Numerical methods for deterministic and stochastic fractional partial differential equations

Thesis submitted in accordance with the requirements of the University of Chester for the degree of Doctor in Philosophy by

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Abstract

In this thesis we will explore the numerical methods for solving deterministic and stochastic space and time fractional partial differential equations. Firstly we consider Fourier spectral methods for solving some linear stochastic space fractional partial differential equations perturbed by space-time white noises in one dimensional case. The space fractional derivative is defined by using the eigenvalues and eigenfunctions of Laplacian subject to some boundary conditions. We approximate the space-time white noise by using piecewise constant functions and obtain the approximated stochastic space fractional partial differential equations. The approximated stochastic space fractional partial differential equations are then solved by using Fourier spectral methods.

Secondly we consider Fourier spectral methods for solving stochastic space fractional partial differential equation driven by special additive noises in one dimensional case. The space fractional derivative is defined by using the eigenvalues and eigenfunctions of Laplacian subject to some boundary conditions. The space-time noise is approximated by the piecewise constant functions in the time direction and by appropriate approximations in the space direction. The approximated stochastic space fractional partial differential equation is then solved by using Fourier spectral methods.

Thirdly, we will consider the discontinuous Galerkin time stepping methods for solving the linear space fractional partial differential equations. The space fractional derivatives are defined by using Riesz fractional derivative. The space variable is discretized by means of a Galerkin finite element method and the time variable is discretized by the discontinous Galerkin method. The approximate solution will be sought as a piecewise polynomial function in t of degree at most $q-1, q \geq 1$, which is not necessarily continuous at the nodes of the defining partition. The error estimates in the fully discrete case are obtained and the numerical examples are given.

Finally, we consider error estimates for the modified L1 scheme for solving time fractional partial differential equation. Jin et al. (2016, An analysis of the L1 scheme for the subdiffiffusion equation with nonsmooth data, IMA J. of Number. Anal., 36, 197-221)

established the $O(k)$ convergence rate for the L1 scheme for both smooth and nonsmooth initial data. We introduce a modified L1 scheme and prove that the convergence rate is $O(k^{2-\alpha}), 0 < \alpha < 1$ for both smooth and nonsmooth initial data. We first write the time fractional partial differential equations as a Volterra integral equation which is then approximated by using the convolution quadrature with some special generating functions. A Laplace transform method is used to prove the error estimates for the homogeneous time fractional partial differential equation for both smooth and nonsmooth data. Numerical examples are given to show that the numerical results are consistent with the theoretical results.

Declaration

No part of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other institution of learning. However some parts of the materials contained herein have been published previously.

Publications

- Y. Liu, M. Khan, Y. Yan, [60] Fourier spectral methods for solving some linear stochastic space fractional partial differential equations, Mathematics, 4 (2016), 45.

- F. Liu, M. Khan, Y. Yan, [59] Fourier spectral methods for solving stochastic space fractional partial differential equation driven by special additive noises, Journal of Computational Analysis and Applications, 24 (2018), 290-309.

- Y. Liu, M. Khan, Y. Yan, [61] Discontinuous Galerkin time stepping method for solving linear space fractional partial differential equations, Applied Numerical Mathematics, 115 (2017), 200-213.

- Y. Yan, M. Khan, N. J. Ford, [96] An analysis of the modified L1 scheme for the time fractional partial differential equations with nonsmooth data, SIAM J. Numer. Anal., 56 (2018), 210-227.

Conference presentations

- Finite difference methods for solving space fractional partial differential equations; Faculty of Applied Sciences Post-graduate Research Conference, 21th March 2015, University of Chester.

- Fourier spectral methods for solving some linear stochastic space fractional partial differential equations, 26th Biennial Numerical Analysis Conference, June 23-26, 2016, Glasgow.

- Fourier spectral methods for solving some linear stochastic space fractional partial differential equations, 7th Conference on Finite Difference Methods: Theory and Applications, June 18-23, 2018, Lozenetz, Bulgaria.

Poster presentations

- Finite difference methods for solving space fractional partial differential equations; Faculty of Applied Sciences Post-graduate Research Conference, June 27, 2013, University of Chester.

- Fourier spectral methods for solving some linear stochastic space fractional partial differential equations; Sixth Conference on Finite Difference Methods: Theory and Applications, June 18-23, 2015, Lozenetz, Bulgaria.

- Fourier spectral methods for solving some linear stochastic space fractional partial differential equations; Faculty of Applied Sciences Post-graduate Research Conference, June 21, 2017, Thornton, University of Chester.

- Fourier spectral methods for solving stochastic space fractional partial differential equation driven by special additive noises; Workshop of "Recent trends in stochastic analysis and partial differential equations ", September 5-6, 2019, Thornton, University of Chester.

Seminar Presentations

- Fourier spectral methods for solving some linear stochastic space fractional partial differential equations; Department of Mathematics, October 16, 2018, Parkgate Road, University of Chester.

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Chapter 1

Introduction

1.1 History of Fractional Calculus

The Fractional Calculus is a generalization of classical calculus concerned with operations of integration and differentiation of non-integer or fractional order. The idea of fractional operators has been introduced almost simultaneously with the development of the classical ones. The reference can be found first known in the correspondence of G. W. Leibniz and Marquis de Hospital in 1695 where the question of meaning of the semi-derivative has been raised. This question consequently attracted the interest of many well-known mathematicians, including Riemann, Grünwald, Letnikov, Euler, Liouville, Laplace, and many others. Since the 19th century, the theory of fractional calculus developed rapidly, mostly as a foundation for a number of applied disciplines, including fractional geometry, fractional differential equations and fractional dynamics. The applications of Fractional Calculus are very wide nowadays. It is safe to say that almost no discipline of modern engineering and science in general, remains untouched by the tools and techniques of fractional calculus. For example, wide and fruitful applications can be found in rheology, viscoelasticity, acoustics, optics, chemical and statistical physics, robotics, control theory, electrical and mechanical engineering, bio-engineering, etc. In fact, one could argue that real world processes are fractional order systems in general. The main reason for the success of fractional calculus applications is that these new fractional-order models are often more accurate than integer-order ones, i.e., there are more degrees of freedom in the fractional order model than in the corresponding classical one. One of the intriguing

beauties of the subject is that fractional derivatives and integrals are not local quantities. The fractional operators consider the entire history of the physical process and can be used to model the non-local and distributed effects often encountered in natural and technical phenomena. Fractional calculus is therefore an excellent set of tools for describing the memory and hereditary properties of various materials and processes.

However the interest in the specific topic of fractional calculus surged only at the end of the last century. Fractional differential equations, that is, those involving real and complex order derivatives, have assumed an important role in modeling the anomalous dynamics of many processes related to complex systems in the most diverse areas of science and engineering. There has been a spectacular increase in the use of fractional differential models to simulate the dynamics of many different anomalous processes, especially those involving ultra-slow diffusion. The following table is only based on the scopus database, but it reflects this state of affairs clearly: [6]

Words in title or abstract	1960-1980	1981-1990	1991-2000	2001-2010
Fractional Brownian Motion	2	38	532	1295
Anomalous Diffusion	185	261	626	1205
Anomalous Relaxation	21	23	70	61
Superdiffusion or Subdiffusion	0	22	121	521
Fractional Models, Kinetics, Dynamics	11	24	128	443
Fractional Differential Equations	1	1	74	943

Table 1.1.1: Evolution in the number of publications on fractional differential equations and their applications.

1.2   Definitions of Fractional Derivatives

There are several definitions given to fractional derivatives. In this section we will give some important definitions of fractional integral and derivative.

Riemann-Liouville fractional integral: The Riemann-Liouville fractional integral of

order $\alpha > 0$ is defined by [27]

$$
{}_0^R D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau. \tag{1.2.1}
$$

Riemann-Liouville fractional derivative: The Riemann-Liouville fractional derivative is defined, with $\alpha > 0$ and $n - 1 < \alpha < n$, $n \in \mathbb{Z}^+$, [27]

$$
{}_0^R D_t^{\alpha} f(t) = D^n D_t^{\alpha-n} f(t) = D^n \frac{1}{\Gamma(n-\alpha)} \int_\alpha^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau. \tag{1.2.2}
$$

Caputo fractional derivative: For $\alpha > 0$, the Caputo fractional derivative is defined, with $n - 1 < \alpha < n$, $n \in \mathbb{Z}^+$, [27]

$$
{}_0^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} D^n f(\tau)\, d\tau. \tag{1.2.3}
$$

When $\alpha = n$, we define

$$
{}_0^R D_t^{\alpha} f(t) = \frac{d^n}{dt^n} f(t). \tag{1.2.4}
$$

Riesz fractional derivative: For $n - 1 < \alpha < n$, $n \in \mathbb{Z}^+$, the Riesz fractional derivative is defined by, [76]

$$
\frac{d^\alpha}{d|x|^\alpha} u(t,x) = -C_\alpha \left({}_0^R D_x^\alpha u(t,x) + {}_x^R D_1^\alpha u(t,x) \right), \tag{1.2.5}
$$

where $C_\alpha = \frac{1}{2\cos(\frac{\pi\alpha}{2})}, \alpha \neq 1$ and

$$
{}_0^R D_x^\alpha u(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{u(\xi,t)}{(x-\xi)^{\alpha+1-n}} d\xi,
$$
$$
{}_x^R D_1^\alpha u(x,t) = \frac{1}{\Gamma(n-\alpha)} (-1)^n \frac{d^n}{dx^n} \int_x^1 \frac{u(\xi,t)}{(\xi-x)^{\alpha+1-n}} d\xi.
$$

There are relations between the different fractional derivatives, see, e.g., Podlubny [76].

1.3 Laplace Transform and Fourier Transform

Laplace transform is an integral transform named after its inventor Pierre-Simon Laplace. It transforms a function of a real variable t to a function of a complex variable s. The Laplace transform has many applications in science and engineering.

The Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, which is a unilateral transform defined by, [8]

$$F(s) = \int_0^\infty f(t)e^{-st}dt,$$

where s is a complex number frequency parameter.

The inverse Laplace Transform of $F(s)$ is defined as

$$f(t) = \frac{1}{2\pi i} \int_\Gamma e^{st}F(s)ds,$$

where Γ is the line with $\Re z = a, a > 0$.

The Fourier transform of a function f is traditionally denoted by $\hat{f}$. There are several common conventions for defining the Fourier transform of an integrable function $f : \mathbb{R} \mapsto \mathbb{C}$. We can write it's Fourier transform, for any real number ξ, [8]

$$\hat{f}(\xi) = \int_{-\infty}^\infty f(x)e^{-2\pi i x \xi}dx.$$

The inverse Fourier transform of $\hat{f}$ is defined by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{f}(\xi)e^{2\pi i x \xi}d\xi.$$

1.4 Stochastic Calculus

Stochastic calculus has come to play an important role in many branches of science and technology where day by day more and more mathematician have encountered in this field. Stochastic calculus is the area of mathematics that deals with processes containing a stochastic component and thus allows the modeling of random systems. Many stochastic processes are based on functions which are continuous, but nowhere differentiable. This rules out differential equations that require the use of derivative terms, since they are unable to be defined on non-smooth functions. Instead, a theory of integration is required where integral equations do not need the direct definition of derivative terms. In quantitative finance, the theory is known as Itô Calculus. We will discuss about stochastic calculus and their properties in Chapter two and Chapter three.

1.5   Contributions of this Work and Outline

In this section, we introduce the topics in each chapter in this thesis.

In Chapter 1, we will discuss the general history of fractional calculus, stochastic calculus.

In Chapter 2, we will discuss the basic notations and properties of stochastic ordinary differential equation. Some of the important results of stochastic calculus are discussed in this chapter. For example probability theory, Brownian motions, stochastic process, stochastic ODE, stochastic integral are discussed here.

In Chapter 3, we will discuss the basic notations and properties of stochastic partial differential equation. We shall introduce Q-Wiener process, Green function, etc. In addition we will present the existence and uniqueness theorems of stochastic partial differential equations.

In Chapter 4, we will discuss Fourier spectral methods for solving parabolic partial differential equations. Here we will consider how to use MATLAB functions "dst.m", "idst.m" and "fft.m", "ifft.m" to solve semilinear parabolic equation by using spectral method.

In Chapter 5, we will discuss Fourier spectral methods for solving some linear stochastic space fractional partial differential equations perturbed by space-time white noises in one dimensional case. The space fractional derivative is defined by using the eigenvalues and eigenfunctions of Laplacian subject to some boundary conditions. We approximate the space-time white noise by using piecewise constant functions and obtain the approximated stochastic space fractional partial differential equations. The approximated stochastic space fractional partial differential equations are then solved by using Fourier spectral methods. Error estimates in L_2-norm are obtained and numerical examples are given.

In Chapter 6, we will discuss Fourier spectral methods for solving stochastic space fractional partial differential equation driven by special additive noises in one dimensional case. The space fractional derivative is defined by using the eigenvalues and eigenfunctions of Laplacian subject to some boundary conditions. The space-time noise is approximated by the piecewise constant functions in the time direction and by appropriate approx-

imations in the space direction. The approximated stochastic space fractional partial differential equations is then solved by using Fourier spectral methods. Error estimates are obtained and numerical examples are given.

In Chapter 7, we will consider the discontinuous Galerkin time stepping methods for solving the linear space fractional partial differential equations. The space fractional derivatives are defined by using Riesz fractional derivative. The space variable is discretized by means of a Galerkin finite element method and the time variable is discretized by the discontinuous Galerkin method. The approximate solution will be sought as a piecewise polynomial function in t of degree at most $q - 1, q \geq 1$, which is not necessarily continuous at the nodes of the defining partition. The error estimates in the fully discrete case are obtained and the numerical examples are given.

In Chapter 8, we consider error estimates for the modified L1 scheme for solving time fractional partial differential equation. Jin et al. (2016, An analysis of the L1 scheme for the subdiffifusion equation with nonsmooth data, IMA J. of Number. Anal., 36, 197-221) established the $O(k)$ convergence rate for the L1 scheme for both smooth and nonsmooth initial data. We introduce a modified L1 scheme and prove that the convergence rate is $O(k^{2-\alpha}), 0 < \alpha < 1$ for both smooth and nonsmooth initial data. We first write the time fractional partial differential equations as a Volterra integral equation which is then approximated by using the convolution quadrature with some special generating functions. The numerical schemes obtained in this way are equivalent to the standard L1 scheme and modified L1 scheme, respectively. A Laplace transform method is used to prove the error estimates for the homogeneous time fractional partial differential equation for both smooth and nonsmooth data. Numerical examples are given to show that the numerical results are consistent with the theoretical results.

Finally in Chapter 9, we outline the summary of the thesis and indicate the further research plans.

Chapter 2

Basics for Stochastic Ordinary Differential Equations

2.1   Introduction

A stochastic differential equation (SDE) is a differential equation in which one or more of the terms are stochastic processes, resulting in a solution which is also a stochastic process. The most common form of SDE in the literature is an ordinary differential equation with the right hand side perturbed by a term dependent on a white noise. Stochastic ODEs are used to model various phenomena such as unstable stock prices or physical systems subject to thermal fluctuations.

2.2   Basic Notations of Probability Theory

Let us consider rolling a die or tossing a coin, with an outcome that changes randomly with each repetition. When the experiment is repeated, the statistical and probabilistic tools are needed to analyze the frequency of the outcome. In particular, we assign a probability to each outcome as a limit of the frequency of occurrence relative to the total number of trials.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ [62, p.137] be the probability space. Here Ω is the sample space and $\mathcal{F}$ is the σ-algebra of Ω and $\mathbb{P}$ is the probability measure defined on $\mathcal{F}$. A probability measure $\mathbb{P}$ on the measurable space $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \mapsto [0, 1]$ such that

1) $\mathbb{P}(\Omega) = 1$;

2) for any disjoint sequence $A_i \subset \mathcal{F}$, that is, $A_i \cap A_j = \phi$, $i \neq j$,

$$\mathbb{P}(\cup A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

If X is a real valued random variable and is integrable with respect to the probability measure $\mathbb{P}$, then the number [62, p.139]

$$\mathbb{E}X = \int_{\Omega} X(\omega)d\mathbb{P}(\omega)$$

is called the expectation of X. The number

$$\mathrm{Var}(X) = \mathbb{E}(X - \mathbb{E}X)^2$$

is called the variance of X.

2.3   Brownian Motions

The name of the Brownian Motion is given to the irregular movement of pollen grains, suspended in water, observed by the Scottish botanist Robert Brown in 1828 [62]. The motion was later explained by the random collisions with the molecules of water. To describe the motion mathematically it is natural to use the concept of a stochastic process $B_t(\omega)$, interpreted as the position of the pollen grain ω at time t. Brownian motion is the actual physical motion of these particles, the Wiener process is the mathematical interpretation of this process. Let us now give the mathematical definition of Brownian motion [62].

Definition 2.3.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\{\mathcal{F}_t\}$. A standard one-dimensional Brownian motion is a real-valued continuous $\{\mathcal{F}_t\}$-adapted process $\{B_t\}_{t \geq 0}$ with the following properties:

1) $B_0 = 0$ a.s;

2) for $0 \leq s < t < \infty$ the increment $B_t - B_s$ is normally distributed with mean zero and variance $t - s$;

3) for $0 \leq s < t < \infty$ the increment $B_t - B_s$ is independent of $\mathcal{F}_s$.

2.4 Stochastic Process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A stochastic process is simply a collection of random variables indexed by time. It will be useful to consider separately the cases of discrete time and continuous time. That is, a discrete time stochastic process $X = \{X_n, n = 0, 1, 2, ...\}$ is a countable collection of random variables indexed by the non-negative integers, and a continuous time stochastic process $X = \{X_t, 0 \leq t < \infty\}$ is an uncountable collection of random variables indexed by the non-negative real numbers [62].

2.5 Stochastic Integral

Now we consider how to define the stochastic integral

$$\int_0^T f(t)dB(t),$$

where $B(t) = B_t$ is the standard Brownian motion. Here $f(t) = f(t, \omega) : [0, \infty) \times \Omega \mapsto \mathbb{R}$ is a measurable function. Since $B(t)$ is not of bounded variation, we cannot define $\int_0^T f(t)dB_t$ by using usual Riemanm-Stieltjes integration method. We need to introduce other way to define the stochastic integral. For example, we may define the integral for a large class of stochastic processes by making use of the properties of Brownian motions. Such integral was first defined by K. Itô in 1949 and now it's known as Itô stochastic integral [62].

2.6 Stochastic Ordinary Differential Equation

We now consider the following stochastic ordinary differential equation [62],

$$du = f(u(t))dt + G(u(t))dB(t), \tag{2.6.1}$$
$$u(0) = u_0.$$

This stochastic ordinary differential equation can be written as the following integral form,

$$u(t) = u_0 + \int_0^t f(u(s))ds + \int_0^t G(u(s))\, dB(s), \tag{2.6.2}$$

where $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a vector-valued function known as the drift, $G : \mathbb{R}^d \mapsto \mathbb{R}^{d \times m}$ is a matrix-valued function, known as the diffusion. Here $B(t) = [B_1(t), B_2(t), \ldots, B_m(t)] \in \mathbb{R}^m$ is a $\mathbb{R}^m$-valued Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, and $\int_0^t G(u(s))B(s)$ denotes the Itô integral, see e.g., [62, page 314].

Assumption 2.6.1. [62, page 325]

Let $d \in \mathbb{Z}^+$. There exists a constant $L > 0$ such that the following linear growth conditions hold:

$$\|f(u)\|_{\mathbb{R}^d}^2 \leq L(1 + \|u\|_{\mathbb{R}^d}^2), \tag{2.6.3}$$

$$\|G(u)\|_{\mathbb{R}^{d \times m}}^2 \leq L(1 + \|u\|_{\mathbb{R}^d}^2), \quad \forall \, u \in \mathbb{R}^d,$$

and the following global Lipschitz conditions hold:

$$\|f(u_1) - f(u_2)\|_{\mathbb{R}^d} \leq L\|u_1 - u_2\|_{\mathbb{R}^d}, \tag{2.6.4}$$

$$\|G(u_1) - G(u_2)\|_{\mathbb{R}^{d \times m}} \leq L\|u_1 - u_2\|_{\mathbb{R}^d}, \quad \forall \, u_1, u_2 \in \mathbb{R}^d.$$

We remark that in this thesis, we will use $\|\cdot\|_{\mathbb{R}^d}$ or $|\cdot|_{\mathbb{R}^d}$ to denote the Euclidean norm in $\mathbb{R}^d$.

Definition 2.6.1. [65] A real valued stochastic process $g = \{g(t)\}_{a \leq t \leq b}$ is called a simple process if there exist a partition $a = t_0 < t_1 < \cdots < t_k = b$ of $[a, b]$ and the bounded random variables $\xi_i, 0 \leq i \leq k - 1$ such that ξ_i are $\mathcal{F}_{t_i}$- measurable and

$$g(t) = \begin{cases} \xi_0, & t_0 \leq t \leq t_1, \\ \xi_1, & t_1 \leq t \leq t_2, \\ \ldots \\ \xi_{k-1}, & t_{k-1} \leq t \leq t_k. \end{cases} \tag{2.6.5}$$

Denote by $\mathcal{M}_0([a, b]; \mathbb{R})$ the family of all such processes defined in (2.6.5).

Definition 2.6.2. [65] For a simple process $g \in \mathcal{M}_0([a, b]; \mathbb{R})$, we define

$$\int_a^b g(t)dB_t = \sum_{i=1}^{k-1} \xi_i(B_{t_{i+1}} - B_{t_i}), \tag{2.6.6}$$

and call it the stochastic integral or Itô integral of g with respect to the Brownian motion $\{B_t\}$.

More precisely, the stochastic integral $\int_a^b g(t)dB_t$ is $\mathcal{F}_b$-measurable. We will now show that it belongs to $L^2(\Omega, \mathbb{R})$.

Lemma 2.6.2. [65] If $g \in \mathcal{M}_0([a, b]; \mathbb{R})$, then we have

$$\mathbb{E} \int_a^b g(t)\, dB_t = 0,$$

$$\mathbb{E} \Big| \int_a^b g(t)dB_t \Big|^2 = \mathbb{E} \int_a^b |g(t)|^2 dt.$$

Proof. Since ξ_i is $\mathcal{F}_{t_i}$ measurable and $(B_{t_{i+1}} - B_{t_i})$ is independent of $\mathcal{F}_{t_i}$, we have

$$\mathbb{E} \int_a^b g(t)dB_t = \sum_{i=0}^{k-1} \mathbb{E}[\xi_i(B_{t_{i+1}} - B_{t_i})] = \sum_{i=0}^{k-1} \xi_i \mathbb{E}(B_{t_{i+1}} - B_{t_i}) = 0.$$

Moreover, we have, noting that $B_{t_{j+1}} - B_{t_j}$ is independent of $\xi_i \xi_j (B_{t_{i+1}} - B_{t_i})$ if $i < j$,

$$\begin{aligned}
\mathbb{E} \Big| \int_a^b g(t)dB_t \Big|^2 &= \sum_{0 \leq i,j \leq k-1} \mathbb{E}[\xi_i \xi_j (B_{t_{i+1}} - B_{t_i})(B_{t_{j+1}} - B_{t_j})] \\
&= \sum_{i=0}^{k-1} \mathbb{E}\xi_i^2 (B_{t_{i+1}} - B_{t_i})^2 = \sum_{i=0}^{k-1} \mathbb{E}\xi_i^2 (B_{t_{i+1}} - B_{t_i})^2 \\
&= \sum_{i=0}^{k-1} \mathbb{E}\xi_i^2 (B_{t_{i+1}} - B_{t_i})^2 = \mathbb{E} \int_a^b |g(t)|^2 dt.
\end{aligned}$$

The proof is complete. $\qquad\square$

We may also define the stochastic integral $\int_a^b f(t)dB_t$ for any $\mathcal{M}^2([a, b]; \mathbb{R})$.

Definition 2.6.3. [65] Let $0 \leq a < b < \infty$. Denote by $\mathcal{M}^2([a, b]; R)$ the space of all real valued measurable $\{\mathcal{F}_t\}$ adapted stochastic process $f = \{f(t)\}_{a \leq t \leq b}$ such that

$$||f||_{a.b}^2 = \mathbb{E} \int_a^b |f(t)|^2 dt < \infty.$$

Definition 2.6.4. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space. Let $H_{2,T}$ be the set of $\mathbb{R}^d$-valued predictable process $\{u(t) : t \in [0, T]\}$ such that

$$||u||_{H_{2,T}} := \sup_{t \in [0,T]} ||u(t)||_{L^2(\Omega, \mathbb{R}^d)} = \sup_{t \in [0,T]} \mathbb{E}\big(|u(t)|_{\mathbb{R}^d}^2\big)^{\frac{1}{2}} < \infty. \tag{2.6.7}$$

Theorem 2.6.3. (Contraction mapping) [62, page 2] Let Y be a non-empty closed subset of the Banach space $(X, \| \ . \ \|)$. Consider a mapping $\mathcal{J} : Y \to Y$ such that, for some $\mu \in (0, 1)$,

$$\| \mathcal{J}u - \mathcal{J}v \| \leq \mu \| u - v \|, \quad \forall u, v \in Y. \tag{2.6.8}$$

Then there exists a unique fixed point of $\mathcal{J}$ in Y, that is, there is a unique $u \in Y$ such that $\mathcal{J}u = u$.

Proof. Fix $u_0 \in Y$ and consider $u_n = \mathcal{J}^n u_0$ (the nth iteration of u_0 under application of $\mathcal{J}$) The sequence u_n is easily shown to be Cauchy in Y using (2.6.8) and therefore converges to a limit $u \in Y$ because Y is complete (as a closed subset of X). Now $u_n \to u$ and hence $u_{n+1} = \mathcal{J}u_n \to \mathcal{J}u$ as $n \to \infty$. We conclude that u_n converges to a fixed point of $\mathcal{J}$. If $u, v \in Y$ are both fixed points of $\mathcal{J}$, then $\mathcal{J}u - \mathcal{J}v = u - v$. But (2.6.8) holds and hence $u = v$ and the fixed point is unique.

The proof of Theorem 2.6.3 is complete. $\square$

2.7 Existence and Uniqueness of SODEs

Theorem 2.7.1. [62, page 325]

Suppose that the Assumption 2.6.1 holds and that $B(t)$ is an $\mathcal{F}_t$ Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. For each $T > 0$ and $u_0 \in \mathbb{R}^d$, there exists an unique $u \in H_{2,T}$ such that for $t \in [0, T]$, u satisfies the stochastic differential equation (2.6.1).

Proof. Consider a random variable $u_0 \in L^2(\Omega, \mathbb{R}^d)$ which is independent of $\mathcal{F}_0$ and hence of the process $B(t)$. For $u \in H_{2,T}$, let

$$\mathcal{J}(u)(t) := u_0 + \int_0^t f(u(s)) ds + \int_0^t G(u(s)) \, dB(s), \quad t \in [0, T]. \tag{2.7.1}$$

If $u \in H_{2,T}$ is the fixed point of $\mathcal{J}$, then it satisfies the integral equation (2.7.1).

Step1. We assume that u_0 is deterministic.

Step2. Define the operator $\mathcal{J} : H_{2,T} \mapsto H_{2,T}$ by

$$\mathcal{J}(u)(t) = u_0 + \int_0^t f(u(s)) ds + \int_0^t G(u(s)) \, dB(s).$$

We shall show the existence of a unique fixed point by applying the contraction mapping theorem on the Banach space $H_{2,T}$. To apply the contraction mapping Theorem we need to show the following:

1) $\mathcal{J}$ maps $H_{2,T}$ into $H_{2,T}$,

2) $\mathcal{J}$ satisfies the contraction condition, i.e.,

$$\|\mathcal{J}(u) - \mathcal{J}(v)\|_{H_{2,T}} \leq \mu \|u - v\|_{H_{2,T}},$$

for some $\mu \in (0, 1)$.

Step3. We shall prove that $\mathcal{J}$ maps from $H_{2,T}$ into $H_{2,T}$. We have

$$\mathcal{J}(u) = u_0 + \int_0^t f(u(s))ds + \int_0^t G(u(s))\, dB(s),$$

$$|\mathcal{J}(u)|_{\mathbb{R}^d}^2 \leq 3|u_0|_{\mathbb{R}^d}^2 + 3\Big|\int_0^t f(u(s))ds\Big|_{\mathbb{R}^d}^2 + 3\Big|\int_0^t G(u(s))\, dB(s)\Big|_{\mathbb{R}^d}^2,$$

$$\mathbb{E}|\mathcal{J}(u)|_{\mathbb{R}^d}^2 \leq 3\mathbb{E}|u_0|_{\mathbb{R}^d}^2 + 3\mathbb{E}\Big|\int_0^t f(u(s))ds\Big|_{\mathbb{R}^d}^2 + 3\mathbb{E}\Big|\int_0^t G(u(s))\, dB(s)\Big|_{\mathbb{R}^d}^2.$$

By Cauchy-Schwarz inequality, we have

$$\Big|\int_0^t f(u(s))ds\Big|_{\mathbb{R}^d}^2 \leq \Big(\int_0^t 1^2\, ds\Big) \cdot \int_0^t |f(u(s))|_{\mathbb{R}^d}^2\, ds = t\int_0^t |f(u(s))|_{\mathbb{R}^d}^2 ds.$$

By the isometry property, we have

$$\mathbb{E}\Big|\int_0^t G(u(s))\, dB(s)\Big|_{\mathbb{R}^d}^2 = \int_0^t \mathbb{E}\|G(u(s))\|_{\mathbb{R}^{d\times m}}^2 ds.$$

Thus we get

$$\mathbb{E}|\mathcal{J}(u)|_{\mathbb{R}^d}^2 \leq 3\mathbb{E}|u_0|_{\mathbb{R}^d}^2 + 3\mathbb{E}t\int_0^t |f(u(s))|_{\mathbb{R}^d}^2\, ds + 3\mathbb{E}\int_0^t \|G(u(s))\|_{\mathbb{R}^{d\times m}}^2\, ds.$$

By the linear growth conditions, we have

$$\mathbb{E}|\mathcal{J}(u)|_{\mathbb{R}^d}^2 \leq 3\mathbb{E}|u_0|_{\mathbb{R}^d}^2 + 3t\mathbb{E}\int_0^t L^2\Big(1 + |u(s)|_{\mathbb{R}^d}^2\Big)\, ds + 3\mathbb{E}\int_0^t L^2\mathbb{E}\Big(1 + |u(s)|_{\mathbb{R}^d}^2\Big)\, ds.$$

Finally, we take the supremum over $t \in [0, T]$ in the last two terms and we get

$$\mathbb{E}|\mathcal{J}(u)|_{\mathbb{R}^d}^2 \leq 3\mathbb{E}|u_0|_{\mathbb{R}^d}^2 + 3L^2 t \cdot t\Big(1 + \sup_{0\leq s\leq t}\mathbb{E}|u(s)|_{\mathbb{R}^d}^2\Big) + 3L^2 t\Big(1 + \sup_{0\leq s\leq t}\mathbb{E}|u(s)|_{\mathbb{R}^d}^2\Big).$$

Hence we see that, noting $u \in H_{2,T}$,

$$\sup_{0\leq t\leq T} \mathbb{E}|\mathcal{J}(u)|_{\mathbb{R}^d}^2 \leq 3\mathbb{E}|u_0|_{\mathbb{R}^d}^2 + 3L^2 T^2\Big(1 + \sup_{0\leq s\leq t}\mathbb{E}|u(s)|_{\mathbb{R}^d}^2\Big)$$
$$+ 3L^2 T\Big(1 + \sup_{0\leq s\leq t}\mathbb{E}|u(s)|_{\mathbb{R}^d}^2\Big) < \infty,$$

which implies that

$$\mathcal{J}(u) \in H_{2,T}, \quad \forall\, u \in H_{2,T}. \tag{2.7.2}$$

Step4. Show that J satisfies the contraction condition, that is,

$$\|\mathcal{J}(u_1) - J(u_2)\|_{H_{2,T}} \leq \mu \|u_1 - u_2\|_{H_{2,T}}.$$

Note that

$$\mathbb{E}\|\mathcal{J}(u_1)(t) - J(u_2)(t)\|_{\mathbb{R}^d}^2$$

$$= \mathbb{E}\| \int_0^t \big[f(u_1(s) - f(u_2(s)) \big]ds + \int_0^t \big[G(u_1(s)) - G(u_2(s)) \big]\, dB(s)\|_{\mathbb{R}^d}^2$$

$$\leq 2\mathbb{E}\| \int_0^t \big[f(u_1(s) - f(u_2(s)) \big]ds\|_{\mathbb{R}^d}^2 + 2\mathbb{E}\| \int_0^t \big[G(u_1(s)) - G(u_2(s)) \big]\, dB(s)\|_{\mathbb{R}^d}^2$$

$$\leq 2t\mathbb{E}\int_0^t \|f(u_1(s)) - f(u_2(s))\|_{\mathbb{R}^d}^2 ds + 2\mathbb{E}\int_0^t \|G(u_1(s)) - G(u_2(s))\|_{\mathbb{R}^{d\times m}}^2 ds.$$

By the Lipschitz condition (2.6.4), we have

$$\mathbb{E}\|\mathcal{J}(u_1)(t) - J(u_2)(t)\|_{\mathbb{R}^d}^2 \leq 2tL^2\mathbb{E}\int_0^t \|u_1(s) - u_2(s)\|_{\mathbb{R}^d}^2 ds + 2L^2\mathbb{E}\int_0^t \|u_1(s) - u_2(s)\|_{\mathbb{R}^d}^2 ds$$

$$\leq 2L^2 t(t+1) \sup_{0 \leq s \leq t} \mathbb{E}\|u_1(s) - u_2(s)\|_{\mathbb{R}^d}^2.$$

Thus we get

$$\sup_{0 \leq t \leq T} \mathbb{E}\|\mathcal{J}(u_1)(t) - J(u_2)(t)\|_{\mathbb{R}^d}^2 \leq 2L^2 T(T+1) \sup_{0 \leq t \leq T} \mathbb{E}\|u_1(t) - u_2(t)\|_{\mathbb{R}^d}^2.$$

Choosing the sufficiently small T such that

$$2L^2 T(T+1) < \frac{1}{2},$$

we then have

$$\sup_{0 \leq t \leq T} \mathbb{E}\|\mathcal{J}(u_1)(t) - J(u_2)(t)\|_{R^d}^2 \leq \frac{1}{2}\sup_{0 \leq t \leq T} \mathbb{E}\|u_1(t) - u_2(t)\|_{R^d}^2,$$

which implies that

$$\|\mathcal{J}(u_1) - J(u_2)\|_{H_{2,T}} \leq \frac{1}{2}\|u_1 - u_2\|_{H_{2,T}}.$$

Thus $\mathcal{J}$ is indeed a contraction on $H_{2,T}$. By contraction mapping theorem, there exists a unique $u \in H_{2,T}$ such that

$$u(t) = u_0 + \int_0^t f(u(s))ds + \int_0^t G(u(s))\, dB(s).$$

The proof of Theorem 2.7.1 is complete. $\qquad\square$

2.8 Numerical Methods for Stochastic ODEs

We shall discuss how to solve stochastic ODEs by Euler-Maruyama method numerically in this section. In Itô calculus, the Euler-Maruyama method (also called the Euler method) is a method for the approximate numerical solution of a stochastic differential equation (SDE). It is a simple generalization of the Euler method for ordinary differential equations to stochastic differential equations. It is named after Leonhard Euler and Gisiro Maruyama [62].

Now let us consider the numerical solution for solving SODEs,

$$du(t) = f(u(t))dt + G(u(t))dW(t), \quad 0 \le t \le T,$$
$$u(0) = u_0,$$

or formally

$$\frac{du(t)}{dt} = f(u(t)) + G(u(t))\frac{dW(t)}{dt}, \quad 0 \le t \le T, \tag{2.8.1}$$
$$u(0) = u_0. \tag{2.8.2}$$

Here we denote the Brownian motion $B(t)$ by $W(t)$. Let $0 = t_0 \le t_1 \le \cdots \le t_N = T$ be a partition of $[0, T]$ and Δt be the stepsize. At $t = t_n$, $0 \le n \le N - 1$, we have

$$\left.\frac{du(t)}{dt}\right|_{t=t_n} = f(u(t))\Big|_{t=t_n} + G(u(t))\frac{dW(t)}{dt}\Big|_{t=t_n}, \quad 0 \le t \le T.$$

We now consider the Euler-Maruyama method, we shall use the following approximations,

$$\left.\frac{du(t)}{dt}\right|_{t=t_n} = \frac{u(t_{n+1}) - u(t_n)}{\Delta t} + O(\Delta t),$$
$$f(u(t))\Big|_{t=t_n} \approx f(u(t_n)),$$
$$G(u(t))\Big|_{t=t_n} \approx G(u(t_n)),$$
$$\left.\frac{dW(t)}{dt}\right|_{t=t_n} \approx \frac{W(t_{n+1}) - W(t_n)}{\Delta t}.$$

Let $U^n \approx u(t_n)$ be the approximate solution of $u(t_n)$, we define the following Euler-Maruyama method,

$$U^{n+1} - U^n = f(U^n)\Delta t + G(U^n)\Delta W^n, \quad n = 0, 1, 2, \ldots, N - 1,$$

where

$$\Delta W^n = W^{n+1} - W^n \approx \sqrt{\Delta t}\,\mathcal{N}(0,1).$$

Here $\mathcal{N}(0,1)$ denotes the normally distributed random variable.

To consider the strong error estimate, we introduce some notations. Let $\bar{U}(t)$ denote the continuous approximation such that

$$\bar{U}(t) = U^n + (t - t_n)f(U^n) + G(U^n)(W(t) - W(t_n)), \quad t \in [t_n, t_{n+1}].$$

It is easy to see $\bar{U}(t)$ is continuous on $[0, T]$ and

$$\bar{U}(t_n) = U^n, \quad \bar{U}(t_{n+1}) = U^{n+1}.$$

Let $U(t)$ denote the piecewise constant function defined by

$$U(t) = \begin{cases} U^0, & [t_0, t_1), \\ U^1, & [t_1, t_2), \\ \vdots & \\ U^{N-1}, & [t_{N-1}, t_N). \end{cases}$$

Then we have

$$\bar{U}(t) = U^0 + \int_0^t f(U(s))ds + \int_0^t G(U(s))dW(s), \quad \forall\, 0 \le t \le T. \tag{2.8.3}$$

For example, with $t = t_1$ and t_2, we have

$$\bar{U}(t_1) = U^0 + \int_0^{t_1} f(U(s))ds + \int_0^{t_1} G(U(s))dW(s)$$

$$= U^0 + f(U^0)(t_1 - t_0) + \int_0^{t_1} G(U^0)dW(s),$$

$$\bar{U}(t_2) = U^0 + \int_0^{t_2} f(U(s))ds + \int_0^{t_2} G(U(s))dW(s)$$

$$= U^0 + f(U^0)(t_1 - t_0) + \int_0^{t_1} G(U^0)dW(s) + f(U^1)(t_2 - t_1) + \int_{t_1}^{t_2} G(U^1)dW(s)$$

$$= U^1 + f(U^1)(t_2 - t_1) + \int_{t_1}^{t_2} G(U^1)dW(s).$$

For the general $t \in (t_n, t_{n+1}]$, we have

$$\bar{U}(t) = U^0 + \int_0^t f(U(s))ds + \int_0^t G(U(s))dW(s)$$

$$= U^0 + \int_0^{t_n} f(U(s))ds + \int_0^{t_n} G(U(s))dW(s) + \int_{t_n}^t f(U(s))ds + \int_{t_n}^t G(U(s))dW(s)$$

$$= U^n + (t - t_n)f(U^n) + G(U^n)\int_{t_n}^t dW(s).$$

We have the following strong convergence error estimate for the Euler method.

Theorem 2.8.1. Let $u(t)$ and $\bar{U}(t)$ be the solutions of (2.8.1) and (2.8.3), respectively. Then we have

$$\mathbb{E} \sup_{0 \leq t \leq T} |\bar{U}(t) - u(t)|^2_{\mathbb{R}^d} = O(\Delta t).$$

Proof. **Step1**. We observe that $u(t)$ satisfies

$$u(t) = u_0 + \int_0^t f(u(s))ds + \int_0^t G(u(s))dW(s), \tag{2.8.4}$$

and $\bar{U}(t)$ satisfies

$$\bar{U}(t) = U^0 + \int_0^t f(U(s))ds + \int_0^t G(U(s))dW(s). \tag{2.8.5}$$

Subtracting (2.8.5) from (2.8.4), we get, since $u_0 = U^0$,

$$\bar{U}(t) - u(t) = \int_0^t [f(U(s)) - f(u(s))]ds + \int_0^t [G(U(s)) - G(u(s))]dW(s),$$

$$|\bar{U}(t) - u(t)|^2 = \left| \int_0^t [f(U(s)) - f(u(s))]ds + \int_0^t [G(U(s)) - G(u(s))]dW(s) \right|^2$$

$$\leq 2 \left| \int_0^t [f(U(s)) - f(u(s))]ds \right|^2 + 2 \left| \int_0^t [G(U(s)) - G(u(s))]dW(s) \right|^2.$$

Taking the expectation, we get,

$$\mathbb{E}|\bar{U}(t) - u(t)|^2 \leq 2\mathbb{E} \left| \int_0^t [f(U(s)) - f(u(s))] \, ds \right|^2 + 2\mathbb{E} \left| \int_0^t [G(U(s)) - G(u(s))]dW(s) \right|^2$$

$$= I_1 + I_2. \tag{2.8.6}$$

For I_1 we have, by using Cauchy-Schwarz inequality and Lipschitz condition,

$$I_1 \leq 2\mathbb{E}[\int_0^t 1^2].ds \int_0^t [f(U(s)) - f(u(s))]^2 \, ds$$

$$\leq C\mathbb{E} \int_0^t |U(s) - u(s)|^2 \, ds.$$

For I_2, we have, by using the isometry property,

$$I_2 = 2\mathbb{E} \int_0^t \|G(U(s) - G(u))\|^2 ds \leq C\mathbb{E} \int_0^t |U(s) - u(s)|^2 ds.$$

Here $\|G(U(s) - G(u))\|$ denotes the matrix norm. Thus we get

$$\mathbb{E}|\bar{U}(t) - u(t)|^2 \leq C\mathbb{E} \int_0^t |U(s) - u(s)|^2 ds$$

$$\leq C\mathbb{E} \int_0^t |U(s) - \bar{U}(s)|^2 ds + C\mathbb{E} \int_0^t |\bar{U}(s) - u(s)|^2 \, ds.$$

Step2. By the Grönwall Lemma, we have

$$\mathbb{E}|\bar{U}(t) - u(t)|^2 \le C\mathbb{E}\int_0^t |U(s) - \bar{U}(s)|^2 \, ds.$$

Step3. Estimate $\mathbb{E}\int_0^t |U(s) - \bar{U}(s)|^2 ds$.

Note that, for $t_n \le s \le t_{n+1}$,

$$U(s) - \bar{U}(s) = U^n - \left(U^n + \int_{t_n}^s f(U(r)) \, dr + \int_{t_n}^s G(U(r)) \, dW(r)\right)$$
$$= -f(U^n)(s - t_n) - G(U^n)(W(s) - W(t_n)).$$

Thus we have

$$|U(s) - \bar{U}(s)|^2 \le C|f(U^n)|^2 \Delta t^2 + C|G(U^n)|^2 |W(s) - W(t_n)|^2.$$

By linear Growth conditions of f and G, we have

$$\mathbb{E}|U(s) - \bar{U}(s)|^2 \le C\mathbb{E}|U^n|^2 \Delta t^2 + C\mathbb{E}|U^n|^2 \Delta t,$$

where we use the fact that

$$\mathbb{E}|W(s) - W(t_n)|^2 = s - t_n \le \Delta t, \quad \text{for } t_n \le s \le t_{n+1}. \tag{2.8.7}$$

Thus we get

$$\mathbb{E}\int_0^t |U(s) - \bar{U}(s)|^2 \, ds = \mathbb{E}\left[\int_0^{t_1} + \int_{t_1}^{t_2} \cdots + \int_{t_l}^t\right]|U(s) - \bar{U}(s)|^2 \, ds \tag{2.8.8}$$

$$= \mathbb{E}\sum_{k=0}^{l-1} \int_{t_k}^{t_{k+1}} |U(s) - \bar{U}(s)|^2 ds + \mathbb{E}\int_{t_l}^t |U(s) - \bar{U}(s)|^2 \, ds$$

$$\le C\sum_{k=0}^{l-1} \int_{t_k}^{t_{k+1}} \left[\mathbb{E}|U^k|^2 \Delta t^2 + \mathbb{E}|U^k|^2 \Delta t\right] ds \tag{2.8.9}$$

$$+ C\int_{t_l}^t [\mathbb{E}|U^l|^2 \Delta t^2 + \mathbb{E}|U^l|^2 \Delta t] \, ds.$$

By using the boundedness assumption $\mathbb{E}[\sup_{0 \le t \le T} |\bar{U}(t)|^2] \le C$, we get

$$\mathbb{E}\int_0^t |U(s) - \bar{U}(s)|^2 ds \le C\Delta t + C\Delta t^2 \le C\Delta t.$$

The proof Theorem 2.8.1 is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now consider the boundedness of the solution u.

Lemma 2.8.2. Let u be the solution of (2.8.1). We have

$$\mathbb{E}|u(t)|^2 \leq C.$$

Proof. We have

$$u(t) = u_0 + \int_0^t f(u(s))ds + \int_0^t G(u(s))dW(s).$$

Note that

$$\mathbb{E}|u(t)|^2 \leq 3\mathbb{E}|u_0|^2 + 3\mathbb{E}|\int_0^t f(u(s))ds|^2 + 3\mathbb{E}|\int_0^t G(u(s))dW(s)|^2$$

$$= 3\mathbb{E}|u_0|^2 + 3\mathbb{E}|\int_0^t f(u(s))ds|^2 + 3\mathbb{E}\int_0^t |G(u(s))|^2 ds.$$

By the linear growth condition, we get

$$\mathbb{E}|u(t)|^2 \leq 3\mathbb{E}|u_0|^2 + C\mathbb{E}\int_0^t |u(s)|^2\, ds + C\mathbb{E}\int_0^t |u(s)|^2\, ds$$

$$\leq 3\mathbb{E}|u_0|^2 + C\mathbb{E}\int_0^t |u(s)|^2\, ds.$$

Hence we have, by Grönwall Lemma,

$$\mathbb{E}|u(t)|^2 \leq C\mathbb{E}|u_0|^2 < \infty.$$

Together these estimates complete the proof of Lemma 2.8.2. $\qquad\square$

We next consider the boundedness of the solution of (2.8.3).

