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ORIGINAL PAPER

Commuting Involution Graphs for Certain Exceptional Groups of Lie Type

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Abstract
Suppose that G is a finite group and X is a G-conjugacy classes of involutions. The commuting involution graph $\mathcal{C}(G,X)$ is the graph whose vertex set is X with $x, y \in X$ being joined if $x \neq y$ and $xy = yx$. Here for various exceptional Lie type groups of characteristic two we investigate their commuting involution graphs.

Keywords Commuting involution graphs · Exceptional groups of Lie type · Disc structure

1 Introduction

Suppose that G is a finite group and X is a subset of G. The commuting graph, $\mathcal{C}(G,X)$, has X as its vertex set and two vertices $x, y \in X$ are joined by an edge if $x \neq y$ and x and y commute. The extensive bibliography in [9] points towards the many varied commuting graphs which have been studied. But here we shall be considering commuting involution graphs—these are commuting graphs $\mathcal{C}(G,X)$ where X is a G-conjugacy class of involutions. From now on X is assumed to be a G-conjugacy class of involutions. Because involutions are often centre stage in the study of non-abelian simple groups, there is a large literature on their commuting involution graphs. Indeed, such graphs have been instrumental in the construction of some of the sporadic simple groups. For example, the three Fischer groups with the conjugacy class being the 3-transpositions were investigated by Fischer [11], resulting in the construction of these groups. Later, also prior to their construction, commuting involution graphs for the Baby Monster ($\{3, 4\}$-transpositions) and the

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Monster (6-transpositions) were analyzed. Recently the commuting involution graphs of the sporadic simple groups have received much attention, see [5, 12, 14, 15, 17]. For those simple groups of Lie type consult [1, 4, 8–10], while an analysis of the commuting involution graphs of finite Coxeter groups may be found in [2, 3].

The aim of this short note is to describe certain features of $\mathcal{C}(G,X)$ when G is one of the exceptional Lie type groups of characteristic two. Specifically we consider G being one of the simple groups ${}^3D_4(2), E_6(2), {}^2F_4(2)'$ and $F_4(2)$.

For $x \in X$ we define the i^{th} disc of x, $\Delta_i(x)$, $(i \in \mathbb{N})$ to be

$$\Delta_i(x) = \{y \in X \mid d(x,y) = i\}$$

where $d(,)$ is the usual distance metric on the graph $\mathcal{C}(G,X)$. Of course, G acting by conjugation on X embeds G in the group of graph automorphisms of $\mathcal{C}(G,X)$ and, evidently, G is transitive on the vertices of $\mathcal{C}(G,X)$. We now choose $t \in X$ to be a fixed vertex of $\mathcal{C}(G,X)$—our main focus is the description of the discs of t in $\mathcal{C}(G,X)$. The diameter of $\mathcal{C}(G,X)$ will be denoted by $\mathrm{Diam}\,\mathcal{C}(G,X)$and we shall rely upon the Atlas [7] for the names of conjugacy classes of G. Our main result is as follows.

Theorem 1 *Let G be isomorphic to one of ${}^3D_4(2), E_6(2), {}^2F_4(2)'$ and $F_4(2)$.*

(i) *The sizes of the discs $\Delta_i(t)$ are listed in Table 1 and the G-conjugacy classes of tx for $x \in \Delta_i(t), i \in \mathbb{N}$ are given in Table 2.*

(ii) *If $(G,X) = (E_6(2), 2A), (E_6(2), 2B), ({}^2F_4(2)', 2A), (F_4(2), 2A), (F_4(2), 2B)$ or $(F_4(2), 2C)$, then* $\mathrm{Diam}\,\mathcal{C}(G,X)$= 2.

(iii) *If $(G,X) = ({}^3D_4(2), 2A), ({}^3D_4(2), 2B), (E_6(2), 2C), ({}^2F_4(2)', 2B)$ or $(F_4(2), 2D)$, then* $\mathrm{Diam}\,\mathcal{C}(G,X)$= 3.

