j^M) \Big|^2. \end{aligned}$$

Note that

$$\frac{d}{dt} E_{\alpha,\alpha+\gamma}(-t^\alpha \lambda_j) = E'_{\alpha,\alpha+\gamma}(-t^\alpha \lambda_j) \frac{d}{dt}(-t^\alpha \lambda_j),$$

we have, by Lemma 2.1,

$$\begin{aligned} E'_{\alpha,\alpha+\gamma}(-t^\alpha \lambda_j) &= \frac{\frac{d}{dt} E_{\alpha,\alpha+\gamma}(-t^\alpha \lambda_j)}{\frac{d}{dt}(-t^\alpha \lambda_j)} \\ &= \frac{t^{-1} E_{\alpha,\alpha+\gamma-1}(-t^\alpha \lambda_j) - (\alpha+\gamma-1) t^{-1} E_{\alpha,\alpha+\gamma}(-t^\alpha \lambda_j)}{-\alpha t^{\alpha-1} \lambda_j} \\ &= \frac{t^{-\alpha} E_{\alpha,\alpha+\gamma-1}(-t^\alpha \lambda_j) - (\alpha+\gamma-1) t^{-\alpha} E_{\alpha,\alpha+\gamma}(-t^\alpha \lambda_j)}{-\alpha \lambda_j}. \end{aligned}$$

Thus, we get, by using the mean-value theorem

$$\begin{aligned} I_4(t) &\le \sum_{j=1}^{M-1} t^{2(\alpha+\gamma-1)} |E_{\alpha,\alpha+\gamma}(-t^\alpha \lambda_j) - E_{\alpha,\alpha+\gamma}(-t^\alpha \lambda_j^M)|^2 \\ &= \sum_{j=1}^{M-1} \Big| t^{\alpha+\gamma-1} E'_{\alpha,\alpha+\gamma}(c) \big(t^\alpha (\lambda_j - \lambda_j^M) \big) \Big|^2 \\ &\le C \sum_{j=1}^{M-1} \Big| t^{\alpha+\gamma-1} E'_{\alpha,\alpha+\gamma}(-t^\alpha \lambda_j) \big(t^\alpha (\lambda_j - \lambda_j^M) \big) \Big|^2 \\ &\le C \sum_{j=1}^{M-1} t^{2(\alpha+\gamma-1)} \frac{t^{-2\alpha}}{\lambda_j^2} \Big[E_{\alpha,\alpha+\gamma-1}(-t^\alpha \lambda_j) - (\alpha+\gamma-1) E_{\alpha,\alpha+\gamma}(-t^\alpha \lambda_j) \Big]^2 |t^\alpha (\lambda_j - \lambda_j^M)|^2 \\ &= C \sum_{j=1}^{M-1} \frac{t^{2(\alpha+\gamma-1)}}{\lambda_j^2} \Big[\Big| E_{\alpha,\alpha+\gamma-1}(-t^\alpha \lambda_j) \Big|^2 + \Big| E_{\alpha,\alpha+\gamma}(-t^\alpha \lambda_j) \Big|^2 \Big] (\lambda_j - \lambda_j^M)^2. \end{aligned}$$

By (1.3), we get, with $0 \leq \gamma_1 \leq 1$,

$$\begin{aligned}
I_4(t) &\leq C \sum_{j=1}^{M-1} \frac{t^{2(\alpha+\gamma-1)}}{\lambda_j^2} \Big| \frac{1}{t^\alpha \lambda_j} \Big|^{2\gamma_1} (\lambda_j - \lambda_j^M)^2 \leq C \sum_{j=1}^{M-1} t^{2(\alpha+\gamma-1)-2\alpha\gamma_1} \frac{1}{j^{4+4\gamma_1}} \frac{j^8}{M^4} \\
&= C t^{2(\alpha+\gamma-1)-2\alpha\gamma_1} \sum_{j=1}^{M-1} \frac{1}{j^{4\gamma_1}} \frac{j^4}{M^4}.
\end{aligned}$$

Case 1. If $2\gamma - 1 < 0$, then we choose $2(\alpha+\gamma-1) = -1+\varepsilon$, that is, $\gamma_1 = 1 + \frac{2\gamma-1}{2\alpha} - \frac{\varepsilon}{2\alpha}$, and obtain

$$\begin{aligned}
I_4(t) &\leq C t^{2(\alpha+\gamma-1)-2\alpha\gamma_1} \sum_{j=1}^{M-1} \frac{1}{j^{4\gamma_1}} \frac{j^4}{M^4} \\
&= C t^{-1+\varepsilon} \sum_{j=1}^{M-1} \frac{1}{j^{4\left(1+\frac{2\gamma-1}{2\alpha}-\frac{\varepsilon}{2\alpha}\right)}} \frac{j^4}{M^4} \qquad \leq C t^{-1+\varepsilon} \frac{1}{M^4} \int_1^M x^{-4\left(1+\frac{2\gamma-1}{2\alpha}-\frac{\varepsilon}{2\alpha}\right)+4} \, \mathrm{d}x \\
&\leq C t^{-1+\varepsilon} \Delta x^{3-\frac{2(1-2\gamma)}{\alpha}-\frac{2\varepsilon}{\alpha}}.
\end{aligned}$$