Lemma 2.8.3. Let $\bar{U}(t)$ be the solution of (2.8.3). We have

$$\mathbb{E}|\bar{U}(t)|^2 \leq C.$$

Proof. We have

$$\bar{U}(t) = u_0 + \int_0^t f(U(s))ds + \int_0^t G(U(s))dW(s).$$

Note that

$$\mathbb{E}|\bar{U}(t)|^2 \leq 3\mathbb{E}|u_0|^2 + 3\mathbb{E}|\int_0^t f(U(s))ds|^2 + 3\mathbb{E}|\int_0^t G(U(s))dW(s)|^2$$

$$\leq 3\mathbb{E}|u_0|^2 + 3\mathbb{E}|\int_0^t f(U(s))ds|^2 + 3\mathbb{E}|\int_0^t G(U(s))dW(s)|^2$$

$$\leq 3\mathbb{E}|u_0|^2 + C\mathbb{E}\int_0^t |U(s)|^2 ds$$

$$\leq 3\mathbb{E}|u_0|^2 + C\mathbb{E}\int_0^t |U(s) - \bar{U}(s)|^2 ds + C\mathbb{E}\int_0^t |\bar{U}(s)|^2 ds.$$

By Grönwall Lemma, we have,

$$\mathbb{E}|\bar{U}(t)|^2 \leq C\mathbb{E}|u_0|^2 + C\mathbb{E}\int_0^t |U(s) - \bar{U}(s)|^2 ds.$$

Note that,

$$\mathbb{E}|U(s) - \bar{U}(s)|^2 \leq C\big(\mathbb{E}|U^n|^2\big)\Delta t^2 + C\big(\mathbb{E}|U^n|^2\big)\Delta t$$

$$\leq C\Delta t(\mathbb{E}|U^n|^2).$$

Thus we have

$$\mathbb{E}|\bar{U}(t)|^2 \leq C\mathbb{E}|u_0|^2 + C\mathbb{E}\int_0^t \Delta t(\mathbb{E}|U^n|^2)ds$$

$$\leq C\mathbb{E}|u_0|^2 + C\int_0^t \Delta t(\mathbb{E}\sup_{0\leq s\leq t}|\bar{U}(s)|^2)\,dr,$$

that is,

$$\mathbb{E}\big[\sup_{0\leq s\leq t}|\bar{U}(s)|^2\big] \leq C\mathbb{E}|u_0|^2 + C\Delta t\int_0^t \mathbb{E}\sup_{0\leq s\leq t}|\bar{U}(s)|^2\,dr.$$

By Grönwall lemma, we get

$$\mathbb{E}\big[\sup_{0\leq s\leq t}|\bar{U}(s)|^2\big] \leq C\mathbb{E}|u_0|^2 < \infty.$$

The proof of Lemma 2.8.3 is complete now. □

Chapter 3

Basics for Stochastic Partial Differential Equations

3.1 Introduction

Stochastic partial differential equations (SPDEs) generalize partial differential equations via random force terms and coefficients, in the same way as ordinary stochastic differential equations generalize ordinary differential equations. The study of SPDEs is an exciting topic which brings together techniques from probability theory, functional analysis, and the theory of partial differential equations.

The solution to a stochastic partial differential equation may be viewed in several manners. One can view a solution as a random field (set of random variables indexed by a multidimensional parameter). In the case where the SPDE is an evolution equation, the infinite dimensional point of view consists in viewing the solution at a given time as a random element in a function space and thus view the SPDEs as a stochastic evolution equation in an infinite dimensional space. In the pathwise point of view, the solution has a meaning for almost every realization of the noise and then view the solution as a random variable on the set of (infinite dimensional) paths thus defined. SPDEs can describe several phenomena (physics, biology, medicine): heat or sound propagation, fluid flow, transport of substances, population dynamics, neuronal activity, traffic modeling [62].

Let H denote a Hilbert space. We consider the following stochastic partial differential equation

$$du + Audt = f(u)dt + G(u)dW, \tag{3.1.1}$$

where A is a linear elliptic operator that generates a semigroup $E(t) = e^{-tA}$. For example $A = -\Delta, \mathcal{D}(A) = H_0^1(\mathcal{D}) \cap H^2(\mathcal{D})$. Here W is the H-valued stochastic Wiener process.[62]

Now we introduce some different types of solutions of stochastic partial differential equation such that strong solution, weak solution, and mild solution.

Definition 3.1.1. A predictable H-valued process $u(t) : t \in [0, T]$ is called a strong solution of (3.1.1) if

$$u(t) = u_0 + \int_0^t [-Au(s) + f(u(s))]ds + \int_0^t g(u(s))dW(s), \quad \forall t \in [0, T]. \tag{3.1.2}$$

Definition 3.1.2. A predictable H-valued process $u(t) : t \in [0, T]$ is called a weak solution of (3.1.1) if

$$\langle u(t), v \rangle = \langle u_0, v \rangle + \int_0^t [-\langle u(s), Av \rangle + \langle f(u(s)), v \rangle]ds$$
$$+ \int_0^t \langle g(u(s))dW(s), v \rangle, \quad \forall t \in [0, T], \ v \in \mathcal{D}(A).$$

Definition 3.1.3. A predictable H-valued process $u(t) : t \in [0, T]$ is called a mild solution of (3.1.1) if

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}f(u(s))ds + \int_0^t e^{-(t-s)A}G(u(s))dW(s), \tag{3.1.3}$$

where e^{-tA} is the semigroup generated by $-A$.

3.2 Q-Wiener Process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered $\mathcal{F}_t$-adapted probability space. A H-valued stochastic process $\{W(t) : t \geq 0\}$ is defined as follows [62, page 436].

Definition 3.2.1. A H-valued stochastic process $\{W(t) : t \geq 0\}$ is a Q-Wiener process if

    1) W(0)=0, a.s.,

    2) W(t) is a continuous function $\mathbb{R}^+ \mapsto H$ for each $\omega \in \Omega$,

3) W(t) is $\mathcal{F}_t$ adapted and $W(t) - W(s)$ is independent of $\mathcal{F}_t$ for $s < t$,

4) $W(t) - W(s) \sim \mathcal{N}(0, (t-s)Q)$, $\forall\ 0 \le s \le t$, where $Q : H \mapsto H$ is a positive, definite bounded operator.

3.3   Green Function

The unique solution of a PDE can be written in a very compact form by introducing an auxiliary function known as the Green's function.

Consider

$$u_t - u'' = f, \quad 0 < x < 1, \tag{3.3.1}$$

$$u(0) = u(1) = 0,$$

$$u(0) = u_0.$$

Let $\{e_j\}_{j=1}^{\infty}$ be the eigenfunctions of $A = -\frac{\partial^2}{\partial x^2}$ and $\mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$.

Lemma 3.3.1. Let $f = 0$. Assume that

$$u(0, x) = u_0 = \sum_{j=1}^{\infty}(u_0, e_j)e_j. \tag{3.3.2}$$

Then the solution of equation (3.3.1) has the form of

$$u(t, x) = \sum_{j=1}^{\infty}(u_0, e_j)e^{-\lambda_j t}e_j(x) = \int_0^1 G(t, x, y)\, u_0(y)\, dy, \tag{3.3.3}$$

where $G(t, x, y)$ is the Green function and

$$G(t, x, y) = \sum_{j=1}^{\infty}e^{-\lambda_j t}e_j(x)e_j(y).$$

Proof. Assume that the solution of equation (3.3.1) has the form of

$$u(t, x) = \sum_{j=0}^{\infty}u_j(t)e_j(x). \tag{3.3.4}$$

Substituting this into the equation (3.3.1), we have,

$$\sum_{j=0}^{\infty}(u_j'(t) - \lambda_j u_j(t))e_j(x) = 0,$$

which implies that

$$u_j'(t) - \lambda_j u_j(t) = 0, \quad \text{with } u_j(0) = (u_0, e_j).$$

Hence we get

$$u_j(t) = (u_0, e_j)e^{-\lambda_j t},$$

and

$$u(t,x) = \sum_{j=1}^{\infty} (u_0, e_j)e^{-\lambda_j t}e_j(x) = \sum_{j=1}^{\infty} \left(\int_0^1 u_0(y)e_j(y)\, dy \right)e^{-\lambda_j t}e_j(x)$$

$$= \int_0^1 G(t,x,y)u_0(y)\, dy,$$

where the function G is called the "Green function". $\qquad\square$

Assumption 3.3.2. [62] There exists a constant $L > 0$ such that the linear growth conditions hold:

$$\|f(u)\|_H^2 \le L(1 + \|u\|_H^2), \tag{3.3.5}$$

$$\|G(u)\|_{L_0^2}^2 \le L(1 + \|u\|_H^2),$$

and the global Lipschitz conditions hold:

$$\|f(u_1) - f(u_2)\|_H \le L(\|u_1 - u_2\|_H), \tag{3.3.6}$$

$$\|G(u_1) - G(u_2)\|_{L_0^2} \le L(\|u_1 - u_2\|_H).$$

Here $L_0^2 = HS(Q^{\frac{1}{2}}H, H)$ defined by

$$L_0^2 = \{\psi : \|\psi Q^{\frac{1}{2}}\|_{HS}^2 = \sum_{j=1}^{\infty} \|\psi Q^{\frac{1}{2}}e_j\|^2 < \infty\}. \tag{3.3.7}$$

Here $\|.\|_{HS}$ denotes the Hilbert-Schmidt norm, where e_j is an orthonormal basis for H. L_0^2 is a Banach space with norm $\|\cdot\|_{L_0^2}$.

Assumption 3.3.3. [62, page 436]

Let $Q \in L(H)$ be non-negative definite and symmetric bounded operator. Further, Q has an orthonormal basis $\{e_j : j \in \mathbb{N}\}$ of eigenfunctions with corresponding eigenvalues $q_j \ge 0$ such that $\sum_{j \in \mathbb{N}} q_j < \infty$ (i.e., Q is of trace class).

Theorem 3.3.4. [62, page 437] Let Q satisfy Assumption 3.3.3. Then $W(t)$ is a Q-Wiener process if and only if

$$W(t) = \sum_{j=1}^{\infty} \sqrt{q_j} e_j \beta_j(t), \quad a.s., \tag{3.3.8}$$

where $\beta_j(t)$ are iid $\mathcal{F}_t$- Brownian motions and the series converges in $L^2(\Omega, H)$. Moreover, (3.3.8) converges in $L^2(\Omega, C([0,T], H))$ for any $T > 0$.

Proof. Let $W(t)$ be a Q-Wiener process and suppose without loss of generality that $q_j > 0$ for all j. Since $e_j : j \in \mathbb{N}$ is an orthonormal basis for H, we may write

$$W(t) = \sum_{j=1}^{\infty} (W(t), e_j) e_j.$$

Let $\beta_j(t) := \frac{1}{\sqrt{q_j}}(W(t), e_j) e_j$, so that (3.3.8) holds. Clearly then, $\beta_j(0) = 0$ a.s. and $\beta_j(t)$ is $\mathcal{F}_t$ adapted and has continuous sample path. The increment

$$\beta_j(t) - \beta_j(s) = \frac{1}{\sqrt{q_j}}(W(t) - W(s), e_j), \quad 0 \le s \le t,$$

is independent of $\mathcal{F}_s$. As $W(t) - W(s) \sim \mathcal{N}(0, (t-s)Q)$, we have

$$\mathrm{Cov}(\beta_j(t) - \beta_j(s), \beta_k(t) - \beta_k(s)) = \frac{1}{\sqrt{q_j q_k}} \mathbb{E}(W(t) - W(s), e_j),$$

and

$$(W(t) - W(s), e_k) = \frac{1}{\sqrt{q_j q_k}}(t-s)(Qe_j, e_k) = (t-s)\delta_{jk}.$$

Then, $\beta_j(t) - \beta_j(s) \sim \mathcal{N}(0, t-s)$ and $\beta_j(t)$ is a $\mathcal{F}_t$ Brownian motion. Any pair of increments forms a multivariate Gaussian and hence β_j and β_k are independent for $j \neq k$.

To show $W(t)$ as defined by (3.3.8) is a Q-Wiener process, we first show the series converges in $L^2(\Omega, H)$ for any fixed $t \ge 0$. Consider the finite sum approximation

$$W^J(t) := \sum_{j=1}^{J} \sqrt{q_j} e_j \beta_j(t),$$

and the difference $W^J(t) - W^M(t)$, for $M < J$. By the orthonormality for the eigenfunctions e_j, we have, using Parseval's identity,

$$\| W^J(t) - W^M(t) \|_H^2 = \sum_{j=M+1}^{J} q_j \beta_j(t)^2.$$

Each $\beta_j(t)$ is a Brownian motion and taking the expectation gives

$$\mathbb{E} \parallel W^J(t) - W^M(t) \parallel_H^2 = \sum_{j=M+1}^J q_j \mathbb{E}[\beta_j(t)^2] = t \sum_{j=M+1}^J q_j.$$

As Q is a trace class, $\sum_{j=1}^\infty q_j < \infty$ and the right-hand side converges to zero as $M, J \to \infty$. Thus the series (3.3.8) is well defined in $L^2(\Omega, H)$. Now we prove W is Q-wiener process. In fact, we have

$$\mathbb{E}(W(t), e_j) = 0,$$
$$\mathbb{E}(W(t) - W(s), e_j)^2 = (t - s)(Qe_j, x_j) = q_j(t - s).$$

The proof of Lemma 3.3.4 is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 3.3.1. [62] Let (λ_j, e_j) be the eigenpairs of A. The fractional power A^α for $\alpha \in \mathbb{R}$ is defined by, with $u = \sum_{j=1}^\infty u_j e_j$, $u_j \in \mathbb{R}$,

$$A^\alpha u := \sum_{j=1}^\infty \lambda_j^\alpha u_j e_j.$$

Let the domain $\mathcal{D}(A^\alpha)$ be the set of

$$\mathcal{D}(A^\alpha) = \{u : \|A^\alpha u\|_{L^2}^2 = \sum_{j=1}^\infty (A^\alpha u, e_j)^2 = \sum_{j=1}^\infty \lambda_j^{2\alpha} u_j^2 < \infty\}. \tag{3.3.9}$$

Theorem 3.3.5. [62, page 445] Let Q satisfy Assumption 3.3.3 and suppose that $\{\psi(s) : s \in [0, T]\}$ is a L_0^2-valued predictable process such that

$$\int_0^T \mathbb{E}[\parallel \psi(s) \parallel_{L_0^2}^2] ds < \infty. \tag{3.3.10}$$

For $t \in [0, T]$, the following stochastic integral is well defined, with $W(s) = \sum_{j=1}^\infty q_j^{\frac{1}{2}} e_j \beta_j(s)$,

$$\int_0^t \psi(s) dW(s) := \sum_{j=1}^\infty \int_0^t \psi(s) \sqrt{q_j} e_j d\beta_j(s). \tag{3.3.11}$$

For $t \in [0, T]$ the following Itô isometry holds:

$$\mathbb{E} \parallel \int_0^t B(s) dW(s) \parallel^2 = \int_0^t \mathbb{E} \parallel B(s) \parallel_{L_0^2}^2 ds.$$

Further $\{\int_0^t B(s) dW(s) : t \in [0, T]\}$ is an H-valued predictable process.

Proof. We have

$$\int_0^t \psi(s)dW(s) = \sum_{j=1}^{\infty} \int_0^t \psi(s)\sqrt{q_j}e_j d\beta_j(s). \tag{3.3.12}$$

To show (3.3.12) is well defined in $L^2(\Omega, H)$ we have, with $I(t) := \int_0^t \psi(s)dW(s)$,

$$\|I(t)\|^2 = \| \sum_{j=1}^{\infty} \int_0^t \psi(s)\sqrt{q_j}e_j d\beta_j(s) \|^2 = \sum_{k=1}^{\infty} \Big(\sum_{j=1}^{\infty} \int_0^t \psi(s)\sqrt{q_j}e_j d\beta_j(s), e_k \Big)^2$$

$$= \sum_{k=1}^{\infty} (\sum_{j=1}^{\infty} \int_0^t \big(\psi(s)\sqrt{q_j}e_j, e_k\big) d\beta_j(s))^2.$$

By Itô isometry, we get

$$\mathbb{E}[\| I(t) \|^2] = \sum_{j,k=1}^{\infty} \int_0^t \mathbb{E}\big(\psi(s)\sqrt{q_j}e_j, e_k\big)^2 ds.$$

Since $Qe_j = q_j e_j$, we have

$$\mathbb{E} \| I(t) \|^2 = \sum_{j,k=1}^{\infty} \int_0^t \mathbb{E}\big(\psi(s)Q^{\frac{1}{2}}e_j, e_k\big)^2 ds = \int_0^t \mathbb{E} \sum_{j,k=1}^{\infty} \big(\psi(s)Q^{\frac{1}{2}}e_j, e_k\big)^2 ds.$$

By the Parseval identity, we have

$$\mathbb{E} \| I(t) \|^2 = \int_0^t \mathbb{E}\Big[\sum_{j=1}^{\infty} \| \psi(s)Q^{\frac{1}{2}}e_j \|^2 \Big] ds = \int_0^t \mathbb{E} \| \psi(s) \|_{L_0^2}^2 \, ds.$$

Together these estimates complete the proof of Theorem 3.3.5. $\qquad\square$

Assumption 3.3.6. [62, page 450] (Lipschitz condition on G) For constants $\zeta \in (0, 2]$ and L, we have that $G : H \to L_0^2$ satisfies

$$\| A^{\frac{\zeta-1}{2}} G(u) \|_{L_0^2} \le L(1+ \| u \|_H),$$

$$\| A^{\frac{\zeta-1}{2}} (G(u_1) - G(u_2)) \|_{L_0^2} \le L(\|u_1 - u_2 \|_H), \quad \forall u, u_1, u_2 \in H.$$

3.4 Existence and Uniqueness of Stochastic PDEs

Theorem 3.4.1. [62, page 450] Assume that $f : H \mapsto H$ satisfies Assumption 3.3.2 and $G : H \to L_0^2$ satisfies Assumption 3.3.6. Suppose that the initial data $u_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$. Then there exists a unique mild solution $u(t)$ on [0,T] to (3.1.1) for any $T > 0$, such that

$$\sup_{t \in [0,T]} \| u(t) \|_{L^2(\Omega,H)} \le C_T(1+ \| u_0 \|_{L^2(\Omega,H)}).$$

Proof. Let $H_{2,T}$ denote the Banach space of H-valued predictable processes $\{u(t) : t \in [0,T]\}$ with norm $||u||_{H_{2,T}} := \sup_{0 \le t \le T} ||u(t)||_{L^2(\Omega, H)}$. For $u \in H_{2,T}$, we define

$$(\mathcal{J}u)(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(u(s)) ds + \int_0^t e^{-(t-s)A} G(u(s)) dW(s). \qquad (3.4.1)$$

A fixed point $u(t)$ of $\mathcal{J}$ is an H-valued predictable process and obeys Definition 3.1.3 and hence is a mild solution of (3.1.1). To show existence and uniqueness of the fixed point, we show $\mathcal{J}$ is a contraction mapping from $H_{2,T}$ to $H_{2,T}$.

Here we only show that $\mathcal{J}$ maps into $H_{2,T}$. $\mathcal{J}u(t)$ is a predictable process because u_0 is $\mathcal{F}_0$- measurable and the stochastic integral is a predictable process. (Theorem 2.6.3). Let us show that $||\mathcal{J}u||_{H_{2,T}} < \infty$. First, we have

$$||e^{-tA}u_0||_{L^2(\Omega),H)} \le ||u_0||_{L^2(\Omega),H)} < \infty.$$

Second, we have

$$\left\| \int_0^t e^{-(t-s)A} f(u(s)) ds \right\|_{L^2(\Omega, H)} \le \int_0^t \| e^{-(t-s)A} f(u(s)) \|_{L^2(\Omega, H)} \, ds$$

$$\le \int_0^t \| f(u(s)) \|_{L^2(\Omega, H)} \, ds$$

$$\le \int_0^t L\big(1 + \| u(s) \|_{L^2(\Omega, H)} \big) \, ds.$$

Third, by the isometry property, we have

$$\left\| \int_0^t e^{-(t-s)A} G(u(s)) dW(s) \right\|_{L^2(\Omega, H)}^2$$

$$= \int_0^t \mathbb{E} \| A^{\frac{(1-\xi)}{2}} e^{-(t-s)A} A^{\frac{(\xi-1)}{2}} G(u(s)) \|_{L_0^2}^2] ds$$

$$\le \left[L^2(1 + \sup_{0 \le s \le t} \| u(s) \|_{L^2(\Omega, H)})^2 \right] \int_0^t \| A^{\frac{(1-\xi)}{2}} e^{-(t-s)A} \|_{L(H)}^2 \, ds.$$

By the smoothing property of the operator e^{-tA}, we have

$$\| \int_0^t e^{-(t-s)A} G(u(s)) dW(s) \|_{L^2(\Omega, H)} \le CL\big(1 + \sup_{0 \le s \le t} \| u(s) \|_{L^2(\Omega, H)} \big).$$

Thus, for $u \in H_{2,T}$, since all three terms are uniformly bounded over $t \in [0, T]$ in $L^2(\Omega, H)$, we get $\| \mathcal{J}u \|_{H_{2,T}} < \infty$. Hence $\mathcal{J}$ maps into $H_{2,T}$. $\qquad \square$

Lemma 3.4.2. (Regularity in time) [62] Let the Assumptions of Theorem 3.4.1 hold and let $u_0 \in L^2(\Omega, \mathcal{F}_0, D(A))$. For $T > 0, \epsilon \in (0, \xi)$, and $\theta_1 := \min(\frac{\xi}{2} - \epsilon, 1/2)$ there exists a positive constant C such that

$$\| u(t_2) - u(t_1) \|_{L^2(\Omega, H)} \leq C(t_2 - t_1)^{\theta_1}, \quad 0 \leq t_1 \leq t_2 \leq T.$$

Further, for $\xi \in [1, 2]$ and $\theta_2 := \frac{\xi}{2} - \epsilon$ there exists $C > 0$ such that

$$\| u(t_2) - u(t_1) \|_{L^2(\Omega, H)} \leq C(t_2 - t_1)^{\theta_2}.$$

Proof. We write $u(t_2) - u(t_1) = I + II + III$, where

$$I := (e^{-t_2 A} - e^{-t_1 A})u_0,$$

$$II := \int_0^{t_2} (e^{-(t_2 - s)A} f(u(s)) ds - \int_0^{t_1} (e^{-(t_1 - s)A} f(u(s)) \, ds,$$

$$III := \int_0^{t_2} (e^{-(t_2 - s)A} G(u(s)) dW(s) - \int_0^{t_1} (e^{-(t_1 - s)A} G(u(s)) \, dW(s).$$

The estimations of I and II are easy to estimate. We only focus on III here and write

$$III = III_1 + III_2,$$

where

$$III_1 := \int_0^{t_1} \left(e^{-(t_2 - s)A} - e^{-(t_1 - s)A} \right) G(u(s) \, dW(s),$$

and

$$III_2 := \int_{t_1}^{t_2} (e^{-(t_2 - s)A} G(u(s)) dW(s).$$

We consider only the case $\xi \in (0, 1)$ and analyze III_1 and III_2 separately. First let us consider III_1. Using Itô isometry property, we get

$$\mathbb{E} \| III_1 \|^2 = \int_0^{t_1} \mathbb{E} \| \left(e^{-(t_2 - s)A} - e^{-(t_1 - s)A} \right) G(u(s)) \|_{L_0^2}^2 \, ds$$

$$= \int_0^{t_1} \mathbb{E} \| A^{\frac{1-\xi}{2}} (e^{-(t_2 - s)A} - e^{-(t_1 - s)A}) A^{\frac{\xi - 1}{2}} G(u(s)) \|_{L_0^2}^2 \, ds.$$

Using Assumption 3.3.6 on G, we obtain

$$\mathbb{E} \| III_1 \|^2 \leq \left(\int_0^{t_1} \| A^{\frac{1-\xi}{2}} (e^{-(t_2 - s)A} - e^{-(t_1 - s)A}) \|_{L(H)}^2 \, ds \right) \left(1 + \sup_{0 \leq s \leq t} \| u(s) \|_{L^2(\Omega, H)} \right)^2.$$

Note that

$$\int_0^{t_1} \| \, (e^{-(t_2-s)A} - e^{-(t_1-s)A})A^{\frac{1-\xi}{2}} \, \|_{L(H)}^2 \, ds$$

$$= \int_0^{t_1} \| \, A^{\frac{1}{2}-\epsilon}e^{-(t_1-s)A}A^{\frac{-\xi}{2}+\epsilon}(I - e^{-(t_2-t_1)A}) \, \|_{L(H)}^2 \, ds$$

$$= \int_0^{t_1} \| \, A^{\frac{1}{2}-\epsilon}e^{-(t_1-s)A} \, \|_{L(H)}^2 \| \, A^{\frac{-\xi}{2}+\epsilon}(I - e^{-(t_2-t_1)A}) \, \|_{L(H)}^2 \, ds.$$

Hence there exists $C_1, C_2 > 0$ such that, noting that $0 < \xi < 1$,

$$\int_0^{t_1} \| \, (e^{-(t_2-s)A} - e^{-(t_1-s)A})A^{\frac{1-\xi}{2}} \, \|_{L(H)}^2 \, ds \leq (C_1^2 \frac{t_1^{2\epsilon}}{\epsilon})(C_2^2(t_2 - t_1)^{\xi-2\epsilon}).$$

Then we have, with $C_1 := K_1 K_2 L T^{\epsilon}/\sqrt{\epsilon}$,

$$\|III_1\|_{L^2(\Omega,H)} = \left(\mathbb{E}\|III_1\|^2\right)^{\frac{1}{2}} \leq C_1(t_2 - t_1)^{\frac{\xi}{2}-\epsilon}\left(1 + \sup_{0 \leq s \leq T} \|u(s)\|_{L^2(\Omega,H)}\right). \qquad (3.4.2)$$

Similarly we may estimate $\mathbb{E}[III_2\|^2]$.

The proof of Lemma 3.4.2 is complete. $\qquad \square$

Chapter 4

Fourier Spectral Methods for Solving Parabolic Partial Differential Equations

4.1 Introduction

In this chapter we will consider how to use MATLAB functions to solve parabolic equations by using spectral methods. There are many ways to solve such equations by using MATLAB functions. Here we will mainly consider how to use MATLAB functions "**dst**.**m**", "**idst**.**m**" and "**fft**.**m**", "**ifft**.**m**" to solve such equations [62].

We will consider how to use the spectral method for solving the following parabolic equation

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(u(x,t)), \quad 0 < x < 1, t > 0, \tag{4.1.1}$$

$$u(0,t) = u(1,t) = 0, \tag{4.1.2}$$

$$u(x,0) = u_0(x). \tag{4.1.3}$$

Here $u_0(x)$ is the initial condition. $f : \mathbb{R} \to \mathbb{R}$ is a smooth function, for example $f(u) = u^3 - u$.

Denote,

$$A = -\frac{\partial^2}{\partial x^2}, \quad \mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1).$$

Then (4.1.1)-(4.1.3) can be written into the abstract form

$$\frac{du(t)}{dt} + Au(t) = f(u(t)), \quad t > 0, \tag{4.1.4}$$

$$u(0) = u_0 \tag{4.1.5}$$

It is well known that the operator $A : \mathcal{D}(A) \to H$ has the eigenvalues and eigenfunctions.

$$\lambda_j = j^2 \pi^2, \quad e_j = \sqrt{2}\sin(j\pi x), \quad 0 < x < 1. \tag{4.1.6}$$

We also know that $\{e_j\}_{j=1}^{+\infty}$ is an orthonormal basis in $H = L^2(0,1)$. Thus for any $v \in H$, we have

$$v = \sum_{j=1}^{+\infty} v_j e_j,$$

where $v_j = (v, e_j) = \int_0^1 v(x)e_j(x)dx$, $j = 1, 2, 3, \cdots$, are called the Fourier coefficients of v. Note that the solution u of (4.1.1)-(4.1.3) is in $L^2(0,1)$. Therefore, the solution u of (4.1.1)-(4.1.3) must have the form, for $t > 0$,

$$u(x, t) = \sum_{j=1}^{+\infty} u_j(t)e_j(x).$$

To find the approximate solution of (4.1.4)-(4.1.5), we will truncate the series of (4.1.4)-(4.1.5).

Denote by $S_N = \mathcal{L}(e_1, e_2, \ldots e_N)$ the subspace spanned by the basis functions $e_1, e_2, \ldots e_N$. The spectral method of (4.1.1)-(4.1.3) is to find, for $t > 0$, $u^N(t) \in S_N$ such that

$$\frac{du^N(t)}{dt} + Au^N(t) = P_N f(u^N(t)), \tag{4.1.7}$$

$$u^N(0) = P_N u_0,$$

where $P_N : H \to S_N$ is defined by

$$P_N v = \sum_{j=1}^{N} (v, e_j)e_j.$$

Assume that

$$u^N(t) = \sum_{j=1}^{N} u_j^N(t)e_j. \tag{4.1.8}$$

Replacing $u^N(t)$ in (4.1.7) by (4.1.8) and approximating $P_N f(u^N(t))$ with $\sum_{j=1}^{N} f[u_j^N(t)]e_j$, we then have

$$\sum_{j=1}^{N} \left[\frac{d}{dt}u_j^N(t)\right]e_j + \sum_{j=1}^{N} \left[\lambda_j u_j^N(t)\right]e_j = \sum_{j=1}^{N} f[u_j^N(t)]e_j,$$

which implies that

$$\frac{d}{dt}u_j^N(t) + \lambda_j u_j^N(t) = f[u_j^N(t)], \quad j = 1, 2, 3 \ldots. \tag{4.1.9}$$

For each j, we can solve ordinary differential equation (4.1.9) to get $u_j^N(t)$ at different time t. Hence we obtain the approximate solution of (4.1.1)-(4.1.3).

We can also use the discrete sine Fourier transform MATLAB functions "**dst.m** "and "**idst.m** "for solving (4.1.1)-(4.1.3). Let $0 = x_0 < x_1 < x_2 < \cdots < x_{J-1} < x_J = 1$ be the space partition of $[0, 1]$ and Δx be the space stesize.

Let $S_{J-1} = L(e_1, e_2, \ldots e_{J-1})$. The spectral method for solving (4.1.1)-(4.1.3) is to find $u^{J-1}(t) \in S_{J-1}$ such that

$$\frac{du^{J-1}(t)}{dt} + Au^{J-1}(t) = P_{J-1}f(u^{J-1}(t)), \quad t > 0, \tag{4.1.10}$$

$$u^{J-1}(0) = P_{J-1}u_0. \tag{4.1.11}$$

Let $0 = t_0 < t_1 < t_2 \cdots < t_m < \cdots < t_M = T$ be a time partition of $[0, T]$ and Δt be the time stepsize. We define the following backward Euler method at $t = t_m$,

$$\frac{U_m^{J-1}(x) - U_{m-1}^{J-1}(x)}{\Delta t} + AU_m^{J-1}(x) = f(U_{m-1}^{J-1}), \tag{4.1.12}$$

$$U_0^{J-1}(x) = P_{J-1}u_0.$$

Assume that

$$U_m^{J-1}(x) = \sum_{j=1}^{J-1} U_m^{J-1}(j)e_j(x), \tag{4.1.13}$$

for some coefficient $U_m^{J-1}(j)$.

Let $x = x_n = n\Delta x = n\frac{1}{J}, n = 1, 2, \ldots, J - 1$, we have

$$U_m^{J-1}(x_n) = \sum_{j=1}^{J-1} U_m^{J-1}(j)\sqrt{2}\sin(j\pi x_n) = \sum_{j=1}^{J-1} U_m^{J-1}(j)\sqrt{2}\sin(\frac{j\pi n}{J}).$$

By using "**dst.m** ", we have

$$\begin{bmatrix} U_m^{J-1}(x_1) \\ U_m^{J-1}(x_2) \\ \vdots \\ U_m^{J-1}(x_{J-1}) \end{bmatrix} = \mathbf{dst} \begin{bmatrix} \sqrt{2}U_m^{J-1}(1) \\ \sqrt{2}U_m^{J-1}(2) \\ \vdots \\ \sqrt{2}U_m^{J-1}(J-1) \end{bmatrix} = \sqrt{2}\,\mathbf{dst} \begin{bmatrix} U_m^{J-1}(1) \\ U_m^{J-1}(2) \\ \vdots \\ U_m^{J-1}(J-1) \end{bmatrix},$$

and

$$\mathbf{idst} \begin{bmatrix} U_m^{J-1}(x_1) \\ U_m^{J-1}(x_2) \\ \vdots \\ U_m^{J-1}(x_{J-1}) \end{bmatrix} = \begin{bmatrix} \sqrt{2}U_m^{J-1}(1) \\ \sqrt{2}U_m^{J-1}(2) \\ \vdots \\ \sqrt{2}U_m^{J-1}(J-1) \end{bmatrix} = \sqrt{2} \begin{bmatrix} U_m^{J-1}(1) \\ U_m^{J-1}(2) \\ \vdots \\ U_m^{J-1}(J-1) \end{bmatrix}.$$

Replacing $U_m^{J-1}(x)$ and $U_{m-1}^{J-1}(x)$ in (4.1.12) by (4.1.13), we get

$$\sum_{j=1}^{J-1} U_m^{J-1}(j)e_j(x) = (I + \Delta t A)^{-1}\left[U_{m-1}^{J-1}(x) + \Delta t f(U_{m-1}^{J-1})\right]$$

$$= (I + \Delta t A)^{-1}\left[\sum_{j=1}^{J-1} U_m^{J-1}(j)e_j(x) + \Delta t \sum_{j=1}^{J-1} f(U_{m-1}^{J-1})(j)e_j(x)\right],$$

which implies that

$$U_m^{J-1}(j) = (I + \Delta t\lambda_j)^{-1}U_{m-1}^{J-1})(j) + (I + \Delta t\lambda_j)^{-1}\Delta t f(U_{m-1}^{J-1}(j)). \qquad (4.1.14)$$

To determine

$$\begin{bmatrix} U_m^{J-1}(x_1) \\ U_m^{J-1}(x_2) \\ \vdots \\ U_m^{J-1}(x_{J-1}) \end{bmatrix},$$

we need to determine the coefficients

$$\begin{bmatrix} U_m^{J-1}(1) \\ U_m^{J-1}(2) \\ \vdots \\ U_m^{J-1}(J-1) \end{bmatrix}$$

which we can be obtained by (4.1.14).

Suppose we know

$$
Y = \begin{bmatrix} U_{m-1}^{J-1}(1) \\ U_{m-1}^{J-1}(2) \\ \vdots \\ U_{m-1}^{J-1}(J-1) \end{bmatrix},
$$

then we see that

$$
\begin{bmatrix} U_{m-1}^{J-1}(x_1) \\ U_{m-1}^{J-1}(x_2) \\ \vdots \\ U_{m-1}^{J-1}(x_{J-1}) \end{bmatrix} = \mathbf{dst}(Y) * \sqrt{2}.
$$

Further we note that

$$
\mathbf{idst}(y) = Y = \begin{bmatrix} U_{m-1}^{J-1}(1) \\ U_{m-1}^{J-1}(2) \\ \vdots \\ U_{m-1}^{J-1}(J-1) \end{bmatrix},
$$

and

$$
\mathbf{idst}(f(y)) = \begin{bmatrix} f(U_{m-1}^{J-1}(1)) \\ f(U_{m-1}^{J-1}(2)) \\ \vdots \\ f(U_{m-1}^{J-1}(J-1)) \end{bmatrix}.
$$

Hence we get

$$
\begin{bmatrix} U_m^{J-1}(1) \\ U_m^{J-1}(2) \\ \vdots \\ U_m^{J-1}(J-1) \end{bmatrix} = \left(\mathbf{idst}(y + \Delta t f(y))./(1 + \Delta t) \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{J-1} \end{bmatrix} \right)/\sqrt{2},
$$

and

$$
\begin{bmatrix} U_m^{J-1}(x_1) \\ U_m^{J-1}(x_2) \\ \vdots \\ U_m^{J-1}(x_{J-1}) \end{bmatrix} = \mathbf{dst} \begin{bmatrix} U_m^{J-1}(1) \\ U_m^{J-1}(2) \\ \vdots \\ U_m^{J-1}(J-1) \end{bmatrix} * \sqrt{2}.
$$

Let us now consider the starting value. Assume that

$$U_0^{J-1}(x) = \sum_{j=1}^{J-1} U_0^{J-1}(j)e_j(x),$$

with some coefficients $U_0^{J-1}(1), U_0^{J-1}(2), \ldots, U_0^{J-1}(J-1)$. For example, we assume that $U_0^{J-1}(x) = e_1(x) + 3e_3(x)$. Then we have

$$Y = \begin{bmatrix} U_0^{J-1}(1) \\ U_0^{J-1}(2) \\ \vdots \\ U_0^{J-1}(J-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and

$$\begin{bmatrix} U_0^{J-1}(x_1) \\ U_0^{J-1}(x_2) \\ \vdots \\ U_0^{J-1}(x_{J-1}) \end{bmatrix} = \sqrt{2}\mathbf{dst} \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} e_1(x_1) + 3e_3(x_1) \\ e_1(x_2) + 3e_3(x_2) \\ \vdots \\ e_1(x_{J-1}) + 3e_3(x_{J-1}) \end{bmatrix}.$$

Now let us calculate

$$\begin{bmatrix} U_1^{J-1}(x_1) \\ U_1^{J-1}(x_2) \\ \vdots \\ U_1^{J-1}(x_{J-1}) \end{bmatrix}.$$

Noting that

$$\begin{bmatrix} U_0^{J-1}(x_1) \\ U_0^{J-1}(x_2) \\ \vdots \\ U_0^{J-1}(x_{J-1}) \end{bmatrix} = \mathbf{dst}(Y) * \sqrt{2},$$

$$
\begin{bmatrix} U_1^{J-1}(1) \\ U_1^{J-1}(2) \\ \vdots \\ U_1^{J-1}(J-1) \end{bmatrix} = \left[\mathbf{idst}(y + \Delta t f(y))./(1 + \Delta t) \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{N-1} \end{bmatrix} \right)/\sqrt{2},
$$

we have

$$
\begin{bmatrix} U_1^{J-1}(x_1) \\ U_1^{J-1}(x_2) \\ \vdots \\ U_1^{J-1}(x_{J-1}) \end{bmatrix} = \mathbf{dst} \begin{bmatrix} U_1^{J-1}(1) \\ U_1^{J-1}(2) \\ \vdots \\ U_1^{J-1}(J-1) \end{bmatrix} * \sqrt{2}.
$$

4.2 Some Matlab Functions

In MATLAB, there is a MATLAB function "**dst.m** "which transforms a vector [62]

$$
\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{J-1} \end{bmatrix}
$$

into the discrete sine coefficients

$$
\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{J-1} \end{bmatrix},
$$

that is, $\vec{y} = \mathbf{dst}(\vec{u})$, where

$$
y_k = \sum_{n=1}^{J-1} u_n \sin \frac{k\pi n}{J}, \quad k = 1, 2, \cdots, J-1.
$$

The inverse discrete sine transform MATLAB function "idst.m" satisfies $\vec{u} = \mathbf{idst}(\vec{y})$, where

$$
u_n = \frac{2}{J} \sum_{k=1}^{J-1} y_k \sin \frac{k\pi n}{J}.
$$

Thus, given a vector

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{J-1} \end{bmatrix}$$

we can find the discrete sine coefficients $\vec{y}$. The inverse discrete sine transform then transforms $\vec{y}$ back to $\vec{u}$. This is very useful when we solve partial differential equations by using the spectral method.

To understand how to use "dst.m" and "idst.m" to solve partial differential equation, let us first consider the following equation,

$$-u''(x) = f(x), \quad 0 < x < 1, \tag{4.2.1}$$

$$u(0) = u(1) = 0.$$

Let

$$A = -\frac{\partial^2}{\partial x^2}, \quad \mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1).$$

It is well-known that A has eigenvalues $\lambda_j = j^2\pi^2$ and eigenfunctions $e_j = \sqrt{2}\sin(j\pi x)$. Further $\{e_j\}_{j=1}^{+\infty}$ is an orthonormal basis in $L^2(0,1)$. Then the equation (4.2.1) can be written into

$$Au = f. \tag{4.2.2}$$

The solution u of (4.2.2) must have the form

$$u(x) = \sum_{j=1}^{+\infty} \hat{u}_j e_j(x), \tag{4.2.3}$$

where $\hat{u}_j, j = 1.2, 3 \cdots$ are the Fourier coefficients. The spectral method of (4.2.2) is to find $u_{J-1}(x) \in S_{J-1}$ such that

$$Au_{J-1} = P_{J-1}f, \tag{4.2.4}$$

where $S_{J-1} = \text{span}\{e_1, e_2 \cdots, e_{J-1}\}$. Here $P_{J-1} : H \rightarrow S_{J-1}$ is the orthogonal projection defined by

$$P_{J-1}v = \sum_{j=1}^{J-1} \hat{v}_j e_j.$$

Let

$$u_{J-1}(x) = \sum_{j=1}^{J-1} \hat{u}_j e_j = \sqrt{2} \sum_{j=1}^{J-1} \hat{u}_j \sin(j\pi x).$$

Hence we have

$$u_{J-1}(x_k) = \sqrt{2} \sum_{j=1}^{J-1} \hat{u}_j \sin\frac{(j\pi k)}{J}, \qquad k = 1, 2, \cdots, J-1,$$

which may be written as

$$u_{J-1}(x_k) = \left(\sqrt{2}\frac{J}{2}\right)\left[\frac{2}{J}\sum_{j=1}^{J-1} \hat{u}_j \sin\frac{(j\pi k)}{J}\right], \qquad k = 1, 2, \cdots, J-1.$$

By using the MATLAB functions "**dst**.m " and "**idst**.m ", we have

$$\begin{bmatrix} u_{J-1}(x_1) \\ u_{J-1}(x_2) \\ \vdots \\ u_{J-1}(x_{J-1}) \end{bmatrix} = (\sqrt{2}\frac{J}{2})\ \mathbf{idst} \begin{bmatrix} \hat{u}(1) \\ \hat{u}(2) \\ \vdots \\ \hat{u}(J-1) \end{bmatrix}.$$

Taking the discrete sine transform in both sides, we get

$$\mathbf{dst}\begin{bmatrix} u_{J-1}(x_1) \\ u_{J-1}(x_2) \\ \vdots \\ u_{J-1}(x_{N-1}) \end{bmatrix} = (\sqrt{2}\frac{J}{2}) \begin{bmatrix} \hat{u}(1) \\ \hat{u}(2) \\ \vdots \\ \hat{u}(J-1) \end{bmatrix}.$$

Thus we obtain the relation between $u_{J-1}(x_k), k = 1, 2, \cdots, J-1$ and its Fourier sine coefficients $\hat{u}_j, j = 1, 2, \cdots, J-1$.

Now let us solve (4.2.4). Assume that

$$u_{J-1} = \sum_{j=1}^{J-1} \hat{u}_j e_j, \quad P_{J-1}f = \sum_{j-1}^{J-1} \hat{f}_j e_j. \tag{4.2.5}$$

Substituting these into (4.2.4), we get

$$\sum_{j-1}^{J-1} \hat{u}_j(Ae_j) = \sum_{j-1}^{J-1} \hat{f}_j e_j, \tag{4.2.6}$$

or

$$\sum_{j-1}^{J-1} (\lambda_j \hat{u}_j)e_j = \sum_{j-1}^{J-1} \hat{f}_j e_j, \tag{4.2.7}$$

which implies that

$$\hat{u}_j = \lambda_j^{-1}\hat{f}_j, \quad j = 1, 2, \cdots, J-1. \tag{4.2.8}$$

Since $f(x)$ is given, we may have $P_{J-1}f(x_k) \approx f(x_k), k = 1, 2, \ldots, J-1$. The Fourier sine cofficients $\hat{f}_j$ can then be obtained by (4.2),

$$\begin{bmatrix} \hat{f}(1) \\ \hat{f}(2) \\ \vdots \\ \hat{f}(J-1) \end{bmatrix} = (\sqrt{2}\frac{J}{2})\ \mathbf{dst} \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{J-1}) \end{bmatrix}.$$

By (4.2.8), we get $\hat{u}_j = \lambda_j^{-1}\hat{f}_j$. Hence we have

$$\begin{bmatrix} u_{J-1}(x_1) \\ u_{J-1}(x_2) \\ \vdots \\ u_{J-1}(x_{J-1}) \end{bmatrix} = (\sqrt{2}\frac{J}{2})\ \mathbf{idst} \begin{bmatrix} \hat{u}(1) \\ \hat{u}(2) \\ \vdots \\ \hat{u}(J-1) \end{bmatrix}.$$

Based on the detailed analysis above, we now give the following algorithm.

Step 1: Calculate

$$f = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{J-1}) \end{bmatrix}.$$

Step 2: Find the Fourier sine coffeicients

$$\hat{f} = \begin{bmatrix} \hat{f}(1) \\ \hat{f}(2) \\ \vdots \\ \hat{f}(J-1) \end{bmatrix}$$

by

$$\hat{f} = (\sqrt{2}\frac{J}{2})^{-1}\ \mathbf{dst} \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{J-1}) \end{bmatrix}.$$

Step 3: Find the Fourier coefficients

$$\hat{u} = \begin{bmatrix} \hat{u}(1) \\ \hat{u}(2) \\ \vdots \\ \hat{u}(J-1) \end{bmatrix}$$

by $\hat{u}(j) = \lambda_j^{-1} \hat{f}(j)$.