Table 1 Disc sizes for $\mathcal{C}(G,X), G \cong {}^3D_4(2), E_6(2), {}^2F_4(2)', F_4(2)$

G	$X = t^G$	$\|\Delta_1(t)\|$	$\|\Delta_2(t)\|$	$\|\Delta_3(t)\|$
${}^3D_4(2)$	2A	18	288	512
	2B	339	11112	57344
$E_6(2)$	2A	127782	4954112	
	2B	285311	8819313408	
	2C	3384671	609992912640	977994252288
${}^2F_4(2)'$	2A	90	1664	
	2B	147	7712	3840
$F_4(2)$	2A	2286	67328	
	2B	2286	67328	
	2C	20944	4364800	
	2D	50511	113896448	236912640

Table 2 The conjugacy class of products tx for $x \in \Delta_i(t)$

G	$X = t^G$	$\Delta_1(t)$	$\Delta_2(t)$	$\Delta_3(t)$
${}^3D_4(2)$	2A	2A	4A	3A
	2B	2AB	3A, 4AC, 6B, 8AB	3B, 6A, 7AD, 9AC, 12A, 13AC, 14AC, 18AC, 21AC, 28AC
$E_6(2)$	2A	2AB	3A, 4B	
	2B	2AC	3AB, 4AF, 4JK, 5A, 6A, 6D, 6F, 8C,12B	
	2C	2AC	3AC, 4AK,6AI, 8AJ, 12A, 12B (123120576^2, 528482304^4), 12CD, 12E (4227858432^3), 12F (4227858432), 12GM, 12P, 16A, 16C, 24A	5A, 7CD, 9AB, 10AB, 12B (528482304^2), 12E (2818572288), 12F (8455716864), 13A, 14GH, 15CD, 17AB, 18AB, 20AB, 21GH, 24BD, 28KL, 30EF
${}^2F_4(2)'$	2A	2AB	4C, 5A	
	2B	2AB	3A, 4AC, 6A, 8CD, 12AB	5A, 13AB
$F_4(2)$	2A	2A, 2C	3A, 4C	
	2B	2B, 2C	3A, 4D	
	2C	2AD	3AB, 4AD, 4F, 4JM, 5A, 6GH	
	2D	2AD	3AC, 4AO, 6AK, 8AF, 8HK, 12AB (294912^4), 12CH, 12IJ (294912^{12}), 12KO	5A, 7AB, 8G, 9AB, 10AC, 12AB (1179648), 12IJ (1179648), 13A,14AB, 15AB, 16AB, 17AB, 18AB, 20AB, 21AB, 24AD, 28AB, 30AB

These results were obtained computationally with the aid of Magma [6] , Gap [16] and the OnLine Atlas [18]. In the course of these calculations we determined the $C_G(t)$-orbits on X (where $C_G(t)$ is acting by conjugation). Representatives, in Magma format, for each of these orbits are to be found as downloadable files at [13], as they may be of value in other investigations of these groups. In Sect. 2 we also collate information on the action of $C_G(t)$ on X. In particular, we give the $C_G(t)$-orbit sizes on each (non-empty) X_C, X_C being defined below.

We observe that some “obvious” groups are missing in this paper. First $G_2(2)'$ being isomorphic to $PSU_3(3)$ means it is covered in [8]. As for $G \cong {}^2 E_6(2)$, the cases $X = 2A$ and $X = 2B$ are done in [1], while there are partial results in the case $X = 2C$. Likewise [1] also has partial results for $E_7(2)$. While $E_8(2)$ is far and away beyond current computational capabilities.

We remark on the graphs studied here. First we note that as the outer automorphism of $F_4(2)$ interchanges the two classes 2A and 2B, we have that $\mathcal{C}(F_4(2), 2A)$ and $\mathcal{C}(F_4(2), 2B)$ are isomorphic graphs. A very noteworthy consequence of the present work is that the distance between t and x in $\mathcal{C}(G, X)$ is almost always determined by the G-class to which tx belongs. The exceptions are $G \cong$

$E_6(2), X = 2C$ with $tx \in 12B \cup 12E \cup 12F$ and $G \cong F_4(2), X = 2D$ and $tx \in 12A \cup 12B \cup 12I \cup 12J$. See Table 2 for more details—for example when $G \cong F_4(2), X = 2D$ and $tx \in 12I \cup 12J$ each of $12I$ and $12J$ breaks into thirteen $C_G(t)$-orbits, 12 of size 294,912 and one of size 1,179,648 with those of size 294,912 being in $\Delta_2(t)$ and the one of size 1,179,648 in $\Delta_3(t)$.

