NEW SELF-DUAL CODES FROM 2×2 BLOCK CIRCULANT MATRICES, GROUP RINGS AND NEIGHBOURS OF NEIGHBOURS

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Abstract. In this paper, we construct new self-dual codes from a construction that involves a unique combination; 2×2 block circulant matrices, group rings and a reverse circulant matrix. There are certain conditions, specified in this paper, where this new construction yields self-dual codes. The theory is supported by the construction of self-dual codes over the rings $\mathbb{F}_2$, $\mathbb{F}_2 + u\mathbb{F}_2$ and $\mathbb{F}_4 + u\mathbb{F}_4$. Using extensions and neighbours of codes, we construct 32 new self-dual codes of length 68. We construct 48 new best known singly-even self-dual codes of length 96.

1. Introduction

Linear block codes, and specifically self-dual codes, have rapidly evolved since its introduction in the 1970's ([2, 20, 28, 29]). The double circulant construction (introduced in [5, 24]) is one of the most extensively used techniques to construct self-dual codes. The double circulant construction considers generator matrices of the form $(I|A)$ where A is a circulant matrix. Another useful method of constructing self-dual codes is considering generator matrices of the form $(I|A)$ where A is a block circulant matrix [15]. Furthermore, self-dual codes can be constructed from group rings [3]. In recent years, group rings have been used to construct self-dual codes [9, 10], using some interesting construction methods as extensions of double circulant modifications [17]. In this paper we construct self-dual codes by considering generator matrices as a unique combination of 2×2 block circulant constructions, group rings and reverse circulant matrices. Specifically, we construct self-dual codes from generator matrices of the form:

$$\left[\begin{array}{c|cc} & A & B+C \\ I & & \\ & B+C & A \end{array}\right]$$

where A and B are matrices that arise from a group ring construction and C is a reverse circulant matrix.

The remainder of this paper is set out as follows; firstly, we will introduce fundamental definitions and theorems required for further sections. In section 2, we describe the construction itself. We present the structure of the generator matrix and discuss associated theory in order to put some restrictions on unknowns. These restrictions aim to maximise the practicality of the construction method by reducing the search field. Following the theory, we look at the numerical results from certain groups of order 4, 8 and 17. We then apply extensions and consider neighbours of codes as methods of finding new codes. Finally, we apply the construction directly over $\mathbb{F}_4 + u\mathbb{F}_4$ to construct new codes of length 96.

2. Preliminaries

Firstly, we describe some essential definitions in coding theory. A code over a finite commutative ring R is defined as any subset C of R^n, where an element of C is called a codeword. If a code, C,

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satisfies $C = C^{\perp}$ then C is said to be self-dual, alternatively if $C \subseteq C^{\perp}$ then the code is said to be self-orthogonal. The Hamming weight enumerator of a code is defined as:

$$W_C(x,y) = \sum_{\mathbf{c} \in C} x^{n-wt(\mathbf{c})} y^{wt(\mathbf{c})}.$$

For binary codes, a self-dual code where all weights are congruent to 0 (mod 4) is said to be Type II, and otherwise, Type I. If a code satisfies $W_C(x,y) = W_{C^{\perp}}(x,y)$ then the code is said to be formally self-dual. The bounds on the minimum distances, $d(n)$ for Type I and Type II codes are ([30])

$$d(n) \leq 4\lfloor \frac{n}{24} \rfloor + 4$$

and

$$d(n) \leq \begin{cases} 4\lfloor \frac{n}{24} \rfloor + 4 \text{ if } n \not\equiv 22 \pmod{24} \\ 4\lfloor \frac{n}{24} \rfloor + 6 \text{ if } n \equiv 22 \pmod{24} \end{cases}$$

If these bounds are met for self-dual codes, they are called extremal. Although the theoretical results are based around finite Frobenius rings of characteristic 2, the numerical results are based on the rings $\mathbb{F}_2$, $\mathbb{F}_2 + u\mathbb{F}_2$ and $\mathbb{F}_4 + u\mathbb{F}_4$.

Now consider the commutative ring $\mathbb{F}_2 + u\mathbb{F}_2 := \mathbb{F}_2[X]/(X^2)$, where u satisfies $u^2 = 0$. Note that this ring is also defined as R_1 since $R_k = \mathbb{F}_2[u_1, u_2, \ldots, u_k]/\langle {u_i}^2, u_i u_j - u_j u_i \rangle$. The elements of the ring may be written as $0, 1, u$ and $1 + u$, where 1 and $1 + u$ are the units of $\mathbb{F}_2 + u\mathbb{F}_2$. Secondly, we consider $\mathbb{F}_4 + u\mathbb{F}_4$; the commutative ring of size 16, which can be viewed as an extension of $\mathbb{F}_2 + u\mathbb{F}_2$. Therefore, we can express any element of $\mathbb{F}_4 + u\mathbb{F}_4$ in the form $\omega a + (1+\omega)b$, where $a, b \in \mathbb{F}_2 + u\mathbb{F}_2$. These rings are generalised in [12] and [13]. The most effective way of displaying these results is to use the hexadecimal system. This is achieved by use of the ordered basis $\{u\omega, \omega, u, 1\}$:

$$\begin{array}{rcl} 0 & \leftrightarrow & 0000,\ 1 \leftrightarrow 0001,\ 2 \leftrightarrow 0010,\ 3 \leftrightarrow 0011, \\ 4 & \leftrightarrow & 0100,\ 5 \leftrightarrow 0101,\ 6 \leftrightarrow 0110,\ 7 \leftrightarrow 0111, \\ 8 & \leftrightarrow & 1000,\ 9 \leftrightarrow 1001,\ A \leftrightarrow 1010,\ B \leftrightarrow 1011, \\ C & \leftrightarrow & 1100,\ D \leftrightarrow 1101,\ E \leftrightarrow 1110,\ F \leftrightarrow 1111. \end{array}$$

The following Gray Maps were introduced in [8, 14, 27];

$$\begin{array}{rcl} \psi_{\mathbb{F}_4} & : & a\omega + b\overline{\omega} \mapsto (a,b),\ \ a,b \in \mathbb{F}_2^n \\ \varphi_{\mathbb{F}_2+u\mathbb{F}_2} & : & a + bu \mapsto (b, a+b),\ \ a,b \in \mathbb{F}_2^n \\ \psi_{\mathbb{F}_4+u\mathbb{F}_4} & : & a\omega + b\overline{\omega} \mapsto (a,b),\ \ a,b \in (\mathbb{F}_2 + u\mathbb{F}_2)^n \\ \varphi_{\mathbb{F}_4+u\mathbb{F}_4} & : & a + bu \mapsto (b, a+b),\ \ a,b \in \mathbb{F}_4^n \end{array}$$

These Gray maps preserve orthogonality in the respective alphabets, [25, 27]. The binary codes $\varphi_{\mathbb{F}_2+u\mathbb{F}_2} \circ \psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ and $\psi_{\mathbb{F}_4} \circ \varphi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ are equivalent to each other.