Step 4: Find the solution $u_{J-1}(x_k), k = 1, 2, \cdots, J-1$ by

$$\begin{bmatrix} u_{J-1}(x_1) \\ u_{J-1}(x_2) \\ \vdots \\ u_{J-1}(x_{N-1}) \end{bmatrix} = (\sqrt{2}\frac{J}{2}) \, \mathbf{idst} \begin{bmatrix} \hat{u}(1) \\ \hat{u}(2) \\ \vdots \\ \hat{u}(J-1). \end{bmatrix}.$$

For the reader's convenience, below we include the codes for solving partial differential equation by using the spectral method.

```
clear
T=1; Dt=0.01; M1-T/Dt;
A=1;
%epsilon=1e-03;
J=512; h=a/J; x=[h:h:(J-1)*h];
lambda=pi*[1:(J-1)];
M=lambda.^2;
EE=1./(1+dT*m);
u0=sin(pi*x);


for m=1;M1
u0_hat=(sqrt(2)*J/2)^(-1)*DST(U0);
%F_U0=U0-U0.^3; %f(u) = u-u^3
f_u0=exp(m*Dt)*sin(pi*x) + exp(m*Dt)*sin(pi*x)*(pi^2);
f_u0_hat=(sqrt(2)*J/2)^(-1)*dst(f_u0);
u1_hat=(u0_hat + Dt*f_u0_hat).*EE;
u1=(sqrt(2)*J/2)*idst(u1_hat);
```

```
u0=u1;

end

u_exact=exp(1)*sin(pi*x);

error=u0-u_exact;

figure(1)

plot(x,u0)

title('approximate solution')

xlabel('x')

ylabel('exact solution')

figure(2)

plot(x,error)

title(error')

xlabel('x')

ylabel('error')
```

Chapter 5

Fourier Spectral Methods for Some Linear Stochastic Space Fractional Partial Differential Equations

5.1 Introduction

Recently stochastic space fractional partial differential equations attract a lot of attention in view of their modeling applications. Fractional derivative is a powerful instrument for the description of memory and hereditary properties of different substance. Here we consider numerical methods for solving some linear stochastic space fractional partial differential equation in 1- dimensional case.

Fourier spectral methods for solving some linear stochastic space fractional partial differential equations perturbed by space-time white noises in one dimensional case are introduced and analyzed. The space fractional derivative is defined by using the eigenvalues and eigenfunctions of Laplacian subject to some boundary conditions. We approximate the space-time white noise by using piecewise constant functions and obtain the approximated stochastic space fractional partial differential equations. The approximated stochastic space fractional partial differential equations are then solved by using Fourier spectral methods. Error estimates in L_2-norm are obtained, and numerical examples are given.

We will consider now Fourier spectral methods for solving linear stochastic space fractional partial differential equation:

$$\frac{\partial u(t,x)}{\partial t} + (-\Delta)^{\alpha} u(t,x) = \frac{\partial^2 W(t,x)}{\partial t \partial x}, \qquad 0 < t < T, \quad 0 < x < 1, \tag{5.1.1}$$

$$u(t,0) = u(t,1) = 0, \qquad 0 < t < T, \tag{5.1.2}$$

$$u(0,x) = u_0(x), \qquad 0 < x < 1. \tag{5.1.3}$$

Here $(-\Delta)^{\alpha}$, $\frac{1}{2} < \alpha \leq 1$, is the fractional Laplacian and $\frac{\partial^2 W(t,x)}{\partial t \partial x}$ is the mixed second order derivative of the Brownian sheet [94]. It is well known that the Laplacian $-\Delta$ has eigenpairs (λ_j, e_j) with $\lambda_j = j^2 \pi^2, e_j = \sqrt{2} \sin j\pi x, j = 1, 2, 3 \cdots$ subject to the homogeneous Dirichlet boundary conditions on $(0,1), i.e., e_j(0) = e_j(1) = 0$ and

$$-\Delta e_j = \lambda_j e_j, \qquad j = 1, 2, 3, \ldots.$$

Let $H = L^2(0,1)$ with inner product $(\cdot, \cdot)$ and norm $|| \cdot ||$. For any $r \in \mathbb{R}$, we denote:

$$H_0^r := \{ v : v = \sum_{j=1}^{\infty} (v, e_j) e_j, \ \sum_{j=1}^{\infty} \lambda_j^r (v, e_j)^2 < \infty \}, \tag{5.1.4}$$

with norm:

$$|v|_r = \Big(\sum_{j=1}^{\infty} \lambda_j^r (v, e_j)^2 \Big)^{\frac{1}{2}}. \tag{5.1.5}$$

Then, for any $v \in H_0^{2\alpha}(0,1), \frac{1}{2} < \alpha \leq 1$, we have

$$(-\Delta)^{\alpha} v = \sum_{j=1}^{\infty} (v, e_j) \lambda_j^{\alpha} e_j. \tag{5.1.6}$$

Space fractional partial differential equations are widely used to model complex phenomena, for example, quasi-geostrophic flows, fast roating fluids, the dynamics of the frontogenesis in meteorology, diffusion in fractal or disordered medium, pollution problems, mathematical finance and transport problems.

Let us here consider two examples, which apply the fractional Laplacian in the physical models. The first example is the surface quasi-geostrophic (SQG) equation,

$$\partial_t \theta + \vec{u}.\nabla \theta + k(-\Delta)^{\alpha} \theta = 0,$$

where $k \geq 0$ and $\alpha > 0, \theta = \theta(x_1, x_2, t)$ denotes the potential temperature, $\vec{u} = (u_1, u_2)$ is the velocity field determined by θ. When $k > 0$, the SQG equation takes into account

the dissipation generated by a fractional Laplacian. The SQG equation with $k > 0$ and $\alpha = \frac{1}{2}$ arises in geophysical studies of strongly-rotating fluids. For the dissipative SQG equation, $\alpha = \frac{1}{2}$ appears to be a critical index. In the sub-critical case with $\alpha > \frac{1}{2}$, the dissipation is sufficient to control the nonlinearity and global regularity is a consequence of global a priori bound. In the critical case with $\alpha = \frac{1}{2}$, the global regularity issue is more delicate. There are few theoretical results for the supercritical case $\alpha < \frac{1}{2}$ in the literature [20].

The second example is about the wave propagation in complex solids, especially viscoelastic materials (for example, polymers) [12]. In this case, the relaxation function has the form $k(t) = ct^{-v}, 0 < v < 1, c \in \mathbb{R}$, instead of the exponential form known in the standard models. This polynomial relaxation is due to the non-uniformity of the material. The far field is then described by a Burgers equation with the leading operator $(-\Delta)^{\frac{1+v}{2}}$ instead of the Laplacian,

$$\partial_t u = -(-\Delta)^{\frac{1+v}{2}} u + \partial_x(u^2).$$

This equation also describes the far-field evolution of acoustic waves propagating in a gas-filled tube with a boundary layer.

Frequently, the initial value or the coefficients of the equation are random; therefore, it is natural to consider the stochastic space fractional partial differential equations. The existence, uniqueness and regularities of the solution of stochastic space fractional partial differential equations have been extensively studied, see, e.g., [11], [16], [12], [23]. In this work, we will focus on the case $\frac{1}{2} < \alpha \le 1$, since the existence, uniqueness and the regularity of the solution in this case is well understood in the literature, see [23, Theorem 1.3]. However, the numerical methods for solving space fractional stochastic partial differential differential equations are quite restricted even for the case $\frac{1}{2} < \alpha \le 1$. Debbi and Dozzi [23] introduced a discretization of the fractional Laplacian and used it to obtain an approximation scheme for fractional heat equation perturbed by multiplicative cylindrical white noise.To the best of our knowledge, [23] is the only existing paper in the literature that deals with the numerical approach of this kind of problems. In this work, we will use the ideas developed in [2] to consider the numerical methods for solving stochastic space fractional partial differential equation, see also [17], [51]. We first approximate

the space-time white noise by using piecewise constant functions and then obtain the approximate solution $\hat{u}(t)$ of the exact solution $u(t)$. Finally, we provide error estimates in the L^2- norm for $u(t) - \hat{u}(t)$.

For the deterministic space fractional partial differential equations, many numerical methods are available in the literature. There are two approaches to define the fractional Laplacian. One approach is by using the eigenvalues and eigenfunctions of Laplacian $-(\Delta)^\alpha v, \frac{1}{2} < \alpha \le 1$ subject to the boundary conditions as in (5.1.6). Another approach is by using the left-handed, right-handed Riemann-Liouville fractional derivatives. For the deterministic space fractional partial differential equations defined by the Riemann-Liouville fractional derivatives, many numerical methods are available, finite difference method, finite element method and spectral methods. In this work, we will use Fourier spectral method to solve the stochastic space fractional partial differential equations. The main advantage of this approach is that it gives a full diagonal representation of the fractional operator, being able to achieve spectral convergence regardless of the fractional power in the problem. By using the integral for the function $\bar{v}$, where $\bar{v}$ is defined on the whole real line $\mathbb{R}$ and is the extension function of v,

$$\bar{v}(x) = \begin{cases} v(x), & 0 < x < 1, \\ 0, & x \notin (0,1), \end{cases}$$

we define

$$(\Delta)^\alpha \bar{v}(x) = C_\alpha \int_{\mathbb{R}-\{0\}} \frac{2\bar{v}(x) + \bar{v}(x+y) - \bar{v}(x-y)}{|y|^{1+2\alpha}} dy, \quad x \in \mathbb{R},$$

where C_α is a positive constant depending on α. We then define, [81], [50],

$$(\Delta)^\alpha \bar{v}(x) = \mathcal{F}^{-1}\Big(|\xi|^{2\alpha} (\mathcal{F}(\bar{v}))(\xi) \Big), \quad x \in \mathbb{R},$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denotes the Fourier and inverse Fourier transforms, respectively. For $v(x), x \in (0,1)$ we define the fractional Laplacian by

$$(-\Delta)^\alpha v(x) = (-\Delta)^\alpha \bar{v}(x).$$

It is easy to show that for some suitable function $\omega(x), x \in \mathbb{R}$ [98],

$$(-\Delta)^{\alpha}\omega(x) = \mathcal{F}^{-1}\Big(|\xi|^{2\alpha}\mathcal{F}(\omega)(\xi)\Big) = \frac{1}{2\cos(\pi\alpha)}\Big(\underset{-\infty}{\overset{R}{}}D_x^{2\alpha}w(x) + \underset{x}{\overset{R}{}}D_\infty^{2\alpha}w(x)\Big),$$

where $\underset{-\infty}{\overset{R}{}}D_x^{\beta}w(x)$ and $\underset{x}{\overset{R}{}}D_\infty^{\beta}w(x), 1 < \beta < 2$ are called Riemann-Liouville fractional derivatives define by

$$\underset{-\infty}{\overset{R}{}}D_x^{\beta}w(x) = \frac{1}{\Gamma(2-\beta)}\frac{d^2}{dx^2}\int_{-\infty}^{x}(x-y)^{1-\beta}\omega(y)dy,$$

$$\underset{x}{\overset{R}{}}D_\infty^{\beta}w(x) = \frac{1}{\Gamma(2-\beta)}\frac{d^2}{dx^2}\int_{x}^{\infty}(y-x)^{1-\beta}\omega(y)dy.$$

Hence, for the function $v(x)$ defined on the bounded interval $(0,1)$, we have

$$(-\Delta)^{\alpha}v(x) = \frac{1}{2\cos(\pi\alpha)}\Big(\underset{0}{\overset{R}{}}D_x^{2\alpha}v(x) + \underset{x}{\overset{R}{}}D_1^{2\alpha}v(x)\Big), \quad x \in (0,1), \tag{5.1.7}$$

which is also called the Riesz fractional derivative.

We note that Definitions (5.1.6) and (5.1.7) are equivalent [81]. For the deterministic space fractional partial differential equations where the space fractional derivative is defined by (5.1.7), or the Riemann-Liouville space fractional deriavative, or the Caputo space fractional derivative, many numerical methods are available: finite difference methods, [41], [69], [91], [89], finite element methods [31], [45] and spectral methods [57], [58].

For the deterministic space fractional partial differential equations, where the space fractional derivative is defined by (5.1.6), some numerical methods are also available: the matrix transfer method (MTT), [41], [15] and the Fourier spectral method [14]. In this work, we will use Fourier spectral methods to solve the approximated stochastic space fractional partial differential equations. The main advantage of this approach is that it gives a full diagonal representation of the fractional operator, being able to achieve spectral convergence regardless of the fractional power in the problem.

Let $0 = x_0 < x_1 < x_2 < \cdots < x_J = 1$ be the space partition of (0,1) and h the space step size. Let $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ be the time partition of (0,T) and k the time step size. To find the approximate solution of (5.1.1)-(5.1.3), we first approximate the space-time white noise $\frac{\partial^2 W(t,x)}{\partial t \partial x}$ by using a piecewise constant function $\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ defined by, with $n = 1, 2, 3, \ldots, N, \ j = 1, 2, \ldots, J$, [2],

$$\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x} := \frac{\eta_{n,j}}{\sqrt{kh}}, \quad t_{n-1} \le t \le t_n, \ x_{j-1} \le x \le x_j, \tag{5.1.8}$$

where $\eta_{nj} \in \mathcal{N}(0,1)$ is an independently and identically distributed random variable and

$$\eta_{n,j} = \frac{1}{\sqrt{kh}} \int_{t_{n-1}}^{t_n} \int_{x_{j-1}}^{x_j} dW(t,x). \tag{5.1.9}$$

Hence we have

$$\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x} := \frac{1}{\sqrt{kh}} \int_{t_{n-1}}^{t_n} \int_{x_{j-1}}^{x_j} dW(t,x), \quad \text{on } [t_{n-1}, t_n] \times [x_{j-1}, x_j]. \tag{5.1.10}$$

We also note that [2]

$$\int_{t_{n-1}}^{t_n} \int_{x_{j-1}}^{x_j} d\widehat{W}(t,x) = \int_{t_{n-1}}^{t_n} \int_{x_{j-1}}^{x_j} \frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x} \, dx dt.$$

The solution $u(t,x)$ of (5.1.1)-(5.1.3) can be approximated by $\hat{u}(t,x)$, which solves the following:

$$\frac{\partial \hat{u}(t,x)}{\partial t} + (-\Delta)^\alpha \hat{u}(t,x) = \frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}, \quad 0 < t < T, \ 0 < x < 1, \tag{5.1.11}$$

$$\hat{u}(t,0) = \hat{u}(t,1) = 0, \quad 0 < t < T, \tag{5.1.12}$$

$$\hat{u}(0,x) = u_0(x), \quad 0 < x < 1. \tag{5.1.13}$$

Note that $\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ is a function in $L^2((0,T) \times (0,1))$ and therefore we can solve (5.1.11)-(5.1.13) by using any appropriate numerical method for deterministic space fractional partial differential equations. In Theorem 5.2.2 we prove that, if $\frac{1}{2} < \alpha \leq 1$, then we have

$$\mathbb{E} \int_0^T \int_0^1 (u(t,x) - \hat{u}(t,x))^2 \, dx dt \leq C(k^{1-\frac{1}{2\alpha}} + h^2 k^{\frac{2\alpha-3}{2\alpha}}). \tag{5.1.14}$$

Let us now introduce the Fourier spectral method for solving (5.1.11)-(5.1.13). Let J be a positive integer, and denote $S_J = \text{span}\{e_1, e_2 \ldots, e_J\}$.

Define by $P_J : H \to S_J$ the projection from H to S_J,

$$P_J v := \sum_{j=1}^J (v, e_j) e_j. \tag{5.1.15}$$

The Fourier spectral method for solving (5.1.11)-(5.1.13) is to find $\hat{u}_J(t) \in S_J$, such that

$$\frac{\partial \hat{u}_J(t,x)}{\partial t} + (-\Delta)^\alpha \hat{u}_J(t,x) = P_J \frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}, \quad 0 < t < T, \ 0 < x < 1, \tag{5.1.16}$$

$$\hat{u}_J(t,0) = \hat{u}_J(t,1) = 0, \quad 0 < t < T, \tag{5.1.17}$$

$$\hat{u}_J(0,x) = P_J u_0(x), \quad 0 < x < 1. \tag{5.1.18}$$

In Theorem 5.3.1 we prove that, with $\frac{1}{2} < \alpha \leq 1$,

$$||\hat{u}(t) - \hat{u}_J(t)|| \leq C||u_0 - P_J u_0|| + C\frac{1}{(J+1)^\alpha}\left(\int_0^t ||\hat{f}(s)||^2 ds\right)^{\frac{1}{2}}, \qquad (5.1.19)$$

where $||\cdot||$ denotes the norm in $L_2(0,1)$ space.

Combining Theorems 5.2.2 with 5.3.1, we have

$$\mathbb{E}\int_0^T \int_0^1 (u(t,x) - \hat{u}_J(t,x))^2 \, dxdt \leq C(k^{1-\frac{1}{2\alpha}} + h^2 k^{\frac{2\alpha-3}{2\alpha}}) + C||u_0 - P_J u_0||^2$$

$$+ C\frac{1}{(J+1)^{2\alpha}}(k^{-1-\frac{1}{2\alpha}} + h^{-1}k^{-1}).$$

5.2 Approximate the Noise and Regularity of the Solution

Consider the stochastic space fractional partial differential equation:

$$\frac{\partial u(t,x)}{\partial t} + (-\Delta)^\alpha u(t,x) = f(t,x), \quad 0 < t < T, \, 0 < x < 1, \qquad (5.2.1)$$

$$u(t,0) = u(t,1) = 0, \quad 0 < t < T, \qquad (5.2.2)$$

$$u(0,x) = u_0(x), \quad 0 < x < 1, \qquad (5.2.3)$$

where $f(t,x) = \frac{\partial^2 W(t,x)}{\partial t \partial x}$ denotes the mixed second order derivative of the Brownian sheet [2]. There is no strong solution of (5.2.1)-(5.2.3) since $f(t,x) = \frac{\partial^2 W(t,x)}{\partial t \partial x} \notin L^2((0,T) \times (0,1))$.

It is well known that the mild solution of (5.2.1)-(5.2.3) has the following form, [23], [78],

$$u(t,x) = \int_0^1 G_\alpha(t,x,y)u_0(y)dy + \int_0^1 \int_0^t G_\alpha(t-s,x,y)dW(s,y),$$

where

$$G_\alpha(t,x,y) = \sum_{j=1}^\infty e^{-\lambda_j^\alpha t}e_j(x)e_j(y),$$

and the stochastic integral $\int_0^t \int_0^1 G_\alpha(t-s,x,y)\,dW(s,y)$ is well-defined. Such stochastic integral has the following properties.

First, if $S = \{(s, y) : a \leq s < b, \ c \leq y < d\}$ is a rectangle, then

$$\int_c^d \int_a^b dW(s, y) = \int_c^d \int_a^b \frac{\partial^2 W}{\partial s \partial y}(s, y) \, ds dy = W(S)$$

$$= W(b, d) - W(a, d) - W(b, c) + W(a, c),$$

where $W(S)$ is Gaussian with zero means and variance $|S|$ and $|S|$ is the area of S.

Second, if χ_s is the characteristic function of rectangle S, then

$$\int_0^T \int_a^b \chi_s \, dW(s, y) = W(S), \quad \text{for } S \subset (0, T) \times (a, b).$$

Third, if $\mathbb{E}\left(\int_0^T \int_a^b f(s, y) \, ds dy \right) < \infty$, then

$$\mathbb{E}\left(\int_0^T \int_a^b f(s, y) \, dW(s, y) \right)^2 = \mathbb{E}\left(\int_0^T \int_a^b f^2(s, y) \, ds dy \right).$$

We have the following existence and uniqueness theorem, see, e.g., [23], [24].

Theorem 5.2.1. [24, Theorem 1.3] Let $\frac{1}{2} < \alpha \leq 1$ and $\beta > 0$. Let u_0 be a $H_0^\beta(0, 1)$-valued $\mathcal{F}_0$-measurable function, such that

$$\mathbb{E}\|u_0\|_{H_0^\beta(0,1)}^p < \infty$$

for some $p > \frac{4\alpha}{2\alpha - 1}$. Then (5.2.1)-(5.2.3) has a unique mild solution u such that, for any $0 \leq \theta < \min\{\frac{2\alpha - 1}{2} - \frac{2\alpha}{p}, \beta\}$,

$$\mathbb{E}\sup_{0 \leq t \leq T}\|u(t)\|_{H_0^\theta(0,1)}^p < \infty.$$

Our strategy is to approximate the solution $u(t, x)$ of (5.2.1)-(5.2.3) by $\hat{u}(t, x)$, which satisfies the following problem:

$$\frac{\partial \hat{u}(t, x)}{\partial t} + (-\Delta)^\alpha \hat{u}(t, x) = \hat{f}(t, x), \quad 0 < t < T, \ 0 < x < 1, \tag{5.2.4}$$

$$\hat{u}(t, 0) = \hat{u}(t, 1) = 0, \quad 0 < t < T, \tag{5.2.5}$$

$$\hat{u}(0, x) = u_0(x), \quad 0 < x < 1. \tag{5.2.6}$$

Here $\hat{f}(t, x) = \frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ is defined by (5.1.8). The solution of (5.2.4)-(5.2.6) has the form of, see, e.g., [2],

$$\hat{u}(t, x) = \int_0^1 G_\alpha(t, x, y) u_0(y) dy + \int_0^t \int_0^1 G_\alpha(t - s, x, y) d\widehat{W}(s, y). \tag{5.2.7}$$

Theorem 5.2.2. Let u and $\hat{u}$ be the solutions of (5.2.1)-(5.2.3) and (5.2.4)-(5.2.6) respectively. Assume that $u_0 \in H$ and $\frac{1}{2} < \alpha \leq 1$. Then we have

$$\mathbb{E} \int_0^T \int_0^1 (u(t,x) - \hat{u}(t,x))^2 dx dt \leq C(k^{1-\frac{1}{2\alpha}} + h^2 k^{\frac{2\alpha-3}{2\alpha}}).$$

Remark 1. When $\alpha = 1$, we obtain the same estimates in [2], [28], that is,

$$\mathbb{E} \int_0^T \int_0^1 (u(t,x) - \hat{u}(t,x))^2 dx dt \leq C(k^{\frac{1}{2}} + h^2 k^{-\frac{1}{2}}).$$

We shall prove Theorem 5.2.2. To do this, we need the following lemmas.

Lemma 5.2.3. Let $1 < \beta \leq 2$. We have

$$\sum_{n=1}^{\infty} e^{-n^\beta k} n^\beta \leq Ck^{-1-\frac{1}{\beta}}, \tag{5.2.8}$$

$$\sum_{n=1}^{\infty} \frac{1 - e^{-n^\beta k}}{n^\beta} \leq Ck^{1-\frac{1}{\beta}}, \tag{5.2.9}$$

$$\sum_{n=1}^{\infty} \frac{e^{-n^\beta k}}{n^{\beta-2}} \leq Ck^{\frac{\beta-3}{\beta}}, \tag{5.2.10}$$

$$\sum_{n=1}^{\infty} (1 - e^{-n^\beta k})^2 \leq Ck^{-\frac{1}{\beta}}, \tag{5.2.11}$$

$$\sum_{n=1}^{\infty} \frac{(1 - e^{-n^\beta k})^2}{n^{2\beta}} \leq Ck^{\frac{2\beta-1}{\beta}}, \tag{5.2.12}$$

$$\sum_{l=0}^{j-2} e^{-n^\beta}(t_j - t_{l+1}) \leq Ck^{-1}n^{-\beta}, \quad j \geq 2. \tag{5.2.13}$$

Proof. We have, with the variable change $x^\beta k = y^\beta$,

$$\sum_{n=1}^{\infty} e^{-n^\beta k} n^\beta \leq C \int_1^{\infty} e^{-x^\beta k} x^\beta dx = C \int_{k^{\frac{1}{\beta}}}^{\infty} e^{-y^\beta}(k^{-1}y^\beta)k^{-\frac{1}{\beta}} dy \tag{5.2.14}$$

$$\leq C \int_0^{\infty} e^{-y^\beta} k^{-1-\frac{1}{\beta}} y^\beta dy \leq Ck^{-1-\frac{1}{\beta}}. \tag{5.2.15}$$

Similarly we can show (5.2.9)-(5.2.12). For (5.2.13), noting that $1 + x < e^x, x > 0$, we derive

$$\sum_{l=0}^{j-2} e^{-n^\beta}(t_j - t_{l+1}) \leq e^{-n^\beta k} + (e^{-n^\beta k})^2 + \cdots \leq e^{-n^\beta k}(1 + e^{-n^\beta k} + \cdots) \qquad (5.2.16)$$

$$\leq e^{-n^\beta k}\frac{1}{1 - e^{-n^\beta k}} = \frac{1}{e^{n^\beta k} - 1} \leq C(n^\beta k)^{-1} \leq Ck^{-1}n^{-\beta}. \quad (5.2.17)$$

The proof of Lemma 5.2.3 is now complete. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

We also need the following isometry property for the approximated space-time white noise $W(s,y)$, see, e.g.,[94].

Lemma 5.2.4. We have

$$\mathbb{E}\left| \int_0^T \int_0^1 f(s,y)dW(s,y)\right|^2 = \mathbb{E}\int_0^T \int_0^1 f^2(s,y)dsdy.$$

Similarly, we have the following isometry property for the approximated space-time white noise $\hat{W}(s,y)$, see [2].

Lemma 5.2.5. We have

$$\mathbb{E}\left| \int_0^T \int_0^1 f(s,y)d\widehat{W}(s,y)\right|^2 = \mathbb{E}\int_0^T \int_0^1 f^2(s,y)dsdy.$$

Proof. We have, by (5.1.10), Lemma 5.2.4 and the Cauchy-Schwarz inequality,

$$\mathbb{E}|\int_0^T \int_0^1 f(s,y)d\widehat{W}(s,y)|^2 = \mathbb{E}|\int_0^T \int_0^1 f(s,y)\frac{\partial^2 \widehat{W}(s,y)}{\partial s \partial y}dyds|^2$$

$$= \mathbb{E}[\sum_{j=0}^{N-1}\sum_{i=0}^{M-1}\int_{t_i}^{t_{i+1}}\int_{x_j}^{x_{j+1}}f(s,y)\frac{1}{kh}\int_{t_i}^{t_{i+1}}\int_{x_j}^{x_{j+1}}dW(r,z)dsdy]^2$$

$$= \mathbb{E}|\sum_{j=0}^{N-1}\sum_{i=0}^{M-1}\int_{t_i}^{t_{i+1}}\int_{x_j}^{x_{j+1}}\Big(\frac{1}{kh}\int_{t_i}^{t_{i+1}}\int_{x_j}^{x_{j+1}}f(s,y)dsdy\Big)dW(r,z)]^2$$

$$= \mathbb{E}\sum_{j=0}^{N-1}\sum_{i=0}^{M-1}\int_{t_i}^{t_{i+1}}\int_{x_j}^{x_{j+1}}\Big(\frac{1}{kh}\int_{t_i}^{t_{i+1}}\int_{x_j}^{x_{j+1}}f(s,y)dsdy\Big)^2 dr\,dz$$

$$\leq \mathbb{E}\sum_{j=0}^{N-1}\sum_{i=0}^{M-1}\int_{t_i}^{t_{i+1}}\int_{x_j}^{x_{j+1}}\Big[\frac{1}{kh}\int_{t_i}^{t_{i+1}}\int_{x_j}^{x_{j+1}}f(s,y)dsdy\Big]^2 drdz$$

$$\leq \mathbb{E}\sum_{j=0}^{N-1}\sum_{i=0}^{M-1}\int_{t_i}^{t_{i+1}}\int_{x_j}^{x_{j+1}}\Big[\frac{1}{kh}\int_{t_i}^{t_i+1}\int_{x_j}^{x_j+1}f^2(s,y)dsdy\Big]drdz$$

$$\leq \mathbb{E}\sum_{j=0}^{N-1}\sum_{i=0}^{M-1}\int_{t_i}^{t_{i+1}}\int_{x_j}^{x_{j+1}}f^2(s,y)dsdy$$

$$= \mathbb{E}\int_0^T \int_0^1 f^2(s,y)\,dsdy.$$

The proof of Lemma 5.2.5 is complete. □

Proof of Theorem 5.2.2. We shall prove

$$\mathbb{E}\int_0^T \int_0^1 (u(t,x) - \hat{u}(t,x))^2 dxdt \leq C(k^{1-\frac{1}{2\alpha}} + h^2 k^{\frac{2\alpha-3}{2\alpha}}). \tag{5.2.18}$$

Note that

$$u(t,x) = \int_0^1 G_\alpha(t,x,y)u_0(y)\,dy + \int_0^t \int_0^1 G_\alpha(t-s,x,y)\,dW(s,y), \tag{5.2.19}$$

$$\hat{u}(t,x) = \int_0^1 G_\alpha(t,x,y)u_0(y)\,dy + \int_0^t \int_0^1 G_\alpha(t-s,x,y)d\widehat{W}(s,y). \tag{5.2.20}$$

Here

$$G_\alpha(t,x,y) = \sum_{j=1}^{+\infty}e^{-\lambda_j^\alpha t}e_j(x)e_j(y).$$

Subtracting equation (5.2.20) from (5.2.19), we get

$$\mathbb{E}\int_0^T \int_0^1 |u(x,t) - \hat{u}(x,t)|^2 dxdt$$

$$= \mathbb{E}\int_0^T \int_0^1 \Big[\int_0^t \int_0^1 G_\alpha(t-s,x,y)dW(s,y) - \int_0^t \int_0^1 G_\alpha(t-s,x,y)d\widehat{W}(s,y)\Big]^2 dxdt.$$

By using inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$, $\forall a, b, c \in \mathbb{R}$, we get,

$$\mathbb{E} \int_0^T \int_0^1 |u(x,t) - \hat{u}(x,t)|^2 dx dt$$

$$\leq 3\mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left\{ \left[\int_0^t \int_0^1 G_\alpha(t-s,x,y)dW(s,y) - \int_0^{t_j} \int_0^1 G_\alpha(t_j-s,x,y)dW(s,y) \right]^2 \right.$$

$$+ \left[\int_0^{t_j} \int_0^1 G_\alpha(t_j-s,x,y)dW(s,y) - \int_0^{t_j} \int_0^1 G_\alpha(t_j-s,x,y)d\hat{W}(s,y) \right]^2$$

$$\left. + \left[\int_0^{t_j} \int_0^1 G_\alpha(t_j-s,x,y)d\hat{W}(s,y) - \int_0^t \int_0^1 G_\alpha(t_j-s,x,y)d\hat{W}(s,y) \right]^2 \right\} dx dt$$

$$= 3(I + II + III).$$

We first estimate II. Using the approximation of the space-time white noise (5.1.8), we have, taking also account (5.1.10),

$$II = \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\int_0^{t_j} \int_0^1 G_\alpha(t_j-s,x,y)dW(s,y) \right.$$

$$\left. - \int_0^{t_j} \int_0^1 G_\alpha(t_j-s,x,y)d\hat{W}(s,y) \right]^2 dx dt$$

$$= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\sum_{l=0}^{j-1} \sum_{i=0}^{M-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} G_\alpha(t_j-r,x,z)dW(r,z) \right.$$

$$\left. - \sum_{l=0}^{j-1} \sum_{i=0}^{M-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} G_\alpha(t_j-s,x,y) \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} dW(r,z)dy ds \right]^2 dx dt.$$

$$= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\sum_{l=0}^{j-1} \sum_{i=0}^{M-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \right.$$

$$\left. \left(G_\alpha(t_j-r,x,z)dW(r,z) - \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} G_\alpha(t_j-s,x,y)dy ds \right) dW(r,z) \right]^2 dx dt$$

$$= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\sum_{l=0}^{j-1} \sum_{i=0}^{M-1} \int_{t_l}^{t_{l+1}} \right.$$

$$\left. \int_{x_i}^{x_{i+1}} \left(\frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \left(G_\alpha(t_j-r,x,z) - G_\alpha(t_j-s,x,y) \right) dy ds \right) dW(r,z) \right]^2 dx dt.$$

By using the following isometry property and Cauchy-Swartz Inequality

$$\mathbb{E}\left[\int_0^t f(s)dW(s) \right]^2 = E \int_0^t f^2(s)ds,$$

$$\left[\int_a^b f(x)g(x)dx \right]^2 \leq \left[\int_a^b f^2(x)dx \right]\left[\int_a^b g^2(x)dx \right],$$

we get

$$
\begin{aligned}
II =& \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \sum_{l=0}^{j-1} \sum_{i=0}^{M-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \\
& \left(\frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} [G_\alpha(t_j - r, x, z) - G_\alpha(t_j - s, x, y)) \, dy \, ds \right)^2 dz \, dr \, dx \, dt \\
=& \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \sum_{l=0}^{j-1} \sum_{i=0}^{M-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \\
& \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \Big(G_\alpha(t_j - r, x, z) - G_\alpha(t_j - s, x, y) \Big)^2 dy \, ds \, dz \, dr \, dx \, dt.
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
II =& \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \sum_{i=0}^{M-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \\
& \sum_{n=1}^\infty \left(e^{-\lambda_n^\alpha(t_j - r)} e_n(z) - e^{-\lambda_n^\alpha(t_j - s)} e_n(y) \right)^2 dy \, ds \, dz \, dr \, dt \\
=& \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \Big[\sum_{l=0}^{j-1} \sum_{i=0}^{M-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \\
& \sum_{n=1}^\infty e^{-2\lambda_n^\alpha t_j} \left(e^{\lambda_n^\alpha r} e_n(z) - e^{\lambda_n^\alpha s} e_n(y) \right)^2 dy \, ds \, dz \, dr \, dt \\
\leq& 2 \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \sum_{i=0}^{M-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \\
& \sum_{n=1}^\infty e^{-2\lambda_n^\alpha t_j} (e_n(z) - e_n(y))^2 e^{\lambda_n^\alpha r} dy \, ds \, dz \, dr \, dt \\
& + 2 \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \sum_{i=0}^{M-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \\
& \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \sum_{n=1}^\infty e^{-2\lambda_n^\alpha t_j} e_n^2(y) (e^{\lambda_n^\alpha r} - e^{\lambda_n^\alpha s})^2 \, dy \, ds \, dz \, dr \, dt \\
=& \, 2 II_1 + 2 II_2.
\end{aligned}
$$

For II_2, since $e_n^2(y) \leq 1$ and $\sum_{i=0}^{J-1} \int_{x_i}^{x_{i+1}} dx = 1$, we have

$$II_2 \leq \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \sum_{i=0}^{M-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}}$$

$$\frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} (e^{\lambda_n^\alpha r} - e^{\lambda_n^\alpha s})^2 dy\, ds\, dz\, dr\, dt$$

$$= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \int_{t_l}^{t_{l+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} (e^{\lambda_n^\alpha r} - e^{\lambda_n^\alpha s})^2 ds\, dr\, dt$$

$$= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \left[\int_{t_l}^{r} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} (e^{\lambda_n^\alpha r} - e^{\lambda_n^\alpha s})^2 ds \right] dr\, dt$$

$$+ \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \left[\int_{r}^{t_{l+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} \left(e^{\lambda_n^\alpha r} - e^{\lambda_n^\alpha s}\right)^2 ds \right] dr\, dt$$

$$= II_{21} + II_{22}.$$

For II_{21}, we have

$$II_{21} = \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \left[\int_{t_l}^{r} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha (t_j - r)} \left(1 - e^{-\lambda_n^\alpha (r-s)}\right)^2 ds \right] dr dt$$

$$\leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \left[\int_{t_l}^{r} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha (t_j - r)} \left(1 - e^{-\lambda_n^\alpha k}\right)^2 ds \right] dr dt$$

$$\leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \left[\int_{t_l}^{t_{l+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha (t_j - r)} (1 - e^{-\lambda_n^\alpha k})^2 ds \right] dr dt$$

$$= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \left[\int_{0}^{t_j} \sum_{n=1}^{\infty} e^{-2\lambda^\alpha (t_j - r)} (1 - e^{-\lambda_n^\alpha k})^2 dr \right] dt.$$

We will show that

$$\sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_{0}^{t_j} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha (t_j - r)} (1 - e^{\lambda_n^\alpha k})^2 \, dr dt \leq C k^{1 - \frac{1}{2\alpha}}. \tag{5.2.21}$$

Assume (5.2.21) holds at the moment, we then have

$$II_{21} \leq C k^{1 - \frac{1}{2\alpha}}.$$

We now show (5.2.21). Note that $1 - e^{-x} \leq Cx$ for $x > 0$ and $1 - e^{-x} \leq 1$ for $x > 0$, we

obtain

$$\sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_j} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)}(1-e^{-\lambda_n^\alpha k})^2 \, dr \, dt$$

$$= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_{j-1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)}(1-e^{-\lambda_n^\alpha k})^2 \, dr \, dt$$

$$+ \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_{t_{j-1}}^{t_j} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)}(1-e^{-\lambda_n^\alpha k})^2 \, dr \, dt$$

$$\leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_{j-1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)}(\lambda_n^\alpha k)^2 \, dr \, dt$$

$$+ \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_{t_{j-1}}^{t_j} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)}.1^2 dr dt$$

$$\leq C \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{e^{-2\lambda_n^\alpha(k)} - e^{-2\lambda_n^\alpha t_j}}{2\lambda_n^\alpha}(\lambda_n^\alpha k)^2 dt + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{1-e^{-2\lambda_n^\alpha k}}{2\lambda_n^\alpha} \, dt$$

$$\leq C \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{e^{-2\lambda_n^\alpha k}}{2\lambda_n^\alpha}(\lambda_n^\alpha k)^2 \, dt + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{1-e^{-2\lambda_n^\alpha k}}{2\lambda_n^\alpha} dt.$$

Applying (5.2.8) and (5.2.9), we get

$$\sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_j} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)}(1-e^{-\lambda_n^\alpha k})^2 \, dr \, dt \leq Ck^{1-\frac{1}{2\alpha}},$$

which is (5.2.21).

For II_{22}, we have

$$II_{22} = \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \Big[\int_r^{t_{l+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j}(e^{\lambda_n^\alpha r} - e^{\lambda_n^\alpha s})^2 ds \Big] dr dt$$

$$= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \Big[\int_r^{t_{l+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-s)}(1-e^{-\lambda_n^\alpha(s-r)})^2 ds \Big] dr dt$$

$$\leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \Big[\int_r^{t_{l+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-s)}(1-e^{-\lambda_n^\alpha k})^2 ds \Big] dr dt$$

$$\leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \Big[\int_{t_l}^{t_{l+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-s)}(1-e^{-\lambda_n^\alpha k})^2 ds \Big] dr dt$$

$$= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-s)}(1-e^{-\lambda_n^\alpha k})^2 \, ds dt$$

$$= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_j} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-s)}(1-e^{-\lambda_n^\alpha k})^2 \, ds dt.$$

and hence, by (5.2.21), we drive

$$II_{22} \le Ck^{1-\frac{1}{2\alpha}}.$$

For II_1, we have

$$II_1 = \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \sum_{i=0}^{M-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}}$$

$$\sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} (e_n(z) - e_n(y))^2 e^{2\lambda_n^\alpha r} dydsdzdrdt$$

$$\le \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_{j-1}} \sum_{i=0}^{M-1} \int_{x_i}^{x_{i+1}} \frac{1}{h} \int_{x_i}^{x_{i+1}}$$

$$\sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} (e_n(z) - e_n(y))^2 e^{2\lambda_n^\alpha r} dydzdrdt$$

$$+ \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_{j-1}^{t_j} \sum_{i=0}^{M-1} \int_{x_i}^{x_{i+1}} \frac{1}{h} \int_{x_i}^{x_{i+1}}$$

$$\sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} (e_n(z) - e_n(y))^2 e^{2\lambda_n^\alpha r} dydzdrdt.$$

Noting that $e_n(z) = \sqrt{2}\sin(n\pi z), |\sin x - \sin y| \le |x-y|$ and $|\sin x - \sin y| \le 2$, we have

$$II_1 \le \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_{j-1}} \sum_{i=0}^{M-1} \int_{x_i}^{x_{i+1}} \frac{1}{h} \int_{x_i}^{x_{i+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} (2n\pi h)^2 e^{2\lambda_n^\alpha r} \, dydzdrdt$$

$$+ \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_{j-1}^{t_j} \sum_{i=0}^{M-1} \int_{x_i}^{x_{i+1}} \frac{1}{h} \int_{x_i}^{x_{i+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} (8e^{2\lambda_n^\alpha r}) \, dydzdrdt$$

$$\le C \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_{j-1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)} (n\pi h)^2 \, dr \, dt + C \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_{j-1}^{t_j} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)} \, drdt$$

$$= C \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{e^{-2\lambda_n^\alpha k} - e^{-2\lambda_n^\alpha t_j}}{\lambda_n^\alpha} (n\pi h)^2 dt + C \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{1 - e^{-2\lambda_n^\alpha k}}{\lambda_n^\alpha} \, dt$$

$$= C \sum_{n=1}^{\infty} \frac{e^{-2\lambda_n^\alpha k}}{\lambda_n^{\alpha-1}} h^2 + C \sum_{n=1}^{\infty} \frac{1 - e^{-2\lambda_n^\alpha k}}{\lambda_n^\alpha}.$$

Applying (5.2.10) and (5.2.9), we finally get

$$II_1 \le C(k^{1-\frac{1}{2\alpha}} + h^2 k^{\frac{2\alpha-3}{2\alpha}}). \tag{5.2.22}$$

For I, we have

$$
\begin{aligned}
I = \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \Big[& \int_0^t \int_0^1 G_\alpha(t-s,x,y) dW(s,y) \\
& - \int_0^{t_j} \int_0^1 G_\alpha(t_j-s,x,y) dW(s,y) \Big]^2 dx\, dt
\end{aligned}
$$

$$
\leq 2\mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \Big[\int_0^{t_j} \int_0^1 G_\alpha(t-s,x,y) - G_\alpha(t_j-s,x,y) dW(s,y) \Big]^2 dx\, dt
$$

$$
+ 2\mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \Big[\int_{t_j}^t \int_0^1 G_\alpha(t-s,x,y) dW(s,y) \Big]^2 dx\, dt
$$

$$
\leq 2\mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \Big[\int_0^{t_j} \int_0^1 (G_\alpha(t-s,x,y) - G_\alpha(t_j-s,x,y)) \Big]^2 dy\, ds\, dx\, dt
$$

$$
+ 2\mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{t_j}+1} \int_0^1 \Big[\int_{t_j}^t \int_0^1 G_\alpha(t-s,x,y) \Big]^2 dy\, ds\, dx\, dt
$$

$$
= 2I_1 + 2I_2.
$$

Now for I_1, we have, by using isometry property and noting that $(e_n, e_m) = \delta_{nm}, n, m = 1, 2, \ldots,$

$$
I_1 = \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \int_0^{t_j} \int_0^1 \Big(G_\alpha(t-s,x,y) - G_\alpha(t_j-s,x,y) \Big)^2 dy\, ds\, dx\, dt
$$

$$
= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \int_0^{t_j} \int_0^1 \Big[\sum_{n=1}^\infty (e^{-\lambda_n^\alpha(t-s)} - e^{-\lambda_n^\alpha(t_j-s)}) e_n(x) e_n(y) \Big]^2 dy\, ds\, dx\, dt
$$

$$
= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_j} \sum_{n=1}^\infty (e^{-\lambda_n^\alpha(t-s)} - e^{-\lambda_n^\alpha(t_j-s)})^2 ds\, dt
$$

$$
= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_j} \sum_{n=1}^\infty e^{-2\lambda_n^\alpha(t-s)} (1 - e^{-\lambda_n^\alpha(t_j-t)})^2 ds\, dt
$$

$$
= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^\infty \frac{e^{-2\lambda_n^\alpha(t-t_j)} - e^{-2\lambda_n^\alpha t}}{2\lambda_n^\alpha} \Big(1 - e^{-\lambda_n^\alpha(t_j-t)} \Big)^2 dt
$$

$$
= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^\infty \frac{e^{-2\lambda_n^\alpha(t-t_j)} - e^{-2\lambda_n^\alpha t}}{2\lambda_n^\alpha} e^{-2\lambda_n^\alpha(t_j-t)} (e^{-\lambda_n^\alpha(t-t_j)} - 1)^2 dt
$$

$$
= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^\infty \frac{1 - e^{-2\lambda_n^\alpha t_j}}{2\lambda_n^\alpha} \Big(e^{-\lambda_n^\alpha(t-t_j)} - 1 \Big)^2 dt.
$$

Applying (5.2.9) and noting that $1 - e^{-2\lambda_n^\alpha t_j} \leq 1$ and $1 - e^{-\lambda_n^\alpha k} \leq 1$, we get

$$I_1 \leq \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{1}{2\lambda_n^\alpha}(1 - e^{-\lambda_n^\alpha k})^2 \, dt$$

$$\leq \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{1}{2\lambda_n^\alpha}(1 - e^{-\lambda_n^\alpha k})^2 \, dt \leq Ck^{1-\frac{1}{2\alpha}}. \tag{5.2.23}$$

Moreover, for I_2 by (5.2.9) and noting that $(e_n, e_m) = \delta_{nm}, n, m = 1, 2, \cdots$, we have

$$I_2 = \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\int_{t_j}^t \int_0^1 G_\alpha(t - s, x, y) dW(s, y) \right]^2 dx \, dt$$

$$= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \int_{t_j}^t \int_0^1 G_\alpha(t - s, x, y)^2 \, dy \, ds \, dx \, dt$$

$$= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \int_{t_j}^t \int_0^1 \left[\sum_{n=1}^{\infty} e^{-\lambda_n^\alpha(t-s)} e_n(x) e_n(y) \right]^2) \, dy \, ds \, dx \, dt$$

$$= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_{t_j}^t \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t-s)} \, ds \, dt = \mathbb{E} \sum_{j=1}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{(1 - e^{-2\lambda_n^\alpha}(t - t_j)}{2\lambda_n^\alpha} \, dt$$

$$= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{(1 - e^{-2\lambda_n^\alpha(k)})}{2\lambda_n^\alpha} \, dt \leq \sum_{n=1}^{\infty} \frac{(1 - e^{-2\lambda_n^\alpha k})}{2\lambda_n^\alpha} \, dt \leq Ck^{1-\frac{1}{2\alpha}}.$$