A word or two about the information in our tables is required. As mentioned we employ the class names given in the ATLAS though we make some modifications. First we suppress the "slave" notation. So, for example, the classes $7B * 2, 7C * 4$ of ${}^3D_4(2)$ are just written as $7B$, $7C$, respectively. Secondly we compress the letter part of a class name when we mean the union of these classes and their letters are in alphabetical sequence. As an example, in Table 2, for $G \cong F_4(2)$ and $X = 2D$, $8AF$ is short-hand for $8A \cup 8B \cup 8C \cup 8D \cup 8E \cup 8F$.

Let C be a G-conjugacy class and define

$$X_C = \{x \in X \mid tx \in C\}.$$

It is clear that X_C will either be empty or be a union of certain $C_G(t)$-orbits of X (where G acts upon X by conjugation). In locating which discs of t contain the vertices in X_C we sometimes need to determine how X_C breaks into $C_G(t)$-orbits. Also of interest to us is the size of X_C which leads us to class structure constants. Class structure constants are the sizes of sets

$$\{(g_1, g_2) \in C_1 \times C_2 \mid g_1 g_2 = g\}$$

where C_1, C_2, C_3 are G-conjugacy classes and g is a fixed element of C_3. Now these constants can be calculated directly from the complex character table of G which are recorded in the ATLAS and are available electronically in the standard libraries of the computer algebra package GAP [16]. If we take $C_1 = C$, $C_2 = X = C_3$ and $g = t$, then in this case

$$|X_C| = \frac{|G|}{|C_G(t)||C_G(h)|} \sum_{r=1}^{k} \frac{\chi_r(h)\chi_r(t)\overline{\chi_r(t)}}{\chi_r(1)},$$

where h is a representative from C and $\chi_1, \ldots, \chi_k$ the complex irreducible characters of G.

2 $C_G(t)$-Orbits on X

As promised, we tabulate the sizes of the $C_G(t)$-orbits in their action upon X_C where C is a G-conjugacy class for which X_C is non-empty. In the ensuing tables we use an exponential notation to indicate the multiplicity of a particular size. Thus in the table for $G \cong {}^3D_4(2)$ with $X = 2B$ the entry $4^6, 24^{12}$ next to $2B$ is telling us that X_{2B} is the union of eighteen $C_G(t)$-orbits, six of which have size 4 and twelve of which have size 24. Still looking at the same table, the entry 512, 1536 next to $9AC$ indicates that each of X_{9A}, X_{9B} and X_{9C} is the union of two $C_G(t)$-orbits of sizes 512 and 1536. We give details of the permutation ranks in Table 3.

2.1 $G \cong {}^3D_4(2)$

$X = 2A$

2A	18	3A	512	4A	288

$X = 2B$

2A	3, 24	2B	$4^6, 24^{12}$	3A	384	3B	512
4A	$24^5, 192$	4B	$24^{10}, 192$	4C	384^6	6A	1536
6B	384^6	7AC	512	7D	3072	8A	384^6
8B	384^8	9AC	512, 1536	12A	1536^2	13AC	3072
14AC	1536	18AC	1536^2	21AC	3072	28AC	1536^2

2.2 $G \cong E_6(2)$

$X = 2A$

2A	2790	2B	124992	3A	2097152	4B	2856960

$X = 2B$

2A	$63, 2160^2$	2B	$56, 4320, 30240^2, 30720^2, 64512, 120960$	2C	$60480^2, 725760^2, 967680$
3A	2359296	3B	16777216	4A	774144
4B	$725760, 967680^2, 2211840^2$	4C	$1935360^4, 3870720^4, 4423680^4, 7741440^2$	4D	$7864320^2, 8847360$
4E	46448640^2	4F	$2064384^2, 61931520^4$	4J	123863040^2
4K	743178240	5A	939524096	6A	70778880^2
6D	990904320	6F	1056964608	8C	990904320^2
12B	1132462080^2				