Proposition 2.1. ([27]) *Let C be a code over $\mathbb{F}_4 + u\mathbb{F}_4$. If C is self-orthogonal, so are $\psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ and $\varphi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$. C is a Type I (resp. Type II) code over $\mathbb{F}_4 + u\mathbb{F}_4$ if and only if $\varphi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ is a Type I (resp. Type II) $\mathbb{F}_4$-code, if and only if $\psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ is a Type I (resp. Type II) $\mathbb{F}_2 + u\mathbb{F}_2$-code. Furthermore, the minimum Lee weight of C is the same as the minimum Lee weight of $\psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ and $\varphi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$.*

Corollary 2.2. *Suppose that C is a self-dual code over $\mathbb{F}_4 + u\mathbb{F}_4$ of length n and minimum Lee distance d. Then $\varphi_{\mathbb{F}_2+u\mathbb{F}_2} \circ \psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ is a binary $[4n, 2n, d]$ self-dual code. Moreover, C and $\varphi_{\mathbb{F}_2+u\mathbb{F}_2} \circ \psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ have the same weight enumerator. If C is Type I (Type II), then so is $\varphi_{\mathbb{F}_2+u\mathbb{F}_2} \circ \psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$.*

Theorem 2.3. *([26]) Let C be a self-dual code of length n over a commutative Frobenius ring with identity R and $G = (r_i)$ be a $k \times n$ generator matrix for C, where r_i is the i-th row of G, $1 \leq i \leq k$. Let c be a unit in R such that $c^2 = -1$ and X be a vector in S^n with $\langle X, X \rangle = -1$. Let $y_i = \langle r_i, X \rangle$ for $1 \leq i \leq k$. The following matrix*

$$\left[\begin{array}{cc|c} 1 & 0 & X \\ \hline y_1 & cy_1 & r_1 \\ \vdots & \vdots & \vdots \\ y_k & cy_k & r_k \end{array}\right],$$

generates a self-dual code D over R of length $n+2$.

Two self-dual binary codes of length $2n$ are said to be neighbours of each other if their intersection has dimension $n-1$. Let $x \in \mathbb{F}_2^{2n} \setminus \mathcal{C}$ then $\mathcal{D} = \left\langle \langle x \rangle^{\perp} \cap \mathcal{C}, x \right\rangle$ is a neighbour of $\mathcal{C}$.

Terminology discussed in this paper required for group rings are as follows: Let G be a finite group of order n, then the group ring RG consists of $\sum_{i=1}^{n} \alpha_i g_i$, $\alpha_i \in R$, $g_i \in G$. Addition in the group ring is defined as:

$$\sum_{i=1}^{n} \alpha_i g_i + \sum_{i=1}^{n} \beta_i g_i = \sum_{i=1}^{n} (\alpha_i + \beta_i) g_i. \tag{1}$$

The product of two elements in a group ring is defined as:

$$\left(\sum_{i=1}^{n} \alpha_i g_i\right)\left(\sum_{j=1}^{n} \beta_j g_j\right) = \sum_{i,j} \alpha_i \beta_j g_i g_j. \tag{2}$$

It follows, that the coefficient of g_k in the product is $\sum_{g_i g_j = g_k} \alpha_i \beta_j$. Note that, e_G denotes the identity element of any group G.

The following construction of a matrix was first given by Hurley [22]. This was utilised to provide a link between the Automorphism group of a code and the underlying group under a certain construction in [10] among other results. Let R be a finite commutative Frobenius ring and let $G = \{g_1, g_2, \ldots, g_n\}$ be the elements of a group of order n in a given listing. Let $v = \sum_{i=1}^{n} \alpha_{g_i} \in RG$. We define the matrix $\sigma(v) \in M_n(R)$ to be $\sigma(v) = (\alpha_{g_i^{-1} g_j})$ where $i, j \in \{1, 2, \cdots, n\}$.

In this work, we refer to two special types of matrices. A circulant $n \times n$ matrix is denoted $\mathrm{circ}(\alpha_1, \alpha_2, \cdots, \alpha_n)$, where each row vector is rotated one element to the left relative to the preceding row vector [7]. Additionally, a reverse circulant $n \times n$ matrix is denoted $\mathrm{rcir}(\alpha_1, \alpha_2, \cdots, \alpha_n)$, where each row vector is rotated one element to the right relative to the preceding row vector. The notation $CIR(A_1, A_2, \cdots, A_m)$ denotes the block circulant matrix where the first row of block matrices are $A_1, \ldots, A_n$. If $v = \sum_{i=0}^{n-1} \alpha_i x^i \in RC_n$, then $\sigma(v) = \mathrm{circ}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ where $C_n = \langle x \mid x^n = 1 \rangle$ and $\alpha_i \in R$. We will now look at the structure of the matrix $\sigma(v)$ where v is an element of $C_{2 \cdot p}$.

Let $C_{2 \cdot p} = \langle x \mid x^{2p} = 1 \rangle$ and

$$v = \sum_{i=0}^{p-1} \sum_{j=0}^{1} \alpha_{i+pj+1} x^{2i+j} \in RC_{2 \cdot p}$$

then,

$$\sigma(v) = \begin{pmatrix} A_1 & A_2 \\ A_2' & A_1 \end{pmatrix}$$

where $A_j = \mathrm{circ}(\alpha_{(j-1)p+1}, \alpha_{(j-1)p+2}, \ldots, \alpha_{jp})$ and $A_j' = \mathrm{circ}(\alpha_{jp}, \alpha_{(j-1)p+1}, \ldots, \alpha_{jp-1})$.