Finally, we consider III. We have

$$III = \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\int_0^{t_j} \int_0^1 G_\alpha(t_j - s, x, y) d\widehat{W}(s, y) - \int_0^t \int_0^1 G_\alpha(t - s, x, y) d\widehat{W}(s, y) \right]^2 dx \, dt$$

$$= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\int_0^{t_j} \int_0^1 G_\alpha(t_j - s, x, y) d\widehat{W}(s, y) \right.$$

$$\left. - \int_0^{t_j} \int_0^1 G_\alpha(t - s, x, y) d\widehat{W}(s, y) + \int_{t_j}^t \int_0^1 G_\alpha(t - s, x, y) d\widehat{W}(s, y) \right]^2 dx \, dt$$

$$\leq 2\mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\int_0^{t_j} \int_0^1 G_\alpha(t_j - s, x, y) - G_\alpha(t - s, x, y) d\widehat{W}(s, y) \right]^2 dx \, dt$$

$$+ 2\mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\int_{t_j}^t \int_0^1 G_\alpha(t - s, x, y) d\widehat{W} s, y) \right]^2 dx \, dt$$

$$= 2III_1 + 2III_2.$$

For III_1, using the isometry property and the estimates for I_1, we get

$$III_1 = \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \Big[\int_0^{t_j} \int_0^1 G_\alpha(t_j - s, x, y) d\widehat{W}(s, y)$$

$$- \int_0^{t_j} \int_0^1 G_\alpha(t - s, x, y) d\widehat{W}(s, y) \Big]^2 dx dt$$

$$= \sum_{j=1}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \int_0^{t_j} \int_0^1 \Big[G_\alpha(t_j - s, x, y) - G_\alpha(t - s, x, y) \Big]^2 dy ds dx dt$$

$$\leq C k^{1 - \frac{1}{2\alpha}}.$$

Further, for III_2, again by the isometry property and the estimates for I_2, we have

$$III_2 = \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \Big[\int_{t_j}^t \int_0^1 G_\alpha(t - s, x, y) d\widehat{W}(s, y) \Big]^2 dx\, dt$$

$$= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \int_{t_j}^t \int_0^1 (G_\alpha(t - s, x, y))^2 \, ds\, dy\, dx\, dt$$

$$\leq C k^{1 - \frac{1}{2\alpha}}.$$

Together these estimate complete the proof of Theorem 5.2.2. $\qquad\square$

Theorem 5.2.6. Let $\frac{1}{2} < \alpha \leq 1$. Let $\hat{u}$ be the solution of (5.2.4)-(5.2.6). Then we have

$$\mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \hat{u}_t^2(t, x) dx dt \leq C(k^{-\frac{1}{2\alpha}} + h^{-1}), \tag{5.2.24}$$

and

$$\mathbb{E} \int_0^1 \Big| (-\Delta)^\alpha \hat{u}(t, x) \Big|^2 dx \leq C(k^{1 - \frac{1}{2\alpha}} + h^{-1} k^{-1}). \tag{5.2.25}$$

Proof. We only prove (5.2.24), the proof of (5.2.25) is similar. Note that

$$\hat{u}(t, x) = \int_0^1 G_\alpha(t, x, y) u_0(y) dy + \int_0^t \int_0^1 G_\alpha(t - s, x, y) d\widehat{W}(s, y) \tag{5.2.26}$$

$$= \int_0^1 G_\alpha(t, x, y) u_0(y) dy + \int_0^t \Big[\int_0^1 G_\alpha(t - s, x, y) \frac{\partial^2 \widehat{W}(s, y)}{\partial s \partial y} dy \Big] ds,$$

and

$$\hat{u}_t(t, x) = \int_0^1 \frac{\partial}{\partial t} G_\alpha(t, x, y) u_0(y) dy + \int_0^t \Big[\int_0^1 \frac{\partial}{\partial t} G_\alpha(t - s, x, y) \frac{\partial^2 \widehat{W}(s, y)}{\partial s \partial y} dy \Big] ds$$

$$+ \int_0^1 G_\alpha(0, x, y) \frac{\partial^2 \widehat{W}(t, y)}{\partial s \partial y} dy. \tag{5.2.27}$$

Since $w(t, x) = \int_0^1 G_\alpha(t, x, y) w_0(y) dy$ is the solution of the following equation

$$\frac{\partial w(t, x)}{\partial t} + (-\Delta)^\alpha w(t, x) = 0, \qquad 0 < x < 1, 0 < t < T,$$

$$w(t, 0) = w(t, 1) = 0, \qquad 0 < t < T,$$

$$w(0, x) = w_0(x),$$

we therefore have

$$w_0(x) = w(0, x) = \int_0^1 G_\alpha(0, x, y) w_0(y) dy. \tag{5.2.28}$$

Choose $w_0(y) = \frac{\partial^2 \widehat{W}(t, y)}{\partial s \partial y}$ for fixed t, we have

$$\int_0^1 G_\alpha(0, x, y) \frac{\partial^2 \widehat{W}(t, y)}{\partial t \partial y} dy = \frac{\partial^2 \widehat{W}(t, x)}{\partial t \partial x}.$$

Hence, by (5.2.27),

$$\hat{u}(t, x) = \int_0^1 \frac{\partial}{\partial t} G_\alpha(t, x, y) u_0(y) dy + \int_0^t \int_0^1 \frac{\partial}{\partial t} G_\alpha(t - s, x, y) d\widehat{W}(s, y) + \frac{\partial^2 \widehat{W}(t, x)}{\partial t \partial x}.$$

Using the inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$, $\forall a, b, c \in \mathbb{R}$, we have

$$\mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \hat{u}_t^2(t, x) dx dt$$

$$\leq 3\mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\int_0^t \int_0^1 \frac{\partial}{\partial t} G_\alpha(t - s, x, y) d\widehat{W}(s, y) \right]^2 dx dt$$

$$+ 3\mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\frac{\partial^2 \widehat{W}(t, x)}{\partial t \partial x} \right]^2 dx dt + 3\mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\int_0^1 \frac{\partial}{\partial t} G_\alpha(t, x, y) u_0(y) dy \right]^2 dx dt$$

$$= 3(I + II + III).$$

Using the inequality $(a + b)^2 \leq 2(a^2 + b^2)$, $\forall a, b, \in \mathbb{R}$, we have

$$I \leq 2\mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\int_0^{t_{j-1}} \int_0^1 \frac{\partial}{\partial t} G_\alpha(t - s, x, y) d\widehat{W}(s, y) \right]^2 dx dt$$

$$+ 2\mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\int_{t_{j-1}}^t \int_0^1 \frac{\partial}{\partial t} G_\alpha(t - s, x, y) d\widehat{W}(s, y) \right]^2 dx dt$$

$$= 2(I_1 + I_2).$$

For I_1, with $\eta_{kl} = \mathcal{N}(0,1)$, $k = 0,1,2,\ldots,j-1$, $l = 0,1,2,\ldots,j-2$, $j \geq 2$, we have

$$I_1 = \mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \Big[\int_0^{t_{j-1}} \int_0^1 \frac{\partial}{\partial t} G_\alpha(t-s,x,y) d\widehat{W}(s,y) \Big]^2 dxdt$$

$$= \mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \Big[\sum_{l=0}^{j-2} \int_{t_l}^{t_{l+1}} \sum_{k=0}^{M-1} \int_{x_k}^{x_{k+1}} \frac{\partial}{\partial t} G_\alpha(t-s,x,y) \eta_{kl} dy ds \Big]^2 dxdt$$

$$= \mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \Big[\sum_{l=0}^{j-2} \sum_{k=0}^{M-1} \Big(\int_{t_l}^{t_{l+1}} \int_{x_k}^{x_{k+1}} \frac{\partial}{\partial t} G_\alpha(t-s,x,y) dy ds \Big) \eta_{kl} \Big]^2 dxdt$$

$$= \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \sum_{l=0}^{j-2} \sum_{k=0}^{M-1} \Big(\int_{t_l}^{t_{l+1}} \int_{x_k}^{x_{k+1}} \frac{\partial}{\partial t} G_\alpha(t-s,x,y) dy ds \Big)^2 dxdt$$

$$= \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \Big[\sum_{l=0}^{j-2} \sum_{k=0}^{M-1} \Big(\int_{t_l}^{t_{l+1}} \int_{x_k}^{x_{k+1}} \sum_{n=1}^\infty \lambda_n^\alpha e^{-\lambda_n^\alpha(t-s)} e_n(x) e_n(y) dy ds \Big)^2 dxdt$$

$$= C \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \Big[\sum_{l=0}^{j-2} \sum_{k=0}^{M-1} \Big(\sum_{n=1}^\infty \lambda_n^\alpha \frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi}$$

$$e_n(x) \frac{e^{-\lambda_n^\alpha(t-t_{l+1})} - e^{-\lambda_n^\alpha(t-t_l)}}{\lambda_n^\alpha} \Big)^2 dxdt.$$

Note that $(e_n, e_m) = \delta_{nm}, n, m = 1.2.\ldots$, we have

$$I_1 = C \int_{t_j}^{t_{j+1}} \frac{1}{kh} \Big[\sum_{l=0}^{J-2} \sum_{k=0}^{M-1} \sum_{n=1}^\infty \Big(\frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} \Big)^2 \Big(\frac{e^{-\lambda_n^\alpha(t-t_{l+1})} - e^{-\lambda_n^\alpha(t-t_l)}}{1} \Big)^2 dt$$

$$= C \int_{t_j}^{t_{j+1}} \frac{1}{kh} \Big[\sum_{l=0}^{J-2} \sum_{k=0}^{M-1} \sum_{n=1}^\infty \Big(\frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} \Big)^2 e^{-2\lambda_n^\alpha(t-t_{l+1})} \Big(1 - e^{-\lambda_n^\alpha(t_{l+1}-t_l)} \Big)^2 dt$$

$$= C \frac{1}{kh} \sum_{l=0}^{J-2} \sum_{k=0}^{M-1} \sum_{n=1}^\infty \Big(\frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} \Big)^2$$

$$\frac{e^{-\lambda_n^\alpha(t_j-t_{l+1})} - e^{-\lambda_n^\alpha(t_{j+1}-t_{l+1})}}{\lambda_n^\alpha} \Big(1 - e^{-\lambda_n^\alpha(t_{l+1}-t_l)} \Big)^2$$

$$= C \frac{1}{kh} \Big[\sum_{k=0}^{M-1} \sum_{n=1}^\infty \Big(\frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} \Big)^2 \frac{(1 - e^{-\lambda_n^\alpha k})^2}{\lambda_n^\alpha} \sum_{l=0}^{j-2} e^{-2\lambda_n^\alpha(t_j-t_{l+1})}$$

$$= C \frac{1}{kh} \sum_{n=1}^\infty \frac{(1 - e^{-\lambda_n^\alpha k})^2}{\lambda_n^{\alpha+1}} \Big(\sum_{k=0}^{J-1} \cos n\pi x_{k+1} - \cos n\pi x_k \Big)^2 \sum_{l=0}^{J-2} e^{-2\lambda_n^\alpha(t_j-t_{l+1})}$$

Since $|\cos(n\pi x_{k+1}) - \cos(n\pi x_k)| \leq (n\pi h)^2$, we have

$$\sum_{k=0}^{J-1} (\cos n\pi x_{k+1} - \cos n\pi x_k)^2 \leq C \sum_{k=0}^{J-1} (n\pi h)^2 = C\lambda_n h.$$

Hence we get, by (5.2.13) and (5.2.12),

$$
\begin{aligned}
I_1 &= C\frac{1}{kh}\sum_{n=1}^{\infty}\frac{(1-e^{-\lambda_n^{\alpha}k})^2}{\lambda_n^{\alpha+1}}(\lambda_n h)\sum_{l=0}^{j-2}e^{-2\lambda_n^{\alpha}(t_j-t_{l+1})} \\
&= C\frac{1}{kh}\sum_{n=1}^{\infty}\frac{(1-e^{-\lambda_n^{\alpha}k})^2}{\lambda_n^{\alpha+1}}(\lambda_n h)(k^{-1}\lambda_n^{-\alpha}) \\
&= Ck^{-2}\sum_{n=1}^{\infty}\frac{(1-e^{-\lambda_n^{\alpha}k})^2}{\lambda_n^{2\alpha}} \le Ck^{-2}k^{\frac{4\alpha-1}{2\alpha}} \le Ck^{-\frac{1}{2\alpha}}.
\end{aligned}
$$

We remark that I_1 can also be estimated by using the following alternative way.

$$
\begin{aligned}
I_1 &= \mathbb{E}\int_{t_j}^{t_{j+1}}\int_0^1\Big[\int_0^{t_{j-1}}\int_0^1\frac{\partial}{\partial t}G_{\alpha}(t-s,x,y)d\widehat{W}(s,y)\Big]^2 dxdt \\
&= \int_{t_j}^{t_{j+1}}\int_0^1\Big[\int_0^{t_{j-1}}\int_0^1\Big(\frac{\partial}{\partial t}G_{\alpha}(t-s,x,y)\Big)^2 dyds\Big]^2 dxdt \\
&= \int_{t_j}^{t_{j+1}}\int_0^1\int_0^{t_{j-1}}\int_0^1\Big(\sum_{n=1}^{\infty}\lambda_n^{\alpha}e^{-\lambda_n^{\alpha}(t-s)}e_n(x)e_n(y)\Big)^2 dydsdxdt \\
&= \int_{t_j}^{t_{j+1}}\int_0^{t_{j-1}}\sum_{n=1}^{\infty}\lambda_n^{2\alpha}e^{-2\lambda_n^{\alpha}(t-s)}dsdt \\
&= \int_{t_j}^{t_{j+1}}\sum_{n=1}^{\infty}\lambda_n^{2\alpha}\frac{e^{-\lambda_n^{\alpha}(t-t_{j-1})}-e^{-\lambda_n^{2\alpha}t}}{2\lambda_n^{\alpha}}dt.
\end{aligned}
$$

Note that $t \ge t_j$, we then have, by using (5.2.8),

$$
I_1 \le C\int_{t_j}^{t_{j+1}}\sum_{n=1}^{\infty}\lambda_n^{\alpha}e^{-2\lambda_n^{\alpha}k}dt = Ck\sum_{n=1}^{\infty}\frac{\lambda_n^{\alpha}}{e^{2\lambda_n^{\alpha}k}} \le Ck^{-\frac{1}{2\alpha}}.
$$

For I_2, we have

$$
\begin{aligned}
I_2 &\le 2\mathbb{E}\int_{t_j}^{t_{j+1}}\int_0^1\Big[\int_{t_{j-1}}^{t_j}\int_0^1\frac{\partial}{\partial t}G_{\alpha}(t-s,x,y)d\widehat{W}(s,y)\Big]^2 dxdt \\
&\quad + 2\mathbb{E}\int_{t_j}^{t_{j+1}}\int_0^1\Big[\int_{t_j}^t\int_0^1\frac{\partial}{\partial t}G_{\alpha}(t-s,x,y)d\widehat{W}(s,y)\Big]^2 dxdt \\
&= 2I_{21} + 2I_{22}.
\end{aligned}
$$

Here I_{21} can be estimated as follows:

$$I_{21} \le \mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \sum_{k=0}^{M-1} \Big[\int_{t_{j-1}}^{t_j} \int_{x_k}^{x_{k+1}} \frac{\partial}{\partial t} G_\alpha(t-s,x,y) \eta_{kj} dy ds \Big]^2 dx dt$$

$$= \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \sum_{k=0}^{M-1} \Big[\int_{t_{j-1}}^{t_j} \int_{x_k}^{x_{k+1}} \frac{\partial}{\partial t} G_\alpha(t-s,x,y) dy ds \Big]^2 dx dt$$

$$= \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \sum_{k=0}^{M-1} \Big[\int_{t_{j-1}}^{t_j} \int_{x_k}^{x_{k+1}} \sum_{n=1}^\infty \lambda_n^\alpha e^{-\lambda_n^\alpha(t-s)} e_n(x) e_n(y) dy ds \Big]^2 dx dt$$

$$= \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \sum_{k=0}^{M-1} \Big[\sum_{n=1}^\infty \lambda_n^\alpha \frac{e^{-\lambda_n^\alpha(t-t_j)} - e^{-\lambda_n^\alpha(t-t_{j-1})}}{\lambda_n^\alpha} e_n(x) \frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} \Big]^2 dx dt$$

$$= \int_{t_j}^{t_{j+1}} \frac{1}{kh} \sum_{k=0}^{M-1} \Big[\sum_{n=1}^\infty \Big(\frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} \Big)^2 e^{-2\lambda_n^{\alpha(t-t_j)}} (1 - e^{-\lambda_n^\alpha k})^2 \Big] dt$$

$$= \frac{1}{kh} \sum_{k=0}^{M-1} \Big[\sum_{n=1}^\infty \Big(\frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} \Big)^2 \frac{e^{-2\lambda_n^\alpha(t-t_j)} - e^{-2\lambda_n^\alpha(t_{j+1}-t_j)}}{\lambda_n^\alpha} (1 - e^{-\lambda_n^\alpha k})^2 \Big]$$

$$= \frac{1}{kh} \sum_{n=1}^\infty \frac{(1-e^{-\lambda_n^\alpha k})^2}{\lambda_n^{\alpha+1}} \Big(\sum_{k=0}^{M-1} \cos n\pi x_{k+1} - \cos n\pi x_k)^2 \Big) \cdot 1$$

$$\le \frac{1}{kh} \sum_{n=1}^\infty \frac{(1-e^{-\lambda_n^\alpha k})^2}{\lambda_n^{\alpha+1}} \lambda_n h = \frac{1}{k} \sum_{n=1}^\infty \frac{(1-e^{-\lambda_n^\alpha k})^2}{\lambda_n^\alpha} \le \frac{1}{k} \sum_{n=1}^\infty \frac{(1-e^{-\lambda_n^\alpha k})}{\lambda_n^\alpha},$$

which implies, by (5.2.9),

$$I_{21} \le \frac{1}{k} k^{1-\frac{1}{2\alpha}} = k^{-\frac{1}{2\alpha}}.$$

For I_{22}, we have

$$I_{22} = \mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \Big[\int_{t_j}^t \int_0^1 \frac{\partial}{\partial t} G_\alpha(t-s,x,y) d\widehat{W}(s,y) \Big]^2 dx dt$$

$$= \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \sum_{k=0}^{M-1} \Big[\int_{t_j}^t \int_{x_k}^{x_{k+1}} \sum_{n=1}^\infty \lambda_n^\alpha e^{-\lambda_n^\alpha(t-s)} e_n(x) e_n(y) dy ds \Big]^2 dx dt$$

$$= \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \sum_{k=0}^{M-1} \Big[\sum_{n=1}^\infty \lambda_n^\alpha \frac{e^{-\lambda_n^\alpha(t-t_j)} - e^{-\lambda_n^\alpha(t-t_j)}}{\lambda_n^\alpha} e_n(x) \frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} dy ds \Big]^2 dx dt$$

$$= \int_{t_j}^{t_{j+1}} \frac{1}{kh} \sum_{k=0}^{M-1} \sum_{n=1}^\infty \Big(\frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} \Big)^2 (1 - e^{-\lambda_n^\alpha k})^2 \Big] dt$$

$$= k \frac{1}{kh} \sum_{k=0}^{M-1} \sum_{n=1}^\infty \Big(\cos n\pi x_{k+1} - \cos n\pi x_k \Big)^2 \frac{(1-e^{-\lambda_n^\alpha k})^2}{\lambda_n}.$$

Moreover, applying (5.2.11) and taking into account $|\cos(n\pi x_{k+1}) - \cos(n\pi x_k)| \le n\pi h,$

we derive

$$I_{22} \leq \frac{1}{h}\Big(\sum_{k=0}^{M-1} n^2 \pi^2 h^2\Big) \sum_{n=1}^{\infty} \frac{(1-e^{-\lambda_n^\alpha k})^2}{\lambda_n}$$

$$\leq \sum_{n=1}^{\infty}(1-e^{-\lambda_n^\alpha k})^2 \leq Ck^{-\frac{1}{2\alpha}}.$$

For II we have, with $\eta_{kj} = \mathcal{N}(0,1)$,

$$II = \mathbb{E}\int_{t_j}^{t_{j+1}} \int_0^1 \Big(\frac{\partial \widehat{W}(t,x)}{\partial t \partial x}\Big)^2 dxdt = \mathbb{E}\int_{t_j}^{t_{j+1}} \sum_{k=0}^{M-1} \int_{x_k}^{x_{k+1}} \frac{1}{kh}\eta_{kj}^2 dxdt$$

$$= \frac{1}{kh}\int_{t_j}^{t_{j+1}} \sum_{k=0}^{M-1} \int_{x_k}^{x_{k+1}} dxdt = \frac{1}{kh}k = h^{-1}.$$

Similarly, we can estimate III.

Together these estimates complete the proof of Theorem 5.2.6. $\qquad\square$

5.3   Fourier Spectral Method

We will consider a Fourier spectral method for solving the deterministic space fractional partial differential equation:

$$\frac{\partial \hat{u}(t,x)}{\partial t} + (-\Delta)^\alpha \hat{u}(t,x) = \hat{f}(t,x), \quad 0 < t < T, \ 0 < x < 1, \tag{5.3.1}$$

$$\hat{u}(t,0) = \hat{u}(t,1) = 0, \quad 0 < t < T, \tag{5.3.2}$$

$$\hat{u}(0,x) = u_0(x), \quad 0 < x < 1. \tag{5.3.3}$$

Here $\hat{f}(t,x) = \frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ is defined by (5.1.8), and $\hat{f} \in L^2((0,T) \times (0,1))$.

Denote $A = -\Delta$ with $\mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$. For any $s > 0$ and $v \in H_0^{2s}(0,1)$, we have $A^s v = \sum_{j=1}^{\infty} \lambda_j^s(v, e_j)e_j$. It is obvious that

$$|v|_r = \|A^{\frac{r}{2}}v\| = \Big(\sum_{j=1}^{\infty} \lambda_j^r(v, e_j)^2\Big)^{\frac{1}{2}}, \quad \forall v \in H_0^r(0,1), \ r > 0.$$

Further, we denote $E_\alpha(t) = e^{-tA^\alpha}$, $\frac{1}{2} < \alpha \leq 1$. Then the solution of (5.3.1)-(5.3.3) can be written as the following operator form:

$$\hat{u}(t) = E_\alpha(t)\hat{u}_0 + \int_0^t E_\alpha(t-s)\hat{f}(s)ds, \quad \hat{u}(0) = u_0. \tag{5.3.4}$$

The spectral method of (5.3.1)-(5.3.3) consists of finding $\hat{u}_J(t) \in S_J$ such that

$$\frac{\partial \hat{u}_J(t, x)}{\partial t} + P_J(-\Delta)^\alpha \hat{u}_J(t, x) = P_J \hat{f}(t, x), \quad 0 < t < T, \ 0 < x < 1, \tag{5.3.5}$$

$$\hat{u}_J(t, 0) = \hat{u}_J(t, 1) = 0, \quad 0 < t < T, \tag{5.3.6}$$

$$\hat{u}_J(0, x) = P_J u_0(x), \quad 0 < x < 1, \tag{5.3.7}$$

where $P_J : H \mapsto S_J$ is defined by (5.1.15).

Similarly, the solution of (5.3.5)-(5.3.7) has the form of, with $E_{\alpha,J}(t) = e^{-tP_J A^\alpha}$,

$$\hat{u}_J(t) = E_{\alpha,J}(t) P_J u_0 + \int_0^t E_{\alpha,J}(t - s) P_J \hat{f}(s) ds, \qquad \hat{u}_J(0) = P_J u_0.$$

Note that $E_\alpha(t) v_J = E_{\alpha,J}(t) v_J$ for any $v_J \in S_J$. We may have

$$\hat{u}_J(t) = E_\alpha(t) P_J u_0 + \int_0^t E_\alpha(t - s) P_J \hat{f}(s) ds, \qquad \hat{u}_J(0) = P_J u_0. \tag{5.3.8}$$

Theorem 5.3.1. Assume that $\hat{u}$ and $\hat{u}_J$ are the solutions of (5.3.4) and (5.3.8), respectively. Let $0 \le r < \frac{1}{2}$, and assume that $u_0 \in H_0^r(0, 1)$. Then, there exists a positive constant C such that

$$\|\hat{u}(t) - \hat{u}_J(t)\|_r \le C\|u_0 - P_J u_0\|_r + C\frac{1}{(J + 1)^{\alpha(1 - \frac{r}{\alpha})}} \left(\int_0^t \|\hat{f}(s)\|^2 ds \right)^{\frac{1}{2}}. \tag{5.3.9}$$

In particular, we have, with $r = 0$,

$$\|\hat{u}(t) - \hat{u}_J(t)\| \le C\|u_0 - P_J u_0\| + C\frac{1}{(J + 1)^\alpha} \left(\int_0^t \|\hat{f}(s)\|^2 ds \right)^{\frac{1}{2}}. \tag{5.3.10}$$

To prove Theorem 5.4.1, we need the following smoothing property for the solution operator $E_\alpha(t)$.

Lemma 5.3.2. (1). Let $s > 0$. We have

$$\|A^s E_\alpha(t)\| \le C t^{-\frac{s}{\alpha}} e^{-\delta t}, \quad t > 0, \quad \text{with } \frac{1}{2} < \alpha \le 1, \tag{5.3.11}$$

for some constant C and δ which depend on s and α.

(2). Let $P_J : H \mapsto S_J$ be defined by (5.1.15) then we have

$$\|E_\alpha(t)(I - P_J)\| \le e^{-t\lambda_{J+1}^\alpha} \|v\|, \quad t > 0, \quad \text{with } \frac{1}{2} < \alpha \le 1. \tag{5.3.12}$$

Proof. Note that A is a positive definite operator with eigenvalues $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$. For any function $g(\cdot)$, we have

$$\|g(A)\| = \sup_{\lambda > 0} |g(\lambda)|.$$

Hence, with $\delta = \frac{1}{2}\lambda_1^\alpha$,

$$\|A^s E_\alpha(t)\| = \|A^s E_\alpha(\frac{t}{2}) E_\alpha(\frac{t}{2})\| \le \|A^s E_\alpha(\frac{t}{2})\| \|E_\alpha(\frac{t}{2})\|$$

$$= \sup_{\lambda > \lambda_1}(\lambda^s e^{-\frac{t}{2}\lambda^\alpha}) . \sup_{\lambda > \lambda_1}(e^{-\frac{t}{2}\lambda^\alpha}) = \sup_{\lambda > \lambda_1}\left(\frac{(\frac{t}{2}\lambda^\alpha)^{\frac{s}{\alpha}}}{e^{\frac{t}{2}\lambda^\alpha}}(\frac{t}{2})^{-\frac{s}{\alpha}}\right)e^{-\frac{t}{2}\lambda_1^\alpha}$$

$$\le C(\frac{t}{2})^{-\frac{s}{\alpha}}e^{-\delta t} \le Ct^{-\frac{s}{\alpha}}e^{-\delta t},$$

which is (5.3.11). To show (5.3.12), we note that

$$\|E_\alpha(t)(I - P_J)v\| = \left(\sum_{j=J+1}^{\infty} e^{-2t\lambda_j^\alpha}(v, e_j)^2\right)^{\frac{1}{2}} \le e^{-t\lambda_{N+1}^\alpha}\|v\|.$$

The proof of Lemma 5.3.2 is complete.

Proof of Theorem 5.3.1. Subtracting (5.3.8) from (5.3.4), we get

$$|\hat{u}(t) - \hat{u}_J(t)| = E_\alpha(t)(u_0 - P_J u_0) + \int_0^t E_\alpha(t - s)(f(s) - P_J f(s))ds = I + II.$$

$$(5.3.13)$$

For I, we have, with $0 \le r < \frac{1}{2}$,

$$|I|_r^2 = |E_\alpha(t)(u_0 - P_J u_0)|_r^2 = \|A^{\frac{r}{2}}E_\alpha(t)(u_0 - P_J u_0)\|^2$$

$$= \left(\sum_{j=J+1}^{\infty} e^{-2t\lambda_j^\alpha}\lambda_j^r(u_0, e_j)^2\right) \le e^{-t\lambda_{j+1}^\alpha}|u_0 - P_J u_0|_r.$$

For II, by virtue of Lemma 5.3.2, for some $\gamma \in (0, 1)$, we get

$$|II|_r = |\int_0^t E_\alpha(t - s)(\hat{f}(s) - P_J\hat{f}(s))ds|_r = \|\int_0^t A^{\frac{r}{2}}E_\alpha(t - s)(I - P_J)\hat{f}(s)ds\|$$

$$= \|\int_0^t A^{\frac{r}{2}}E_\alpha(1 - \gamma)(t - s)\mathbb{E}(\gamma(t - s))(I - P_J)\hat{f}(s)ds\|$$

$$\le C\int_0^t (t - s)^{-\frac{r}{2\alpha}}e^{-k_\alpha(t - s)}\|\hat{f}(s)ds\|,$$

where $k_\alpha = \delta(1 - \gamma) + \lambda_{\lambda+1}^\alpha\gamma$.

By the Cauchy-Schwarz inequality, we have

$$|II|_r \leq \left(\int_0^\infty ((t-s)^{-\frac{r}{2\alpha}} e^{-k_\alpha(t-s))^2} ds \right)^{\frac{1}{2}} \left(\int_0^t \|(\hat{f}(s)\|^2 ds \right)^{\frac{1}{2}}.$$

Note that $r < \alpha$ and $\lambda_{J+1} = (J+1)^2 \pi$ imply

$$\int_0^\infty \frac{e^{-2k_\alpha s}}{s^{\frac{r}{\alpha}}} ds \leq \frac{\int_0^\infty s^{-\frac{r}{\alpha}} e^{-2s} ds}{k_\alpha^{1-\frac{r}{\alpha}}} \leq C \frac{1}{k_\alpha^{1-\frac{r}{\alpha}}} \leq C \frac{1}{(\lambda_{J+1}^\alpha)^{1-\frac{r}{\alpha}}} \leq C \frac{1}{(J+1)^{2\alpha(1-\frac{r}{\alpha})}}.$$

Thus we get

$$|II|_r \leq C \frac{1}{(J+1)^{2\alpha(1-\frac{r}{\alpha})}} \left(\int_0^t \|\hat{f}(s)\|^2 ds \right)^{\frac{1}{2}}.$$

Together these estimates complete the proof of Theorem 5.3.1. $\qquad\square$

Combining Theorem 5.2.2 with Theorem 5.1.11, we obtain the following Theorem. $\quad\square$

Theorem 5.3.3. Let u and $\hat{u}_J$ be the solutions of (5.2.1)-(5.2.3) and (5.3.5)-(5.3.7), respectively. Assume that $u_0 \in H$. Then we have

$$\mathbb{E} \int_0^T \int_0^1 (u(t,x) - \hat{u}_J(t,x))^2 dx dt \leq C(k^{1-\frac{1}{2\alpha}} + h^2 k^{\frac{2\alpha-3}{2\alpha}}) + C\|u_0 - P_J u_0\|^2$$
$$+ C \frac{1}{(J+1)^{2\alpha}} (k^{-1-\frac{1}{2\alpha}} + h^{-1}k^{-1}), \quad \text{for } \frac{1}{2} < \alpha \leq 1.$$

Proof. Note that

$$\mathbb{E} \int_0^T \int_0^1 (u(t,x) - \hat{u}_J(t,x))^2 dx dt$$
$$\leq 2\mathbb{E} \int_0^T \int_0^1 (u(t,x) - \hat{u}(t,x))^2 dx dt + 2\mathbb{E} \int_0^T \int_0^1 (\hat{u}(t,x) - \hat{u}_J(t,x))^2 dx dt$$
$$= 2I + 2II.$$

For I, by Theorem 5.2.2, we have

$$I \leq C(k^{1-\frac{1}{2\alpha}} + h^2 k^{\frac{2\alpha-3}{2\alpha}}).$$

For II, we have

$$II = \mathbb{E} \int_0^T \|\hat{u}(t) - \hat{u}_J(t))\|^2 dt \leq C\|u_0 - P_J u_0\|^2 + C \frac{1}{(J+1)^{2\alpha}} \mathbb{E} \int_0^T \int_0^t \|\hat{f}(s)\|^2 ds dt.$$

Note that $\hat{f}(s) = \hat{u}_s(s) - (-\Delta)^\alpha \hat{u}(s)$, and hence, by virtue of Theorem 5.2.6, we have

$$\mathbb{E} \int_0^T \int_0^t \|\hat{f}(s)\|^2 \, dsdt \leq \mathbb{E} \int_0^T \int_0^1 \|\hat{u}_s(s) - (-\Delta)^\alpha \hat{u}(s)\|^2 \, dsdt$$

$$\leq C\mathbb{E} \int_0^T \int_0^T \int_0^1 (\hat{u}_s^2(s,x) + |(-\Delta)^\alpha \hat{u}(s,x)|^2 \, dxdsdt$$

$$\leq C \sum_{j=0}^N (k^{-\frac{1}{2\alpha}} + h^{-1}) \leq C(k^{-1-\frac{1}{2\alpha}} + h^{-1}k^{-1}).$$

Together these estimates complete the proof of Theorem 5.3.3. $\qquad\square$

Remark 2. In Theorem 5.3.3, the convergence error bounds depend on the step sizes k, h and the dimension J of the spectral approximate space S_J. In general J is independent of h. So we may choose sufficiently small J and get some convergence rates in Theorem 5.3.3.

5.4 Numerical Simulations

In this section, we will present the computational issues for solving the following stochastic space fractional parabolic partial differential equation by using the spectral method developed in the previous section, with $\frac{1}{2} < \alpha \leq 1$,

$$\frac{\partial u(t,x)}{\partial t} + \epsilon(-\Delta)^\alpha u(t,x) = f(u(t,x)) + \frac{\partial^2 W(t,x)}{\partial t \partial x}, \quad 0 < x < 1, \ 0 < t \leq T, \quad (5.4.1)$$

$$u(t,0) = u(t,1) = 0, \quad 0 < t \leq T, \quad (5.4.2)$$

$$u(0,x) = u_0(x), \quad 0 < x < 1, \quad (5.4.3)$$

where $(-\Delta)^\alpha$ is the fractional Laplacian defined by using the eignvalues and eigenfunctions of the Laplacian operator $-\Delta$ subject to the periodic boundary conditions. Here $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function and $\epsilon > 0$ denotes the diffusion efficient. In our numerical example, we will use the discrete sine transform MATLAB functions **dst** and **idst**. We also include the nonlinear term f, although the error estimates in the previous sections are only proven for $f = 0$. In our future work, we will consider the error estimates for solving the nonlinear stochastic space fractional partial differential equations with multiplicative noise by using the spectral method.

Let $x_0 < x_1 < \cdots < x_J = 1$ be a space partition of $[0,1]$ and $\Delta x = h$ be the space step size. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a time partition of $[0,T]$ and $\Delta t = k$ be the time step size. The space-time noise $\frac{\partial^2 W(t,x)}{\partial t \partial x}$ is approximated by using the following piecewise constant function $\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ where

$$\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x} = \frac{\eta_{n,j}}{\sqrt{\Delta t \Delta x}}, \quad t_{n-1} \le t \le t_n, \ x_{j-1} \le x \le x_j. \tag{5.4.4}$$

For convenience, we will denote $\widehat{G}(t,x) = \frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ below.

Equations (5.4.1)-(5.4.3) can be approximated by the following, with $\frac{1}{2} < \alpha \le 1$,

$$\frac{\partial \hat{u}(t,x)}{\partial t} + \epsilon(-\Delta)^\alpha \hat{u}(t,x) = f(\hat{u}(t,x)) + \widehat{G}(t,x), \quad 0 < x < 1, 0 < t \le T, \tag{5.4.5}$$

$$\hat{u}(t,0) = \hat{u}(t,1) = 0, \quad 0 < t \le T, \tag{5.4.6}$$

$$\hat{u}(0,x) = u_0(x), \quad 0 < x < 1. \tag{5.4.7}$$

Denote $A = -\frac{\partial^2}{\partial x^2}$, with $\mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$. Then operator $A : \mathcal{D}(A) \to H$ has the eigenvalues λ_j and eigenfunctions e_j where,

$$\lambda_j = j^2 \pi^2, \quad e_j = \sqrt{2} \sin(j\pi x), \ j \in \mathbb{Z}^+. \tag{5.4.8}$$

That is

$$Ae_j = \lambda_j e_j, \quad j \in \mathbb{Z}^+. \tag{5.4.9}$$

Then the equations (5.4.5)-(5.4.7) can be written into the abstract form: find $\hat{u}(t) \in H_0^1(0,1) \cap H^2(0,1)$, such that

$$\frac{d\hat{u}(t)}{dt} + A\hat{u}(t) = f(\hat{u}(t)) + \widehat{G}(t), \quad 0 < t \le T, \tag{5.4.10}$$

$$\hat{u}(0) = u_0.$$

Let $S_{J-1} := \operatorname{span}\{e_1, e_2, \cdots, e_{J-1}\}$. The spectral method for solving (5.4.5)-(5.4.7) is to find $u_{J-1}(t) \in S_{J-1}$, such that, with $0 < t \le T$,

$$\frac{d\hat{u}_{J-1}(t)}{dt} + A_{J-1}\hat{u}_{J-1}(t) = P_{J-1}f(u_{J-1}(t)) + P_{J-1}\widehat{G}(t), \tag{5.4.11}$$

$$\hat{u}_{J-1}(0) = P_{J-1}u_0,$$

where $P_{J-1} : H \mapsto S_{J-1}$ is the orthogonal projection operator defined by

$$P_{J-1}v = \sum_{j=1}^{J-1} \bar{v}_j e_j, \quad \bar{v}_j = (v, e_j),$$

where $A_{J-1} = P_{J-1}A : S_{J-1} \mapsto S_{J-1}$. We remark that we use S_{J-1} (not S_J), since we will apply the MATLAB functions "**dst**" and "**idst**" in our numerical algorithms below.

The semi-implicit Euler method for solving (5.4.5)-(5.4.7) is to find $u_{J-1,n} \approx u_{J-1}(t_n)$, such that:

$$\frac{\hat{u}_{J-1,n+1} - \hat{u}_{J-1,n}}{\Delta t} + A_{J-1}\hat{u}_{J-1,n+1} = P_{J-1}f(\hat{u}_{J-1,n}) + P_{J-1}\widehat{G}(t_n), \tag{5.4.12}$$

$$\hat{u}_{J-1,0} = P_{J-1}u_0.$$

Assume that

$$\hat{u}_{J-1,n} = \sum_{j=1}^{J-1} \bar{u}_{j,n} e_j \in S_{J-1}. \tag{5.4.13}$$

It is easy to see that Fourier coefficients $\bar{u}_{j,n}$ satisfy, with $j = 1, 2 \cdots, J-1$,

$$\bar{u}_{j,n+1} = (1 + \Delta t \lambda_j)^{-1} \left(\bar{u}_{j,n} + \Delta t \bar{f}_j(\hat{u}_{J-1,n}) + \Delta t \widehat{G}_{j,n} \right), \tag{5.4.14}$$

$$\bar{u}_{j,0} = (P_{J-1}u_0, e_j), \tag{5.4.15}$$

where

$$P_{J-1}\widehat{G}(t_n) = \sum_{j=1}^{J-1} \bar{G}_{j,n} e_j \in S_{J-1}, \quad P_{J-1}f(\hat{u}_{J,n}) = \sum_{j=1}^{J-1} \bar{f}_j(\hat{u}_{J,n}) e_j.$$

Here $\bar{u}_{J,n}, \bar{G}_{j,n}, \bar{f}_j(\hat{u}_{J,n})$ denote the Fourier coefficients of $\hat{u}_{J-1,n}, \hat{G}(t_n)$ and $f(\hat{u}_{J-1,n})$, respectively. We may use the following steps to describe how to solve (5.4.5)-(5.4.7) numerically by using the spectral method.

Step1. Given initial value $\hat{u}_0(x)$ and f, we get the approximation $u_{J-1,0}(x) = P_{J-1}u_0 \approx u_0$ and $P_{J-1}f(u_{J-1,0}) \approx f(u_0(x))$.

Step2. Find the Fourier coefficients $\bar{u}_{j,0}$ and $\bar{f}_j(u_{J-1,0})$ by

$$\begin{bmatrix} \bar{u}_{1,0} \\ \bar{u}_{2,0} \\ \vdots \\ \bar{u}_{J-1,0} \end{bmatrix} = (\sqrt{2})^{-1}\left(\frac{J}{2}\right)^{-1} \mathbf{dst} \begin{bmatrix} u_0(x_1) \\ u_0(x_2) \\ \vdots \\ u_0(x_{J-1}) \end{bmatrix},$$

and

$$
\begin{bmatrix} \bar{f}_1(u_{J-1,0}) \\ \bar{f}_2(u_{J-1,0}) \\ \vdots \\ \bar{f}_{J-1}(u_{J-1,0}) \end{bmatrix} = (\sqrt{2})^{-1}\left(\frac{J}{2}\right)^{-1} \mathbf{dst} \begin{bmatrix} f(u_0(x_1)) \\ f(u_0(x_2)) \\ \vdots \\ f(u_0(x_{J-1})) \end{bmatrix},
$$

and

$$
\begin{bmatrix} \bar{G}_{1,0} \\ \bar{G}_{2,0} \\ \vdots \\ \bar{G}_{J-1,0} \end{bmatrix} = (\sqrt{2})^{-1}\left(\frac{J}{2}\right)^{-1} \mathbf{dst} \begin{bmatrix} \bar{G}(t_0, x_1) \\ \bar{G}(t_0, x_2) \\ \vdots \\ \bar{G}(t_0, x_{J-1}) \end{bmatrix}.
$$

Here,

$$
\begin{bmatrix} \bar{G}(t_0, x_1) \\ \bar{G}(t_0, x_2) \\ \vdots \\ \bar{G}(t_0, x_{J-1}) \end{bmatrix} = \hat{W}(1,:),
$$

where $\hat{W}$ is generated by

$$
\hat{W} = \frac{1}{\sqrt{\Delta t \Delta x}} * \mathbf{randn}(N, J-1). \tag{5.4.16}
$$

Step3. Find the Fourier coefficients $\bar{u}_{j,1}, j = 1, 2 \cdots J-1$ by

$$
\begin{bmatrix} \bar{u}_{1,1} \\ \bar{u}_{2,1} \\ \vdots \\ \bar{u}_{J-1,1} \end{bmatrix} = GG./EE
$$

where ./ denotes the element-wise division and

$$
GG = (\sqrt{2})^{-1}\left(\frac{J}{2}\right)^{-1} \mathbf{dst} \begin{bmatrix} u_0(x_1) \\ u_0(x_2) \\ \vdots \\ u_0(x_{J-1}) \end{bmatrix} + \Delta t (\sqrt{2})^{-1}\left(\frac{J}{2}\right)^{-1} \mathbf{dst} \begin{bmatrix} f(u_0(x_1)) \\ f(u_0(x_2)) \\ \vdots \\ f(u_0(x_{J-1})) \end{bmatrix}
$$

$$+\Delta t(\sqrt{2})^{-1}\left(\frac{J}{2}\right)^{-1}\mathbf{dst}\begin{bmatrix}\bar{G}(t_0x_1)\\\bar{G}(t_0x_2)\\\vdots\\\bar{G}(t_0x_{J-1})\end{bmatrix},$$

and, with $\lambda_j = \pi j$,

$$EE = \begin{bmatrix}1+\Delta t * \lambda_1^2\\1+\Delta t * \lambda_2^2\\\vdots\\1+\Delta t * \lambda_{J-1}^2\end{bmatrix}.$$

Step4. Find the Fourier coefficients $\bar{u}_{j,2}, j = 1, 2, \cdots, J-1$ by

$$\hat{u}_{j,2} = (1+\Delta t\lambda_j)^{-1}(\bar{u}_{j,1} + \Delta t\bar{f}_j(\hat{u}_{J-1,1}) + \Delta t\bar{G}_{j,1}).$$

Here

$$\begin{bmatrix}\bar{f}_1(u_{J-1,1})\\\bar{f}_2(u_{J-1,1})\\\vdots\\\bar{f}_{J-1}(u_{J-1,1})\end{bmatrix} = (\sqrt{2})^{-1}\left(\frac{J}{2}\right)^{-1}\mathbf{dst}\begin{bmatrix}f(\hat{u}_{J-1,1}(x_1))\\f(\hat{u}_{J-1,1}(x_2))\\\vdots\\f(\hat{u}_{J-1,1}(x_{J-1}))\end{bmatrix},$$

and

$$\begin{bmatrix}\bar{G}_{1,1}\\\bar{G}_{2,1}\\\vdots\\\bar{G}_{J-1,1}\end{bmatrix} = (\sqrt{2})^{-1}\left(\frac{J}{2}\right)^{-1}\mathbf{dst}\begin{bmatrix}\bar{G}(t_1,x_1)\\\bar{G}(t_1,x_2)\\\vdots\\\bar{G}(t_1,x_{J-1})\end{bmatrix},$$

where

$$\begin{bmatrix}\bar{G}(t_1,x_1)\\\bar{G}(t_1,x_2)\\\vdots\\\bar{G}(t_1,x_{J-1})\end{bmatrix} = \hat{W}(2,:),$$

and $\hat{W}$ is defined by (5.4.16).

Step5. Find $\bar{u}_{j,2}(x_k), k = 1, 2, \cdots, J-1$ by

$$
\begin{bmatrix}
\hat{u}_{J-1,2}(x_1) \\
\hat{u}_{J-1,2}(x_2) \\
\vdots \\
\hat{u}_{J-1,2}(x_{J-1})
\end{bmatrix}
= (\sqrt{2})^{-1}\left(\frac{J}{2}\right).\mathbf{dst}
\begin{bmatrix}
\bar{u}_{1,2} \\
\bar{u}_{2,2} \\
\vdots \\
\bar{u}_{J-1,2}
\end{bmatrix}.
$$

Step6. Repeating Step 3-5, we obtain all $\hat{u}_{J-1,n}(x_k), k = 1, 2, \cdots, J-1$.