$X = 2C$					
2A	$3, 84, 1536, 2016$	2B	$168, 224, 2016, 5376, 8064^2, 10752^2, 16128, 32256^2, 43008, 86016$	2C	$96^2, 5376, 16128^3, 32256^4$ $36864^4, 64512^4, 86016^4$ $129024^3, 25048^3, 1032192$
3A	$917504, 1572864$	3B	29360128	3C	134217728
4A	$1536, 21504, 32256$ $36864^3, 64512^3, 86016$ $786432, 1032192$	4B	$1536^2, 16128, 32256^4$ $36864, 43008, 64512^4$ $86016, 129024^3, 258048^2$ $786432, 1032192^3$	4C	$64512^4, 129024^6, 258048^2$ $516096^{12}, 688128^2, 1032192^6$ $2064384^8, 4128768^4$
4D	$1032192, 1376256, 2752512^2, 11010048, 16515072$	4E	$258048^4, 516096^{10}, 1032192^{10}, 2064384^{12}, 4128768^{22}, 8257536^2$ 33030144	4F	$1032192^2, 2064384^2, 2752512^2, 4128768^8, 5505024^2, 16515072^8, 33030144^6$
4G	$4128768^2, 8257536^2, 16515072^4, 33030144^3, 66060288^2$	4H	$3748736^2, 66060288^2$	4I	$11010048^2, 16515072^2, 33030144^6, 66060288^7, 88080384, 264241152$
4J	$1376256^2, 2064384^2, 4128768^6, 8257536^{16}, 16515072^{10}, 33030144^{22}$ 66060288^{10}	4K	$4128768, 8257536^6, 16515072^{13}, 33030144^{12}, 66060288^8, 264241152$	5A	$234881024, 1409286144$
6A	$2752512, 33030144^3, 44040192^2$	6B	402653184	6C	$528482304, 704643072$
6D	$1835008, 66060288^4, 88080384^3, 132120576^4, 176160768, 264241152, 528482304$	6E	$37748736^2, 66060288^2, 88080384^2, 132120576^2, 528482304^2$	6F	$88080384, 352321536$ $528482304, 704643072$ 1056964608

continued

6G	2818572288	6H	$1056964608^2, 4227858432$	6I	8455716864
7C	805306368	7D	3221225472	8A	$1572864^2, 33030144^2, 37748736^2, 66060288^2, 88080384^2, 132120576^2$ 528482304^2
8B	$37748736^2, 44040192^2, 66060288^2, 88080384^2, 132120576^2, 528482304^2$	8C	$16515072^2, 33030144^2, 66060288^8, 88080384^2, 132120576^{12}, 264241152^{10}$	8D	$132120576^2, 264241152^{20}$ 1056964608^2
8E	$176160768^2, 528482304^4, 2113929216^2$	8F	2113929216^5	8G	$26441152^4, 528482304^4, 105664608^{12}$
8H	2113929216^3	8I	427858432^6	8J	$1056964608^2, 2113929216^6, 4227858432^4$
9A	22548578304	9B	3221225472, 9663676416	10A	$2818572288^2, 4227858432$
10B	$2818572288, 4227858432^2$ 8455716864, 16911433728	12A	402653184^2	12B	$132120576^2, 528482304^6$
12C	$264241152^8, 528482304^4$ 1056964608^{16}	12D	$1409286144^2, 2113929216^4, 4227858432$	12E	$2818572288, 4227858432^3$
12F	4227858432, 8455716864	12G	5637144576	12H	$352321536^2, 2113929216^6$ 4227858432^4
12I	8455716864^2	12J	$1056964608^2, 2113929216^6, 4227858432^8$	12K	$1409286144^2, 4227858432^4$ 8455716864^4
12L	16911433728	12M	16911433728^2	12P	16911433728^2
13A	19327352832	14G	16911433728	14H	9663676416
15C	22548578304	15D	7516192768, 22548578304	16A	8455716864^4

continued

16C	16911433728^4	17A	45097156608	17B	45097156608
18A	9663676416^2	18B	67645734912	20A	16911433728^2
20B	33822867456^4	21G	19327352832	21H	45097156608
24A	8455716864^8	24B	16911433728^4	24C	33822867456^2
24D	33822867456^2	28K	9663676416^2	28L	33822867456
30E	22548578304^2	30F	67645734912		