Recall the canonical involution $* : RG \to RG$ on a group ring RG is given by $v^* = \sum_g \alpha_g g^{-1}$, for $v = \sum_g \alpha_g g \in RG$. If v satisfies $vv^* = 1$, then we say that v is a unitary unit in RG. Furthermore, note that $\sigma(v^*) = \sigma(v)^T$.

Following these fundamental theorems and definitions, we will now introduce a new construction and present the theory allowing this method to construct new self-dual codes.

3. Construction

Consider the matrix $M(\sigma)$, where v_1 and v_2 are distinct group ring elements from the same group ring RG where R is a finite Frobenius commutative ring of characteristic 2 and G is a finite group of order n. $\sigma(v)$ is a matrix generated from a group ring element and A denotes a reverse circulant matrix. The construction is given as:

$$M(\sigma) = \left[\begin{array}{c|cc} & \sigma(v_1) & \sigma(v_2)+A \\ I_{2n} & & \\ & \sigma(v_2)+A & \sigma(v_1) \end{array}\right]$$

Let C_σ be the code generated by the matrix $M(\sigma)$. Clearly, C_σ has length $4n$. We will now establish conditions when C_σ is a self-dual code. We will also create a link between unitary units in RG and the above construction yielding self-dual codes. The first Lemma shows a generalisation of our construction and the conditions for a self-dual code.

Lemma 3.1. *Let R be a finite commutative Frobenius ring of characteristic 2 and let B and C be $n \times n$ matrices over R. Then, the matrix*

$$M = \left[\begin{array}{c|cc} & B & C \\ I_{2n} & & \\ & C & B \end{array}\right]$$

generates a self-dual code iff $(B+C)(B+C)^T = I_n$ and $BC^T = CB^T$.

Proof. Clearly, the code generated by M has free rank $2n$, as the left-hand side of the matrix M is the $2n \times 2n$ identity matrix. The code generated by M is self-dual iff the code generated by M is self-orthogonal. Now,

$$MM^T = I_{2n} + \begin{pmatrix} B & C \\ C & B \end{pmatrix}\begin{pmatrix} B^T & C^T \\ C^T & B^T \end{pmatrix} = \begin{pmatrix} I_n + BB^T + CC^T & BC^T + CB^T \\ CB^T + BC^T & I_n + CC^T + BB^T \end{pmatrix}$$

and $MM^T = 0$ iff $I_n + BB^T + CC^T = 0$ and $BC^T + CB^T = 0$. Adding these equations, we obtain

$$I_n + BB^T + CC^T + BC^T + CB^T = 0 \Longleftrightarrow (B+C)(B+C)^T = I_n.$$

■

Using this result, we can consider the matrix $M(\sigma)$ and the conditions for C_σ, the code generated by $M(\sigma)$, to be self-dual.

Theorem 3.2. *Let R be a finite commutative Frobenius ring of characteristic 2 and let G be a finite group of order n. Then, C_σ generates a self-dual code of length $4n$ iff $(\sigma(v_1+v_2)+A)(\sigma((v_1+v_2)^*)+A) = I_n$ and $\sigma(v_1)(\sigma((v_1+v_2)^*)+A) = (\sigma(v_1+v_2)+A)\sigma(v_1^*)$.*

Proof. By the previous result, C_σ generates a self-dual code iff

$$(\sigma(v_1)+\sigma(v_2)+A)(\sigma(v_1)+\sigma(v_2)+A)^T = I_n \text{ and } \sigma(v_1)(\sigma(v_2)+A)^T = (\sigma(v_2)+A)\sigma(v_1)^T.$$

Now, $\sigma(v_1)+\sigma(v_2)+A = \sigma(v_1+v_2)+A$ and

$$\begin{aligned}(\sigma(v_1)+\sigma(v_2)+A)^T &= \sigma(v_1)^T + \sigma(v_2)^T + A^T \\ &= \sigma(v_1^*) + \sigma(v_2^*) + A \\ &= \sigma(v_1^* + v_2^*) + A \\ &= \sigma((v_1+v_2)^*) + A.\end{aligned}$$

Clearly, $\sigma(v_1)(\sigma(v_2)+A)^T = (\sigma(v_2)+A)\sigma(v_1)^T$ is equivalent to

$$\sigma(v_1)\sigma(v_1)^T + \sigma(v_1)(\sigma(v_2)+A)^T = \sigma(v_1)\sigma(v_1)^T + (\sigma(v_2)+A)\sigma(v_1)^T.$$

Considering the left-and right-hand sides separately, we obtain:

$$\begin{aligned}\sigma(v_1)\sigma(v_1)^T + \sigma(v_1)(\sigma(v_2)+A)^T &= \sigma(v_1)\sigma(v_1^*) + \sigma(v_1)(\sigma(v_2)^T + A^T)\\ &= \sigma(v_1)\sigma(v_1^*) + \sigma(v_1)\sigma(v_2^*) + \sigma(v_1)A\\ &= \sigma(v_1)(\sigma(v_1^*) + \sigma(v_2^*) + A)\\ &= \sigma(v_1)(\sigma(v_1^* + v_2^*) + A)\\ &= \sigma(v_1)(\sigma((v_1+v_2)^*) + A).\end{aligned}$$

and

$$\begin{aligned}(\sigma(v_2)+A)\sigma(v_1)^T + \sigma(v_1)\sigma(v_1)^T &= \sigma(v_1)\sigma(v_1^*) + (\sigma(v_2)+A)\sigma(v_1^*)\\ &= \sigma(v_1)\sigma(v_1^*) + \sigma(v_2)\sigma(v_1^*) + A\sigma(v_1^*)\\ &= (\sigma(v_1) + \sigma(v_2) + A)\sigma(v_1^*)\\ &= (\sigma(v_1+v_2) + A)\sigma(v_1^*).\end{aligned}$$

■

In addition to the main theorem of this paper, we will now present some interesting results regarding circulant and reverse circulant matrices over rings and the structure of $\sigma(v)$.