Let us now introduce the MATLAB codes to solve our problem. Let u_0 denote the initial value vector, that is $u_0 = [u(x_1,), u_0(x_2) \cdots, u_0(x_{J-1})]$. Let u denote the approximate solution vector at time T, that is $u = [u(x_1, T), u(x_2, T) \ldots, u(x_{J-1}, T)]$. We may use the following MATLAB function to get the approximate solution u at T for any function f. Here we choose $f(u) = u - u^3$.

Let $x = [x_1, x_2 \cdots, x_{J-1}]$, epsilon $= 1$, kappa $= 1$. We can obtain the approximate solution u at time T at the different $x_k, k = 1, 2, \cdots, J-1$, by the following MATLAB function.

```
function [u]=spde_oned_Gal(u0,x,T,N,kappa,W1,J, epsilon)
dt=T/N; Dt=kappa*dt; % kappa for the different time steps
N/Dt;
lambda=pi*[1:(J-1)]'; M= epsilon*lambda. ^2; EE=1./(1+Dt*M);
for n=1;N
u0_hat=(sqrt(2)*J/2)^(-1)*dst(u0);
f_u0=u0-u0.^3;%f(u)=u-u^3
f_u0_hat=(sqrt(2)*J/2)^(-1)*dst(f_u0);
W=W1(kappa*(n-1)+1,:); W=W';  % kappa for the different time steps
G_hat=(sqrt(2)*J/2)^(-1)*dst(W);
u1_hat=(u0_hat + Dt*f_u0_hat + Dt*G_hat).*EE;
u1=(sqrt(2)*J/2)*idst(u1_hat);
u0=u1;
end
u=u1;
```

where **W1** denotes the Brownian sheet generated by:

$$W1 = \frac{1}{\sqrt{\Delta t * \Delta x}} * \mathbf{randn}(N, J - 1).$$

Example 1. Consider, with $0 < x < 1, 0 < t \leq T$, [2], [28],

$$\frac{\partial u(t, x)}{\partial t} + \epsilon(-\Delta)^\alpha u(t, x) = f(u(t, x)) + h(t, x) + \frac{\partial^2 W(t, x)}{\partial t \partial x}, \qquad (5.4.17)$$

$$u(t, 0) = u(t, 1) = 0, \qquad (5.4.18)$$

$$u(0, x) = u_0(x), \qquad (5.4.19)$$

where $\epsilon = 1, f(u) = -bu, b = 0.5$ and $u_0(x) = 10x^2(1 - x)^2$ and

$$h(t, x) = 10(1 + b)x^2(1 - x)^2 e^t - 10(2 - 12x + 12x^2)e^t.$$

Allen, Novosel and Zhang [2] and Du and Zhang provided [28] the numerical approximation of $\mathbb{E}(u(t, x))$ and $\mathbb{E}(u(t, x))^2$ with $\alpha = 1$ at time $t = 1$ and $x = 0.5$ by using the finite element method and the finite difference method. In Table 5.4.1 we obtain similar approximation values as in their papers for different pair $(\Delta t, \Delta x)$ by using spectral method. In our experience, for each pair $(\Delta t, \Delta x), 1000$ runs are performed. In Table 5.4.1 $u(1, 0.5)$ denotes the approximation of $u(t, x)$ at $t = 1$ and $x = 0.5$. The computational results converge as Δt and Δx approach zero.

In Figure 5.4.1, we plot a piecewise constant approximation of the noise $\hat{G}(t, x)$ with $J = 2^4$ and $N = 2^6$ on $0 \leq t \leq 1$ and $0 \leq x \leq 1$.

In Figure 5.4.2, we plot an approximation sample path of $u(t, x)$ with $J = 2^4$ and $N = 2^6$ on $0 \leq t \leq 1$ and $0 \leq x \leq 1$.

In Figure 5.4.3, we consider the convergence rate against the different time steps. Choose the fixed $J = 64$; we than consider the different time steps. The reference solution is obtained by using the time step $\Delta \text{tref} = T/N\text{ref}$ with $N\text{ref} = 10^4$. Let kappa $= [20, 50, 100, 150, 200, 250, 300]$; we will consider the approximate solutions with the different time steps $\Delta t_i = \Delta \text{tref} * \text{kappa}(i), i = 1, 2, \ldots, 7$.

In our experiment, for saving the computation time, we will consider the error estimates $\|\hat{u}_N(t_n) - u(t_n)\|_{L^2(\Omega, H)}$ at time t_n. We hope to observe the same convergence order as in Theorem 5.3.3.

Figure 5.4.1: Piecewise constant approximation of the noise $\hat{G}(t,x)$ with $J = 2^4$ and $N = 2^6$, $T = 1$.

Figure 5.4.2: An approximation sample path of $u(t,x)$ with $J = 2^4$ and $N = 2^6$, $T = 1$.

Figure 5.4.3: A plot of the error at $T = 1$ against $\log 2(\Delta t)$.

Figure 5.4.4: Piecewise constant approximation of the noise $\hat{G}(t, x)$ with $J = 2^4$ and $N = 2^6$, $T = 3$.

Figure 5.4.5: An approximation sample path of $u(t, x)$ with $J = 2^4$ and $N = 2^6$, $T = 3$.

Figure 5.4.6: A plot of the error at $T = 3$ against $\log 2(\Delta t)$.
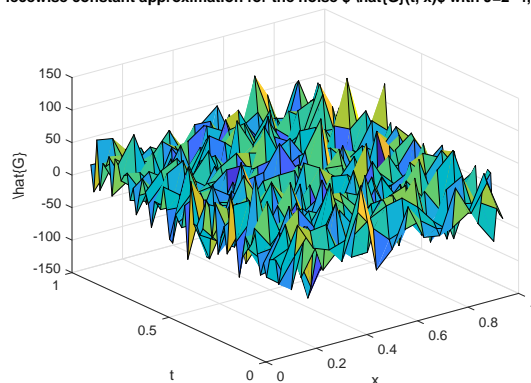
Figure 5.4.7: Piecewise constant approximation of the noise $\hat{G}(t,x)$ with $J = 2^4$ and $N = 2^6$, $T = 5$.
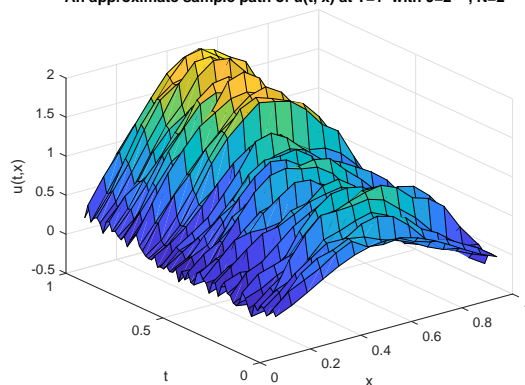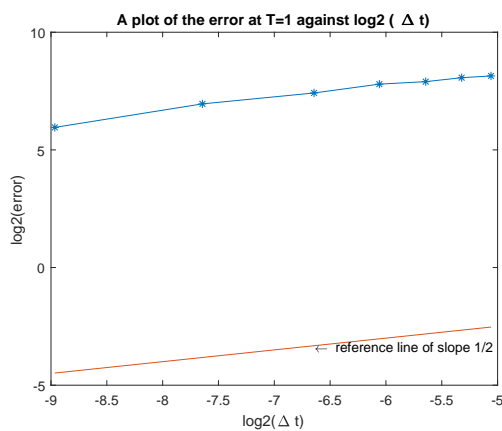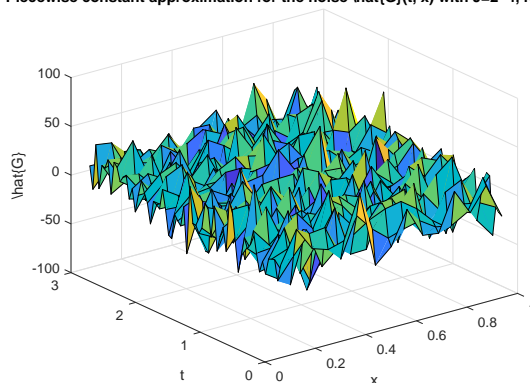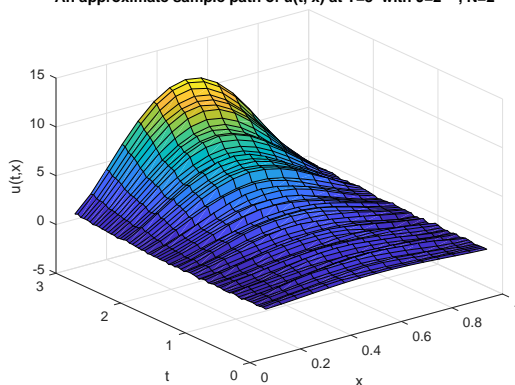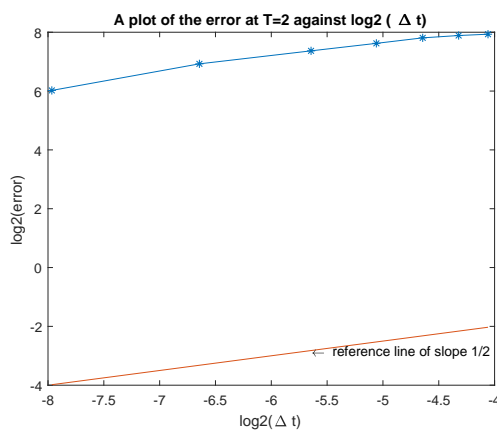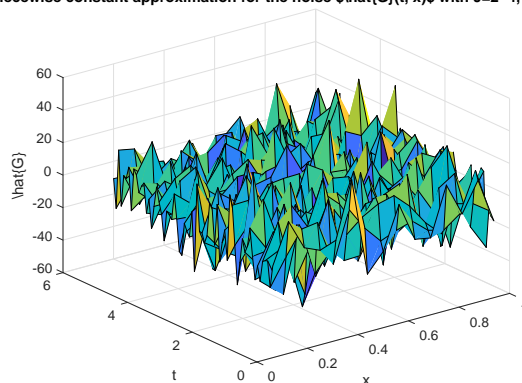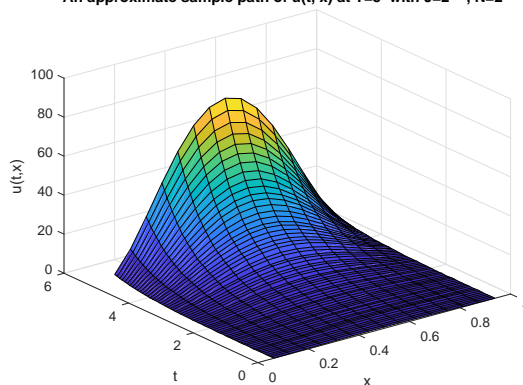

Figure 5.4.8: An approximation sample path of $u(t,x)$ with $J = 2^4$ and $N = 2^6$, $T = 5$.
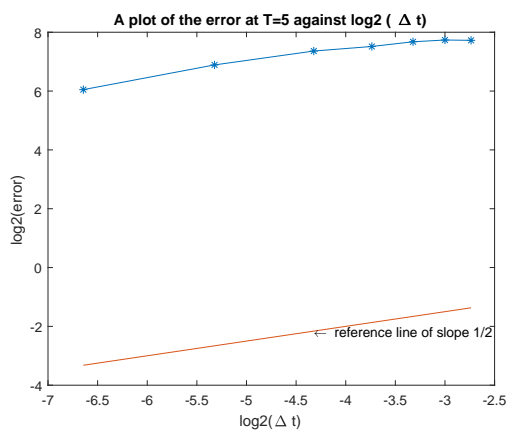
Figure 5.4.9: A plot of the error at $T = 5$ against $\log 2(\Delta t)$.

Δx	Δt	$\mathbb{E}u(1, 0.5)$	$\mathbb{E}(u(1, 0.5))^2$
1/4	1/4	1.6108	2.6386
1/4	1/8	1.7003	2.9883
1/4	1/16	1.9051	3.6534
1/4	1/32	1.9051	3.6534
1/8	1/4	1.4838	2.5923
1/8	1/8	1.6574	2.7709
1/8	1/16	1.7323	2.7585
1/8	1/32	1.6676	2.8153
1/16	1/4	1.4681	2.3333
1/16	1/8	1.6097	2.6420
1/16	1/16	1.6110	2.5681
1/16	1/32	1.6133	2.8737
1/32	1/4	1.3605	2.4143
1/32	1/8	1.6099	2.6095
1/32	1/16	1.6839	2.7930
1/32	1/32	1.7061	2.8747

Table 5.4.1: The approximation of $\mathbb{E}u(1, 0.5)$ and $\mathbb{E}(u(1, 0.5))^2$

To do this, we consider $M = 100$ simulations. For each simulation $\omega_m, m = 1, 2, \cdots, M$, we compute $\hat{u}_N(t_n) \sim \hat{u}(t_n)$ at time $t_n = 1$ by using the different time steps. We then compute the following L^2 norm of the error at $t_n = 1$ for the simulations $\omega_m, m = 1, 2, \ldots, M$,

$$\epsilon(\Delta t_i, \omega_m) = \epsilon((\Delta t_i, \omega_m, t_n) = \|\hat{u}_N(t_n, \omega_m) - \mathbf{uref}(t_n, \omega_m)\|^2,$$

where the reference (or "true") solution uref (t_n, ω_m) is approximated by the step $\Delta \text{tref} = T/N\text{ref}$. We then average $\epsilon(\Delta t_i, \omega_m)$ with respect to ω_m to obtain the following approximation of $\|\hat{u}_N(t_n, \omega_m) - \text{uref}(t_n)\|_{L^2(\Omega, H^r)}$ with respect to the different time step Δt_i,

$$S(\Delta t_i) = \Big(\frac{1}{M} \sum_{m=1}^{M} \epsilon(\Delta t_i, \omega_m)\Big)^{\frac{1}{2}} = \Big(\frac{1}{M} \sum_{m=1}^{M} \|\hat{u}(t_n, \omega_m) - \mathbf{uref}(t_n, \omega_m)\|^2\Big)^{\frac{1}{2}}.$$

Since the convergence rate with respect to the time step is $O(\Delta t^{\frac{1}{2}})$ which implies that $\log(S(\Delta t_i)) \approx \frac{1}{2}\log(\Delta t_i), i = 1, 2, \ldots, 7$.

In Figure 5.4.3, we plot the points $\log(S(\Delta t_i)) \approx \frac{1}{2}\log(\Delta t_i), i = 1, 2, \ldots, 7$ and we see that the points are parallel to the reference line, which has the slope $\frac{1}{2}$, as we expected in our theoretical results.

In Table 5.4.2, we list the error $S(\Delta t_i)$ against the different time steps Δt_i.

Δt_i	$2e-03$	$5e-03$	$1e-02$	$1.5e-02$	$2e-02$	$2.5e-02$	$3e-02$
L^2-error	0.2775	0.5355	0.7116	0.9249	1.0306	1.1159	1.1742

Table 5.4.2: The L^2 norm error at $T = 1$ against Δt

.

In this chapter, we present a Fourier spectral method for solving space fractional partial differential equations. The space-time white noise is approximated by using piecewise constant functions. For the liner problem, we obtain the exact error estimates in the L_2-norm and find the relations between the convergence order and the fractional power $\alpha, \frac{1}{2} < \alpha \leq 1$. For the nonlinear problem, we introduce the numerical algorithm and the MATLAB code for solving such problem based on the discrete sine transform and inverse discrete sine transform MATLAB functions dst.m and idst.m. The MATLAB code in this paper can be easily modified to solve other nonlinear stochastic fractional partial differential equations with Dirichlet boundary conditions.

Chapter 6

Fourier Spectral Methods for Stochastic Space Fractional Partial Differential Equations Driven by Special Additive Noises

6.1 Introduction

Fourier spectral methods for solving stochastic space fractional partial differential equation driven by special additive noises in one dimensional case are introduced and analyzed in this chapter. The space fractional derivative is defined by using the eigenvalues and eigenfunctions of Laplacian subject to some boundary conditions. The space-time noise is approximated by the piecewise constant functions in the time direction and by some appropriate approximations in the space direction. The approximated stochastic space fractional partial differential equations is then solved by using Fourier spectral methods.

For the linear problem, we obtain the precise error estimates in the L_2 norm and find the relation between the error bounds and the fractional powers. For the nonlinear problem, we introduce the numerical algorithms and MATLAB codes based on the FFT transforms. Our numerical algorithms can be adapted easily to solve other stochastic space fractional partial differential equations with multiplicative noises. Numerical examples for

the semilinear stochastic space fractional partial differential equations are given.

Fourier spectral methods for solving the following stochastic space fractional partial differential equation are considered in this work, with $\frac{1}{2} < \alpha \leq 1$,

$$\frac{du(t)}{dt} + A^\alpha u(t) = f(u(t)) + \frac{dW(t)}{dt}, \qquad 0 < t < T, \tag{6.1.1}$$

$$u(0) = u_0. \tag{6.1.2}$$

Here A is an unbounded positive self-adjoint operator, u_0 is an initial value and $f(u)$ is a nonlinear term. The space-time white noise $W(t)$ will be defined below.

Let H be a separable Hilbert space and $||\cdot||$, $(\cdot, \cdot)$ denote the norm and inner product in H, respectively. Let $A : D(A) \subset H \to H$ be a positive selfjoint operator such that A^{-1} is compact on H. Assume that A has the eigenpair $\{\lambda_k, e_k\}, k = 1, 2, \cdots$.

Using the basis $\{e_k\}$ we may also define the fractional powers of A. Given $\frac{1}{2} < \alpha \leq 1$, we define

$$H^{2\alpha} := \mathcal{D}(A^\alpha) = \{v \in H : \sum_k \lambda_k^{2\alpha} |(v, e_k)|^2 < \infty\},$$

and

$$A^\alpha v := \sum_k \lambda_k^\alpha (v, e_k) e_k, \quad v \in D(A^\alpha), \tag{6.1.3}$$

with the associated norm defined by

$$||A^\alpha v||^2 = \sum_k \lambda_k^{2\alpha} (v, e_k)^2.$$

The special space-time noise considered in this work is

$$\frac{dW(t)}{dt} = \sum_{k=1}^\infty \sigma_k(t) \dot{\beta}_k(t) e_k, \tag{6.1.4}$$

where $\dot{\beta}_k(t) = \frac{d\beta_k(t)}{dt}$, $k = 1, 2, \cdots$ is the derivative of the standard Brownian motions $\beta_k(t), k = 1, 2, \cdots$ and $\sigma_k(t), k = 1, 2, \cdots$ are some appropriate functions of t. In particular, when $\sigma_k(t) = \bar{\gamma}_k^{\frac{1}{2}}, \bar{\gamma}_k > 0$, the noise (6.1.4) reduces to

$$\frac{dW(t)}{dt} = \sum_{k=1}^\infty \bar{\gamma}_k^{\frac{1}{2}} \dot{\beta}_k(t) e_k,$$

which is so called H-valued Wiener process with the covariance operator Q and the linear operator $Q : H \rightarrow H$ is a trace class operator, that is $Tr(Q) = \sum_{k=1}^{\infty} \bar{\gamma}_k < \infty$ where $Qe_k = \bar{\gamma}_k e_k, k = 1, 2 \dots$.

Let us here give two possible operators in (6.1.1)-(6.1.2). One is $A = -\Delta$ with the homogeneous Dirichlet boundary condition, $\mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$, where $\Delta = \frac{\partial^2}{\partial x^2}$ denotes the Laplacian. In this case, A has eigenvalues $\lambda_k = k^2 \pi^2$ and eigenfunctions $e_k = \sqrt{2} \sin k\pi x, k = 1, 2, \dots$ Our error estimates in this work are based on these eigenvalues and eigenfunctions. Another one is $A = I - \Delta$ with periodic boundary conditions, $\mathcal{D}(A) = H_{per}^2(-\pi, \pi)$. Here $H_{per}^2(-\pi, \pi)$ denotes the completion with respect to the $H^2(-\pi, \pi)$ norm of the set of $u \in C^\infty([-\pi, \pi])$ such that the pth derivative $u^{(p)}(-\pi) = u^{(p)}(\pi)$ for $p = 0, 1, 2, \dots$. It is well known that $H_{per}^2(-\pi, \pi)$ is a Hilbert space with the $H^2(-\pi, \pi)$ inner product, [62, Definition 1.47]. In this case, A has the eigenvalues $\lambda_1 = 1, \lambda_{2k} = 1 + k^2, \lambda_{2k+1} = 1 + k^2$ and eigenfunctions $e_1(x) = \frac{1}{\sqrt{2\pi}}, e_{2k}(x) = \frac{1}{\sqrt{\pi}} \sin kx, e_{2k+1}(x) = \frac{1}{\sqrt{\pi}} \cos kx, k = 1, 2 \dots$, see, e.g., [62, Example 1.84].

We obtain the detailed error estimates, i.e., Theorems 6.2.1, 6.3.1, 6.3.2 below for the linear stochastic space fractional partial differential equation subject to the Dirichlet boundary conditions. More precisely, we shall consider the error estimates for the following linear problem, with $\frac{1}{2} < \alpha \leq 1$,

$$\frac{\partial u(t,x)}{\partial t} + (-\Delta)^\alpha u(t,x) = \frac{\partial^2 W(t,x)}{\partial t \partial x}, \quad 0 < t < T, \ 0 < x < 1, \tag{6.1.5}$$

$$u(t,0) = u(t,1) = 0, \quad 0 < t < T, \tag{6.1.6}$$

$$u(0,x) = u_0(x), \quad 0 < x < 1. \tag{6.1.7}$$

Here the space-time noise $\frac{\partial^2 W(t,x)}{\partial t \partial x} = \frac{dW(t)}{dt}$ is defined by (6.1.4).

For the linear stochastic space fractional partial differential equation subject to the periodic boundary conditions, we may obtain the similar error estimates as in Theorems 6.2.1, 6.3.1, 6.3.2. The stochastic partial differential equations driven by the white noise (the co-variance operator $Q = I$) often have poor regularity estimates. In the physical world, to take into account short and long range correlations of the stochastic effects, both white noise and colored noises may be considered. There are many situations where colour noises model the reality more closely, and there are also instances where the important stochastic effects are the noises acting on a few selected frequencies. For example one

may choose $\sigma_k(t) = \frac{\cos t}{k^3}$, see e.g., [28].

Space fractional partial differential equations are widely used to model complex phenomena, for example, quasi-geostrophic flows, fast roating fluids, dynamic of the frontogenesis in meteorology, diffusion in fractal or disordered medium, pollution problems, mathematical finance and transport problem, [9], [11], [16], [101].

Let $N_t \in \mathbb{N}$ and let $0 = t_0 < t_1 < t_2 < \cdots < t_{N_t} = T$ be the time partition of $[0, T]$ and Δt the time step size. To find the approximation solution of (6.1.5)-(6.1.7) we approximate the noise $\frac{\partial^2 W(t,x)}{\partial t \partial x}$ by the piecewise constant functions in the time direction, with $l = 1, 2, \cdots N_t$ [28],

$$\frac{\partial^2 \widehat{W}(t, x)}{\partial t \partial x} = \sum_{k=1}^{\infty} \sigma_k^M(t) e_k(x) \Big(\sum_{l=1}^{N_t} \frac{1}{\sqrt{\Delta t}} \eta_{kl} \chi_l(t) \Big), \tag{6.1.8}$$

where

$$\eta_{k,l} = \frac{1}{\sqrt{\Delta t}} \int_{t_{l-1}}^{t_l} d\beta_k(t) = \frac{1}{\sqrt{\Delta t}} \Big(\beta_k(t_l) - \beta_k(t_{l-1}) \Big) \in \mathcal{N}(0, 1), \tag{6.1.9}$$

and

$$\chi_l(t) = \begin{cases} 1, & t \in [t_{l-1}, t_l], \quad l = 1, 2, \cdots, N, \\ 0, & \text{otherwise.} \end{cases} \tag{6.1.10}$$

Here $\sigma_k^M(t)$ is the approximation of $\sigma_k(t)$ in the space direction. For example, we can choose with some positive integer $M > 0$,

$$\sigma_k(t) = \frac{\cos t}{k^3}, \qquad \sigma_k^M(t) = \begin{cases} \sigma_k(t), & k \le M, \\ 0, & k > M. \end{cases} \tag{6.1.11}$$

More precisely, replacing $\sigma_k(t)$ by $\sigma_k^M(t)$ we get the noise approximation in space, and replacing $\dot{\beta}_k(t)$ by $\sum_{j=1}^{N_t} \frac{1}{\sqrt{\Delta t}} \eta_{k,j} \chi_l(t)$, we get the noise approximation in time.

Substituting $\frac{\partial^2 W(t,x)}{\partial t \partial x}$ with $\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ in (6.1.5)-(6.1.7), we get

$$\frac{\partial \hat{u}(t, x)}{\partial t} + (-\Delta)^\alpha \hat{u}(t, x) = \frac{\partial^2 \widehat{W}(t, x)}{\partial t \partial x}, \quad 0 < t < T, \ 0 < x < 1, \tag{6.1.12}$$

$$\hat{u}(t, 0) = \hat{u}(t, 1) = 0, \quad 0 < t < T, \tag{6.1.13}$$

$$\hat{u}(0, x) = u_0(x), \quad 0 < x < 1. \tag{6.1.14}$$

Note that $\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ now is a function in $L^2((0,T) \times (0,1))$ and therefore we can solve (6.1.12)-(6.1.14) by using any numerical methods for deterministic space fractional partial differential equations. Assume that $\{\sigma_k(t)\}$ and its derivative are uniformly bounded, [28]

$$|\sigma_k(t)| \le \beta_k, \qquad |\sigma'_k(t)| \le \gamma_k, \qquad \forall t \in [0,T], \tag{6.1.15}$$

and the coefficients $\{\sigma_k^M\}$ are constructed such that

$$|\sigma_k(t) - \sigma_k^M(t)| \le \alpha_k^M, |\sigma_k^M(t)| \le \beta_k^M, \quad |(\sigma_k^M)'(t)| \le \gamma_k^M, \quad \forall t \in [0,T], \tag{6.1.16}$$

with positive sequences $\{\alpha_k^M\}$ being arbitrarily chosen, $\{\beta_k^M\}$ and $\{\gamma_k^M\}$ being related to $\{\beta_k\}$ and $\{\gamma_k\}$. Further we assume that

$$\beta_k^M \le k^{-\bar{\alpha}}, \quad \text{for some} \quad 0 \le \bar{\alpha} < \frac{1}{2}. \tag{6.1.17}$$

Let $\mathbb{E}$ denote the expectation, in the Theorem 6.2.1, we prove that, with $\frac{1}{2} < \alpha \le 1$ and $0 \le \bar{\alpha} < \frac{1}{2}$,

$$\mathbb{E} \int_0^T \int_0^1 (u(t,x) - \hat{u}(t,x))^2 dx dt \tag{6.1.18}$$

$$\le C\Big(\sum_{k=1}^\infty \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k=1}^\infty \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M \Big)^2 + \Delta t^{1+\frac{\bar{\alpha}}{\alpha} - \frac{1}{2\alpha}} \Big). \tag{6.1.19}$$

Let $J \in \mathbb{N}$, we denote

$S_J = \text{span}\{e_1, e_2, \cdots e_J\}$,

and define by $P_J : H \to S_J$ the projection from H to S_J,

$$P_J v = \sum_{j=1}^J (v, e_j) e_j. \tag{6.1.20}$$

The Fourier spectral method of (6.1.12)-(6.1.14) is to find $\hat{u}_J(t) \in S_J$ such that, with $\hat{g}(t,x) := \frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$,

$$\frac{\partial \hat{u}_J(t,x)}{\partial t} + (-\Delta)^\alpha \hat{u}_J(t,x) = P_J \hat{g}(t,x), \quad 0 < t < T, \, 0 < x < 1, \tag{6.1.21}$$

$$\hat{u}_J(t,0) = \hat{u}_J(t,1) = 0, \quad 0 < t < T, \tag{6.1.22}$$

$$\hat{u}_J(0,x) = P_J u_0(x), \quad 0 < x < 1. \tag{6.1.23}$$

In Theorem 6.3.1, we prove that, with $\frac{1}{2} < \alpha \leq 1$,

$$||\hat{u}(t) - \hat{u}_J(t)||^2 \leq C||u_0 - P_J u_0||^2 + C\frac{1}{(J+1)^{2\alpha}}\int_0^t ||\hat{g}(s)||^2 ds. \qquad (6.1.24)$$

Combining Theorem 6.2.1 with 6.3.1 we have, with $u_0 \in \mathcal{D}(A^\alpha), \frac{1}{2} < \alpha \leq 1$, and $0 \leq \bar{\alpha} < \frac{1}{2}$,

$$\mathbb{E}\int_0^T \int_0^t (u(t,x) - \hat{u}_J(t,x))^2 dxdt$$

$$\leq C\Big(\sum_{k=1}^\infty \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k=1}^\infty (\lambda_k^\alpha \beta_k^M + \gamma_k^M)^2 + \Delta t^{1 + \frac{\bar{\alpha}}{\alpha} - \frac{1}{2\alpha}}\Big)$$

$$+ C\mathbb{E}||u_0 - P_J u_0||^2 + C\frac{1}{(J+1)^{2\alpha}}\Big(\Delta t \mathbb{E}||A^\alpha u_0||^2 + \Delta t \sum_{k=1}^\infty (\lambda_k^\alpha \beta_k^M)^2 + \sum_{k=1}^\infty (\beta_k^M)^2\Big).$$

6.2   Approximate Noise and Regularity of the Solution

It is well known that the mild solution of (6.1.5)-(6.1.7) has the following form.

$$u(t,x) = \int_0^1 G_\alpha(t,x,y)u_0(y)dy + \int_0^t \int_0^1 G_\alpha(t-s,x,y)dW(s,y), \qquad (6.2.1)$$

where

$$G_\alpha(t,x,y) = \sum_{j=1}^\infty e^{-\lambda_j^\alpha t}e_j(x)e_j(y),$$

and the stochastic integral $\int_0^t \int_0^1 G_\alpha(t-s,x,y)dW(s,y)$ is well-defined. The existence and uniqueness of the solutions of (6.1.12)-(6.1.14) are discussed in, e.g., [23], [24], [78] and the references cited therein.

Similarly the mild solution of (6.1.12)-(6.1.14) has the form of, see, e.g., [28]

$$\hat{u}(t,x) = \int_0^1 G_\alpha(t,x,y)u_0(y)dy + \int_0^t \int_0^1 G_\alpha(t-s,x,y)d\widehat{W}(s,y). \qquad (6.2.2)$$

Theorem 6.2.1. Let u and $\hat{u}$ be the solutions of (6.1.5)-(6.1.7) and (6.1.12)-(6.1.14) respectively. Assume that the Assumptions (6.1.15)-(6.1.17) hold . Then we have, with $0 \leq \bar{\alpha} < \frac{1}{2}$,

$$\mathbb{E}\int_0^T \int_0^t (u(t,x) - \hat{u}(t,x))^2 dxdt$$

$$\leq C\Big(\sum_{k=1}^\infty \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k=1}^\infty \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M\Big)^2 + \Delta t^{1 + \frac{\bar{\alpha}}{\alpha} - \frac{1}{2\alpha}}\Big). \qquad (6.2.3)$$

To prove Theorem 6.2.1, we need the following Lemma.

Lemma 6.2.2. Let $\frac{1}{2} < \alpha \leq 1$ and $0 \leq \bar{\alpha} < \frac{1}{2}$. We have

$$\int_0^\infty x^{-2(\bar{\alpha}+\alpha)}(1 - e^{-x^{2\alpha}\Delta t})dx \leq C\Delta t^{1+\frac{\bar{\alpha}}{\alpha}-\frac{1}{2\alpha}}. \tag{6.2.4}$$

Proof. With the variable change $y = x^{2\alpha}\Delta t$, we have

$$\int_0^\infty x^{-2(\bar{\alpha}+\alpha)}(1 - e^{-x^{2\alpha}\Delta t})dx \leq C\Delta t^{1+\frac{\bar{\alpha}}{\alpha}-\frac{1}{2\alpha}}\Big(\int_0^1 + \int_1^\infty\Big)\frac{1-e^{-y}}{y^{2+\frac{\bar{\alpha}}{\alpha}-\frac{1}{2\alpha}}}\,dy.$$

It is easy to see that, with $\frac{1}{2} < \alpha \leq 1$ and $0 \leq \bar{\alpha} < \frac{1}{2}$,

$$\int_1^\infty \frac{1-e^{-y}}{y^{2+\frac{\bar{\alpha}}{\alpha}-\frac{1}{2\alpha}}}dy \leq C.$$

Further, we have, with $\frac{1}{2} < \alpha \leq 1$, $0 \leq \bar{\alpha} < \frac{1}{2}$,

$$\Big|\int_0^1 \frac{1-e^{-y}}{y^{2+\frac{\bar{\alpha}}{\alpha}-\frac{1}{2\alpha}}}dy\Big| \leq C\int_0^1 \frac{y}{y^{2+\frac{\bar{\alpha}}{\alpha}-\frac{1}{2\alpha}}}dy \leq C\int_0^1 \frac{1}{y^{1+\frac{\bar{\alpha}}{\alpha}-\frac{1}{2\alpha}}}dy < \infty.$$

Together these estimates complete the proof of Lemma 6.2.2. $\qquad\square$

Now we turn to the proof of Theorem 6.2.1.

Proof of Theorem 6.2.1. Subtracting (6.2.2) from (6.2.1), we have

$$
\begin{aligned}
&u(t,x) - \hat{u}(t,x) \\
&= \int_0^t \int_0^1 G_\alpha(t-s,x,y)dW(s,y) - \int_0^t \int_0^1 G_\alpha(t-s,x,y)d\widehat{W}(s,y) \\
&= \Big[\int_0^t \int_0^1 G_\alpha(t-s,x,y)dW(s,y) - \int_0^t \int_0^1 G_\alpha(t-s,x,y)d\overline{W}(s,y)\Big] \\
&\quad + \Big[\int_0^t \int_0^1 G_\alpha(t-s,x,y)d\overline{W}(s,y) - \int_0^t \int_0^1 G_\alpha(t-s,x,y)d\widehat{W}(s,y)\Big] \\
&= F_1(t,x) + F_2(t,x),
\end{aligned}
$$

where with $\eta_{k,l}$ and $\chi_l(t)$ defined as in (6.1.9),

$$dW(s,y) = \frac{\partial^2 W(s,y)}{\partial s \partial y}dsdy = \Big[\sum_{k=1}^\infty \sigma_k(s)e_k(y)\Big]d\beta_k(s)dy,$$

$$d\overline{W}(s,y) = \frac{\partial^2 \overline{W}(s,y)}{\partial s \partial y}dsdy = \Big[\sum_{k=1}^\infty \sigma_k^M(s)e_k(y)\Big]d\beta_k(s)dy,$$

$$d\widehat{W}(s,y) = \frac{\partial^2 \widehat{W}(s,y)}{\partial s \partial y}dsdy = \Big[\sum_{k=1}^\infty \sigma_k^M(s)\Big(\sum_{l=1}^{N_t} \frac{\eta_{k,l}}{\sqrt{\Delta t}}\chi_l(s)\Big)e_k(y)\Big]dsdy.$$

Thus we have

$$\mathbb{E} \int_0^T \int_0^1 |u(t,x) - \hat{u}(t,x)|^2 dxdt$$

$$\leq C\mathbb{E} \int_0^T \int_0^1 F_1^2(t,x)dxdt + C\mathbb{E} \int_0^T \int_0^1 F_2^2(t,x)dxdt = C(I + II).$$

For I, we have, by using isometry property and (6.1.16),

$$I = \mathbb{E} \int_0^T \int_0^1 \Big[\int_0^t \int_0^1 G_\alpha(t-s,x,y)dW(s,y) - \int_0^t \int_0^1 G_\alpha(t-s,x,y)d\overline{W}(s,y) \Big]^2 dxdt$$

$$= \int_0^T \int_0^1 \int_0^t \Big[\int_0^1 G_\alpha(t-s,x,y) \Big(\sum_{k=1}^\infty (\sigma_k(s) - \sigma_k^M(s))e_k(y) \Big) dy \Big]^2 dsdxdt$$

$$= \int_0^T \int_0^1 \sum_{k=1}^\infty e^{-2(t-s)\lambda_k^\alpha}(\sigma_k^M)^2 dsdt = \int_0^T \sum_{k=1}^\infty \frac{1 - e^{-2t\lambda_k^\alpha}}{2\lambda_k^\alpha}(\sigma_k^M)^2 dt$$

$$\leq C \sum_{k=1}^\infty \frac{1}{2\lambda_k^\alpha}(\sigma_k^M)^2.$$

For II, we have

$$II = \mathbb{E} \int_0^T \int_0^1 \Big\{ \Big[\int_0^t \int_0^1 G_\alpha(t-s,x,y)d\overline{W}(s,y) - \int_0^t \int_0^1 G_\alpha(t-s,x,y)d\widehat{W}(s,y) \Big]^2 \Big\} dxdt$$

$$\leq 3\mathbb{E} \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^1 \Big\{ \Big[\int_0^t \int_0^1 [G_\alpha(t-s,x,y)d\overline{W}(s,y) - \int_0^{t_j} \int_0^1 G_\alpha(t_j-s,x,y)d\overline{W}(s,y) \Big]^2$$

$$+ \Big[\int_0^{t_j} \int_0^1 G_\alpha(t_j-s,x,y)d\overline{W}(s,y) - \int_0^{t_j} \int_0^1 G_\alpha(t_j-s,x,y)d\widehat{W}(s,y) \Big]^2$$

$$+ \Big[\int_0^{t_j} \int_0^1 G_\alpha(t_j-s,x,y)d\widehat{W}(s,y) - \int_0^t \int_0^1 G_\alpha(t_j-s,x,y)d\widehat{W}(s,y) \Big]^2 \Big\} dxdt$$

$$\leq 3(II_1 + II_2 + II_3).$$

For II_2, we have, using the isometry property,

$$
\begin{aligned}
II_2 = \mathbb{E} \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^1 & \Big[\sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \int_0^1 G_\alpha(t_j-s,x,y) \Big(\sum_{k=1}^\infty \sigma_k^M(s)e_k(y)dy \Big) d\beta_k(s) \\
& - \sum_{l=0}^{j-1} \int_{t_j}^{t_{j+1}} \int_0^1 G_\alpha(t_j-\bar{s},x,y) \sum_{k=1}^\infty \sigma_k^M(\bar{s})e_k(y)dyd\bar{s} \Big(\frac{1}{\Delta t} \int_{t_j}^{t_{j+1}} d\beta_k(s) \Big) \Big]^2 dxdt \\
= \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^1 \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} & \Big\{ \int_0^1 G_\alpha(t_j-s,x,y) \Big(\sum_{k=1}^\infty \sigma_k^M(s)e_k(y) \Big) dy \\
& - \frac{1}{\Delta t} \int_{t_l}^{t_{l+1}} \int_0^1 G_\alpha(t_j-\bar{s},x,y) \Big(\sum_{k=1}^\infty \sigma_k^M(\bar{s})e_k(y)dyd\bar{s} \Big\}^2 dsdxdt \\
= \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^1 \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} & \Big\{ \frac{1}{\Delta t} \int_{t_l}^{t_{l+1}} \Big[\int_0^1 G_\alpha(t_j-s,x,y) (\sum_{k=1}^\infty \sigma_k^M(s)e_k(y)dy \\
& - \int_0^1 G_\alpha(t_j-\bar{s},x,y) \Big(\sum_{k=1}^\infty \sigma_k^M(\bar{s})e_k(y)dy]d\bar{s} \Big\}^2 dsdxdt \\
= \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} [\sum_{l=0}^{j-1} \int_{t_j}^{t_{j+1}} & \sum_{k=1}^\infty \Big\{ \frac{1}{\Delta t} \int_{t_l}^{t_{l+1}} [e^{-\lambda_k^\alpha(t_j-s)}\sigma_k^M(s) - e^{-\lambda_k^\alpha(t_j-\bar{s})}\sigma_k^M(\bar{s})]d\bar{s} \Big\}^2 dsdt \\
= \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} & \sum_{k=1}^\infty \frac{e^{-2\lambda_k^\alpha t_j}}{\Delta t^2} \Big\{ \int_{t_l}^{t_{l+1}} [e^{\lambda_k^\alpha(s)}\sigma_k^M(s) - e^{\lambda_k^\alpha(\bar{s})}\sigma_k^M(\bar{s})]d\bar{s} \Big\}^2 dsdt.
\end{aligned}
$$

By (6.1.16) we have, with ξ_l^1, ξ_l^2 lying between s and $\bar{s}$,

$$
\begin{aligned}
\Big| e^{\lambda_k^\alpha(s)}\sigma_k^M(s) - e^{\lambda_k^\alpha(\bar{s})}\sigma_k^M(\bar{s}) \Big| &= \Big| (e^{\lambda_k^\alpha s} - e^{\lambda_k^\alpha(\bar{s})})\sigma_k^M(s) + e^{\lambda_k^\alpha(\bar{s})}(\sigma_k^M(s) - \sigma_k^M(\bar{s}) \Big| \\
&\leq \Big| (\lambda_k^\alpha e^{\lambda_k^\alpha \xi_l^1}\Delta t)\sigma_k^M(s) + e^{\lambda_k^\alpha \bar{s}}(\sigma_k^M)'(\xi_l^2)\Delta t \Big| \\
&\leq \Big| (\lambda_k^\alpha e^{\lambda_k^\alpha t_{l+1}}\beta_k^M \Delta t) + e^{\lambda_k^\alpha t_{l+1}}(\gamma_k^M)\Delta t \Big| \\
&\leq e^{\lambda_k^\alpha t_{l+1}}(\lambda_k^\alpha \beta_k^M + \gamma_k^M)\Delta t.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
II_2 &\leq \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \sum_{k=1}^\infty \frac{e^{-2\lambda_k^\alpha t_j}}{\Delta t^2} \Big[e^{\lambda_k^\alpha t_{l+1}}(\lambda_k^\alpha \beta_k^M + \gamma_k^M)\Delta t^4 \Big] dsdt \\
&\leq \Delta t^2 \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^{t_j} \sum_{k=1}^\infty \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M \Big)^2 dsdt \leq C\Delta t^2 \sum_{k=1}^\infty \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M \Big)^2,
\end{aligned}
$$

where we use the inequality $e^{-2\lambda_k^\alpha(t_j-t_{l+1})} \leq 1$ for $l = 0, 1, 2, \cdots, j-1$.