Table 3 Class sizes and permutation rank

G	$X = t^G$	$\|X\|$	Permutation rank
$^3D_4(2)$	2A	819	4
	2B	68796	27
$E_6(2)$	2A	5081895	5
	2B	8822169720	62
	2C	1587990549600	719
$^2F_4(2)'$	2A	1755	5
	2B	11700	30
$F_4(2)$	2A	69615	5
	2B	69615	5
	2C	4385745	33
	2D	350859600	1002

2.3 $G \cong {}^2F_4(2)'$

$X = 2A$

2A	10	2B	80	4C	640	5A	1024

$X = 2B$

2A	3, 12	2B	$12^3, 48^2$	3A	256^2	4A	192^2
4B	96^2	4C	$96, 192^2$	5A	768	6A	768^2
8CD	384^2	12AB	768^2	13AB	1536		

2.4 $G \cong F_4(2)$

$X = 2A$

2A	270	2C	2016	3A	32768	4C	34560

$X = 2B$

2B	270	2C	2016	3A	32768	4D	34560

$X = 2C$

2AB	30	2C	$32^2, 180, 1920^2$	2D	$720^2, 960^4, 11520$	3AB	32768
4AB	15360	4CD	11520	4F	1024^2	4JK	30720^2
4L	737280	4M	184320^2	5A	1048576	6GH	983040

$X = 2D$

2AB	$3, 12, 72^2, 192$	2C	$9, 12^2, 24^2, 72^4,$ $144^7, 192^2, 576^4$	2D	$24^4, 144^{29}, 576^{24}$ $1152^{16}, 9216$
3AB	2048, 6144, 24576	3C	262144	4AB	$192, 576^8, 1152^4$ 9216, 12288
4CD	$144^4, 192^3, 288^4$ $576^{13}, 1152^2, 2304^4,$ $4608^4, 12288$	4EF	$576^4, 1536^4$ $2304^4, 9216^8$	4GH	$2304^4, 4608^6, 9216^2$ $18432^4, 73728$
4I	$9216^{14}, 18432^8$ $36864^4, 73728^2$	4JK	$1152^4, 1536^4, 2304^4$ $4608^{20}, 9216^{16}, 18432^{22}$	4L	$9216^9, 36864^8$ 147456^2
4M	$2304^2, 4608^{12}, 9216^{30}$ $18432^{36}, 36864^2$	4N	147456^4	4O	$36864^{12}, 147456^4$
5A	$196608^2, 589824$	6AB	$6144^2, 24576^2, 73728^3$	6CD	$36864^2, 49152^2$ $73728^3, 294912$
6EF	786432	6GH	$12288, 36864^2, 49152^2$ $73728^7, 147456^2, 294912$	6IJ	$73728^8, 147456^4, 294912^4$
6K	2359296	7AB	1572864	8A	294912^4
8B	$147456^8, 294912^4$	8CF	$24576^2, 73728^{10}$ $147456^4, 294912^4$	8G	589824^2
8HI	294912^6	8J	589824^{16}	8K	589824^6
9AB	1572864, 4718592	10AB	$589824^2, 1179648^2$	10C	$589824^2, 1179648^4$
12AB	$294912^4, 1179648$	12CD	786432^2	12EH	$98304^2, 294912^4, 589824^4$
12IJ	$294912^{12}, 1179648$	12KL	2359296^2	12MN	589824^{14}
12O	2359296^4	13A	9437184	14AB	4718592
15AB	1572864, 4718592	16AB	2359296^4	17AB	9437184
18AB	4718592^2	20AB	2359296^4	21AB	9437184
24AD	2359296^4	28AB	4718592^2	30AB	4718592^2

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