Lemma 3.3. *Let R be a finite commutative Frobenius ring of characteristic 2, A be an $n \times n$ reverse circulant over R and V be an $n \times n$ circulant matrix over R. Then,*

$$AV^T + VA^T = 0. \tag{3}$$

Proof. Let $V = \mathrm{circ}(v_1, v_n, v_{n-1}, \ldots, v_3, v_2)$. Clearly, $V = v_1 I_n + v_2 P + v_3 P^2 + \cdots + v_n P^{n-1}$ where $P = \mathrm{circ}(0, 0, \ldots, 0, 1)$ and $A = \mathrm{rcirc}(a_1, a_2, \ldots, a_{n-1}, a_n)$. Now,

$$\begin{aligned}V^T &= v_1 I_n^T + v_2 P^T + v_3 (P^2)^T + \cdots + v_n (P^{n-1})^T\\ &= v_1 I_n + v_2 P^T + v_3 (P^T)^2 + \cdots + v_n (P^T)^{n-1}.\end{aligned}$$

As $A = A^T$, it remains to show that $AP^T + PA = 0$. Finally,

$$PA = \mathrm{circ}(0, 0, \ldots, 0, 1) \cdot \mathrm{rcirc}(a_1, a_2, \ldots, a_{n-1}, a_n) = \mathrm{rcirc}(a_n, a_1, \ldots, a_{n-1})$$

and

$$AP^T = \mathrm{rcirc}(a_1, a_2, \ldots, a_{n-1}, a_n) \cdot \mathrm{circ}(0, 1, \ldots, 0, 0) = \mathrm{rcirc}(a_n, a_1, \ldots, a_{n-1}).$$

■

Lemma 3.4. *Let R be a commutative ring and let $G = \{g_1 = e, \ldots, g_n\}$ be a finite group of order $n > 1$. The $\sigma(v)$ is symmetric for any $v \in RG$ if and only if G is abelian group of exponent 2.*

Proof. Clearly, $\sigma(v)$ is symmetric for any $v \in RG$ if and only if $\alpha_{g_i^{-1} g_j} = \alpha_{g_j^{-1} g_i}$ $(i, j = 1, \ldots, n)$ for any $v = \sum_{g \in G} \alpha_g g \in RG$. Furthermore, we have $g_i^{-1} g_j = g_j^{-1} g_i$ $(i = 1, \ldots, n)$ $(i, j = 1, \ldots, n)$ or $xy = y^{-1}x^{-1}$ for any $x, y \in G$. Note that for an abelian group of exponent 2, $yxy = x^{-1}$ or $xyxy = e$ or $(xy)^2 = e$ for any $x, y \in G$. Therefore, we have that $g^2 = e$ for any $g \in G$; thus, G has exponent 2.

It is interesting to note that any group of exponent 2 is abelian because $xyxy = e$ and $xxyy = ee = e$ since x and y are commutative for any $x, y \in G$. ■

Lemma 3.5. *Let R be a commutative ring. An $n \times n$-matrix X satisfies $XA = AX^T$ for any $n \times n$ reverse circulant matrix A over R if and only if X is a circulant matrix.*

Proof. This proof follows from lemma 3.3. Let X be an $n \times n$-matrix which satisfies $XA = AX^T$. Then

$$XA = A^T X^T$$

and

$$XA = (XA)^T$$

for any $n \times n$ reverse circulant matrix A over R. This implies that XA is symmetric. Let $D = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix} = \text{rcirc}(0, 0, \dots, 0, 1)$, $X = (x_{i,j})$. Clearly, we have $D^2 = I_n$ and $XDDA$ is symmetric for any $n \times n$ reverse circulant matrix A over R. Therefore, $(x_{i,n-j})DA$ is symmetric.

So we have $(x_{i,n-j})B$ is symmetric for any $n \times n$ circulant matrix B over R. This is equivalent to the fact that $(x_{i,n-j})P^k$ is symmetric for any $k \in \{1, \dots, n\}$ and $n \times n$ matrix $P = \text{circ}(0, 0, \dots, 0, 1)$. Thus, $(x_{i,(k-j) \bmod n+1})$ is symmetric for any $k \in \{1, \dots, n\}$. We have

$$x_{i,(k-j) \mod n+1} = x_{j,(k-i) \mod n+1} \; i, j, k \in \{1, \dots, n\}$$

It is easy to see that $j' = (k - j) \mod (n + 1)$ equivalent to $j = (k - j') \mod (n + 1)$ where $i, j, j', k \in \{1, \dots, n\}$. So

$$x_{i,j'} = x_{(k-j') \mod n+1,(k-i) \bmod n+1} \; i, j', k \in \{1, \dots, n\}$$

Thus $((k - j') \mod (n + 1)) - ((k - i) \mod (n + 1)) \equiv i - j \pmod n$. Therefore, we have that $x_{i,j'}$ is constant if $(i - j) \bmod n$ is fixed. Thus, X is circulant. ■

Lemma 3.6. *Let R be a finite commutative Frobenius ring of characteristic 2 and let G be a finite abelian group of order n of exponent 2. Then, C_σ generates a self-dual code of length $4n$ if $\sigma(v_1), \sigma(v_2)$ are circulant matrices, $\sigma((v_1 + v_2)^2) + A^2 = I_n$.*

Proof. We note that $A\sigma(v_1^*) = \sigma(v_1)A$, $A\sigma(v_2^*) = \sigma(v_2)A$ by lemma 3.3. By lemma 3.4 for any $v \in RG$ $\sigma(v)$ is symmetric, so $\sigma(v^*) = \sigma(v)^T = \sigma(v)$. We also know by Theorem 3.2 that C_σ generates a self-dual code iff

$$(\sigma(v_1) + \sigma(v_2) + A)(\sigma(v_1) + \sigma(v_2) + A)^T = I_n \text{ and } \sigma(v_1)(\sigma(v_2) + A)^T = (\sigma(v_2) + A)\sigma(v_1)^T.$$

Now,

$$\begin{aligned}(\sigma(v_1) + \sigma(v_2) + A)(\sigma(v_1) + \sigma(v_2) + A)^T &= (\sigma(v_1 + v_2) + A)(\sigma((v_1 + v_2)^*) + A) \\ &= \sigma(v_1 + v_2)\sigma((v_1 + v_2)^*) + [\sigma(v_1 + v_2)A + A\sigma((v_1 + v_2)^*)] + A^2 \\ &= \sigma((v_1 + v_2)(v_1 + v_2)^*) + A^2 = \sigma((v_1 + v_2)^2) + A^2 = I_n.\end{aligned}$$

and

$$\begin{aligned}\sigma(v_1)(\sigma(v_2) + A)^T + (\sigma(v_2) + A)\sigma(v_1)^T &= \sigma(v_1)\sigma(v_2^*) + [\sigma(v_1)A + A\sigma(v_1^*)] + \sigma(v_2)\sigma(v_1^*) \\ &= \sigma(v_1 v_2) + \sigma(v_2 v_1) \\ &= \sigma(v_1 v_2) + \sigma(v_1 v_2) = 0.\end{aligned}$$