For II_1, we have

$$II_1 = \mathbb{E} \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\int_0^t \int_0^1 G_\alpha(t-s,x,y)d\overline{W}(s,y) - \int_0^{t_j} \int_0^1 G_\alpha(t-s,x,y)d\overline{W}(s,y) \right]^2 dxdt$$

$$\leq 2\mathbb{E} \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\int_0^{t_j} \int_0^1 [G_\alpha(t-s,x,y) - G_\alpha(t_j-s,x,y)d\overline{W}(s,y) \right]^2 dxdt$$

$$+ 2\mathbb{E} \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\int_{t_j}^t \int_0^1 G_\alpha(t-s,x,y)d\overline{W}(s,y) \right]^2 dxdt = 2(II_1^1 + II_1^2).$$

For II_1^1, we have, by the isometry property and (6.1.16),

$$II_1^1 = \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^{t_j} \sum_{k=1}^\infty \left(e^{-\lambda_k^\alpha(t-s)} - e^{-\lambda_k^\alpha(t_j-s)} \right)^2 (\sigma_k^M(s))^2 dsdt$$

$$\leq \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \sum_{k=1}^\infty (\beta_k^M)^2 \int_0^{t_j} \left(e^{-\lambda_k^\alpha(t-s)} - e^{-\lambda_k^\alpha(t_j-s)} \right)^2 dsdt.$$

Note that

$$\int_0^{t_j} \left(e^{-\lambda_k^\alpha(t-s)} - e^{-\lambda_k^\alpha(t_j-s)} \right)^2 ds = \int_0^{t_j} (e^{-2\lambda_k^\alpha(t-s)} \left(1 - e^{-\lambda_k^\alpha(t_j-t)} \right)^2 ds$$

$$= \left(1 - e^{-\lambda_k^\alpha(t_j-t)} \right)^2 \frac{e^{-2\lambda_k^\alpha(t-t_j)} - e^{-2\lambda_k^\alpha t}}{2\lambda_k^\alpha} \leq \frac{\left(1 - e^{\lambda_k^\alpha(t-t_j)} \right)^2}{2\lambda_k^\alpha}.$$

We have

$$II_1^1 \leq \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \left(\sum_{k=1}^\infty (\beta_k^M)^2 \right) \frac{(1 - e^{-\lambda_k^\alpha(t-t_j)})^2}{2\lambda_k^\alpha} dt \leq C \sum_{k=1}^\infty (\beta_k^M)^2 \frac{(1 - e^{-\lambda_k^\alpha \Delta t})^2}{2\lambda_k^\alpha}.$$

By (6.1.17) and Lemma 6.2.2, we obtain

$$II_1^1 \leq C \sum_{k=1}^\infty k^{-2\bar\alpha} \frac{(1 - e^{-\lambda_k^\alpha \Delta t})^2}{2\lambda_k^\alpha} \leq C \int_1^\infty x^{-2(\bar\alpha+\alpha)}(1 - e^{-x^{2\alpha}\Delta t}dx \leq C\Delta t^{1 + \frac{\bar\alpha}{\alpha} - \frac{1}{2\alpha}}.$$

For II_1^2, we have, by isometry property and (6.1.16) and (6.1.17),

$$II_1^2 = \mathbb{E} \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\int_{t_j}^t \int_0^1 G_\alpha(t-s,x,y)d\overline{W}(s,y) \right]^2 dxdt$$

$$= \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_{t_j}^t \sum_{k=1}^\infty e^{-2\lambda_k^\alpha(t-s)}(\sigma_k^M(s))^2 dsdt \leq \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_{t_j}^t \sum_{k=1}^\infty \left(k^{-2\bar\alpha} e^{-2\lambda_k^\alpha(t-s)} \right) dsdt$$

$$\leq C \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \sum_{k=1}^\infty \left[k^{-2\bar\alpha} \frac{(1 - e^{2\lambda_k^\alpha \Delta t})}{\lambda_k^\alpha} \right] dt = C \sum_{k=1}^\infty \left[k^{-2\bar\alpha} \frac{(1 - e^{-\lambda_k^\alpha \Delta t})}{\lambda_k^\alpha} \right]$$

$$\leq C \int_0^\infty \frac{1 - e^{-2x^{2\alpha}\Delta t}}{x^{2\alpha+2\bar\alpha}} dx \leq C \int_0^\infty x^{-2(\bar\alpha+\alpha)}(1 - e^{-x^{2\alpha}\Delta t})dx.$$

By Lemma 6.2.2, we obtain

$$II_1^2 \leq C\Delta t^{1+\frac{\bar{\alpha}}{\alpha}-\frac{1}{2\alpha}}. \tag{6.2.5}$$

Similarly we may show, with $0 \leq \bar{\alpha} < \frac{1}{2}$,

$$II_3 \leq C\Delta t^{1+\frac{\bar{\alpha}}{\alpha}-\frac{1}{2\alpha}}. \tag{6.2.6}$$

Together these estimates complete the proof of Theorem 6.2.1. $\qquad\square$

Theorem 6.2.3. Let $\hat{u}$ be the solution of (6.1.12)-(6.1.14). Assume that the Assumptions (6.1.15)-(6.1.17) hold . Further assume that $u_0 \in \mathcal{D}(A^\alpha)$, $\frac{1}{2} < \alpha \leq 1$ and $\mathbb{E}||A^\alpha u_0||^2 < \infty$. Then we have

$$\mathbb{E}\int_{t_j}^{t_{j+1}} \int_0^1 \left|\frac{\partial \hat{u}(t,x)}{\partial t}\right|^2 dxdt \leq C\Big(\Delta t\mathbb{E}||A^\alpha u_0||^2 + \Delta t\sum_{k=1}^\infty \lambda_k^\alpha(\beta_k^M)^2 + \sum_{k=1}^\infty (\beta_k^M)^2\Big), \tag{6.2.7}$$

and

$$\mathbb{E}\int_{t_j}^{t_{j+1}} \int_0^1 |A^\alpha\hat{u}(t,x)|^2 dxdt \leq C\Big(\Delta t\mathbb{E}||A^\alpha u_0||^2 + \Delta t\sum_{k=1}^\infty \lambda_k^\alpha(\beta_k^M)^2\Big). \tag{6.2.8}$$

Proof. Assume that, with $0 < t \leq t_{j+1}$,

$$\hat{u}(t,x) = \sum_{k=1}^\infty \hat{u}_k(t)e_k(x), \tag{6.2.9}$$

and with $\hat{u}_k(0) = (u_0, e_k), k = 1, 2, \cdots$,

$$\hat{u}(0,x) = u_0(x) = \sum_{k=1}^\infty \hat{u}_k(0)e_k(x).$$

Substituting (6.2.9) into (6.1.12) we get, with $0 < t \leq t_{j+1}$,

$$\frac{d\hat{u}_k(t)}{dt} + \lambda_k^\alpha\hat{u}_k(t) = \sigma_k^M(t)\Big(\sum_{l=1}^{j+1} \frac{1}{\sqrt{\Delta t}}\eta_{k,l}\chi_l(t)\Big), \tag{6.2.10}$$

which implies that, with $0 < t \leq t_{j+1}$,

$$\hat{u}_k(t) = e^{-\lambda_k^\alpha t}\hat{u}_k(0) + \int_0^t e^{-\lambda_k^\alpha(t-s)}\sigma_k^M(s)\Big(\sum_{l=1}^{j+1} \frac{1}{\sqrt{\Delta t}}\eta_{k,l}\chi_l(s)\Big)ds. \tag{6.2.11}$$

Let us first show (6.2.7). Note that $\{e_k\}$ is an orthonormal basis in $H = L^2(0,1)$, we have, by (6.2.10),

$$\mathbb{E}\int_{t_j}^{t_{j+1}}\int_0^1 \left|\frac{\partial \hat{u}(t,x)}{\partial t}\right|^2 dxdt = \mathbb{E}\sum_{k=1}^\infty \int_{t_j}^{t_{j+1}}\left|\frac{d\hat{u}_k(t)}{dt}\right|^2 dt$$

$$\leq 2\mathbb{E}\sum_{k=1}^\infty \left(\int_{t_j}^{t_{j+1}}|\lambda_k^\alpha \hat{u}_k(t)|^2 dt + \int_{t_j}^{t_{j+1}}\left|\frac{\sigma_k^M(t)}{\sqrt{\Delta t}}\sum_{l=1}^{j+1}\eta_{k,l}\chi_l(t)\right|^2 dt\right)$$

$$= 2\mathbb{E}\sum_{k=1}^\infty \lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}|\hat{u}_k(t)|^2 dt + 2\mathbb{E}\sum_{k=1}^\infty \int_{t_j}^{t_{j+1}}\left|\frac{\sigma_k^M(t)}{\sqrt{\Delta t}}\eta_{k,j+1}\chi_{j+1}(t)\right|^2 dt$$

$$= 2(I + II).$$

For I, we have, by (6.2.11), with $t_l^\star = t_l, 1 \leq l \leq j$ and $t_l^\star = t, l = j+1$,

$$I \leq 2\mathbb{E}\sum_{k=1}^\infty \lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}\left|e^{-\lambda_k^\alpha t}\hat{u}_k(0)\right|^2 dt$$

$$+ 2\mathbb{E}\sum_{k=1}^\infty \lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}\left|\sum_{l=1}^{j+1}\frac{\eta_{k,l}}{\sqrt{\Delta t}}\int_{t_{l-1}}^{t_l^\star}e^{-\lambda_k^\alpha(t-s)}\sigma_k^M(s)ds\right|^2 dt$$

$$= 2\mathbb{E}\sum_{k=1}^\infty \int_{t_j}^{t_{j+1}}e^{-2\lambda_k^\alpha t}(A^\alpha u_0, e_k)^2 dt$$

$$+ 2\sum_{k=1}^\infty \lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}\sum_{l=1}^{j+1}\frac{1}{\Delta t}\left(\int_{t_{l-1}}^{t_l^\star}e^{-\lambda_k^\alpha(t-s)}\sigma_k^M(s)ds\right)^2 dt$$

$$\leq 2\mathbb{E}\sum_{k=1}^\infty (A^\alpha u_0, e_k)^2 \Delta t + 2\sum_{k=1}^\infty \lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}$$

$$\sum_{l=1}^{j+1}\frac{1}{\Delta t}\left(\int_{t_{l-1}}^{t_l^\star}e^{-2\lambda_k^\alpha(t-s)}(\sigma_k^M(s))^2 ds\right)\left(\int_{t_{l-1}}^{t_l^\star}1^2 ds\right)dt$$

$$\leq 2\mathbb{E}\sum_{k=1}^\infty (A^\alpha u_0, e_k)^2 \Delta t + 2\sum_{k=1}^\infty \lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}\left(\int_0^t e^{-2\lambda_k^\alpha(t-s)}\sigma_k^M(s))^2 ds\right)dt$$

$$\leq 2\mathbb{E}\sum_{k=1}^\infty (A^\alpha u_0, e_k)^2 \Delta t + 2\sum_{k=1}^\infty \lambda_k^{2\alpha}(\beta_k^M)^2 \int_{t_j}^{t_{j+1}}\frac{1 - e^{-2\lambda_k^\alpha t}}{2\lambda_k^\alpha}dt$$

$$\leq 2\mathbb{E}||A^\alpha u_0||^2 \Delta t + \Delta t\sum_{k=1}^\infty \lambda_k^\alpha(\beta_k^M)^2,$$

where in the last inequality, we use the inequality $1 - e^{-2\lambda_k^\alpha t} \leq 1$.

For II, we have

$$II = \mathbb{E}\sum_{k=1}^\infty \int_{t_j}^{t_{j+1}}\left|\frac{\sigma_k^M(t)}{\sqrt{\Delta t}}\eta_{k,j+1}\chi_{j+1}(t)\right|^2 dt = \sum_{k=1}^\infty \int_{t_j}^{t_{j+1}}\left(\frac{\sigma_k^M(t)}{\sqrt{\Delta t}}\right)^2 dt \leq \sum_{k=1}^\infty (\beta_k^M)^2.$$

Combining I with II, we get (6.2.7).

Similarly we have

$$\mathbb{E}\int_{t_j}^{t_{j+1}}\int_0^1 |A^\alpha \hat{u}(t)|^2 dxdt = \mathbb{E}\int_{t_j}^{t_{j+1}} ||A^\alpha \hat{u}(t,x)||^2 dt$$

$$= \mathbb{E}\int_{t_j}^{t_{j+1}} \Big(\sum_{k=1}^\infty \lambda_k^{2\alpha}\hat{u}_k^2(t)\Big)dt = \mathbb{E}\sum_{k=1}^\infty \lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}} |\hat{u}_k(t))|^2\, dt = I,$$

which implies (6.2.8) also holds.

Together these estimates complete the proof of the Theorem 6.2.3. $\qquad\qquad\square$

6.3 Fourier Spectral Method

Denote $E_\alpha(t) = e^{-tA^\alpha}, \frac{1}{2} < \alpha \le 1$, where A^α is defined by (6.1.3). The mild solution of (6.1.12)-(6.1.14) has the form of, with $\hat{g}(t) = \frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$,

$$\hat{u}(t) = E_\alpha(t)\hat{u}(0) + \int_0^t E_\alpha(t-s)\hat{g}(s)ds, \quad \hat{u}(0) = u_0. \qquad (6.3.1)$$

Similarly the solution of (6.1.21)-(6.1.23) has the form of

$$\hat{u}_J(t) = E_\alpha(t)P_J\hat{u}(0) + \int_0^t E_\alpha(t-s)P_J\hat{g}(s)ds, \quad \hat{u}(0) = u_0. \qquad (6.3.2)$$

Theorem 6.3.1. Assume that $\hat{u}$ and $\hat{u}_J$ are the solutions of (6.1.12)-(6.1.14) and (6.1.21)-(6.1.23), respectively. Let $0 \le r < \frac{1}{2}$ and let $u_0 \in H$. Then we have, with $\frac{1}{2} < \alpha \le 1$,

$$||A^{\frac{r}{2}}(\hat{u}(t) - \hat{u}_J(t))||^2 \le C||A^{\frac{r}{2}}(u_0 - P_J u_0)||^2 + C\frac{1}{(J+1)^{2\alpha(1-\frac{r}{\alpha})}}\int_0^t ||\hat{g}(s)||^2 ds.$$

$$(6.3.3)$$

In particular, with $r = 0$,

$$||(\hat{u}(t) - \hat{u}_J(t))||^2 \le C||u_0 - P_J u_0||^2 + C\frac{1}{(J+1)^{2\alpha}}\int_0^t ||\hat{g}(s)||^2 ds. \qquad (6.3.4)$$

Proof. Subtracting (6.3.2) from (6.3.1), we get

$$\hat{u}(t) - \hat{u}_J(t) = E_\alpha(t)(u_0 - P_J u_0) + \int_0^t E_\alpha(t-s)(\hat{g}(s) - P_J\hat{g}(s))ds = I + II.$$

$$(6.3.5)$$

For I, we have, with $0 \le r < \frac{1}{2}$,

$$\|A^{\frac{r}{2}} I\| = \|A^{\frac{r}{2}} E_\alpha(t)(u_0 - P_J u_0)\|$$
$$= \Big(\sum_{j=J+1}^{\infty} e^{-2t\lambda_j^\alpha} \lambda_j^r (u_0, e_j)^2 \Big)^{\frac{1}{2}} \le e^{-t\lambda_{j+1}^\alpha} \|A^{\frac{r}{2}}(u_0 - P_J u_0)\|.$$

For II we have, for some $\gamma \in (0,1)$,

$$\|A^{\frac{r}{2}} II\| = \| \int_0^t A^{\frac{r}{2}} E_\alpha(t-s)(I - P_J) \hat{g}(s) ds\|$$
$$= \| \int_0^t \Big[A^{\frac{r}{2}} E_\alpha(1-\gamma)(t-s) \Big] \Big[E_\alpha(\gamma(t-s))(I - P_J) \Big] \hat{g}(s) ds\|$$
$$\le C \int_0^t (t-s)^{\frac{-r}{2\alpha}} e^{-k_\alpha(t-s)} \|\hat{g}(s)\| ds,$$

where $k_\alpha = \delta(1-\gamma) + \lambda_{J+1}^\alpha \gamma$.

By Cauchy-Schwarz inequality, we have

$$\|A^{\frac{r}{2}} II\| \le C \Big(\int_0^t ((t-s)^{\frac{-r}{2\alpha}} e^{-k_\alpha(t-s)})^2 ds \Big)^{\frac{1}{2}} \Big(\int_0^t \|\hat{g}(s)\|^2 ds \Big)^{\frac{1}{2}}.$$

Note that $r < \alpha$, we obtain, with $\lambda_{J+1} = (J+1)^2 \pi^2$,

$$\int_0^t \frac{e^{-2k_\alpha s}}{s^{\frac{r}{s}}} ds \le \int_0^\infty \frac{e^{-2k_\alpha s}}{s^{\frac{r}{s}}} ds \le \frac{\int_0^\infty s^{\frac{-r}{\alpha}} e^{-2s} ds}{k_\alpha^{1-\frac{r}{\alpha}}} \le C \frac{1}{k_\alpha^{1-\frac{r}{\alpha}}}$$
$$\le C \frac{1}{(\lambda_{J+1}^\alpha)^{1-\frac{r}{\alpha}}} \le C \frac{1}{(J+1)^{2\alpha(1-\frac{r}{\alpha})}}.$$

Hence we have

$$\|A^{\frac{r}{2}} II\| \le C \frac{1}{(J+1)^{2\alpha(1-\frac{r}{\alpha})}} \Big(\int_0^t \|\hat{g}(s)\|^2 ds \Big)^{\frac{1}{2}}.$$

Together these estimates complete the proof of Theorem 6.3.1. $\qquad \square$

Combining Theorem 6.2.1 with Theorem 6.3.1, we obtain the following Theorem.

Theorem 6.3.2. Let u and $\hat{u}_J$ be the solutions of (6.1.5)-(6.1.7) and (6.1.21)-(6.1.23) respectively. Assume that the Assumptions (6.1.15)-(6.1.17) hold. Further assume that $u_0 \in \mathcal{D}(A^\alpha), \frac{1}{2} < \alpha \le 1$ and $\mathbb{E}\|A^\alpha u_0\|^2 < \infty$. Then we have

$$\mathbb{E}\int_0^T\int_0^1 (u(t,x)-\hat{u}_J(t,x))^2 dxdt$$

$$\leq C\Big(\sum_{k=1}^\infty \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k=1}^\infty \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M\Big)^2 + \Delta t^{1+\frac{\bar{\alpha}}{\alpha}-\frac{1}{2\alpha}}\Big)$$

$$+ C\mathbb{E}||u_0 - P_J u_0||^2 + C\frac{1}{(J+1)^{2\alpha}}\Big(\Delta t\mathbb{E}||A^\alpha u_0||^2 + \Delta t\sum_{k=1}^\infty \lambda_k^\alpha (\beta_k^M)^2 + \sum_{k=1}^\infty (\beta_k^M)^2\Big).$$

Proof. Note that

$$\mathbb{E}\int_0^T\int_0^1 (u(t,x)-\hat{u}_J(t,x))^2 dxdt$$

$$\leq 2\mathbb{E}\int_0^T\int_0^1 (u(t,x)-\hat{u}(t,x))^2 dxdt + 2\mathbb{E}\int_0^T\int_0^1 (\hat{u}(t,x)-\hat{u}_J(t,x))^2 dxdt$$

$$= 2I + 2II.$$

For I, we have, by Theorem 6.2.1,

$$I \leq C\Big(\sum_{k=1}^\infty \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k=1}^\infty (\lambda_k^\alpha \beta_k^M + \gamma_k^M)^2 + \Delta t^{1+\frac{\bar{\alpha}}{\alpha}-\frac{1}{2\alpha}}\Big).$$

For II, we have

$$II = \mathbb{E}\int_0^T ||\hat{u}(t)-\hat{u}_J||^2 dt \leq C\mathbb{E}||u_0-P_J u_0||^2 + C\frac{1}{(J+1)^{2\alpha}}\int_0^T\int_0^t ||\hat{g}(s)||^2 dsdt.$$

Note that $\hat{g}(s) = \frac{d\hat{u}(s)}{ds} + (-\Delta)^\alpha \hat{u}(s)$, we obtain, by Theorem 6.2.3,

$$\mathbb{E}\int_0^T\int_0^t ||\hat{g}(s)||^2 dsdt \leq \mathbb{E}\int_0^T\int_0^t ||\frac{d\hat{u}(s)}{ds} + (-\Delta)^\alpha \hat{u}(s)||^2 dsdt$$

$$\leq C\mathbb{E}\int_0^T\int_0^T\int_0^t \Big(\Big|\frac{\partial \hat{u}(s,x)}{\partial s}\Big|^2 + \Big|(-\Delta)^\alpha \hat{u}(s,x)\Big|\Big)^2 dxdsdt$$

$$\leq C\Big(\Delta t\mathbb{E}||A^\alpha u_0||^2 + \Delta t\sum_{k=1}^\infty \lambda_k^\alpha (\beta_k^M)^2 + \sum_{k=1}^\infty (\beta_k^M)^2\Big).$$

Together these estimates complete the proof of Theorem 6.3.2. $\qquad\square$

6.4 Numerical Simulations

Here, we will consider the numerical simulation of the Fourier spectral methods for solving the following semilinear stochastic space fractional partial differential equation subject to the periodic boundary conditions, with $\frac{1}{2} < \alpha \leq 1$, $0 < x < 1$, $0 < t \leq T$,

$$\frac{\partial u(t,x)}{\partial t} + \epsilon(-\Delta)^\alpha u(t,x) = f(u(t,x)) + \frac{\partial^2 W(t,x)}{\partial t \partial x}, \tag{6.4.1}$$

$$u(t,0) = u(t,1) = 0, \quad \acute{u}_x(t,0) = \acute{u}_x(t,1), \tag{6.4.2}$$

$$u(0,x) = u_0(x), \tag{6.4.3}$$

where $(-\Delta)^\alpha$ is the fractional Laplacian defined by using the eignvalues and eigenfunctions of the Laplacian $-\Delta$ subject to the periodic boundary conditions. Here $f : \mathbb{R} \to \mathbb{R}$ is a smooth function and $\epsilon > 0$ denotes the diffusion coefficient. Here we consider the problems with the periodic boundary conditions because we want to compare our numerical results with the results in [62, Example 10.39]. Where the algorithms of the spectral methods for stochastic semilinear parabolic equation subject to the periodic boundary conditions are given and discussed. One may also consider the algorithms and MATLAB codes for stochastic space fractional partial differential equations with homogeneous boundary conditions following the approaches in, e.g., [42], [43]. Although the Laplacian is singular in (6.4.1)-(6.4.3) due to the periodic boundary conditions, we expect the error to behave as in Theorem 6.3.1, see the comments in [62, Corollary 10.38].

Denotes, $A = -\frac{\partial^2}{\partial x^2}$ with $\mathcal{D}(A) = H_{per}^2(0,1)$, where $\mathcal{D}(A) = H_{per}^2(0,1)$ is defined in the introduction section. Then the eigenvalues and eigenfunctions of A can also be expressed by

$$\lambda_k = (2\pi k)^2, \qquad e_k = e^{i2\pi kx}, \quad k \in \mathbb{Z}.$$

The noise has the form of

$$\frac{\partial^2 W(t,x)}{\partial t \partial x} = \sum_{k \in \mathbb{Z}} \sigma_k(t) \dot{\beta}_k(t) e_k(x), \tag{6.4.4}$$

where $\dot{\beta}_k(t) = \frac{d\beta_k(t)}{dt}, k \in \mathbb{Z}$ are the derivatives of the standard Brownian motions $\beta_k(t), k \in \mathbb{Z}$ and $\sigma_k(t), k \in \mathbb{Z}$ are some appropriate functions of t. Here $k \in \mathbb{Z}$ since we consider the periodic boundary conditions. When $\sigma_k(t) = \bar{\gamma}_k^{\frac{1}{2}}, \bar{\gamma}_k > 0, k \in \mathbb{Z}$, the noise (6.4.4) reduces to

$$\frac{\partial^2 W(t,x)}{\partial t \partial x} = \sum_{k \in \mathbb{Z}} \bar{\gamma}_k^{\frac{1}{2}} \dot{\beta}_k(t) e_k(x). \tag{6.4.5}$$

The approximation noise $\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ is, with some positive integer $M > 0$,

$$\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x} = \sum_{k \in \mathbb{Z} |k| \leq M} \bar{\gamma}_k^{\frac{1}{2}} e_k(x) \sum_{l=1}^{N_t} \frac{\eta_{k,l}}{\Delta t} \chi_l(t). \tag{6.4.6}$$

In our numerical example below, we assume that, [62, Example 10.8],

$$\bar{\gamma}_0 = 0, \qquad \bar{\gamma}_k = |k|^{-(2r_1+1+\bar{\epsilon})}, \quad k \in \mathbb{Z}, k \neq 0, \tag{6.4.7}$$

where $\bar{\epsilon} > 0$ is a small positive number. When $r_1 = -\frac{1}{2}$ we obtain so-called space-time white noise. When $r_1 = 1$ we obtain the smooth noise.

Let $S_J := \text{span}\{e_0, e_1 \cdots, e_{\frac{J}{2}}, e_{-\frac{J}{2}+1} \cdots, e_{-1}\}$. We assume $J \leq M$ where M is determined in (6.4.5). Here the ordering $0, 1, 2, \cdots, \frac{J}{2}, -\frac{J}{2} + 1, \cdots, -1$ is consistent with the ordering of the **MATLAB** functions **fft** and **ifft** [92]. Let $0 = t_0 < t_1 < t_2 < \cdots < t_{N_t} = T$, $N_t \in \mathbb{N}$ be the partition of $[0,T]$ and Δt the time step size with $T = N_t \Delta t$. We use the semi-implicit Euler method to consider the time discretization.

We will consider the convergence rate against the different type of steps. Choose J=64. The reference solution is obtained by using the time step size $\Delta t_{\text{ref}} = T/N_{\text{ref}}$ with $N_{\text{ref}} = 10^4$. Let **kappa** $= [5, 10, 20, 50, 100, 200, 500]$, we will consider the approximate solutions with the different time step sizes $\Delta t_i = \Delta t_{\text{ref}} * \textbf{kappa}(i), i = 1, 2, \cdots, 7$. By Theorem 6.2.1, we have

$$\mathbb{E} \int_0^T \int_0^1 \Big(u(t,x) - \hat{u}(t,x) \Big)^2 dx dt$$
$$\leq C \Big(\sum_{k \in \mathbb{Z}} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k \in \mathbb{Z}} \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M \Big)^2 + \Delta t^{1+\frac{\bar{\alpha}}{\alpha} - \frac{1}{2\alpha}} \Big). \tag{6.4.8}$$

We remark that here we choose $k \in \mathbb{Z}$ since we consider the periodic boundary conditions. In our numerical example, we will choose, with $\bar{\gamma}_k$ given by (6.4.7),

$$\sigma_k(t) = \bar{\gamma}_k^{\frac{1}{2}}, \bar{\gamma}_k > 0, \quad k \in \mathbb{Z},$$

$$\sigma_k^M(t) = \begin{cases} \sigma_k(t) = \bar{\gamma}_k^{\frac{1}{2}}, & |k| \leq M, \\ 0, & |k| > M, \end{cases}$$

which implies that

$$|\sigma_k^M(t)| \leq \beta_k^M, \text{where} \quad \beta_k^M = \bar{\gamma}_k^{\frac{1}{2}}, |k| \leq M,$$

and

$$|\sigma_k(t) - \sigma_k^M(t) \le \alpha_k^M, \text{where} \quad \alpha_k^M, = \bar{\gamma}_k^{\frac{1}{2}}, \quad |k| \ge M.$$

We first observe that for sufficiently large M the convergence order of the L^2 norm of the errors in (6.4.8) is dominated by $O(\Delta t^{\frac{1}{2}(1+\frac{\bar{\alpha}}{\alpha}-\frac{1}{2\alpha})})$. In fact, we will choose $M = J$ where J is sufficiently large. Then the first term of the right side of (6.4.8) satisfies, with $\lambda_k = (2\pi k)^2, k \in \mathbb{Z}$,

$$\sum_{k \in \mathbb{Z}} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} = \sum_{|k|>M} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} \le C\Big(\frac{1}{\lambda_{M+1}^\alpha} + \frac{1}{\lambda_{M+2}^\alpha} + \cdots\Big)$$

$$\le C\Big(\frac{1}{(M+1)^{2\alpha}} + \frac{1}{(M+2)^{2\alpha}} + \cdots\Big)$$

$$= C\Big(\frac{1}{(J+1)^{2\alpha}} + \frac{1}{(J+2)^{2\alpha}} + \cdots\Big).$$

The second term of the right side of the error in (6.4.8) is $O(\Delta t^2)$. Hence for sufficiently large J, the convergence order of the L^2 norm of the errors in (6.4.8) is $O(\Delta t^{\frac{1}{2}(1+\frac{\bar{\alpha}}{\alpha}-\frac{1}{2\alpha})})$. We now consider two cases $r_1 = -\frac{1}{2}$ and $r_1 = 1$ in (6.4.7). For $r_1 = -\frac{1}{2}$ we may choose $\bar{\alpha} = 0$ which implies that the convergence order of the L^2 norm in (6.4.8) is $O(\Delta t^{\frac{1}{2}(1+\frac{\bar{\alpha}}{\alpha}-\frac{1}{2\alpha})}) = O(\Delta t^{\frac{1}{2}(1-\frac{1}{2\alpha})})$. Indeed $\bar{\alpha} = 0$ satisfies (6.1.17) that is,

$$\beta_k^M = \bar{\gamma}_k^{\frac{1}{2}} = |k|^{-\frac{2r_1+1+\bar{\epsilon}}{2}} = |k|^{-\frac{\bar{\epsilon}}{2}} \le |k|^{-\bar{\alpha}}.$$

For $r_1 = 1$ we may choose $\bar{\alpha} = \frac{1}{2} - \bar{\epsilon}$ (since $0 \le \bar{\alpha} < \frac{1}{2}$) with arbitrarily small positive number $\bar{\epsilon}$ which implies that the convergence order of the L^2 norm in (6.4.8) is $O(\Delta t^{\frac{1}{2}(1+\frac{\bar{\alpha}}{\alpha}-\frac{1}{2\alpha})}) = O(\Delta t^{\frac{1}{2}(1-\frac{\bar{\epsilon}}{\alpha})}) \approx O(\Delta t^{\frac{1}{2}})$. Indeed, in this case, $\bar{\alpha} = \frac{1}{2} - \bar{\epsilon}$ satisfies (6.1.17) that is,

$$\beta_k^M = \bar{\gamma}_k^{\frac{1}{2}} = |k|^{-\frac{2r_1+1+\bar{\epsilon}}{2}} = |k|^{-\frac{3+\bar{\epsilon}}{2}} \le |k|^{-\bar{\alpha}}.$$

Thus we have, by Theorem 6.2.1, the following error estimates, with $\frac{1}{2} < \alpha \le 1$ and $r_1 = -\frac{1}{2}$,

$$||\hat{u} - u||_{L^2(\Omega, L^2(0,T), H))} \le C\Delta t^{\frac{1}{2}(1-\frac{1}{2\alpha})}, \tag{6.4.9}$$

and with $\frac{1}{2} < \alpha \le 1$ and $r_1 = 1$

$$||\hat{u} - u||_{L^2(\Omega, L^2(0,T), H))} \le C(\Delta t^{\frac{1}{2}}), \tag{6.4.10}$$
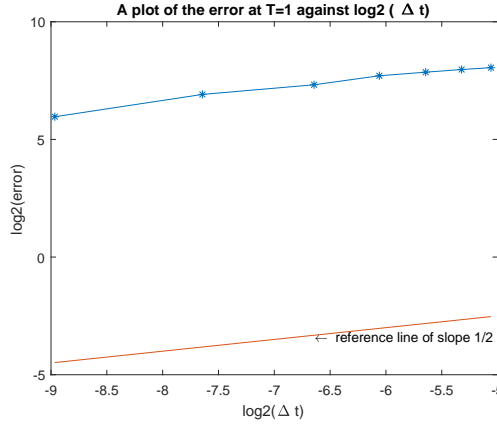
Figure 6.4.1: A plot of the error at $T = 1$ against $\log 2(\Delta t)$ with $\alpha = 0.8, r_1 = -\frac{1}{2}$

where the norm is measured in L^2 both for time and space. In particular, when $\alpha = 1, r_1 = -\frac{1}{2}$, we have

$$\|\hat{u} - u\|_{L^2(\Omega, L^2(0,T),H))} \leq C(\Delta t^{\frac{1}{4}}), \tag{6.4.11}$$

which is consistent with the standard time discretization error for the stochastic heat equation driven by space-time white noise, e.g., [95].

In our numerical experiments below, we choose $f(u) = u - u^3, u_0(x) = \sin(2\pi x)$, and $\epsilon = 1$. See the simulation of this problem for $\alpha = 1$ in [82]. We will consider the error estimates $\|\hat{u}(t_n) - u(t_n)\|_{L^2(\Omega, H)}$ at time t_n. We hope to observe the same convergence order as in (6.4.9) and (6.4.10).

To do this, we consider $\bar{M} = 100$ simulations. For each simulation $\omega_m, m = 1, 2 \cdots, \bar{M}$, we generate J independent Brownian motions $\beta_l, l = 0, 1, 2 \cdots \frac{J}{2}, -\frac{J}{2} + 1 \cdots, -1$ and compute $\hat{u}_J(t_n) \approx \hat{u}(t_n)$ at time $t_n = 1$ by using the different time stepsizes. We then compute the following L^2 norm of the error at $t_n = 1$ for the simulation $\omega_m, m = 1, 2, \cdots, \bar{M}$,

$$\epsilon(\Delta t_i, \omega_m) = \epsilon(\Delta t_i, \omega_m, t_n) = \|\hat{u}_J(t_n, \omega_m) - u_{ref}(t_n, \omega_m)\|^2,$$

where the reference ("true") solution $u_{ref}(t_n, \omega_m)$ is approximated by using the time step $\Delta t_{ref} = T/N_{\mathrm{ref}}$ and $J_{ref} = J$. We then average $\epsilon(\Delta t_i, \omega_m)$ with respect to ω_m to obtain the following approximation of $\|\hat{u}_J(t_n) - u_{ref}(t_n)\|_{L^2(\Omega, H)}$ for the different time step size Δt_i,

$$S(\Delta t_i) = \Big(\frac{1}{\bar{M}} \sum_{m=1}^{\bar{M}} \epsilon(\Delta t_i, \omega_m)\Big)^{\frac{1}{2}} = \Big(\frac{1}{\bar{M}} \sum_{m=1}^{\bar{M}} \|\hat{u}_J(t_n, \omega_m) - u_{ref}(t_n, \omega_m)\|^2\Big)^{\frac{1}{2}}.$$

A plot of the error at T=1 against log2 (Δ t)

Figure 6.4.2: A plot of the error at $T = 1$ against $\log 2(\Delta t)$ with $\alpha = 0.8, r_1 = 1$

For example, in the case $\alpha = 0.8, r_1 = -\frac{1}{2}$, the convergence rate against the time step size is $O(\Delta t^{\frac{1}{2}(1-\frac{1}{2\alpha})}) = O(\Delta t^{\frac{3}{16}})$, i.e., with some positive constant C,

$$S(\Delta t_i) \approx C\Delta t_i^{\frac{3}{16}},$$

which implies that

$$\log(S(\Delta t_i)) \approx \log(C) + \frac{3}{16}\log(\Delta t_i), \qquad i = 1, 2, \cdots 7.$$

In Figure 6.4.1, we consider the case $\alpha = 0.8, r_1 = -\frac{1}{2}$, and plot the points

$(\log((\Delta t_i), \log(S(\Delta t_i)), i = 1, 2, \cdots, 7$ and we observed that the experimentally determined convergence order is higher than the theoretical order in this case. Here the reference line has the slop $\frac{3}{16}$.

In Figure 6.4.2, we consider the case $\alpha = 0.8, r_1 = 1$ and in this case the theoretical convergence order with respect to the time step size is $O(\Delta t^{\frac{1}{2}})$. We plot the points $(\log(\Delta t_i), \log(S(\Delta t_i)), i = 1, 2, \cdots, 7$ and we observe that the experimentally determined convergence order is also higher than the theoretical order in this case. Here the reference line has the slop $\frac{1}{2}$.

Chapter 7

Discontinuous Galerkin Time Stepping Method for Solving Linear Space Fractional Partial Differential Equations

7.1   Introduction

In this chapter, we will consider the discontinuous Galerkin time stepping methods for solving the linear space fractional partial differential equations. The space fractional derivatives are defined by using Riesz fractional derivative. The space variable is discretized by means of a Galerkin finite element method and the time variable is discretized by the discontinuous Galerkin method. The approximate solution will be sought as a piecewise polynomial function in t of degree at most $q - 1, q \geq 1$, which is not necessarily continuous at the nodes of the defining partition. The error estimates in the fully discrete case are obtained and the numerical examples are given.

Now we will consider the discontinuous Galerkin time stepping methods for solving the following linear space fractional partial differential equation, with $\frac{1}{2} < \alpha \leq 1$,

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^{2\alpha} u(t,x)}{\partial |x|^{2\alpha}} = f(t,x), \quad 0 < t < T, \ 0 < x < 1, \tag{7.1.1}$$

$$u(t,0) = u(t,1) = 0, \quad 0 < t < T, \tag{7.1.2}$$

$$u(0,x) = u_0(x), \quad 0 < x < 1, \tag{7.1.3}$$

where the Riesz fractional derivative is defined by, [76], [80]

$$\frac{\partial^{2\alpha} w(x)}{\partial |x|^{2\alpha}} = -\frac{1}{2\cos(\alpha\pi)} \left({}^R_0 D^{2\alpha}_x w(x) + {}^R_x D^{2\alpha}_1 w(x) \right),$$

and ${}^R_0 D^\gamma_x w(x)$ and ${}^R_x D^\gamma_1 w(x)$, $1 < \gamma < 2$ are called the left-sided and right-sided Riemann-Liouville fractional derivatives, respectively,

$$ {}^R_0 D^\gamma_x w(x) = \frac{1}{\Gamma(2-\gamma)} \frac{d^2}{dx^2} \int_0^x (x-y)^{1-\gamma} w(y) dy, \tag{7.1.4}$$

and

$$ {}^R_x D^\gamma_1 w(x) = \frac{1}{\Gamma(2-\gamma)} \frac{d^2}{dx^2} \int_x^1 (y-x)^{1-\gamma} w(y) dy. \tag{7.1.5}$$

Space fractional partial differential equations are widely used to model complex phenomena, for example, in quasi-geostrophic flow, the fast rotating fluids, the dynamic of the frontogenesis in meteorology, the diffusion in fractal or disordered medium, the pollution problems, the mathematical finance and the transport problems, soil contamination and underground water flow, see, e.g.,[9], [11], [20], [72].

In recent years, many authors consider the numerical methods for solving space fractional partial differential equations, e.g., finite difference methods [4], [5], [69], [70] finite element methods [25], [26] and spectral methods [14], [15], matrix transfer technique (MTT) [41]. Recently, Jin et al. [45] considered the finite element method for solving the linear space fractional parabolic equation where the space fractional derivative is defined as left-handed Riemann-Liouville derivative, see also [44]. The estimates in [45] are for both smooth and nonsmooth initial data, and are expressed directly in terms of the smoothness of the initial data.

The Riesz space fractional partial differential equations are firstly proposed by Chaves [18] to investigate the mechanism of super-diffusion. Benson et al. [9], [10] considered the fractional order governing equation of Levy motion. Zhang et al. [103] considered a

finite element method in space and backward difference method in time for solving Riesz space fractional partial differential equation. Sousa [86] studied a second order numerical method for Riesz space fractional convection-diffusion equation. Bu et al. [13] considered a finite element method in space and Crank-Nicolson method in time for solving Riesz space fractional partial differential equations in two-dimensional case. Duan et al. [22] studied a finite element method in space and backward Euler method in time for solving Riesz space fractional partial differential equations in two-dimentional case.

In this chapter, we will consider a finite element method in space and discontinuous Galerkin method in time for solving Riesz space fractional partial differential equation. When the approximating functions are piecewise constant in time, we proved the error is $O(h^{2(r-\alpha)} + k_n)$ and the bound contains the terms $||u||_{r,J_n}$ and $||u_t||_{r,J_n}$, see Theorem 7.3.1 below. When the approximating functions are piecewise linear in time, we proved the error is $O(h^{2(r-\alpha)} + k_n^3)$ and the bound contain the terms $||u||_{r,J_n}$ and $||u_{tt}||_{r,J_n}$, see Theorem 7.3.3 below. The advantages of the discontinuous Galerkin method is that, e.g., variable coefficients and nonlinearities present no complication in principle. We obtain precise error estimates for the discontinuous Galerkin method which make it possible to construct the adaptive methods based on the automatic time-step control.

Definition 7.1.1. [31], [57] For any $\sigma > 0$, we define the space ${}^l H_0^\sigma(0,1)$ and ${}^r H_0^\sigma(0,1)$ to be the the closure of $C_0^\infty(0,1)$ with respect to the norms $||v||_{{}^l H_0^\sigma(0,1)}$ and $||v||_{{}^r H_0^\sigma(0,1)}$, respectively, where

$$||v||^2_{{}^l H_0^\sigma(0,1)} := ||v||^2_{L^2(0,1)} + ||{}^R_0 D_x^\sigma v||^2_{L^2(0,1)}, \tag{7.1.6}$$

and

$$||v||^2_{{}^r H_0^\sigma(0,1)} := ||v||^2_{L^2(0,1)} + ||{}^R_x D_1^\sigma v||^2_{L^2(0,1)}. \tag{7.1.7}$$

The semi-norms are defined by $|v|_{{}^l H_0^\sigma(0,1)} := ||{}^R_0 D_x^\sigma v||_{L^2(0,1)}$ and $|v|_{{}^r H_0^\sigma(0,1)} := ||{}^R_x D_1^\sigma v||_{L^2(0,1)}$, respectively.

Remark 3. In Definition 7.1.1, $|v|_{{}^l H_0^\sigma(0,1)}, \sigma > 0$ is a semi-norm (not a norm) since $|v|_{{}^l H_0^\sigma(0,1)} = 0$ does not imply $v = 0$. For example, when $0 < \sigma < 1$, let $w(x) = x^{\sigma-1}$, we

have $w(x) \neq 0$ and

$$
\begin{aligned}
{}^R_0 D^\sigma_x w(x) &= \frac{1}{\Gamma(1-\sigma)} \frac{d}{dx} \int_0^x (x-y)^{-\sigma} w(y) dy = \frac{1}{\Gamma(1-\sigma)} \frac{d}{dx} \int_0^x (x-y)^{-\sigma} y^{\sigma-1} dy, \\
&= \frac{1}{\Gamma(1-\sigma)} \frac{d}{dx} \int_0^1 t^{-\alpha} (1-t)^{\sigma-1} dt = 0,
\end{aligned}
$$

which implies that $|w|_{{}^l H^\sigma_0(0,1)} = ||{}^R_0 D^\sigma_x w||_{L^2(0,1)} = 0$. The similar comments are for the semi-norm $|v|_{{}^r H^\sigma_0(0,1)}$ and the semi-norm in Definitions 7.1.2 below.

Definition 7.1.2. [31], [57] For any $\sigma > 0, \sigma \neq n - \frac{1}{2}, n \in \mathbb{Z}^+$, we define the space ${}^c H^\sigma_0(0,1)$ to be the closure of $C^\infty_0(0,1)$ with respect to the norm $||v||_{{}^c H^\sigma_0(0,1)}$, where

$$
||v||^2_{{}^c H^\sigma_0(0,1)} := ||v||^2_{L^2(0,1)} + \left| \left({}^R_0 D^\sigma_x v, {}^R_x D^\sigma_1 v \right) \right|.
$$

The semi-norm is defined by $|v|^2_{{}^c H^\sigma_0(0,1)} := |({}^R_0 D^\sigma_x v, {}^R_x D^\sigma_1 v)|$.

Definition 7.1.3. [31], [57] For any $\sigma > 0$, let $H^\sigma(\mathbb{R})$ denote the fractional Sobolev space defined in the whole line $\mathbb{R}$. We define

$$
H^\sigma(0,1) = \{v \in L^2(0,1) : \tilde{v}|_{(0,1)} = v, \text{ where } \quad \tilde{v} \in H^\sigma(\mathbb{R})\},
$$

with the norm

$$
||v||_{H^\sigma(0,1)} = \inf_{\tilde{v} \in H^\sigma(\mathbb{R}), \tilde{v}|_{(0,1)} = v} ||\tilde{v}||_{H^\sigma(\mathbb{R})},
$$

where

$$
||\tilde{v}||_{H^\sigma(\mathbb{R})} = ||(1 + |w|^2)^{\frac{\sigma}{2}} \mathcal{F}(\tilde{v})(w)||_{L^2(\mathbb{R})}, \tag{7.1.8}
$$

and $\mathcal{F}(\tilde{v})$ denotes the Fourier transform of $\tilde{v}$ and the corresponding semi-norm is defined by $|\tilde{v}|_{H^\sigma(\mathbb{R})} = |||w|^\sigma \mathcal{F}(\tilde{v})||_{L^2(\mathbb{R})}$. Further we define the Sobolev space $H^\sigma_0(0,1)$ to be closure of $C^\infty_0(0,1)$ with respect to the norm $||v||_{H^\sigma(0,1)}$ and the semi-norm in $H^\sigma_0(0,1)$ is denoted by $|v|_{H^\sigma_0(0,1)}$.

Lemma 7.1.1. [31, Theorem 2.12, 2.13], [57, Lemma 2.4, 2.5] Let $\sigma > 0$, $\sigma \neq n - \frac{1}{2}$, $n \in \mathbb{Z}^+$. The semi-norms and norms in spaces ${}^l H^\sigma_0(0,1)$, ${}^r H^\sigma_0(0,1)$, ${}^c H^\sigma_0(0,1)$ and $H^\sigma_0(0,1)$ are equivalent.

Lemma 7.1.2. Let $\sigma > 0$, $\sigma \neq n - \frac{1}{2}$, $n \in \mathbb{Z}^+$, we have

$$({}_0^R D_x^\sigma v, \, {}_x^R D_1^\sigma v) = \cos(\pi\sigma) \|{}_0^R D_x^\sigma v\|^2, \quad \forall v \in H_0^\sigma(0, 1).$$

In particular, $({}_0^R D_x^\sigma v, \, {}_x^R D_1^\sigma v)$ is negative when $\frac{1}{2} < \sigma \leq 1$.

Proof. It is sufficient to prove

$$({}_0^R D_x^\sigma \varphi, \, {}_x^R D_1^\sigma \varphi) = \cos(\pi\sigma) \|{}_0^R D_x^\sigma \varphi\|^2, \quad \forall \varphi \in C_0^\infty(0, 1).$$

In fact, we have, for any $\varphi \in C_0^\infty(0, 1)$, [57],

$$({}_0^R D_x^\sigma \varphi, \, {}_x^R D_1^\sigma \varphi) = ({}_{-\infty}^R D_x^\sigma \tilde{\varphi}, \, {}_x^R D_\infty^\sigma \tilde{\varphi})_{L^2(\mathbb{R})} = \cos(\pi\sigma) \|{}_{-\infty}^R D_x^\sigma \tilde{\varphi}\|_{L^2(\mathbb{R})}^2 = \cos(\pi\sigma) \|{}_0^R D_x^\sigma \varphi\|^2,$$

where $\tilde{\varphi}$ is the extension of φ by zero outside of $(0, 1)$. $\qquad \square$

Lemma 7.1.3. Let $\frac{1}{2} < \alpha \leq 1$. We have, see [57],

$$({}_0^R D_x^{2\alpha} w, v) = ({}_0^R D_x^\alpha w, \, {}_x^R D_1^\alpha v), \quad \forall w, v \in H_0^\alpha(0, 1),$$

$$({}_x^R D_1^{2\alpha} w, v) = ({}_x^R D_1^\alpha w, \, {}_0^R D_x^\alpha v), \quad \forall w, v \in H_0^\alpha(0, 1).$$

We also have the following fractional Poincáre inequality:

Lemma 7.1.4. [31], [32], [57] Let $u \in H_0^\alpha(0, 1)$, $\frac{1}{2} < \alpha \leq 1$. We have

$$\|u\|_{L^2(0,1)} \leq C|u|_{H_0^\alpha(0,1)},$$

and for $0 < s < \mu$, $s \neq n - \frac{1}{2}$, $n \in \mathbb{Z}^+$,

$$|u|_{H_0^s(0,1)} \leq C|u|_{H_0^\mu(0,1)}.$$

Multiplying $v \in H_0^\alpha(0, 1)$ in both sides of the equation (7.1.1) and integrating on $(0, 1)$ we get, by Lemma 7.1.3,

$$(u_t, v) + B_\alpha(u, v) = (f, v), \quad \forall v \in H_0^\alpha(0, 1), \tag{7.1.9}$$

$$u(0) = u_0, \tag{7.1.10}$$

where the bilinear form $B_\alpha(., .)$ is defined by

$$B_\alpha(u, v) = \frac{1}{2\cos(\alpha\pi)} \left(({}_0^R D_x^\alpha u, \, {}_x^R D_1^\alpha v) + ({}_x^R D_1^\alpha u, \, {}_0^R D_x^\alpha v) \right). \tag{7.1.11}$$

By Lemmas 7.1.1, 7.1.2, and 7.1.4, it is easy to show that the bilinear form $B_\alpha(.,.)$ is symmetric, continuous and coercive on $H_0^\alpha(0,1)$, $\frac{1}{2} < \alpha \leq 1$.

Let $S_h \subset H_0^\alpha(0,1)$, $\frac{1}{2} < \alpha \leq 1$ be the piecewise continuous linear finite element space. The finite element method of (7.1.1)-(7.3.12) is to find $u_h(t) \in S_h$ such that

$$(u_{h,t}, \chi) + B_\alpha(u_h, \chi) = (f, \chi), \quad \forall \chi \in S_h, \tag{7.1.12}$$

$$u_h(0) = v_h, \tag{7.1.13}$$

where $v_h \in S_h$ is some appropriate approximation of $u_0 \in L^2(0,1)$.