Case 2. If $2\gamma - 1 \geq 0$, then we choose $\gamma_1 = 1 - \frac{\varepsilon}{2\alpha}$ and we get

$$\begin{aligned}
I_4(t) &= C t^{2(\alpha+\gamma-1)-2\alpha\gamma_1} \sum_{j=1}^{M-1} \frac{1}{j^{4\gamma_1}} \frac{j^4}{M^4} \\
&= C t^{2(\alpha+\gamma-1)-2\alpha\left(1-\frac{\varepsilon}{2\alpha}\right)} \sum_{j=1}^{M-1} \frac{1}{j^{4\left(1-\frac{\varepsilon}{2\alpha}\right)}} \frac{j^4}{M^4} \\
&= C t^{(2\gamma-1)-1+\varepsilon} \frac{1}{M^4} \sum_{j=1}^{M-1} \frac{1}{j^{-\frac{2\varepsilon}{\alpha}}} \\
&\leq C t^{(2\gamma-1)-1+\varepsilon} \frac{1}{M^4} \int_1^M x^{\frac{2\varepsilon}{\alpha}} \, \mathrm{d}x \leq C t^{(2\gamma-1)-1+\varepsilon} \Delta x^{3-\frac{2\varepsilon}{\alpha}}.
\end{aligned}$$

Thus we get

$$I_4(t) \leq \begin{cases} C t^{-1+\varepsilon} \Delta x^{3-\frac{2(1-2\gamma)}{\alpha}-\frac{2\varepsilon}{\alpha}}, & 2\gamma-1<0, \\ C t^{(2\gamma-1)-1+\varepsilon} \Delta x^{3-\frac{2\varepsilon}{\alpha}}, & 2\gamma-1 \geq 0. \end{cases}$$

Note that the convergence order of $I_4(t)$ is higher than the order Δx^{r_3}, we therefore have

$$I_4(t) \leq C t^{-1+\varepsilon} \Delta x^{r_3},$$

where r_3 is defined by (1.9).

Thus we get

$$\int_0^1 |G_3(t,x,y) - G_3^M(t,x,y)|^2 \, \mathrm{d}y \leq C t^{-1+\varepsilon} \Delta x^{r_3}.$$

For (5.17), we have

$$\int_0^t \int_0^1 |G_3(s,x,y) - G_3^M(s,x,y)|^2 \, \mathrm{d}y \mathrm{d}s \leq C t^{\delta} \Delta x^{r_3}.$$

Together these estimates complete the proof of Lemma 5.6.
□

5.3. Green functions $G_2(t,x,y)$ and its approximation $G_2^M(t,x,y)$. In this subsection, we will consider the bounds of $G_2(t,x,y)$ and its approximation $G_2^M(t,x,y)$ and the error bounds of $G_2(t,x,y)-G_2^M(t,x,y)$ in some suitable norms. We obtain the following Lemmas 5.7- 5.9 by choosing $\gamma=0$ in Lemmas 5.7- 5.9, respectively.

LEMMA 5.7. *Let $0<\alpha\leq 1$. There exist some positive constant $\delta>0$ and sufficiently small $\varepsilon>0$ such that, with $t>0$,*

$$\int_0^1 |G_2(t,x,y)|^2\,\mathrm{d}y \leq Ct^{-\frac{\alpha}{2}}, \quad 0\leq x\leq 1, \tag{5.34}$$

$$\int_0^t\int_0^1 |G_2(s,x,y)|^2\,\mathrm{d}y\mathrm{d}s \leq Ct^{\delta}, \quad ,0\leq x\leq 1, \quad \textit{if } \frac{2}{\alpha}<3, \tag{5.35}$$

$$\int_0^1 |G_2(t,y,z)-G_2(t,k_M(y),z)|^2\,\mathrm{d}z \leq Ct^{-1+\varepsilon}\Delta x^{r_2}, \quad 0\leq y\leq 1, \tag{5.36}$$

$$\int_0^t\int_0^1 |G_2(s,y,z)-G_2(s,k_M(y),z)|^2\,\mathrm{d}z\mathrm{d}s \leq Ct^{\delta}\Delta x^{r_2}, \quad 0\leq y\leq 1. \tag{5.37}$$

LEMMA 5.8. *Let $0<\alpha\leq 1$. There exist some positive constant $\delta>0$ and sufficiently small $\varepsilon>0$ such that, with $t>0$,*

$$\int_0^1 |G_2^M(t,x,y)|^2\,\mathrm{d}y \leq Ct^{-\frac{\alpha}{2}}, \quad 0\leq x\leq 1, \tag{5.38}$$

$$\int_0^t\int_0^1 |G_2^M(s,x,y)|^2\,\mathrm{d}y\mathrm{d}s \leq Ct^{\delta}, \quad ,0\leq x\leq 1, \quad \textit{if } \frac{2}{\alpha}<3, \tag{5.39}$$

$$\int_0^1 |G_2^M(t,y,z)-G_2^M(t,k_M(y),z)|^2\,\mathrm{d}z \leq Ct^{-1+\varepsilon}\Delta x^{r_2}, \quad 0\leq y\leq 1, \tag{5.40}$$

$$\int_0^t\int_0^1 |G_2^M(s,y,z)-G_2^M(s,k_M(y),z)|^2\,\mathrm{d}z\mathrm{d}s \leq Ct^{\delta}\Delta x^{r_2}, \quad 0\leq y\leq 1. \tag{5.41}$$

LEMMA 5.9. *Let $0<\alpha\leq 1$. There exist some positive constant $\delta>0$ and sufficiently small $\varepsilon>0$ such that, with $t>0$,*

$$\int_0^1 |G_2(t,x,y)-G_2^M(t,x,y)|^2\,\mathrm{d}y \leq Ct^{-1+\varepsilon}\Delta x^{r_2}, \quad 0\leq x\leq 1, \tag{5.42}$$

$$\int_0^t\int_0^1 |G_2(s,x,y)-G_2^M(s,x,y)|^2\,\mathrm{d}y\mathrm{d}s \leq Ct^{\delta}\Delta x^{r_3}, \quad 0\leq x\leq 1. \tag{5.43}$$

Data Availability Statement. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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