■

Lemma 3.7. *Let R be a finite commutative Frobenius ring of characteristic 2 and let G be a finite cyclic group of order n. Then, C_σ generates a self-dual code of length $4n$ iff $\sigma((v_1 + v_2)(v_1 + v_2)^*) + A^2 = I_n$ and $v_1 v_2^* = v_2 v_1^*$.*

Proof. We note that $A\sigma(v^*) = \sigma(v)A$ for all $v \in RG$ by the previous result. We also know that C_σ generates a self-dual code iff

$$(\sigma(v_1) + \sigma(v_2) + A)(\sigma(v_1) + \sigma(v_2) + A)^T = I_n \text{ and } \sigma(v_1)(\sigma(v_2) + A)^T = (\sigma(v_2) + A)\sigma(v_1)^T.$$

Now,

$$\begin{aligned}(\sigma(v_1)+\sigma(v_2)+A)(\sigma(v_1)+\sigma(v_2)+A)^T &= (\sigma(v_1+v_2)+A)(\sigma((v_1+v_2)^*)+A)\\ &= \sigma(v_1+v_2)\sigma((v_1+v_2)^*)+[\sigma(v_1+v_2)A+A\sigma((v_1+v_2)^*)]+A^2\\ &= \sigma((v_1+v_2)(v_1+v_2)^*)+A^2=I_n\end{aligned}$$

and

$$\begin{aligned}\sigma(v_1)(\sigma(v_2)+A)^T+(\sigma(v_2)+A)\sigma(v_1)^T &= \sigma(v_1)\sigma(v_2^*)+[\sigma(v_1)A+A\sigma(v_1^*)]+\sigma(v_2)\sigma(v_1^*)\\ &= \sigma(v_1v_2^*)+\sigma(v_2v_1^*)\\ &= \sigma(v_1v_2^*+v_2v_1^*).\end{aligned}$$

Finally, $\sigma(v_1v_2^*+v_2v_1^*)=0$ iff $v_1v_2^*=v_2v_1^*$. ■

Lemma 3.8. *Let R is a finite commutative Frobenius ring of characteristic 2 and let G be a finite abelian group of order n. Let C_σ be self-dual. If $A=0$, then v_1+v_2 is unitary.*

Proof. If C_σ is self-dual and $A=0$, then $\sigma((v_1+v_2)(v_1+v_2)^*)=I_n$ and $(v_1+v_2)(v_1+v_2)^*=1$. ■

This concludes the theoretical part of this paper. We will now show the numerical results.

4. Numerical Results

In this section, we construct 32 new self-dual codes of length 68 and 48 new self-dual codes of length 96. We begin with the construction of self-dual codes of length 64 from groups of order 4 and 8. Using Theorem 2.3, we construct new self-dual codes of length 68. Next, we construct codes of length 68 from groups of order 17. We then find new self-dual codes of length 68 by finding neighbours of these codes and neighbours of these neighbours. We conclude this section by constructing new self-dual codes of 96 from groups of order 6. Magma ([4]) was used to construct all of the codes throughout this section.

4.1. **New codes of length 68.** The possible weight enumerators for a self-dual Type I $[64,32,12]$-code are given in [6, 11] as:

$$\begin{aligned}W_{64,1} &= 1+(1312+16\beta)\,y^{12}+(22016-64\beta)\,y^{14}+\cdots, 14\le\beta\le 284,\\ W_{64,2} &= 1+(1312+16\beta)\,y^{12}+(23040-64\beta)\,y^{14}+\cdots, 0\le\beta\le 277.\end{aligned}$$

Extremal singly even self-dual codes with weight enumerators $W_{64,1}$ are known ([1, 18, 32]):

$$\beta\in\left\{\begin{array}{c}14,16,18,19,20,22,24,25,26,28,29,30,32,34,\\ 35,36,38,39,44,46,49,53,54,58,59,60,64,74\end{array}\right\}$$

and extremal singly even self-dual codes with weight enumerator $W_{64,2}$ are known for:

$$\beta\in\left\{\begin{array}{c}0,...,40,41,42,44,45,46,47,48,49,50,51,52,54,55,56,57,\\ 58,60,62,64,69,72,80,88,96,104,108,112,114,118,120,184\end{array}\right\}\setminus\{31,39\}.$$

The weight enumerator of a self-dual $[68,34,12]_2$ code is in one of the following forms:

$$\begin{aligned}W_{68,1} &= 1+(442+4\beta)y^{12}+(10864-8\beta)y^{14}+\ldots,\\ W_{68,2} &= 1+(442+4\beta)y^{12}+(14960-8\beta-256\gamma)y^{14}+\ldots\ ,\end{aligned}$$

where β and γ are parameters and $0\le\gamma\le 9$.

The existence of codes in $W_{68,1}$ are known for ([9]) β =104, 105, 112, 115, 117, 119, 120, 122, 123, 125,..., 284, 287, 289,291, 294, 301, 302, 308, 313, 315, 322, 324, 328,..., 336, 338, 339, 345, 347, 350, 355, 379 and 401.

The first examples of codes with a $\gamma=7$ in $W_{68,2}$ are constructed in [33]. Together with these, the existence of the codes in $W_{68,2}$ is known for the following parameters (see [18, 33]):

$\gamma = 0,\ \beta \in \{2m | m = 0, 7, 11, 14, 17, 21, \ldots, 99, 102, 105, 110, 119, 136, 165\}$; or
$\beta \in \{2m + 1 | m = 3, 5, 8, 10, 15, 16, 17, 20, \ldots, 82, 87, 93, 94, 101, 104, 110, 115\}$;
$\gamma = 1,\ \beta \in \{2m | m = 19, 22, \ldots, 99\}$; or $\beta \in \{2m + 1 | m = 24, \ldots, 85\}$;
$\gamma = 2,\ \beta \in \{2m | m = 29, \ldots, 100, 103, 104\}$; or$\beta \in \{2m + 1 | m = 32, \ldots, 81, 84, 85, 86\}$;
$\gamma = 6$ with $\beta \in \{2m | m = 69, 77, 78, 79, 81, 88\}$
$\gamma = 7$ with $\beta \in \{7m | m = 14, \ldots, 39, 42\}$.