7.2 The Backward Euler Method

In this section, we will consider the error estimates of the backward Euler method for solving (7.1.9)-(7.1.10). Let us first consider the error estimates for solving (7.1.9)-(7.1.10) in the semidiscrete case.

To do this, we need to introduce the regularity assumption for the following fractional elliptic problem, with $\frac{1}{2} < \alpha \leq 1$, $g \in L^2(0,1)$.

$$-\frac{\partial^{2\alpha} w(x)}{\partial|x|^{2\alpha}} = \frac{1}{2\cos(\alpha\pi)}\left({}_0^R D_x^{2\alpha} w(x) + {}_x^R D_1^{2\alpha} w(x) \right) = g(x), \quad 0 < x < 1, \tag{7.2.1}$$

$$w(0) = w(1) = 0. \tag{7.2.2}$$

The variational form of (7.2.1)-(7.2.2) is to find $w \in H_0^\alpha(0,1)$ such that

$$B_\alpha(\omega, \varphi) = (g, \varphi), \quad \forall \varphi \in H_0^\alpha(0,1). \tag{7.2.3}$$

Assumption 7.2.1. Let $\frac{1}{2} < \alpha \leq 1$. For ω solving (7.2.3) with $g \in L^2(0,1)$, there exists some $r \in [\alpha, 2\alpha]$, such that

$$||\omega||_{H_0^r(0,1)} \leq C||g||_{L^2(0,1)}.$$

Remark 4. Suppose that the equation (7.2.1) only contains the left-sided Riemann-Liouville derivative, Jin et al. [45, Lemma 4.2] and [44, Theorem 4.4] shows that $r = 2\alpha - 1 + \beta$, $0 \leq \beta < \frac{1}{2}$ for $\frac{1}{2} < \alpha \leq 1$ in the Assumption 7.2.1. For the equation (7.2.1) with the Riesz fractional derivative, we have at least $\omega \in H_0^\alpha(0,1)$. Further we assume that, by the Assumption 7.2.1, there exists $r \in [\alpha, 2\alpha]$ such that $\omega \in H^r(0,1) \cap H_0^\alpha(0,1)$. This similar assumption was also used in [31, Assumption 6.3.1].

We next introduce the fractional Ritz projection $R_{h,\alpha}$ on S_h.

Definition 7.2.1. Let $\frac{1}{2} < \alpha \le 1$ and let $v \in H_0^\alpha(0,1)$. We define $R_{h,\alpha} : H_0^\alpha(0,1) \to S_h$ by

$$B_\alpha(R_{h,\alpha}v, \chi) = B_\alpha(v, \chi), \quad \forall \chi \in S_h, \quad v \in H_0^\alpha(0,1). \tag{7.2.4}$$

It is easy to see that $R_{h,\alpha} : H_0^\alpha(0,1) \to S_h$ is well defined since $B_\alpha(.,.)$ is symmetric, continuous and coercive on S_h. Further we have, see [31],

Lemma 7.2.2. Let $v \in H^r(0,1) \cap H_0^\alpha(0,1)$, $\frac{1}{2} < \alpha \le 1$, $\alpha \le r \le 2\alpha$ and let $R_{h,\alpha} : H_0^\alpha(0,1) \to S_h$ be the fractional Ritz projection onto S_h defined as in (7.2.4). Then, under Assumption 7.2.1, there exists a constant $C = C(\alpha)$ such that

$$\|R_{h,\alpha}v - v\| + h^{r-\alpha}|R_{h,\alpha}v - v|_{H_0^\alpha(0,1)} \le Ch^{2(r-\alpha)}\|v\|_{H^r(0,1)}. \tag{7.2.5}$$

Theorem 7.2.3. Let u_h and u be the solutions of (7.1.12)-(7.1.13) and (7.1.9)-(7.1.10), respectively. Let $\alpha \le r \le 2\alpha$, $\frac{1}{2} < \alpha \le 1$. Let $u_0 \in H^r(0,1)$. Then under the Assumption 7.2.1, there exists a constant $C = C(\alpha)$ such that

$$\|u_h(t) - u(t)\| \le \|v_h - u_0\| + Ch^{2(r-\alpha)}\left(\|u_0\|_{H^r(0,1)} + \int_0^t \|u_t(s)\|_{H^r(0,1)}ds\right). \tag{7.2.6}$$

Proof. We have

$$u_h(t) - u(t) = \theta(t) + \rho(t),$$

where $\theta(t) = u_h(t) - R_{h,\alpha}u(t)$ and $\rho(t) = R_{h,\alpha}u(t) - u(t)$.

By Lemma 7.2.2, we have, with $\frac{1}{2} < \alpha \le 1$,

$$\|\rho(t)\| \le Ch^{2(r-\alpha)}\|u(t)\|_{H^r(0,1)}.$$

Note that

$$u(t) = u(0) + \int_0^t u_t(s)ds,$$

we get

$$\|u_t\|_{H^r(0,1)} \le \|u_0\|_{H^r(0,1)} + \int_0^t \|u_t(s)\|_{H^r(0,1)}ds.$$

Hence we have

$$\|\rho(t)\| \le Ch^{2(r-\alpha)}\left(\|u_0\|_{H^r(0,1)} + \int_0^t \|u_t(s)\|_{H^r(0,1)}ds\right).$$

We next consider the estimates for $\theta(t)$. Note that $\theta(t)$ satisfies

$$
\begin{aligned}
(\theta_t, \chi) + B_\alpha(\theta, \chi) &= (u_{h,t}, \chi) + B_\alpha(u_h, \chi) - (R_{h,\alpha} u_t, \chi) - B_\alpha(u, \chi) \\
&= (f, \chi) - (R_{h,\alpha} u_t, \chi) - B_\alpha(u, \chi) = (u_t - R_{h,\alpha} u_t, \chi) \\
&= (-\rho_t, \chi), \quad \forall \chi \in S_h.
\end{aligned}
$$

Choose $\chi = \theta$, we get

$$
(\theta_t, \theta) + B_\alpha(\theta, \theta) = -(\rho_t, \theta),
$$

which implies, by Lemma 7.1.1,

$$
\frac{1}{2} \frac{d}{dt} ||\theta||^2 + C|\theta|^2_{H_0^\alpha(0,1)} \leq -(\rho_t, \theta) \leq ||\rho_t|| ||\theta||.
$$

Note that $|\theta|^2_{H_0^\alpha(0,1)} > 0$, we get

$$
\frac{1}{2} \frac{d}{dt} ||\theta||^2 \leq -(\rho_t, \theta) \leq ||\rho_t|| ||\theta||,
$$

which implies that

$$
\frac{d}{dt} ||\theta(t)|| \leq ||\rho_t(t)||.
$$

Hence

$$
\begin{aligned}
||\theta(t)|| &\leq ||\theta|(0)|| + \int_0^t ||\rho_t(s)|| ds \leq ||u_h(0) - R_{h,\alpha} u(0)|| + \int_0^t Ch^{2(r-\alpha)} ||u_t(s)||_{H^r(0,1)} ds. \\
&\leq ||u_h(0) - u(0)|| + Ch^{2(r-\alpha)} ||u(0)||_{H^r(0,1)} + \int_0^t Ch^{2(r-\alpha)} ||u_t(s)||_{H^r(0,1)} ds.
\end{aligned}
$$

Together these estimates complete the proof of Theorem 7.2.3. $\qquad\square$

We now introduce the backward Euler method for solving (7.1.9)-(7.1.10). Let $0 = t_0 < t_1 < t_2 < \cdots < T_N = T$ be a partition of $[0, T]$ and k be the time step size. Let $U^n \approx u_h(t_n)$ be the approximation of $u_h(t_n)$. The backward Euler method for solving (7.1.9)-(7.1.10) is to find $U^n \in S_h$, such that

$$
\left(\frac{U^n - U^{n-1}}{k}, \chi \right) + B_\alpha(U^n, \chi) = (f(t_n), \chi), \quad \forall \chi \in S_h, \tag{7.2.7}
$$

$$
U^0 = v_h, \tag{7.2.8}
$$

or

$$(U^n, \chi) + kB_\alpha(U^n, \chi) = (U^{n-1} + kf(t_n), \chi), \qquad (7.2.9)$$

$$U^0 = v_h. \qquad (7.2.10)$$

Theorem 7.2.4. Let U^n and $u(t_n)$ be the solutions of (7.2.7) and (7.1.9) respectively. Let $\alpha \le r \le 2\alpha, \frac{1}{2} < \alpha \le 1$. Assume that $u_0 \in H^r(0,1)$ and

$$||v_h - u_0|| \le Ch^{2(r-\alpha)}||u_0||_{H^r(0,1)}.$$

We have, under the Assumption 7.2.1 with $n = 1, 2, \cdots, N$,

$$||U^n - u(t_n)|| \le Ch^{2(r-\alpha)}\Big(||u_0||_{H^r(0,1)} + \int_0^{t_n} ||u_t||_{H^r(0,1)}ds\Big) + k\int_0^{t_n} ||u_{tt}||ds.$$

Proof. We write

$$U^n - u(t_n) = (U^n - R_{h,\alpha}u(t_n)) + (R_{h,\alpha}u(t_n) - u(t_n)) = \theta^n + \rho^n.$$

Hence $\rho^n = \rho(t_n)$ is bounded by

$$||\rho^n|| = ||R_{h,\alpha}u(t_n) - u(t_n)|| \le Ch^{2(r-\alpha)}||u(t_n)||_{H^r(0,1)} \le Ch^{2(r-\alpha)}||u_0 + \int_0^t u_t(s)ds||_{H^r(0,1)}$$

$$\le Ch^{2(r-\alpha)}\Big(||u_0||_{H^r(0,1)} + \int_0^t ||u_t||_{H^r(0,1)}ds\Big).$$

We next estimate θ^n. Note that θ^n satisfies, by (7.2.7) and (7.1.9),

$$\Big(\frac{\theta^n - \theta^{n-1}}{k}, \chi\Big) + B_\alpha(\theta^n, \Delta\chi) \qquad (7.2.11)$$

$$= \Big(\frac{U^n - U^{n-1}}{k}, \chi\Big) - \Big(R_{h,\alpha}\frac{u(t_n) - u(t_{n-1})}{k}, \chi\Big) + B_\alpha(U^n, \chi) - B_\alpha(R_{h,\alpha}u(t_n), \chi)$$

$$= (u_t(t_n), \chi) - \Big(R_{h,\alpha}\frac{u(t_n) - u(t_{n-1})}{k}, \chi\Big)$$

$$= \Big(u_t(t_n) - \frac{u(t_n) - u(t_{n-1})}{k}, \chi\Big) + \Big(\frac{u(t_n) - u(t_{n-1})}{k} - R_{h,\alpha}\frac{u(t_n) - u(t_{n-1})}{k}, \chi\Big)$$

$$= -(w^n, \chi),$$

where $w^n = w_1^n + w_2^n$,

$$w_1^n = (R_{h,\alpha} - I)\frac{u(t_n) - u(t_{n-1})}{k}, \quad w_2^n = \frac{u(t_n) - u(t_{n-1})}{k} - u_t(t_n).$$

Choose $\chi = \theta^n$ in (7.2.11), we have

$$(\theta^n, \theta^n) - (\theta^{n-1}, \theta^n) + kB_\alpha(\theta^n, \theta^n) = -k(w^n, \theta^n).$$

Note that, by the coercivity property, $B_\alpha(\theta^n, \theta^n) \geq 0$, we have

$$||\theta^n||^2 - (\theta^{n-1}, \theta^n) \leq k||w^n||||\theta^n||,$$

or

$$||\theta^n||^2 \leq ||\theta^{n-1}||||\theta^n|| + k||w^n||||\theta^n||.$$

Cancelling $||\theta^n||$, we get

$$||\theta^n|| \leq ||\theta^{n-1}|| + k||w^n||.$$

By repeated application, we have

$$||\theta^n|| \leq ||\theta(0)|| + k\sum_{j=1}^n ||w^j|| \leq ||\theta(0)|| + k\sum_{j=1}^n ||w_1^j|| + k\sum_{j=1}^n ||w_2^j||.$$

Noting that

$$\begin{aligned} w_1^j &= (R_{h,\alpha} - I)\frac{u(t_j) - u(t_{j-1})}{k}, \\ &= (R_{h,\alpha} - I)k^{-1}\int_{t_{j-1}}^{t_j} u_t(s)ds = k^{-1}\int_{t_{j-1}}^{t_j} (R_{h,\alpha} - I)u_t(s)ds, \end{aligned}$$

we have

$$\begin{aligned} k\sum_{j=1}^n ||w_1^j|| &\leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} ||(R_{h,\alpha} - I)u_t(s)||ds, \\ &\leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} Ch^{2(r-\alpha)}||u_t||_{H^r(0,1)}ds \leq Ch^{2(r-\alpha)}\int_0^{t_n} ||u_t||_{H^r(0,1)}ds. \end{aligned}$$

Further, we have

$$kw_2^j = u(t_j) - u(t_{j-1}) - ku_t(t_j) = -\int_{t_{j-1}}^{t_j} (s - t_{j-1})u_{tt}(s)ds.$$

We therefore obtain

$$k\sum_{j=1}^n ||w_2^j|| \leq \sum_{j=1}^n ||\int_{t_{j-1}}^{t_j} (s - t_{j-1})u_{tt}(s)||ds \leq k\sum_{j=1}^n \int_{t_{j-1}}^{t_j} ||u_{tt}(s)||ds = k\int_0^{t_n} ||u_{tt}(s)||ds.$$

Together these estimates complete the proof of Theorem 7.2.4. $\qquad\square$

7.3 The Discontinuous Galerkin Time Stepping Method

In Section 7.2, we obtain the error estimates for solving (7.1.9)-(7.1.10) by using the finite element method in space and backward Euler method in time. The error is $O(h^{2(r-\alpha)} + k), \alpha \leq r \leq 2\alpha, \frac{1}{2} < \alpha \leq 1$ and the bounds contain the terms $\int_0^{t_n} ||u_t(s)||_{H^r(0,1)} ds$ and $\int_0^{t_n} ||u_{tt}(s)|| ds$. In this section, we will consider the discontinuous Galerkin time stepping method for solving (7.1.9). When the approximating functions are piecewise constant in time, we proved that, in Theorem 7.3.1, the error is $O(h^{(r-\alpha)} + k_n)$ and error bounds contain the terms $||u||_{r,J_n}$ and $||u_t||_{\alpha,J_n}$. When the approximating functions are piecewise linear in time, we prove that the error is $O(h^{2(r-\alpha)} + k_n^3)$ and the bounds contain the terms $||u||_{r,J_n}$ and $||u_{tt}||_{\alpha,J_n}$, see Theorem 7.3.3.

Let $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n < \cdots < t_N = T$ be the time partition of $[0,T]$. Let $k_n = t_n - t_{n-1}, n = 1, 2, 3 \ldots N$ be the time step size. Denote $J_n = (t_{n-1}, t_n]$. Define

$$S_{kh} = \{X; [0,T] \mapsto S_h, X|_{J_n} = \sum_{j=0}^{q-1} X_j t^j, X_j \in S_h\},$$

where q is a given positive integer and $X = X(t) \in S_{kh}$ is left continuous at the discretization point t_n, but not necessarily right continuous at t_{n-1} on each subinterval $J_n = (t_{n-1}, t_n], n = 1, 2, \ldots, N$. Denote $X_-^n = X(t_n-) = \lim_{t \to t_n^-} X(t)$ and $X_+^{n-1} = X(t_{n-1}+) = \lim_{t \to t_{n-1}} + X(t)$. We then have $X_-^n = X(t_n) = X^n$. Further, let S_{kh}^n denote the restriction of S_{kh} on $J_n = (t_{n-1}, t_n]$.

The discontinuous Galerkin time stepping method of (7.1.9)-(7.1.10) is to find $U = U(t) \in S_{kh}^n$ such that, with $n = 1, 2, \ldots, N$, and $\forall X \in S_{kh}^n$,

$$\int_{t_{n-1}}^{t_n} \Big[(U_t, X) + B_\alpha(U, X) \Big] dt + (U_+^{n-1}, X_+^{n-1}) = (U_-^{n-1}, X_+^{n-1}) + \int_{t_{n-1}}^{t_n} (f, X) dt,$$

$$\tag{7.3.1}$$

$$U(t_{n-1}) = U_-^{n-1}, \tag{7.3.2}$$

or

$$\int_{t_0}^{t_N} \Big[(U_t, X) + B_\alpha(U, X) \Big] dt + \sum_{n=1}^{N-1} ([U]_n, X_+^n) + (U_+^0, X_+^0) = (U_-^0, X_+^0) + \int_{t_0}^{t_N} (f, X) dt,$$

$$U(0) = U_-^0 = v_h.$$

Here $[U]_n = U_+^n - U_-^n$ denotes the jump of U at $t_n, n = 1, 2, \cdots N - 1$.

Denote

$$\bar{B}_\alpha(U, X) = \int_{t_0}^{t_N} [(U_t, X) + B_\alpha(U, X)]dt + \sum_{n=1}^{N-1} ([U]_n, X_+^n) + (U_+^0, X_+^0).$$

Then the discontinuous Galerkin time stepping method of (7.1.9)-(7.1.10) is to find $U \in S_{kh}$ such that

$$\bar{B}_\alpha(U, X) = (U_-^0, X_+^0) + \int_{t_0}^{t_N} (f, X)dt, \quad \forall X \in S_{kh}. \tag{7.3.3}$$

We remark that in the case $q = 1$, (7.3.1)-(7.3.2) reduces to the following modified backward Euler method

$$(U^n, \chi) + k_n B_\alpha(U^n, \chi) = (U^{n-1}, \chi) + \left(\int_{t_{n-1}}^{t_n} f(t)dt, \chi \right), \qquad \forall \chi \in S_h. \tag{7.3.4}$$

Note that the $f^n = f(t_n)$ occurring the standard backward Euler method (7.2.9)-(7.2.10) has been replaced by an average of f over (t_{n-1}, t_n). The standard backward Euler method may thus be interpreted as resulting from (7.3.4) after quadrature. We have the following Theorem.

Theorem 7.3.1. Assume that $k_{n+1}/k_n \geq c > 0$ for $n \geq 1$ and let $q = 1$. Let U^n and $u(t_n)$ be the solutions of (7.3.1)-(7.3.2) and (7.1.9)-(7.1.10), respectively. Let $\alpha \leq r \leq 2\alpha, \frac{1}{2} < \alpha \leq 1$. Then we have, under the Assumption 7.2.1 with $v_h = P_h u_0, u_0 \in L^2(0,1)$,

$$\|U^N - u(t_N)\| \leq CL_N \max_{n \leq N}(h^{r-\alpha}\|u\|_{r,J_n} + k_n\|u_t\|_{\alpha,J_n}), \tag{7.3.5}$$

where $L_N = 1 + (\log \frac{t_N}{k_N})^{\frac{1}{2}}$ and $\|\varphi\|_{s,J_n} = \sup_{t \in J_n} \|\varphi(t)\|_{H^s(0,1)}, s = \alpha, r$.

Denote $A_\alpha : \mathcal{D}(A_\alpha) \to L^2(0,1)$ by

$$B_\alpha(\varphi, \psi) = (A_\alpha \varphi, \psi), \qquad \forall \varphi \in \mathcal{D}(A_\alpha), \psi \in H_0^\alpha(0,1). \tag{7.3.6}$$

We may consider the following backward homogeneous problem

$$- z_t + A_\alpha z = 0, \quad \text{for } t < t_N, \tag{7.3.7}$$

$$z(t_N) = \varphi. \tag{7.3.8}$$

We next introduce the discrete fractional elliptic operator $A_{h,\alpha} : S_h \to S_h$ by, with $\frac{1}{2} < \alpha \leq 1$,

$$(A_{h,\alpha}\psi, \chi) = \frac{1}{2cos(\pi\alpha)}\left[\left({}_0^R D_x^\alpha \psi, \; {}_x^R D_1^\alpha \chi \right) + \left({}_x^R D_1^\alpha \psi, \; {}_0^R D_x^\alpha \chi \right)\right], \quad \forall \psi, \chi \in S_h.$$

The semidiscreat problem of (7.3.7)-(7.3.8) is then to find $z_h \in S_h$ such that

$$- z_{h,t} + A_{h,\alpha} z_h = 0, \quad \text{for } t < t_N, \tag{7.3.9}$$

$$z_h(t_N) = P_h \varphi. \tag{7.3.10}$$

The discontinous Galerkin time stepping method for (7.3.9)-(7.3.10) is to find $Z_h \in S_{kh}^n$ such that

$$\int_{t_{n-1}}^{t_n} \left[(X_h, -Z_{h,t} + B_\alpha(X_h, A_{h,\alpha} Z_h)) \right] dt + (X_h(t_n-), Z_h(t_n-)) \tag{7.3.11}$$

$$= (X_h(t_n-), Z_h(t_n+)), \qquad \forall X_h \in S_{kh}^n, \tag{7.3.12}$$

$$Z_h(t_N+) = Z_h(t_N) = P_h \varphi. \tag{7.3.13}$$

Here we use the fact that $B_\alpha(X_h, Z_h) = (A_{h\alpha} X_h, Z_h) = (X_h, A_{h,\alpha} Z_h)$.

We remark that (7.3.12)-(7.3.13) are obtained by transforming (7.3.9)-(7.3.10) into the forward homogeneous problem and then apply the discontinuous Galerkin time stepping method (7.3.1)-(7.3.2) to this forward homogenous problem. In fact, let $t = T - s$, we assume

$$z_h(t) = z_h(T - s) = \bar{z}_h(s)$$

which implies that

$$z_{h,t} = -\bar{z}_{h,s}, \quad z_h(t_N) = \bar{z}_h(0).$$

Here (7.3.9)-(7.3.10) is equivalent to the following forward homogeneous problem

$$\bar{z}_{h,t} + A_{h,\alpha} \bar{z}_h = 0, \quad \text{for } t \leq t_N, \tag{7.3.14}$$

$$\bar{z}_h(0) = P_h \varphi. \tag{7.3.15}$$

The discontinuous Galerkin time stepping method of (7.3.14)-(7.3.15) is to find $\bar{Z}_h \in S_{kh}^n$ such that

$$\int_{t_N - t_n}^{t_N - t_{n-1}} [(\bar{Z}_{h,s}, \bar{X}_h) + (A_{h,\alpha} \bar{Z}_h, \bar{X}_h)] ds + \left(\bar{Z}((t_N - t_n)+), \bar{X}_h((t_N - t_n)+) \right)$$

$$= (\bar{Z}(t_N - t_n)-), \bar{X}_h((t_N - t_n)-) \Big), \qquad \forall \bar{X}_h \in S_{kh}^n,$$

which implies that, with $s = t_N - t$, $\bar{Z}_h(s) = Z_h(t)$, $\bar{Z}_{h,s}(s) = -Z_{h,t}(t)$,

$$\int_{n-1}^{t_n} \left[(X_h, -Z_{h,t}) + (X_h, A_{h,\alpha} Z_h) \right] dt + (X_h(t_N - (t_N - t_n)+), Z_h(t_N - (t_N - t_n)+))$$

$$= (X_h(t_N - (t_N - t_n)-), Z_h(t_N - (t_N - t_n)-), \qquad \forall X_h \in S_{kh}^n,$$

or

$$\int_{t_{n-1}}^{t_n} \left[(X_h, -Z_{h,t}) + (X_h, A_{h,\alpha} Z_h) \right] dt + (X_h(t_n-), Z_h(t_n-))$$

$$= (X_h(t_n-), Z_h(t_n+)), \qquad \forall X_h \in S_{kh}^n,$$

which is (7.3.12)-(7.3.13).

By summation with $n = 1, 2, \ldots, N$, we get

$$\int_{t_0}^{t_N} \left[X_h, -Z_{h,t}) + (X_h, A_{h,\alpha} Z_h) \right] dt - \sum_{n=1}^{N-1} (X_h(t_n-), [Z_h]_n) + (X_h(t_N-), Z_h(t_N-))$$

$$= (X_h(t_N-), Z_h(t_N+)) = (X_h(t_N-), P_h \varphi), \quad \forall X_h \in S_{kh}.$$

It is easy to show that, by integration by parts with respect to t,

$$\bar{B}_\alpha(X_h, Z_h) = \int_{t_0}^{t_N} \left[(X_h, -Z_{h,t}) + (X_h, A_{h,\alpha} Z_h) \right] dt \tag{7.3.16}$$

$$- \sum_{n=1}^{N-1} (X_h(t_n-), [Z_h]_n) + (X_h(t_N-), Z_h(t_N-)).$$

Hence we see that the solution $Z_h \in S_{kh}$ of (7.3.12)-(7.3.13) satisfies

$$\bar{B}_\alpha(X_h, Z_h) = (X_h(t_N-), P_h \varphi) = (X_h(t_N-), \varphi), \quad \forall X_h \in S_{kh}. \tag{7.3.17}$$

Lemma 7.3.2. Assume that $k_{n+1}/k_n \geq c \geq 0, n \geq 1$. Then we have, for the solution of (7.3.17),

$$\int_0^{t_N} \left(||Z_{h,t}|| + ||A_{h,\alpha} Z_h|| \right) dt + \sum_{n=1}^{N} ||[Z_h]_n|| \leq C L_N ||\varphi||, \tag{7.3.18}$$

where $L_N = 1 + (\log \frac{t_N}{k_N})^{\frac{1}{2}}$.

The proof is similar to the proof of [93, Lemma 12.3]. We omit the proof here.

Proof of Theorem 7.3.1. Let $\bar{u}$ denote the piecewise constant function (with respect to t) defined by $\bar{u}(t) = u(t_n)$, for $t \in (t_{n-1}, t_n]$, we write

$$e = U - u = (U - R_{h,\alpha} \bar{u}) + (R_{h,\alpha} \bar{u} - u) = \theta + \rho,$$

where $R_{h,\alpha}$ is defined by (7.2.4). For ρ, we have, noting that $\bar{u}(t_N) = u(t_N)$,

$$||\rho^N|| = ||R_{h,\alpha}\bar{u}(t_N) - u(t_N)|| = ||R_{h,\alpha}u(t_N) - u(t_N)|| \leq Ch^{2(r-\alpha)}||u||_{H^r(0,1)}.$$

For θ, we have, with $\varphi \in L^2(0,1)$, by (7.3.17),

$$\bar{B}_\alpha(\theta, Z_h) = (\theta^N, \varphi).$$

Thus we have

$$(\theta^N, \varphi) = \bar{B}_\alpha(\theta, Z_h) = \bar{B}_\alpha(e - \rho, Z_h) = \bar{B}_\alpha(e, Z_h) - \bar{B}_\alpha(\rho, Z_h).$$

Note that

$$\bar{B}_\alpha(e, X_h) = \bar{B}_\alpha(U - u, X_h) = 0, \qquad \forall X_h \in S_{kh}.$$

In fact, we have, by (7.3.3),

$$\bar{B}_\alpha(U, X_h) = (U^0_-, X_h(0+)) + \int_{t_0}^{t_N}(f, X_h)dt, \qquad \forall X_h \in S_{kh}.$$

Further we have

$$\bar{B}_\alpha(u, X_h) = \int_{t_0}^{t_N}\left[(u_t, X_h) + (A_{\alpha,u}, X_h)\right]dt + \sum_{n=1}^{N-1}([u]_n, X_h(t_n+)) + (u^0_-, X_h(0+))$$
$$= \int_{t_0}^{t_N}(f, X_h)dt + (u^0_-, X_h(0+)).$$

Thus we obtain

$$\bar{B}_\alpha(u, X_h) = (U^0_- - u^0_-, X_h(0+)) = (P_h u_0 - u_0, X_h(t_0+)) = 0.$$

Hence we have

$$(\theta^N, \varphi) = -\bar{B}_\alpha(\rho, Z_h) = -\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\left[(\rho, -Z_{h,t}) + B_\alpha(\rho, Z_h)\right] + \sum_{n=1}^{N-1}(\rho^n[Z_h]_n) - (\rho^N, P_h\varphi).$$

$$(7.3.19)$$

Noting that

$$B_\alpha(\rho, Z_h) = B_\alpha(R_{h,\alpha}\rho, Z_h) = (R_{h,\alpha}A_{h,\alpha}\rho, Z_h),$$

and $\rho^n = 0, n = 1, 2, \cdots N$, we get

$$(\theta^N, \varphi) = -\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (R_{h,\alpha}\rho, A_{h,\alpha}Z_h)dt + \sum_{n=1}^{N-1}(\rho^n[Z_h]_n) - (\rho^N, P_h\varphi)$$

$$\leq \max_{n\leq N}\left(||\rho||_{J_n} + ||R_{h,\alpha}\rho||_{J_n}\right)\left[\int_0^{t_N}||A_{h,\alpha}Z_h||dt + \sum_{n=1}^{N-1}||[Z_h]_n|| + ||\varphi||\right].$$

By (7.2.5) with $r = \alpha$, we have

$$||R_{h,\alpha}\rho||_{J_n} \leq ||R_{h,\alpha}\rho - \rho||_{J_n} + ||\rho||_{J_n} \leq Ch^0||\rho||_{\alpha,J_n} + ||\rho||_{J_n} \leq C||\rho||_{\alpha,J_n}. \quad (7.3.20)$$

We therefore have

$$||\theta^N|| \leq CL_N \max_{n\leq N}||\rho||_{\alpha,J_n}.$$

Note that,

$$||\rho||_{\alpha,J_n} = ||R_{h,\alpha}\bar{u} - u||_{\alpha,J_n} \leq ||(R_{h,\alpha} - I)\bar{u}||_{\alpha,J_n} + ||\bar{u} - u||_{\alpha,J_n}$$

$$= ||(R_{h,\alpha} - I)u(t_n)||_{\alpha,J_n} + ||\bar{u} - u||_{\alpha,J_n} \leq Ch^{r-\alpha}||u(t_n)||_{H^r(0,1)} + Ck_n||u_t||_{\alpha,J_n}$$

$$\leq Ch^{r-\alpha}||u||_{r,J_n} + Ck_n||u_t||_{\alpha,J_n}.$$

Together these estimates complete the proof of Theorem 7.3.1 $\qquad\qquad \square$

Theorem 7.3.3. Let $q = 2$ and assume that $k_{n+1}/k_n \geq c > 0$ for $n \geq 1$. Let U^n and $u(t_n)$ be the solutions of (7.3.1)-(7.3.2) and (7.1.9)-(7.1.10), respectively. Let $\alpha \leq r \leq 2\alpha, \frac{1}{2} < \alpha \leq 1$. Then we have, under the Assumption 7.2.1 with $v_h = P_h u_0, u_0 \in L^2(0,1)$,

$$||U^N - u(t_N)|| \leq CL_N \max_{n\leq N}(h^{2(r-\alpha)}||u||_{r,J_n} + k_n^3||u_{tt}||_{\alpha,J_n}),$$

where $L_N = 1 + (\log \frac{t_N}{k_N})^{\frac{1}{2}}$ and $||\varphi||_{s,J_n} = \sup_{t\in J_n}||\varphi(t)||_{H^s(0,1)}, \quad s = \alpha, r$.

Proof. Let $J_n = (t_{n-1}, t_n], n \geq 1$ and let $\bar{u} \in S_k$ denote the piecewise linear interpolation defined by

$$\bar{u}(t_n) = u(t_n), n \geq 0,$$
$$\int_{J_n}(\bar{u}(t) - u(t))dt = 0, n \geq 1,$$

i.e., $\bar{u}$ interpolates at the nodal points, and the interpolation error is orthogonal to any constant on J_n. This interpolation is uniquely defined and the error estimates read as follows, see [93, (12.10) pp.186],

$$|\bar{u}(t) - u(t)|_{H_0^j(0,1)} \leq Ck_n^3 \int_{J_n} |u_{tt}(s)|_{H_0^j(0,1)} ds, \quad \text{for } t \in J_n, \, j = 0,1. \tag{7.3.21}$$

This time we find instead of (7.3.19),

$$(\theta^N, \varphi) = -\sum_{n=1}^N \int_{J_n} (-(\rho, Z_{h,t}) + B_\alpha(\rho, Z_h))dt + \sum_{n=1}^{N-1}(\rho^n, [Z_h]_n) - (\rho^N, P_h\varphi).$$

Here we have, using the definition of $\bar{u}$,

$$\int_{J_n} (\rho, Z_{h,t})dt = \int_{J_n} \left(R_{h,\alpha}\bar{u} - u, Z_{h,t}\right)dt = \int_{J_n} (R_{h,\alpha}u - u, Z_{h,t})\, dt.$$

By Lemma 7.3.2, we have

$$\left|\sum_{n=1}^N \int_{J_n} (R_{h,\alpha}u - u, Z_{h,t})dt\right| \leq \max_{n \leq N} ||R_{h,\alpha}u - u||_{J_n} \int_0^{t_N} ||Z_{h,t}||dt$$

$$\leq CL_N h^{2(r-\alpha)} \max_{n \leq N} ||u||_{r,J_n} ||\varphi||,$$

and similarly

$$\left|\sum_{n=1}^{N-1}(\rho^n, [Z_h]_n| + |(\rho^N, P_h\varphi)\right| \leq \max_{n \leq N} ||R_{h,\alpha}u - u)(t_n)||\left(\sum_{n=1}^{N-1} ||[Z_h]_n|| + ||P_n\varphi||\right)$$

$$\leq CL_N h^{2(r-\alpha)} \max_{n \leq N} ||u||_{r,J_n} ||\varphi||.$$

Finally, by the definition of $R_{h,\alpha}$,

$$\sum_{n=1}^N \int_{J_n} B_\alpha(\rho, Z_h)dt = \sum_{n=1}^N \int_{J_n} B_\alpha(\bar{u} - u, Z_h)dt$$

$$= -\sum_{n=1}^N \int_{J_n} (A_\alpha(\bar{u} - u), Z_h)dt = \sum_{n=1}^N K_n.$$

By the Assumption 7.2.1 and definition of the interpolant $\bar{u}$, we have

$$|K_n| \leq k_n ||\bar{u} - u||_{r,J_n} \int_{J_n} ||Z_{h,t}||dt.$$

Thus we have

$$\sum_{n=1}^N |K_n| \leq \max_{n \leq N}(k_n ||\bar{u} - u||_{r,J_n}) \sum_{n=1}^N \int_{J_n} ||Z_{h,t}||dt \leq CL_N \max_{n \leq N}(k_n^3 ||u_{tt}||_{r,J_n})||\varphi||.$$

Hence we get the following estimates

$$(\theta^N, \varphi) \leq CL_N \max_{n \leq N}(k_n^3 ||u_{tt}||_{r,J_n} + h^{2(r-\alpha)})||\varphi||.$$

Together these estimates complete the proof of Theorem 7.3.3. □

7.4 Numerical Simulations

In this section, we will consider two numerical examples.

Example 5. *Consider the following linear space fractional partial differential equation, with $\frac{1}{2} < \alpha \le 1$,*

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^{2\alpha} u(t,x)}{\partial |x|^{2\alpha}} = f(t,x), \quad 0 < t < T, \ 0 < x < 1, \tag{7.4.1}$$

$$u(t,0) = u(t,1) = 0, \quad 0 < t < T, \tag{7.4.2}$$

$$u(0,x) = u_0(x), \quad 0 < x < 1, \tag{7.4.3}$$

where $u_0(x) = 0$ and $f(t,x) = 2tx^2(1-x)^2 - t^2(2 - 12x + 12x^2)$. When $\alpha = 1$, the exact solution is $u(t,x) = t^2 x^2 (1-x)^2$.

In the numerical simulation, we use the piecewise constant approximation in time and the linear finite element approximation in space. We consider the experimentally determined orders of convergence ("EOC") for the different α at $t_n = 1$. We choose $k = 0.001$ and the different step size $h_i = \frac{1}{2^i}, i = 1, 2, 3, 4, 5$. Let $e_n^{(i)} = ||u(t_n) - U^n||_{L^2(0,1)}$ denote the L^2 norm at $t_n = 1$ obtained by using the different space step sizes $h_i = \frac{1}{2^i}, i = 1, 2, 3, 4$. Since the exact solution is not available, we will calculate the reference solution (or 'true' solution) $u(t_n)$ by using the very small time step size $k = 0.0001$ and space step size $h = 2^{-10}$. By Theorem 7.3.1, we have, with some $\alpha \le r \le 2\alpha$ and $\frac{1}{2} < \alpha \le 1$,

$$e_n^{(i)} \le C h_i^{r-\alpha}, \tag{7.4.4}$$

which implies that the convergence order $r - \alpha$ satisfies

$$\log_2(\frac{e_n^{(i)}}{e_n^{(i+1)}}) \approx \log_2(\frac{h_i}{h_{i+1}})^{r-\alpha} = r - \alpha.$$

In Table 7.4.1 we observe that the experimentally determined orders of convergence ("EOC") are 2α which is much better that the theoretical convergence order in Theorem 7.3.1.

Example 6. *Consider the following linear space fractional partial differential equation,*

k	h	$EOC(\alpha = 0.6)$	$EOC(\alpha = 0.7)$	$EOC(\alpha = 0.8)$	$EOC(\alpha = 0.9)$
0.001	1/2				
0.001	1/4	2.0132	2.4290	2.4232	2.3176
0.001	1/8	1.3547	1.6634	2.0163	2.1684
0.001	1/16	1.3493	1.3635	1.5023	1.5863

Table 7.4.1: The experimentally determined orders of convergence ("EOC") for the different α at $t_n = 1$ in Example 5

k	h	$EOC(\alpha = 0.6)$	$EOC(\alpha = 0.7)$	$EOC(\alpha = 0.8)$	$EOC(\alpha = 0.9)$
0.001	1/2				
0.001	1/4	1.4233	1.5410	1.5249	1.4240
0.001	1/8	1.0621	1.1559	1.4353	1.6324
0.001	1/16	1.0171	1.1045	1.2011	1.5345

Table 7.4.2: The experimentally determined orders of convergence ("EOC") for the different α at $t_n = 1$ in Example 6

with $\frac{1}{2} < \alpha \leq 1$,

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^{2\alpha} u(t,x)}{\partial |x|^{2\alpha}} = f(t,x) \qquad 0 < t < T, 0 < x < 1, \tag{7.4.5}$$

$$u(t,0) = u(t,1) = 0, \quad 0 < t < T, \tag{7.4.6}$$

$$u(0,x) = u_0(x), \quad 0 < x < 1, \tag{7.4.7}$$

where $u_0(x) = x(1-x)$ and $f(t,x) = 0$.

In Table 7.4.2, we observe that the experimentally determined orders of convergence ("EOC") are also better than our theoretical convergence order $O(h^{r-\alpha})$, $\alpha \leq r \leq 2\alpha$ in Theorem 7.3.1. We will investigate this issue in our future work.

Chapter 8

An Analysis of the Modified L1 Scheme for the Time Fractional Partial Differential Equations with Nonsmooth Data

8.1   Introduction

We consider error estimates for the modified L1 scheme for solving time fractional partial differential equation. Jin et al. (2016, An analysis of the L1 scheme for the subdifffiusion equation with nonsmooth data, IMA J. of Number. Anal., 36, 197-221) established an $O(k)$ convergence rate for the L1 scheme for both smooth and nonsmooth initial data. We introduce a modified L1 scheme and prove that the convergence rate is $O(k^{2-\alpha}), 0 < \alpha < 1$ for both smooth and nonsmooth initial data. We first write the time fractional partial differential equations as a Volterra integral equation which is then approximated by using the convolution quadrature with some special generating functions. The numerical schemes obtained in this way are equivalent to the standard L1 scheme and modified L1 scheme, respectively. A Laplace transform method is used to prove the error estimates for the homogeneous time fractional partial differential equation for both smooth and nonsmooth data. Numerical examples are given to show that the numerical results are

consistent with the theoretical results.

Here we consider the following time fractional partial differential equation, with $0 < \alpha < 1$,

$$\begin{matrix} C \\ 0 \end{matrix} D_t^\alpha u(t) + Au(t) = f(t), \quad \text{for} \quad 0 < t \leq T, \quad \text{with} \quad u(0) = u_0, \tag{8.1.1}$$

where $\begin{smallmatrix} C \\ 0 \end{smallmatrix} D_t^\alpha u(t)$ denotes the Caputo fractional derivative defined by

$$\begin{matrix} C \\ 0 \end{matrix} D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) ds,$$

and $u'(s) = \frac{\partial u}{\partial s}$ and A is a selfadjoint positive definite second order elliptic partial differential operator in $\Omega \subset \mathbb{R}^d, d = 1, 2, 3$, with $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$, where $H_0^1(\Omega) \cap H^2(\Omega)$, denotes the standard Sobolev spaces.

The equation (8.1.1) can be written as

$$\begin{matrix} R \\ 0 \end{matrix} D_t^\alpha (u(t) - u(0)) + Au(t)) = f(t), \quad 0 < t \leq T, \tag{8.1.2}$$

where

$$\begin{matrix} R \\ 0 \end{matrix} D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} u(s) ds, \tag{8.1.3}$$

denotes the Riemann-Liouville fractional derivative.

Our analysis will use Laplace transforms. The assumption that A is positive definite implies that A generates an analytic semigroup, so that for some $\frac{\pi}{2} < \theta_0 < \pi$ and with $C = C_{\theta_0}$ we have the resolvent estimate, see Lubich et al. [63], Thomee [93],

$$||(zI + A)^{-1}|| \leq C|z|^{-1}, \tag{8.1.4}$$

for $z \in \sum_{\theta_0} = \{z \neq 0 : |\arg z| < \theta_0\}$.

In our analysis, we also need to choose $\theta > \frac{\pi}{2}$ close to $\frac{\pi}{2}$ such that $\theta < \theta_0$. Hence $z^\alpha \in \sum_{\theta_0}$ for any $z \in \sum_\theta$ since $0 < \alpha < 1$ implies that $\arg(z^\alpha) = \alpha\theta < \theta < \theta_0$. Hence there exists a constant C which depends only on θ and α such that, see Jin et al.[47, (2.3)] or [46],

$$||(z^\alpha I + A)^{-1}|| \leq C|z|^{-\alpha}, \tag{8.1.5}$$

for $z \in \sum_{\theta_0} = \{z \neq 0 : |\arg z| < \theta\}$.

We also need to choose $\theta > \frac{\pi}{2}$ close to $\frac{\pi}{2}$ such that $z_k^\alpha \in \sum_{\theta_0}$ for $z \in \Gamma$ which implies that $(z_k^\alpha I + A)^{-1}$ exists where z_k is defined in (8.3.21) and $\Gamma = \Gamma_\theta = \{z : |\arg z| = \theta\}$.

Many application problems can be modeled by (8.1.1), for example, thermal-diffusion in media with fractional geometry [74], highly heterogeneous aquifer [1], underground environmental problems [39], random walks [38], [72], etc.

To solve (8.1.1) numerically one needs to approximate the time fractional derivative. There are three predominant approximations in the literature: finite difference methods (including L1 schemes), [52], [56], the Grünwald-Letnikov methods, [100], [105], Diethelm's method [27], [35]. The L1 scheme is obtained by approximating the integer derivative with the finite difference quotients in the definition of the fractional derivative. The Grunwald-Letnikov method is based on the convolution quadrature and finally the Diethelm's method is based on the approximation of the Hadamard finite-part integral.

Let us briefly recall some main results in these three approximations to the time fractional derivative in the literature. Langlands and Henry [52] considered the L1 scheme for the Riemann-Liouville derivative and proved that the convergence order is $O(k^{2-\alpha})$ if $u \in C^2[0, T]$. Lin and Xu [56] studied the L1 scheme for the Caputo fractional derivative and proved that the convergence order is also $O(k^{2-\alpha})$ if $u \in C^2[0, T]$, see also [87]. Gao, Sun and Zhang [37] introduced a new L1-type formula and proved that the convergence order is $O(k^{3-\alpha})$ if $u \in C^3[0, T]$. Li and Ding [54] obtained a finite difference method with order $O(k^2)$ if ${}_{-\infty}^R D_t^{3-\alpha} u \in L^1(0, T)$, see also [67]. Yuste and Acedo [100] considered a Grünwald-Letnikov discretization of the Riemann-Liouville fractional derivative and provided a von Neumann type stability analysis. Zeng et al. [105] introduced two fully discrete schemes with convergence order $O(k^{2-\alpha})$ if $u \in C^2[0, T]$ by using fractional linear multistep method in time to approximate the convolution integral. Diethelm introduced a finite difference scheme to approximate the Riemann-Liouville fractional derivative by using the Hadamard finite-part integral and showed that the truncation error is $O(k^{2-\alpha})$ if $u \in C^2[0, T]$. The scheme is obtained by Hadamard finite-part integral with linear interpolation polynomials. This scheme is actually equivalent to the L1 scheme since the weights of both schemes are the same, see below. Ford, Xiao and Yan applied Diethelm's method for solving the time fractional partial differential equation and proved the convergence order is $O(k^{2-\alpha})$ if $u \in C^2[0, T]$. High order Diethelm schemes are also available

in the literature, see [34], [35] [97].

However, the regularity of the solution of (8.1.1) is restrictive. For example, for the homogenous equation with the initial data $u_0 \in L^2(\Omega)$. We have the following stability estimate [79]

$$||_0^C D_t^\alpha u|| \leq Ct^{-\alpha}||u_0||,$$

where $||.||$ denotes the L^2 norm. That is, the αth order Caputo derivative is already unbounded when $t \mapsto 0$. Hence, the C^2-regularity assumption generally does not hold for (8.1.1) and the case of nonsmooth data is not covered by the existing error analysis. Numerical experiments indicated that the $O(k^{2-\alpha})$ convergence rate actually does not hold uniformly in t even for smooth data u_0 [47]. The purpose of this work is to consider error estimates for approximating (8.1.1) with nonsmooth data.