Note that all binary codes of length 68 with an automorphism of order 17 are classified in [16]. Firstly, we construct self-dual codes of length 64 from C_4 (over $\mathbb{F}_4 + u\mathbb{F}_4$), $C_{2\cdot4}$ (over $\mathbb{F}_2 + u\mathbb{F}_2$) and C_8 (over $\mathbb{F}_2 + u\mathbb{F}_2$). We then construct three self-dual codes of length 68 (Table 4) by applying Theorem 2.3 to the codes constructed in Tables 1,2 and 3. We replace $1 + u \in \mathbb{F}_2 + u\mathbb{F}_2$ with 3 to save space.

TABLE 1. Self-dual code over $\mathbb{F}_4 + u\mathbb{F}_4$ of length 64 from C_4 and C_4.

A_i	$v_1 \in C_4$	$v_2 \in C_4$	r_A	$\lvert Aut(A_i)\rvert$	β
1	(8966)	(0000)	(A617)	2^4	0

TABLE 2. Self-dual code over $\mathbb{F}_2 + u\mathbb{F}_2$ of length 64 from C_8 and C_8.

B_i	$v_1 \in C_8$	$v_2 \in C_8$	r_A	$\lvert Aut(B_i)\rvert$	β
1	(uuu10311)	(uu011uu0)	(u0300013)	2^3	0

TABLE 3. Self-dual code over $\mathbb{F}_2 + u\mathbb{F}_2$ of length 64 from $C_{2\cdot4}$ and $C_{2\cdot4}$.

C_i	$v_1 \in C_{2\cdot4}$	$v_2 \in C_{2\cdot4}$	r_A	$\lvert Aut(C_i)\rvert$	β
1	(uu01u0u1)	(u0u11u31)	(u3u3u3u0)	2^4	48

TABLE 4. Self-dual code of length 68 from extensions of C_1, C_2 and C_3.

D_i	Code	c	X	γ	β	$\lvert Aut(E_i)\rvert$
1	A_1	1	(0133010303011u1001333u01031uuu1u)	4	113	2
2	B_1	$u+1$	(013011030003013301111030uuu13u10)	**2**	**61**	2
3	C_1	$u+1$	(0u10303u110333001103u00130103303)	**1**	**179**	2

We now construct two self-dual codes of length 68 using C_{17} (Table 5). We let $v_2 = 0 \in RC_{17}$. We note that in this case, the construction is equivalent to the usual four circulant construction.

TABLE 5. Self-dual codes over $\mathbb{F}_2$ of length 68 ($W_{68,2}$) from C_{17} and C_{17}.

E_i	$v_1 \in C_{17}$	$v_2 \in C_{17}$	r_A	$\lvert Aut(D_i)\rvert$	γ	β
1	(00000000000011011)	(00000000000000000)	(00100110010110111)	$2^2 \cdot 17$	0	238
2	(00000000110001111)	(00000000000000000)	(00100100101010101)	$2^2 \cdot 17$	0	272

We now construct neighbours of these codes and neighbours of these neighbours.

TABLE 6. New codes of length 68 from neighbours of E_1 and E_2

F_i	E_i	$(x_{35}, x_{36}, ..., x_{68})$	$\lvert Aut(F_i)\rvert$	γ	β	Type
1	2	(01110111001000110000010010001001l0)	2	**0**	**208**	$W_{68,2}$
2	2	(11100000111110000110000111100110000)	1	**0**	**214**	$W_{68,2}$
3	2	(00010001000011101111000010100110l0)	2	**1**	**191**	$W_{68,2}$
4	2	(00101111111111100011110010101110001)	2	**1**	**202**	$W_{68,2}$
5	1	(10011011111011100110001010000l0110)	1	**1**	**210**	$W_{68,2}$
6	2	(01010010001110011000111100110001l01)	1	**1**	**211**	$W_{68,2}$
7	2	(00101011010101001111000000001010001)	1	**1**	**229**	$W_{68,2}$
8	2	(11111111111111111111011101111111111)	1		**317**	$W_{68,1}$

TABLE 7. New codes of length 68 from neighbours of F_7 and F_8

G_i	F_i	$(x_{35}, x_{36}, ..., x_{68})$	$\lvert Aut(G_i)\rvert$	γ	β	Type
1	8	(00010011011100000000000101011001100)	1	**0**	**218**	$W_{68,2}$
2	7	(01100000100010001110001110001000l0)	1	**1**	**193**	$W_{68,2}$
3	7	(10001001010110000110111100110000000)	1	**1**	**195**	$W_{68,2}$
4	7	(01010010100100100001001001010010001)	1	1	233	$W_{68,2}$
5	7	(01110100100010010010000000100101010)	1	**2**	**193**	$W_{68,2}$
6	7	(11000100110000101101110111011011l111)	1	**2**	**195**	$W_{68,2}$

TABLE 8. New codes of length 68 from neighbours of G_5

H_i	G_i	$(x_{35}, x_{36}, ..., x_{68})$	$\lvert Aut(H_i)\rvert$	γ	β	Type
1	5	(00100101100110000000010111001111110)	1	**1**	**197**	$W_{68,2}$
2	5	(01000010110010111010101101110111111)	1	**1**	**199**	$W_{68,2}$
3	5	(11010010111011010111111101111001l1)	1	**2**	**199**	$W_{68,2}$
4	5	(00110000110011100110000001100000001)	1	**2**	**191**	$W_{68,2}$
5	5	(00011001001100100101010000111100100)	1	**2**	**204**	$W_{68,2}$
6	5	(10111010010000011010010101110111101)	1	**2**	**218**	$W_{68,2}$

TABLE 9. Code of length 68 from the neighbours of D_1

I_i	D_i	$(x_{35}, x_{36}, ..., x_{68})$	$\lvert Aut(I_i)\rvert$	γ	β	Type
1	1	(11110001101100111101110010101111101)	1	5	133	$W_{68,2}$

TABLE 10. Code of length 68 from the neighbours of I_1

J_i	I_i	$(x_{35}, x_{36}, ..., x_{68})$	$\lvert Aut(J_i)\rvert$	γ	β	Type
1	1	(00001000010110001110010101000011l00	1	6	141	$W_{68,2}$

TABLE 11. New codes of length 68 from the neighbours of J_1

K_i	J_i	$(x_{35}, x_{36}, ..., x_{68})$	$\lvert Aut(K_i)\rvert$	γ	β	Type
1	1	(11111111010011000101000010000l0100)	1	**6**	**131**	$W_{68,2}$
2	1	(00000011100101111101110011111001111)	1	**7**	**158**	$W_{68,2}$

TABLE 12. New codes of length 68 from the neighbours of K_2

L_i	K_i	$(x_{35}, x_{36}, ..., x_{68})$	$\lvert Aut(L_i)\rvert$	γ	β	Type
1	2	(01101111110101000111010100110101l01)	1	**7**	**155**	$W_{68,2}$
2	2	(01010101010100010010100111101110010)	1	**7**	**156**	$W_{68,2}$
3	2	(00100111010101010101110111101101l0)	1	**7**	**157**	$W_{68,2}$
4	2	(11011111101101110011111101011011l00)	1	**7**	**159**	$W_{68,2}$
5	2	(10010111110001100011111101100101110)	1	**7**	**160**	$W_{68,2}$
6	2	(11000001001000000101001011000110l0)	1	**7**	**162**	$W_{68,2}$
7	2	(10000100000101100001111100100111l1)	1	**7**	**164**	$W_{68,2}$
8	2	(01000010011011111110100001010100l1)	1	**7**	**165**	$W_{68,2}$
9	2	(00111010001000110111010011111011l1)	1	**7**	**167**	$W_{68,2}$

4.2. **New codes of length 96.** The possible weight enumerators of a singly-even binary self-dual $[96, 48, 16]$ code are given in [21] as

$$W_{96,1} = 1 + (\alpha - 5814)x^{16} + (97280 + 64\beta)x^{18} + (1784320 - 16\alpha - 384\beta)x^{20} + (17626112 + 192\beta)x^{22} + \cdots,$$
$$W_{96,2} = 1 + (\alpha - 5814)x^{16} + (97280 + 64\beta)x^{18} + (1694208 - 16\alpha - 384\beta + 4096\gamma)x^{20} + (18969600 + 192\beta - 49152\gamma)x^{22} + \cdots,$$

where $\alpha, \beta, \gamma \in \mathbb{Z}$. Previously known (α, β, γ) values for weight enumerators $W_{96,1}$ and $W_{96,2}$ can be found online at [31] (see [19, 21, 34]).

We construct 48 new codes of length 96 as the binary images of codes of length 24 over $\mathbb{F}_4 + u\mathbb{F}_4$ from C_6 (Table 13).

TABLE 13. New singly-even binary self-dual $[96, 48, 16]$ codes from C_6 and C_6 over $\mathbb{F}_4 + u\mathbb{F}_4$.

$C_{96,i}$	$v_1 \in C_6$	$v_2 \in C_6$	r_A	$\lvert Aut(C_{96,i})\rvert$	α	β	γ	Type
1	$(17DD00)$	$(DC34EB)$	$(7C111C)$	2^4	**11104**	**−68**	**0**	$W_{96,2}$
2	$(C00E11)$	$(C8BDA9)$	$(F656F5)$	2^4	**10208**	**−52**	**0**	$W_{96,2}$
3	$(6482FF)$	$(0D0D0D)$	$(7C111C)$	$2^4 \cdot 3$	**11328**	**−28**	**0**	$W_{96,2}$
4	$(1236FC)$	$(914FD8)$	$(D4DE6E)$	2^4	**11312**	**−108**	**2**	$W_{96,2}$
5	$(3E222F)$	$(8EBA97)$	$(D4DE6E)$	2^4	**11728**	**−100**	**2**	$W_{96,2}$
6	$(C6EB5F)$	$(EA56C1)$	$(7C111C)$	2^4	**11184**	**−84**	**2**	$W_{96,2}$
7	$(B88D66)$	$(99680F)$	$(7C111C)$	2^4	**10592**	**−80**	**2**	$W_{96,2}$
8	$(1D271F)$	$(A7870E)$	$(6B6DBD)$	2^4	**11184**	**−76**	**2**	$W_{96,2}$
9	$(0A7B3D)$	(126325)	$(6B6DBD)$	2^4	**11488**	**−72**	**2**	$W_{96,2}$
10	$(535DD1)$	$(F1CECB)$	$(6B6DBD)$	2^4	**10624**	**−64**	**2**	$W_{96,2}$
11	$(C2F3D9)$	$(1EDF0A)$	$(6B6DBD)$	2^4	**10944**	**−60**	**2**	$W_{96,2}$
12	$(D4787D)$	$(9FCD5D)$	$(6B6DBD)$	2^4	**11224**	**−56**	**2**	$W_{96,2}$
13	$(344A57)$	$(47F231)$	$(7C111C)$	2^4	**10728**	**−48**	**2**	$W_{96,2}$
14	$(D399AB)$	$(6DB3F0)$	$(D4DE6E)$	2^4	**12320**	**−156**	**4**	$W_{96,2}$
15	$(F7A016)$	$(AE0EBF)$	$(D4DE6E)$	2^4	**11104**	**−140**	**4**	$W_{96,2}$
16	$(EF2862)$	$(8867A5)$	$(F656F5)$	2^4	**11528**	**−136**	**4**	$W_{96,2}$
17	$(A56B03)$	(317717)	$(7C111C)$	2^4	**11472**	**−132**	**4**	$W_{96,2}$
18	$(4250B6)$	$(979C73)$	$(D4DE6E)$	2^4	**11728**	**−120**	**4**	$W_{96,2}$
19	$(01A176)$	$(CA0455)$	$(F656F5)$	2^4	**11360**	**−116**	**4**	$W_{96,2}$
20	$(FE26F3)$	$(23B01B)$	$(F656F5)$	2^4	**11160**	**−112**	**4**	$W_{96,2}$
21	$(6C02AE)$	$(6F098F)$	$(6B6DBD)$	2^4	**11328**	**−112**	**4**	$W_{96,2}$
22	$(F79924)$	$(AA77C9)$	$(D4DE6E)$	2^4	**11568**	**−112**	**4**	$W_{96,2}$
23	$(5FFB7B)$	$(4A6DD5)$	$(7C111C)$	2^4	**11088**	**−108**	**4**	$W_{96,2}$
24	$(3522FB)$	$(C05E9F)$	$(6B6DBD)$	2^4	**11488**	**−108**	**4**	$W_{96,2}$
25	$(9E88C6)$	$(07DE86)$	$(7C111C)$	2^4	**11072**	**−104**	**4**	$W_{96,2}$
26	$(088C5F)$	$(77601A)$	$(F656F5)$	2^4	**10672**	**−100**	**4**	$W_{96,2}$
27	(313674)	$(343BD9)$	$(7C111C)$	2^4	**10944**	**−100**	**4**	$W_{96,2}$
28	$(35EA9C)$	(930785)	$(7C111C)$	2^4	**11048**	**−96**	**4**	$W_{96,2}$
29	(505084)	$(57696E)$	$(F656F5)$	2^4	**11064**	**−88**	**4**	$W_{96,2}$
30	$(6D4401)$	$(92206E)$	$(6B6DBD)$	2^4	**11504**	**−84**	**4**	$W_{96,2}$
31	$(58263B)$	$(D98510)$	$(6B6DBD)$	2^4	**10888**	**−80**	**4**	$W_{96,2}$
32	$(9AE7CA)$	$(74D032)$	$(F656F5)$	2^4	**12504**	**−160**	**6**	$W_{96,2}$
33	$(73A8CF)$	$(D46308)$	$(F656F5)$	2^4	**11552**	**−156**	**6**	$W_{96,2}$
34	$(F97D3B)$	$(6B7D82)$	$(6B6DBD)$	2^4	**11872**	**−156**	**6**	$W_{96,2}$
35	$(B4196E)$	$(97B0E5)$	$(D4DE6E)$	2^4	**11376**	**−148**	**6**	$W_{96,2}$

TABLE 13. (continued)

$C_{96,i}$	$v_1 \in C_6$	$v_2 \in C_6$	r_A	$\lvert Aut(C_{96,i})\rvert$	α	β	γ	Type
36	$(47E5CD)$	$(CECECE)$	$(6B6DBD)$	$2^4 \cdot 3$	**11736**	**−148**	**6**	$W_{96,2}$
37	$(6B78E6)$	$(113CD9)$	$(F656F5)$	2^4	**11576**	**−140**	**6**	$W_{96,2}$
38	$(B1C856)$	$(F7452D)$	$(D4DE6E)$	2^4	**12448**	**−140**	**6**	$W_{96,2}$
39	$(FC0863)$	$(18BD3B)$	$(D4DE6E)$	2^4	**11008**	**−132**	**6**	$W_{96,2}$
40	$(DC4A91)$	$(A58C34)$	$(6B6DBD)$	2^4	**11304**	**−132**	**6**	$W_{96,2}$
41	$(8798CD)$	$(FD6017)$	$(7C111C)$	2^4	**11312**	**−120**	**6**	$W_{96,2}$
42	$(9217CF)$	$(DCD676)$	$(7C111C)$	2^4	**12928**	**−192**	**8**	$W_{96,2}$
43	$(C620D5)$	$(EAE546)$	$(7C111C)$	2^4	**11768**	**−172**	**8**	$W_{96,2}$
44	$(3617E2)$	$(19B065)$	$(7C111C)$	2^4	**11272**	**−168**	**8**	$W_{96,2}$
45	$(3BAE33)$	$(5F852E)$	$(7C111C)$	2^4	**11968**	**−168**	**8**	$W_{96,2}$
46	$(E90589)$	$(D62FE2)$	$(D4DE6E)$	2^4	**12896**	**−260**	**12**	$W_{96,2}$
47	$(B89454)$	$(F5F331)$	$(D4DE6E)$	2^4	**12288**	**−244**	**12**	$W_{96,2}$
48	$(E9DA51)$	$(6D030D)$	$(6B6DBD)$	2^4	**12320**	**−244**	**12**	$W_{96,2}$

5. Conclusion

In this work, we introduced a new construction that involved both block circulant matrices and a reverse circulant matrix. We demonstrated the relevance of this new construction by constructing many binary self-dual codes, including new self-dual codes of length 68 and 96. To summarise the numerical results, we construct the following unknown $W_{68,1}$ code:

$$\beta = \{317\}.$$

Furthermore, we construct the following unknown $W_{68,2}$ codes:

$$\begin{aligned}
&(\gamma = 0, \quad \beta = \{208, 214, 218\}),\\
&(\gamma = 1, \quad \beta = \{179, 191, 193, 195, 197, 199, 202, 210, 211, 229\}),\\
&(\gamma = 2, \quad \beta = \{61, 191, 193, 195, 199, 204, 218\}),\\
&(\gamma = 6, \quad \beta = \{131\}),\\
&(\gamma = 7, \quad \beta = \{155, 156, 157, 158, 159, 160, 162, 164, 165, 167\})
\end{aligned}$$

We also construct the following new codes of length 96 with weight enumerator $W_{96,2}$:

$$\begin{aligned}
&(\gamma = 0, \quad (\alpha, \beta) = \{(11104, -68), (10208, -52), (11328, -28)\}),\\
&(\gamma = 2, \quad (\alpha, \beta) = \{(11312, -108), (11728, -100), (11184, -84), (10592, -80),\\
&\qquad (11184, -76), (11488, -72), (10624, -64), (10944, -60), (11224, -56),\\
&\qquad (10728, -48)\}),\\
&(\gamma = 4, \quad (\alpha, \beta) = \{(12320, -156), (11104, -140), (11528, -136), (11472, -132),\\
&\qquad (11728, -120), (11360, -116), (11160, -112), (11328, -112), (11568, -112),\\
&\qquad (11088, -108), (11488, -108), (11072, -104), (10672, -100), (10944, -100),\\
&\qquad (11048, -96), (11064, -88), (11504, -84), (10888, -80)\}),\\
&(\gamma = 6, \quad (\alpha, \beta) = \{(12504, -160), (11552, -156), (11872, -156), (11376, -148),\\
&\qquad (11736, -148), (11576, -140), (12448, -140), (11008, -132),\\
&\qquad (11304, -132), (11312, -120)\}),\\
&(\gamma = 8, \quad (\alpha, \beta) = \{(12928, -192), (11768, -172), (11272, -168), (11968, -168)\}),\\
&(\gamma = 12, \quad (\alpha, \beta) = \{(12896, -260), (12288, -244), (12320, -244)\})
\end{aligned}$$

Regarding this construction, we were restricted to small group rings due to computational limitations. With a higher computational power, it would be possible to investigate larger group rings which would yield more results. One could also consider other families of rings.

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