Jin et al. [47] presented an optimal $O(k)$ convergence rate for the fully discrete scheme based on the $L1$ scheme , i.e., (8.1.13)-(8.1.14) in time and Galerkin finite element method in space for both smooth and nonsmooth data, i.e., $u_0 \in L^2(\Omega)$ and $Au_0 \in L^2(\Omega)(A = -\Delta$ with a homogenous Dirichlet boundary condition), respectively. We will introduce a modified $L1$ scheme (8.1.19)-(8.1.20) to discretize the time fractional derivative in (8.4.1) and discretize the spatial derivative by using the Galerkin finite element method. We shall prove the optimal convergence order $O(k^{2-\alpha})$ with nonsmooth data for such modified L1 scheme. Our estimates are derived by using the techniques developed in Lubich et al. [63] for solving the intergo-differential equation. We will use some delicate estimates for the kernel function which involves the polylogarithmic functions, see Jin et al [47]. To the best of our knowledge, there are no proofs of the convergence order $O(k^{2-\alpha})$ for numerical methods for solving the time fractional partial differential equation (8.4.1) with nonsmooth data in the literature.

The main contributions of this paper are as follows:

($\mathbf{1}$) we introduce the modified L1 scheme for solving time fractional partial differential equation and prove that convergence order of this scheme is $O(k^{2-\alpha}), 0 < \alpha < 1$ for both smooth and nonsmooth data.

($\mathbf{2}$) we find the equivalence of the L1 scheme or modified L1 scheme with the convolution quadrature formula for solving time fractional partial differential equations, i.e.,

Lemmas 8.3.1, 8.3.4.

(**3**) we apply the Laplace transform approach of solving integro-differential equation in Lubich et al [63] to solve time fractional partial differential equation (8.4.1).

With $\beta(t) = t^{\alpha-1}/\Gamma(\alpha), 0 < \alpha < 1$, the equation (8.1.2) can be written in the integral form

$$u(t) - u_0 + \int_0^t \beta(t-s)Au(s)ds = \int_0^t \beta(t-s)f(s)ds, \tag{8.1.6}$$

with $A(\cdot, \cdot)$ denoting the bilinear form associated with A, and $(\cdot, \cdot)$ the inner product in $L^2(\Omega)$. The variational form of (8.1.6) is to find $u(t) \in H_0^1(\Omega)$ such that, for $\forall v \in H_0^1(\Omega)$,

$$(u(t), v) + \int_0^t \beta(t-s)A(u(s), v)ds = \int_0^t \beta(t-s)(f(s), v)ds. \tag{8.1.7}$$

We first consider the case of the discretization in space only. Let S_h denote the piecewise linear functions on a triangulation of Ω of the standard type so that

$$\inf_{x \in s_h}\{||u - \chi|| + h||u - \chi||_1\} \leq Ch^2||u||_2,$$

where $||v||_s = ||A^{\frac{s}{2}}v||$ for $s \geq 0$.

The spatially discrete problem is then to find $u_h(t) \in S_h$ for $0 < t \leq T$ such that, for $\forall \chi \in S_h$,

$$(u_h(t), \chi) - (u_h(0), \chi) + \int_0^t \beta(t-s)A(u_h(s), \chi)ds = \int_0^t \beta(t-s)(f(s), \chi)ds, \tag{8.1.8}$$

$$u_h(0) = u_{0h} \approx u_0, \tag{8.1.9}$$

where $u_{0h} \in S_h$ is some approximation of u_0. Or in the abstract form,

$$u_h(t) - u_{0h} + \int_0^t \beta(t-s)A_h u_h(s)ds = \int_0^t \beta(t-s)P_h f(s)ds, \tag{8.1.10}$$

where $A_h : S_h \mapsto S_h$ denotes the discrete Laplacian defined by

$$(A_h\psi, \chi) = A(\psi, \chi), \quad \forall \psi, \chi \in S_h,$$

and P_h is the L^2-projection onto S_h.

For this problem, we show that, in Theorem 8.2.2, if $u_{0h} = P_h u_0$, then $t > 0$,

$$||u_h(t) - u(t)|| \leq Ch^2(t^{-\alpha}||u_0|| + t^{-\alpha+1}||f(0)|| + \int_0^t (t-s)^{-\alpha+1}||f_t(s)||ds). \tag{8.1.11}$$

We now turn to the discretization in time only. Let $N \geq 1$ be a positive integer and let $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ be a partion of $[0, T]$ and k the time step size.

For the convolution integral in (8.1.6), we apply the following quadrature formula [63],

$$q_n^0(\varphi) = k^\alpha \sum_{j=0}^{n} \beta_{n-j}\varphi^j \approx \int_0^{t_n} \beta(t_n - s)\varphi(s)ds.$$

Here, the weights $\beta_j, j = 0, 1, 2, \cdots$ are determined by some special generating function

$$\bar{\beta}(\zeta) := \sum_{j=0}^{\infty} \beta_j \zeta^j \tag{8.1.12}$$

which we will define in Section 3.

It will prove convenient, however, to modify this formula slightly by omitting the term with $j = 0$ so that [63],

$$q_n(\varphi) = k^\alpha \sum_{j=1}^{n} \beta_{n-j}\varphi^j \approx \int_0^{t_n} \beta(t_n - s)\varphi(s)ds.$$

Let $U^n \approx u(t_n), n = 1, 2, \cdots, N$ be the approximate solution of $u(t_n)$. We may define the following time discretization problem for solving (8.1.6),

$$U^n - U^0 + q_n(AU) = q_n(f), n \geq 1, \tag{8.1.13}$$

$$U^0 = u_0. \tag{8.1.14}$$

Let $V^n = U^n - u_0$, then V^n satisfies

$$V^n + q_n(AV) = -q_n(Au_0) + q_n(f), \quad n \geq 1, \tag{8.1.15}$$

$$V^0 = 0. \tag{8.1.16}$$

Assume that $\bar{\beta}(\zeta) = \bar{w}(\zeta)^{-1}$ where $\bar{w}(\zeta) = \sum_{j=0}^{\infty} w_j \zeta^j$ is defined in (8.3.2) below. Then we prove, in Theorem 8.3.3 the following nonsmooth data error estimate of the equation (8.1.6) with $f = 0$

$$\|U^n - u(t_n)\| \leq Ck t_n^{-1} \|u_0\|. \tag{8.1.17}$$

In order to achieve higher accuracy, we will use the following modification of $q_n(\varphi)$, i.e., [63]

$$q_n^c(\varphi) = k^\alpha \left(\sum_{j=1}^{n} \beta_{n-j}\varphi^j + c_0 \beta_{n-1}\varphi^0 \right), \quad \text{with} \quad c_0 = \frac{1}{2}, \tag{8.1.18}$$

to approximate the integral $\int_0^{t_n} \beta(t_n - s)\varphi(s)ds$. We therefore define the following time discretization problem for solving (8.1.6),

$$U^n - U^0 + q_n^c(AU) = q_n^c(f), \quad n \geq 1, \tag{8.1.19}$$

$$U^0 = u_0. \tag{8.1.20}$$

Let $V^n = U^n - u_0$. Then V^n satisfies

$$V^n + q_n^c(AV) = -q_n^c(Au_0) + q_n^c(f), \quad n \geq 1, \tag{8.1.21}$$

$$V^0 = 0. \tag{8.1.22}$$

Assume that $\bar{\beta}(\zeta) = \bar{w}(\zeta)^{-1}$ where $\bar{w}(\zeta) = \sum_{j=0}^{\infty} w_j \zeta^j$ is defined in (8.3.2) below. Then we prove, in Theorem 8.3.6 the following nonsmooth data error estimate of the equation (8.1.6) with $f = 0$

$$||U^n - u(t_n)|| \leq Ck^{2-\alpha}t_n^{\alpha-2}||u_0||.$$

8.2 Finite Element Method

In this section, we will consider the finite element method for solving (8.1.6). We have the following theorem:

Theorem 8.2.1. Assume that $u(t)$ and $u_h(t)$ are the solutions of (8.1.6) and (8.1.10), respectively. Assume that $f = 0$. Then we have

$$||u_h(t) - u(t)|| \leq Ch^2 t^{-\alpha}||u_0||. \tag{8.2.1}$$

Proof: The estimate (8.2.1) was first proved in Jin et al. [47] with a log factor. Later the log factor was removed using the operator trick in Bazhlekova et al. [7]. The proof in Bazhlekova et al. [7] is for the general inverse operator $(g(z) + A)^{-1}$ in $\hat{E}(z)$ below. In our case, we have $g(z) = z^\alpha$ and we will prove the estimate (8.2.1) below by using the argument in Lubich et al. [63] which is slightly simpler than the argument in Bazhlekova et al. [7] .

Let $\hat{u}(z)$ be the Laplace transform of $u(t)$. Taking the Laplace transform for (8.1.6), we get

$$\hat{u}(z) - u_0 z^{-1} + \hat{\beta}A\hat{u}(z) = 0.$$

We have, noting that $z^\alpha \in \sum_\theta$ by (8.1.5),

$$\hat{u}(z) = (zI + z^{-\alpha+1}A)^{-1}u_0 = \hat{E}(z)u_0,$$

where $\hat{E}(z) = z^{\alpha-1}(z^\alpha + A)^{-1}$, By using the inverse Laplace transform, we have

$$u(t) = \frac{1}{2\pi i}\int_\Gamma e^{tz}\hat{E}(z)u_0 dz = E(t)u_0,$$

where

$$\Gamma = \Gamma_\theta = \{z : |\arg z| = \theta\},$$

for some $\theta > \frac{\pi}{2}$ sufficiently close to $\frac{\pi}{2}$ such that (8.1.5) holds. Here

$$||\hat{E}(z)|| = ||z^{\alpha-1}(z^\alpha + A)^{-1}|| \le \frac{|z|^{\alpha-1}}{|z|^\alpha} \le C|z|^{-1},$$

and $\hat{E}(z)$ is analytic for $z \in \sum_\theta$.

Similarly the solution of (8.1.10) has the form, with $u_{0h} = P_h u_0, f = 0$,

$$u_h(t) = \frac{1}{2\pi i}\int_\Gamma e^{tz}\hat{E}_h(z)u_{0h}dz = E_h(t)P_h u_0,$$

where

$$\hat{E}_h(z) = z^{\alpha-1}(z^\alpha + A_h)^{-1}.$$

Thus we have

$$||u_h(t) - u(t)|| = ||E_h(t)P_h u_0 - E(t)u_0||$$
$$= \left\|\frac{1}{2\pi i}\int_\Gamma e^{tz}(\hat{E}_h(z)P_h - \hat{E}(z))u_0 dz\right\|.$$

We will show that

$$||\hat{E}_h(z)P_h - \hat{E}(z)|| \le Ch^2|z|^{\alpha-1}, z \in \Gamma. \tag{8.2.2}$$

Assume this for the moment, we then have, with some constant $c_1 > 0$,

$$||u_h(t) - u(t)|| \le C\int_\Gamma |e^{tz}||h^2|z|^{\alpha-1}|dz|||u_0|| \le C\int_\Gamma e^{-c_1 t|z|}|h^2|z|^{\alpha-1}|dz|||u_0||$$
$$\le C\int_0^\infty e^{-c_1 tr}h^2 r^{\alpha-1}dr||u_0|| \le Ch^2 t^{-\alpha}\int_0^\infty e^{-x}x^{\alpha-1}dx||u_0|| \le Ch^2 t^{-\alpha}||u_0||.$$

It remains to show (8.2.2). Note that

$$||\hat{E}_h(z)P_h - \hat{E}(z)|| = z^{\alpha-1}\big((z^\alpha I + A_h)^{-1}P_h - (z^\alpha + A)^{-1}\big).$$

With $w = z^\alpha$, we have

$$(wI + A_h)^{-1}P_h - (wI + A)^{-1}$$
$$= P_h((wI + A_h)^{-1} - (wI + A)^{-1})P_h - (I - P_h)(wI + A)^{-1})P_h - (wI + A)^{-1}(I - P_h)$$
$$= I + II + III.$$

For I, we have

$$||I|| = ||P_h((wI + A_h)^{-1} - (wI + A)^{-1})P_h|| \leq ||A_h^{-1}P_h - P_h A^{-1}||$$
$$= ||A_h^{-1}P_h - A^{-1}|| + ||(I - P_h)A^{-1}|| \leq Ch^2.$$

For III, we have, by (8.1.5) and noting that $A(w + A)^{-1} = I - w(w + A)^{-1}$,

$$||III|| = ||(wI + A)^{-1}(I - P_h)|| = ||(wI + A)^{-1}AA^{-1}(I - P_h)||$$
$$\leq C||A^{-1}(I - P_h)|| \leq Ch^2.$$

For II, we have

$$||II|| = ||(I - P_h)(wI + A)^{-1})P_h|| \leq Ch^2.$$

Together these estimates complete the proof of Theorem 8.2.1.

Our next Theorem is the nonsmooth data error estimates for $f \neq 0, u_0 = 0$.

Theorem 8.2.2. Assume that $u(t)$ and $u_h(t)$ are the solutions of (8.1.6) and (8.1.10), respectively. Assume that $f \neq 0, u_0 = 0$. Then we have

$$||u_h(t) - u(t)|| \leq Ch^2\Big(t^{-\alpha+1}||f(0)|| + \int_0^t (t - s)^{-\alpha+1}||f'(s)||ds\Big), \tag{8.2.3}$$

where $f'(s) = \frac{df(s)}{ds}$.

Proof. We first show that, with the sufficient smooth solution $u(t)$ and $u_h(t)$,

$$u_h(t) = \int_0^t E_h(t - s)P_h f(s)ds,$$

and

$$u(t) = \int_0^t E(t-s)f(s)ds,$$

where

$$E_h(t)P_h u_0 = \frac{1}{2\pi i} \int_\Gamma e^{tz} \hat{E}_h(z) P_h u_0 dz,$$

and

$$E(t)u_0 = \frac{1}{2\pi i} \int_\Gamma e^{tz} \hat{E}(z) u_0 \, dz.$$

In fact, by taking the Laplace transform of (8.1.6), we have

$$\hat{u}(z) + z^{-\alpha} A \hat{u}(z) = z^{-\alpha} \hat{f}(z),$$

which implies that

$$\hat{u}(z) = (1 + z^{-\alpha} A)^{-1} z^{-\alpha} \hat{f}(z).$$

Denote $\hat{E}(z) = (1 + z^{-\alpha} A)^{-1} z^{-\alpha}$, we have

$$u(t) = (E * f)(t) = \int_0^t E(t-s)f(s)ds.$$

With $J_h(t) = \int_0^t F_h(s)ds, F_h(s) = E_h(s)P_h - E(s)$, we have

$$u_h(t) - u(t) = \int_0^t (E_h(t-s)P_h - E(t-s))f(s)ds$$

$$= \int_0^t F_h(t-s)f(s)ds = J_h(t)f(0) + \int_0^t J_h(t-s)f'(s)ds.$$

Thus we obtain

$$||u_h(t) - u(t)|| \leq ||J_h(t)f(0)|| + \int_0^t ||J_h(t-s)|| ||f'(s)||ds.$$

Since the Laplace transform $\hat{J}_h(z)$ of $J_h(t)$ is $z^{-1}\hat{F}_h(z)$, with $\hat{F}_h(z) = \hat{E}_h(z)P_h - \hat{E}(z)$, our results follows from, using (8.2.2),

$$||J_h(t)|| = \left|\left|\frac{1}{2\pi i}\int_\Gamma e^{tz}\hat{J}_h(z)dz\right|\right| = ||\frac{1}{2\pi i}\int_\Gamma e^{tz}z^{-1}\hat{F}_h(z)dz||$$

$$\leq Ch^2 \int_\Gamma |z|^{\alpha-2} e^{-c_1 t|z|}|dz| \leq Ch^2 t^{-\alpha+1}.$$

Together these estimates complete the proof of Theorem 8.2.2.

$\square$

8.3 Time Discretization

In this section we will consider the nonsmooth data error estimates of the time discretization scheme for the equation (8.1.6) with $f = 0$.

At $t = t_n, n = 1, 2, \cdots N$, we may use the following L1 scheme to approximate the Caputo fractional derivative, see [47],

$$
{}_0^C D_t^\alpha u(t_n) = k^{-\alpha} \left(b_0 u(t_n) + \sum_{j=1}^{n-1} (b_j - b_{j-1}) u(t_{n-j}) - b_{n-1} u_0 \right) + O(k^{2-\alpha}),
$$

where the weights b_j are given by

$$
b_j = ((j+1)^{1-\alpha} - j^{1-\alpha})/\Gamma(2-\alpha), j = 0, 1, 2 \cdots, n-1.
$$

Rearranging the coefficients, we may write

$$
{}_0^C D_t^\alpha u(t_n) = k^{-\alpha} \sum_{j=0}^{n} w_{j,n} u(t_{n-j}) + O(k^{2-\alpha})
$$

$$
= k^{-\alpha} \sum_{j=0}^{n} w_{n-j,n} u(t_j) + O(k^{2-\alpha}), \tag{8.3.1}
$$

where $w_{j,n}, j = 0, 1, 2 \cdots, n$ are given by

$$
\Gamma(2-\alpha) w_{j,n} = \begin{cases} 1, & \text{for} \quad j = 0, \\ -2j^{1-\alpha} + (j-1)^{1-\alpha} + (j+1)^{1-\alpha}, & \text{for,} \quad j = 1, 2, \cdots, n-1, \\ (j-1)^{1-\alpha} - j^{1-\alpha}, & \text{for} \quad j = n. \end{cases}
$$

We remark that the above weights $w_{j,n}, j = 0, 1, 2 \cdots, n$ can also be obtained by using Diethelm method [27]. In other words, the L1 scheme for approximating the Caputo fractional derivative may be obtained by first approximating the Riemann-Liouville fractional derivative by using Diethelm's method [27] and then applying the relation between the Riemann-Liouville and Caputo fractional derivatives, i.e., ${}_0^C D_t^\alpha u(t) = {}_0^R D_t^\alpha u(t) - u_0)$ for $0 < \alpha < 1$.

For any fixed $n \geq 1, w_{j,n}, j = 0, 1 \cdots, n-1$ only depend on $j = 0, 1, 2 \cdots, n-1$. For example, we have $w_{0,n} = 1/\Gamma(2-\alpha)$ for any $n \geq 1$, $w_{1,n} = 1/\Gamma(2-\alpha)((-2)1^{1-\alpha} + (1-1)^{1-\alpha}) + (1+1)^{1-\alpha})$ for any $n \geq 2$. Therefore, we may write $w_0 = w_{0,n}, w_1 = w_{1,n}, w_2 = w_{2,n}, \cdots, w_{n-1} = w_{n-1,n}$ for any $n \geq 1$. For such defined $w_j, j = 0, 1, 2, \cdots$, we denote

$$\bar{w}(z) := \sum_{j=0}^{\infty} w_j \zeta^j. \tag{8.3.2}$$

Lemma 8.3.1. Assume that $\bar{\beta}(\zeta) = \bar{w}(\zeta)^{-1}$ where $\bar{w}(\zeta) := \sum_{j=0}^{\infty} w_j \zeta^j$ is defined in (8.3.2). Then the time discretization problem (8.1.15)-(8.1.16) is equivalent to the following L1 scheme, with $f = 0$,

$$k^{-\alpha} \sum_{j=0}^{n} w_{n-j,n} V^j + AV^n = -Au_0, \quad n \geq 1, \tag{8.3.3}$$

$$V^0 = 0, \tag{8.3.4}$$

where the weights $w_{n-j,n}, j = 0, 1, 2, \cdots, n$ are given by (8.3.1) and noting that $w_j = w_{j,n}, j = 0, 1, 2 \cdots, n-1$ for all $n \geq 1$.

Proof: The proof is similar to the proof of Lemma 8.3.4 below. We omit the proof here.

Remark 7. Assume that $\bar{\beta}(\zeta) = \bar{w}(\zeta)^{-1}$ where $\bar{w}(\zeta) := \sum_{j=0}^{\infty} w_j \zeta^j$ is defined in (8.3.2). Then the time discretization problem (8.1.13)-(8.1.14) is equivalent to the following L1 scheme, with $f = 0$,

$$k^{-\alpha} \sum_{j=0}^{n} w_{n-j,n} U^j + AU^n = 0, \quad n \geq 1, \tag{8.3.5}$$

$$U^0 = u_0, \tag{8.3.6}$$

where the weights $w_{n-j,n}, j = 0, 1, 2, \cdots, n$ are given by (8.3.1) and noting that $w_j = w_{j,n}, j = 0, 1, 2 \cdots, n-1$ for all $n \geq 1$. In fact, let $V^j = U^j - u_0, j = 0, 1, 2 \cdots, n$ we have, by (8.3.3)

$$k^{-\alpha} \sum_{j=0}^{n} w_{n-j,n} (U^j - u_0) + AU^n = 0, \tag{8.3.7}$$

which implies (8.3.5) by noting the fact

$$\sum_{j=0}^{n} w_{n-j,n} = 0.$$

We then have the following theorem from Jin et al. [47].

Theorem 8.3.2. Assume that $\bar{\beta}(\zeta) = \bar{w}(\zeta)^{-1}$ where $\bar{w}(\zeta) := \sum_{j=0}^{\infty} w_j \zeta^j$ is defined in (8.3.2). Let $u(t_n)$ and U^n be the solutions of (8.1.6) and (8.1.13)-(8.1.14), respectively. Let $u_0 \in L^2(\Omega)$ and $f = 0$. We have, with $0 < \alpha < 1$,

$$\|u(t_n) - U^n\| \le Ckt_n^{-1}\|u_0\|. \tag{8.3.8}$$

Proof. By Remark 7 the time discretization problem (8.1.13)-(8.1.14) is equivalent to the L1 scheme (8.3.3)-(8.3.4). Further we note that (8.3.3)-(8.3.4) is the same scheme as the difference scheme [47, (2.8)] introduced in Jin et al. [47]. Hence (8.3.8) follows from Jin et al. [47, Theorem 3.16]. $\qquad\square$

We next consider the modified L1 scheme (8.1.21)-(8.1.22) with $f = 0$, that is,

$$V^n + q_n^c(AV) = -q_n^c(Au_0), \quad n \ge 1, \tag{8.3.9}$$

$$V^0 = 0. \tag{8.3.10}$$

We have the following lemma.

Lemma 8.3.3. Assume that $\bar{\beta}(\zeta) = \bar{w}(\zeta)^{-1}$, where $\bar{w}(\zeta) := \sum_{j=0}^{\infty} w_j \zeta^j$ is defined in (8.3.2). Then the time discretization problem (8.3.9)-(8.3.10) is equivalent to, with $c_0 = \frac{1}{2}$,

$$k^{-\alpha} \sum_{j=0}^{n} w_{n-j,n} V^j + AV^n = (-Au_0)(1 + c_0), \quad \text{for} \quad n = 1, \tag{8.3.11}$$

$$k^{-\alpha} \sum_{j=0}^{n} w_{n-j,n} V^j + AV^n = (-Au_0), \quad \text{for} \quad n \ge 2, \tag{8.3.12}$$

$$V^0 = 0. \tag{8.3.13}$$

Proof. Denote

$$a_n = \begin{cases} 1 + c_0, & c_0 = \frac{1}{2}, \quad \text{for} \quad n = 1, \\ 1, & \text{for} \quad n \ge 2. \end{cases}$$

The time discretization problem of (8.3.11)-(8.3.13) can then be written as

$$k^{-\alpha} \sum_{j=1}^{n} w_{n-j,n} V^j + AV^n = (-Au_0)a_n.$$

Denote $\bar{V}(\zeta) = \sum_{j=0}^{\infty} V^j \zeta^j$, we have

$$\sum_{n=1}^{\infty} \left(k^{-\alpha} \sum_{j=1}^{n} w_{n-j,n} V^j \right) \zeta^n + \sum_{n=1}^{\infty} (AV^n) \zeta^n = (-Au_0) \sum_{n=1}^{\infty} (a_n \zeta^n).$$

Note that

$$\sum_{n=1}^{\infty} \Big(\sum_{j=1}^{n} w_{n-j,n} V^j \Big) \zeta^n = \Big(\sum_{j=0}^{\infty} w_j \zeta^j \Big)(V^1 \zeta^1 + V^2 \zeta^2 + \cdots), \tag{8.3.14}$$

where we only use the weights $w_{j,n}, j = 0, 1, 2 \cdots, n-1$ for $n \geq 1$ and we do not use $w_{n,n}$. Hence we have

$$k^{-\alpha} \bar{w}(\zeta) \bar{V}(\zeta) + A \bar{V}(\zeta) = (-Au_0)(\frac{\zeta}{1-\zeta} + c_0 \zeta).$$

Note that, by (8.1.12),

$$\bar{V}(\zeta) + k^\alpha \bar{\beta}(\zeta) A \bar{V}(\zeta) = k^\alpha \bar{\beta}(\zeta)(-Au_0)(\frac{\zeta}{1-\zeta} + c_0 \zeta),$$

we have

$$\sum_{n=1}^{\infty} V^n \zeta^n + k^\alpha \sum_{n=1}^{\infty} \Big(\sum_{j=1}^{n} \beta_{n-j} A V^j \Big) \zeta^n$$

$$= -k^\alpha \sum_{n=1}^{\infty} (\sum_{j=1}^{n} \beta_{n-j} A u_0) \zeta^n - k^\alpha \sum_{n=1}^{\infty} (c_0 \beta_{n-1} A u_0) \zeta^n,$$

which implies

$$V^n + k^\alpha \sum_{j=1}^{n} \beta_{n-j} A V^j = -k^\alpha \sum_{j=1}^{n} \beta_{n-j} A u_0 - k^\alpha c_0 \beta_{n-1} A u_0, \quad n \geq 1.$$

Thus we get (8.3.9)-(8.3.10).

Together these estimates complete the proof of Lemma 8.3.3.

Remark 8. Assume that $\bar{\beta}(\zeta) = \bar{w}(\zeta)^{-1}$ where $\bar{w}(\zeta) := \sum_{j=0}^{\infty} w_j \zeta^j$ is defined in (8.3.2). Then the time discretization problem (8.1.19)-(8.1.20) is equivalent to the following modified L1 scheme, with $f = 0$,

$$k^{-\alpha} \sum_{j=0}^{n} w_{n-j,n} U^j + A U^n = (-Au_0)c_0, \quad \text{for } n = 1, \tag{8.3.15}$$

$$k^{-\alpha} \sum_{j=0}^{n} w_{n-j,n} U^j + A U^n = 0, \quad \text{for } n \geq 2, \tag{8.3.16}$$

$$U^0 = u_0, \tag{8.3.17}$$

where the weights $w_{n-j,n}, j = 0, 1, 2 \cdots, n$ are given by (8.3.1) and noting that $w_{n-j,n}, j = 0, 1, 2 \cdots, n-1$ for all $n \geq 1$, which follows by the substitution $V^j = U^j - u_0, j = 0, 1, 2 \cdots, n$ in (8.3.11)-(8.3.13) and the equality $\sum_{j=0}^{n} w_{n-j,n} = 0$.

We are now ready to show our nonsmooth data error estimates of the modified L1 scheme for the homogeneous equation.

Theorem 8.3.4. Assume that $\bar{\beta}(\zeta) = \bar{w}(\zeta)^{-1}$, where $\bar{w}(\zeta) := \sum_{j=0}^{\infty} w_j \zeta^j$ is defined in (8.3.2). Let $u(t_n)$ and U^n be the solutions of (8.1.6) and (8.1.19)-(8.1.20), respectively. Let $u_0 \in L^2(\Omega)$ and $f = 0$. We have, with $0 < \alpha < 1$,

$$||u(t_n) - U^n|| \leq Ck^{2-\alpha} t_n^{\alpha-2} ||u_0||.$$

To prove Theorem 8.3.4 , we need to show that $z_k^\alpha \in \sum_{\theta_0}$ for some $\theta_0 \in (\pi/2, \pi)$ where z_k is defined in (8.3.21) below and θ_0 is introduced in (8.1.4).

Lemma 8.3.5. [47, Lemma 3.7] Let $\theta > \pi/2$ be close to $\pi/2$. Let $z \in \Gamma_k$ with $\Gamma_k = \{z \in \Gamma : |\Im z| \leq \pi/k\}$ and $\Gamma = \{z : |arg \ z| = \theta\}$ (with $\Im z$ running from $-\infty$ to ∞). Let $z_k = \frac{\delta(\zeta)}{k}, \zeta = e^{zk}$ defined by (8.3.21). Then there exists $\theta_0 \in (\pi/2, \pi)$ such that

$$z_k^\alpha \in \sum_{\theta_0}, \text{ for all, } z \in \sum_{\theta}.$$

Remark 9. In Lemma 8.3.5 in Jin et al [47], the authors proved that for all $-\pi \leq \theta < \pi$, there exists $\theta_0 \in (\pi/2, \pi)$, such that $z_k^\alpha \in \sum_{\theta_0}$ for all $z \in \sum_\theta$. Actually in our analysis, we only need to show $z_k^\alpha \in \sum_{\theta_0}$ for all $z \in \sum_\theta$ for some $\theta > \pi/2$ close to $\pi/2$.

Proof of Theorem 8.3.4. Let $V(t) = u(t) - u_0$ and $V^n = U^n - u_0$. It suffices to show

$$||V(t_n) - V^n|| \leq Ck^{2-\alpha} t_n^{\alpha-2} ||u_0||,$$

which we will prove now. Note that, by (8.1.6),

$$V(t) + \int_0^t \beta(t-s)AV(s)ds = -\int_0^t \beta(t-s)Au_0 ds, \tag{8.3.18}$$

$$V^0 = 0. \tag{8.3.19}$$

Taking the Laplace transform in (8.3.18), we have,

$$\hat{V}(z) = -z^{-1}(z^\alpha + A)^{-1}Au_0,$$

which implies that

$$V(t) = -\frac{1}{2\pi i} \int_\Gamma e^{zt} z^{-1}(z^\alpha + A)^{-1}Au_0 dz. \tag{8.3.20}$$

Further we note that $V^n, n = 1, 2, 3 \cdots$ satisfy (8.3.9)-(8.3.10) and therefore

$$\sum_{n=1}^{\infty} V^n \zeta^n + \sum_{n=1}^{\infty} q_n^c(AV)\zeta^n = -\Big(\sum_{n=1}^{\infty} q_n^c(1)\zeta^n\Big) Au_0.$$

Denoting [63]

$$\delta(\zeta)^{-\alpha} = \bar{\beta}(\zeta),$$

we obtain

$$\sum_{n=1}^{\infty} q_n^c(AV)\zeta^n = \sum_{n=1}^{\infty} (k^\alpha \sum_{j=1}^{n} \beta_{n-j}(AV^j) + c_0 k^\alpha \beta_{n-1} AV^0)\zeta^n$$

$$= \sum_{n=1}^{\infty} \Big(k^\alpha \sum_{j=1}^{n} \beta_{n-j}(AV^j)\Big)\zeta^n = \Big(\frac{\delta(\zeta)}{k}\Big)^{-\alpha} A\bar{V}(\zeta),$$

and

$$\sum_{n=1}^{\infty} q_n^c(1)\zeta^n = \sum_{n=1}^{\infty} \Big(k^\alpha \sum_{j=1}^{n} \beta_{n-j}\zeta^n + c_0 k^\alpha \beta_{n-1}\zeta^n\Big)$$

$$= \Big(\frac{\delta(\zeta)}{k}\Big)^{-\alpha}(\zeta + \zeta^2 + \zeta^3 + \cdots) + c_0 k^\alpha(\beta_0\zeta + \beta_1\zeta^2 + \beta_2\zeta^3 + \cdots)$$

$$= \Big(\frac{\delta(\zeta)}{k}\Big)^{-\alpha}\Big(\frac{\zeta}{1-\zeta} + c_0\zeta\Big).$$

Therefore we get

$$\bar{V}(\zeta) + \Big(\frac{\delta(\zeta)}{k}\Big)^{-\alpha} A\bar{V}(\zeta) = -\Big(\frac{\delta(\zeta)}{k}\Big)^{-\alpha}\Big(\frac{\zeta}{1+\zeta} + c_0\zeta\Big).$$

Denote

$$z_k = \frac{\delta(\zeta)}{k}. \tag{8.3.21}$$

By Lemma 8.3.5 we see that $(z_k^\alpha + A)^{-1}$ exists and hence we have

$$\bar{V}(\zeta) = -(z_k^\alpha + A)^{-1}(\frac{\zeta}{1-\zeta} + c_0\zeta)Au_0.$$

Hence we have

$$V^n = -\frac{1}{2\pi i} \int_{|\zeta|=\rho} \zeta^{-n-1}\Big(\frac{\zeta}{1-\zeta} + c_0\zeta\Big)\Big(z_k^\alpha + A\Big)^{-1} Au_0 d\zeta$$

$$= -\frac{1}{2\pi i} \int_{|\zeta|=\rho} \zeta^{-n-1}\Big(\frac{\zeta}{1-\zeta} + c_0\zeta\Big)\Big(\frac{\delta(\zeta)}{k}\Big) z_k^{-1}(z_k^\alpha + A)^{-1} Au_0 d\zeta.$$

Let $\zeta = e^{-zk}$, $z = \frac{1}{k}\log\frac{1}{\rho} + i(-\frac{\theta}{k})|\theta| \le \pi$, we have

$$V^n = -\frac{1}{2\pi i}\int_{\Gamma_k} e^{t_n z}\Big(\frac{\zeta}{1-\zeta} + c_0\zeta\Big)\delta(\zeta)z_k^{-1}(z_k^\alpha + A)^{-1}Au_0 dz,$$

where $\Gamma_k = \{z \in \Gamma : |\Im z| \le \pi/k\}$. For the details of the notation Γ_k, see the proof of Lemma 8.3.2 in [47].

Denoting

$$\mu_2(\zeta) = \Big(\frac{\zeta}{1-\zeta} + c_0\zeta\Big)\delta(\zeta), \tag{8.3.22}$$

we obtain

$$V^n = -\frac{1}{2\pi i}\int_{\Gamma_k} e^{t_n z}\mu_2(\zeta)z_k^{-1}(z_k^\alpha + A)^{-1}Au_0 dz. \tag{8.3.23}$$

Thus we have, subtracting (8.3.20) from (8.3.23),

$$\begin{aligned}
V(t_n) - V^n &= \frac{1}{2\pi i}\int_{\Gamma_k} e^{t_n z}\Big(\mu_2(\zeta)z_k^{-1}(z_k^\alpha + A)^{-1} - z^{-1}(z^\alpha + A)^{-1}\Big)Au_0 dz \\
&\quad + \frac{1}{2\pi i}\int_{\Gamma_k/\Gamma_k} e^{t_n z}z^{-1}(z^\alpha + A)^{-1}Au_0 dz \\
&= I + II.
\end{aligned}$$

Denote

$$\hat{k}(z) = z^{-1}(z^\alpha + A)^{-1}A. \tag{8.3.24}$$

For I, we have, with some suitable constant $\bar{c}_0 > 0$,

$$\begin{aligned}
\|I\| &\le \frac{1}{2\pi}\int_{\Gamma_k} |e^{t_n z}|\|\mu_2(\zeta)\hat{k}(z_k) - \hat{k}(z)\|\|u_0\||dz| \\
&\le \frac{1}{2\pi}\int_{\Gamma_k} |e^{t_n z}|C(k^{2-\alpha}|z|^{1-\alpha})\|u_0\||dz| \\
&\le Ck^{2-\alpha}\int_0^\infty e^{-\bar{c}_0 t_n r}(t_n r)^{1-\alpha}d(rt_n)t_n^{\alpha-1}t_n^{-1}\|u_0\| \\
&\le Ck^{2-\alpha}t_n^{\alpha-2}\|u_0\| \le Ck^{2-\alpha}t_n^{\alpha-2}\|u_0\|.
\end{aligned}$$

For II, we have by (8.1.5) and noting that $(z^\alpha + A)^{-1}A = I - z^\alpha(z^\alpha + A)^{-1}$, with some suitable constant $\bar{c}_0 > 0$,

$$\begin{aligned}
\|II\| &\le \frac{1}{2\pi}\int_{\Gamma/\Gamma_k} |e^{t_n z}|\|u_0\|\|z^{-1}(z^\alpha + A)^{-1}A\|u_0\||dz|\|u_0\| \le C\int_{\frac{1}{k}}^\infty e^{-\bar{c}_0 t_n|z|}|z|^{-1}|dz|\|u_0\| \\
&\le C\int_{\frac{1}{k}}^\infty e^{-\bar{c}_0 t_n|z|}|z|^{-(2-\alpha)}|z|^{-\alpha+1}|dz|\|u_0\| \le Ck^{2-\alpha}\int_{\frac{1}{k}}^\infty e^{-\bar{c}_0 t_n|z|}|z|^{1-\alpha}|dz|\|u_0\| \\
&\le Ck^{2-\alpha}t_n^{\alpha-2}\int_0^\infty e^{-\bar{c}_0 r}r^{-\alpha+1}\,dr\|u_0\| \le Ck^{2-\alpha}t_n^{\alpha-2}\|u_0\|.
\end{aligned}$$

The proof of Theorem 8.3.4 is now complete. □

We close this section by introducing some lemmas which we need in the proof of Theorem 8.3.4.

Lemma 8.3.6. We have the following singularity expansion, with $\zeta = e^{-zk}$,

$$\sum_{j=0}^{\infty} w_j \zeta^j = (zk)^\alpha + c_2(zk)^2 + c_3(zk)^3 + \cdots,$$

for some suitable constants $c_2, c_3 \cdots$

We also need to introduce the polylogorithm function

$$Li_p(z) = \sum_{j=1}^{\infty} \frac{z_j}{j^p}.$$

The polynomial function $Li_p(z)$ is well defined for $|z| < 1$ and $p \in \mathbb{C}$. It can be analytically continued to the split complex plane $\mathbb{C}/[1, +\infty)$; see Flajolet [36]. With $z = 1$, it recovers the Riemann zeta function $\varsigma(p) = Li_p(1)$. We also recall an important singular expansion of the function $Li_p(e^{-z})$ (Flajolet [36, Theorem 1]).

Lemma 8.3.7. [47, Lemma 3.2] For $p \neq 1, 2, \cdots$ the function $Li_p(e^{-z})$ satisfies the singluar expansion

$$Li_p(e^{-z}) \sim \Gamma(I - p)z^{p-1} + \sum_{l=0}^{\infty} (-1)^l \varsigma(p - I)\frac{z^l}{l}, \quad \text{as } z \to 0,$$

where $\varsigma(z)$ denotes the Riemann zeta function.

Lemma 8.3.8. [47, Lemma 3.4], Let $|z| \leq \frac{\pi}{\sin\theta}$ with $\theta \in (\frac{\pi}{2}, \frac{5\pi}{6})$ and $-1 < p < 0$. Then

$$Li_p(e^{-z}) = \Gamma(1 - p)z^{p-1} + \sum_{l=0}^{\infty} (-1)^l \varsigma(p - I)\frac{z^l}{l}$$

converges absolutely.

Proof of Lemma 8.3.6. We have, by the definition of $\bar{w}(z)$ in (8.3.2) with $\zeta = e^{-zk}$,

$$\bar{w}(z) = \sum_{j=0}^{\infty} w_j \zeta^j = \frac{1}{\Gamma(2 - \alpha)}(\zeta^{-1} - 2 + \zeta)\left(\sum_{j=0}^{\infty} j^{1-\alpha}\zeta^j\right)$$

$$= \frac{1}{\Gamma(2 - \alpha)}\left((e^{-zk})^{-1} - 2 + e^{-zk}\right)\left(\sum_{j=1}^{\infty} j^{1-\alpha}\zeta^j\right)$$

$$= \frac{1}{\Gamma(2 - \alpha)}\left((e^{-zk})^{-1} - 2 + e^{-zk}\right)Li_{\alpha-1}(\zeta),$$

where $Li_{\alpha-1}(\zeta)$ denotes the polylogarithm function. We then have, by Lemma 8.3.8,

$$Li_{\alpha-1}(\zeta) = Li_{\alpha-1}(e^{-zk}) = \Gamma(2-\alpha)(zk)^{\alpha-2} + \sum_{l=0}^{\infty}(-1)^l\varsigma(1-\alpha-l)\frac{(zk)^l}{l},$$

where $\varsigma(z)$ denotes the Riemann zeta function.

Hence, with some suitable constants $c_2, d_0, d_1 \cdots$,

$$\begin{aligned}\bar{w}(z) &= \left((zk)^2 + \frac{1}{12}(zk)^4 + \cdots\right)\left((zk)^{\alpha-2} + d_0(zk)^0 + d_1(zk)^1 + \cdots\right), \\ &= (zk)^\alpha + c_2(zk)^2 + \cdots\end{aligned}$$

Together these estimates complete the proof of Lemma 8.3.6. $\qquad\square$

Lemma 8.3.9. Let $\zeta = e^{-zk}$ and $z \in \Gamma_k$. Let $\mu_2(\zeta), z_k$ and $\hat{k}(z)$ be defined as in (8.3.22), (8.3.21), (8.3.24), respectively. We have

$$\mu_2(e^{-zk}) - 1 = O((zk)^{2-\alpha}), \quad as\ zk \to 0, \tag{8.3.25}$$

$$c|z| \le |z_k| \le C|z|, \tag{8.3.26}$$

$$\|\bar{k}(z_k) - \bar{k}(z)\| \le Ck^{2-\alpha}|z|^{-\alpha+1}, \tag{8.3.27}$$

$$\|\mu_2(\zeta)\bar{k}(z_k) - \bar{k}(z)\| \le Ck^{2-\alpha}|z|^{1-\alpha}. \tag{8.3.28}$$

Proof. We first show (8.3.25). It is sufficient to show

$$|\mu_2(e^{-w}) - 1| = O(w^{2-\alpha}), \quad as\ w \to 0. \tag{8.3.29}$$

Note that, by Lemma 8.3.6,

$$\begin{aligned}\mu_2(e^{-w}) - 1 &= \left(\frac{e^{-w}}{1-e^{-w}} + c_0 e^{-w}\right)\left(\sum_{j=0}^{\infty} w_j(e^{-w})^j\right)^{\frac{1}{\alpha}} - 1 \\ &= \left(\frac{e^{-w}}{1-e^{-w}} + c_0 e^{-w}\right)\left(w^\alpha + c_2 w^2 + c_3 w^3 + \cdots\right)^{\frac{1}{\alpha}} - 1 \\ &= (e^{-w} + c_0 e^{-w}(1-e^{-w}))(\frac{w}{1-e^{-w}})\left(1 + c_2 w^{2-\alpha} + c_3 w^{3-\alpha} + \cdots\right)^{\frac{1}{\alpha}} - 1 \\ &= (e^{-w} + c_0 e^{-w}(1-e^{-w}))(\frac{w}{1-e^{-w}})(1 + c_2 w^{2-\alpha} + \cdots)^{\frac{1}{\alpha}} - 1 \\ &= f_1(w)f_2(w)f_3(w) - 1,\end{aligned}$$

where $f_1(w) = e^{-w} + c_0 e^{-w}(1-e^{-w})$, $\quad f_2(w) = \frac{w}{1-e^{-w}}$ and $f_3(w) = 1 + c_2 w^{2-\alpha} + \cdots$ and where $c_2, c_3 \cdots$ denote generic constants, which may differ at different occurrences. Thus

we get

$$\lim_{w \mapsto 0} \frac{\mu_2(e^{-w}) - 1}{w^{2-\alpha}} = \lim_{w \mapsto 0} \frac{F(w) + f_1(w)f_2(w)f_3'(w)}{(2-\alpha)w^{1-\alpha}}$$

$$= \lim_{w \mapsto 0} \frac{F(w) + f_1(w)f_2(w)(c_2 w^{1-\alpha} + \cdots)}{(2-\alpha)w^{1-\alpha}},$$

where

$$F(w) = f_1'(w)f_2(w)f_3(w) + f_1(w)f_2'(w)f_3(w)$$

$$= (e^{-w}(-1) + c_0 e^{-w}(-1)(1 - e^{-w}) + c_0 e^{-w}e^{-w})f_2(w)f_3(w)$$

$$+ (e^{-w} + c_0 e^{-w}(1 - e^{-w}))\Big(\frac{(1 - e^{-w}) - we^{-w}}{(1 - e^{-w})^2}\Big)f_3(w).$$

With $c_0 = 1/2$, it is easy to see that $F(w) = O(w)$, $w \to 0$. Further we have $\lim_{w \to 0} f_1(w)f_2(w) = C$. Thus the following limit exists

$$\lim_{w \to 0} \frac{\mu_2(e^{-w}) - 1}{w^{2-\alpha}} = \lim_{w \to 0} \frac{F(w) + f_1(w)f_2(w)(c_2 w^{1-\alpha} + \cdots)}{(2-\alpha)w^{1-\alpha}},$$

which shows (8.3.29).

Next we show (8.3.26). Note that

$$\frac{|z|}{|z_k|} = \frac{|z|}{|\frac{\delta(e^{-zk})}{k}|} = \frac{|zk|}{|\delta(e^{-zk})|}.$$

To show (8.3.26) it suffices to prove $\frac{|zk|}{|\delta(e^{-zk})|}$ has limit as $|zk| \to 0$, which follows form

$$\lim_{w \to 0} \frac{w}{\delta(e^{-w})} = \lim_{w \to 0} \frac{w}{(\sum_{j=0}^{\infty} w_j(e^{-w})^j)^{\frac{1}{\alpha}}} = \lim_{w \to 0} \frac{w}{(w^\alpha + c_2 w^2 + \cdots)^{\frac{1}{\alpha}}} \tag{8.3.30}$$

$$= \lim_{w \to 0} \frac{1}{(1 + c_2 w^{2-\alpha} + \cdots)^{\frac{1}{\alpha}}} = 1.$$

Hence we have proved, for any fixed constant $M > 0$, there exists a constant C such that

$$\frac{|z|}{|z_k|} \le C, \quad \forall |zk| \le M.$$

Similarly we may show $\frac{|z|}{|z_k|} \le C, \forall |zk| \le M$. Thus we get (8.3.26). We now show (8.3.27). Note that

$$z_k - z = \frac{\delta(e^{-zk})}{k} - z = \frac{\delta(e^{-zk}) - zk}{k} = \frac{(\sum_{j=0}^{\infty} w_j(e^{-zk})^j)^{\frac{1}{\alpha}} - zk}{k}$$

$$= \frac{(zk)^\alpha + c_2 z^2 k^2 + \cdots)^{\frac{1}{\alpha}} - zk}{k} = \frac{(zk)(1 + c_2(zk)^{2-\alpha} + \cdots)^{\frac{1}{\alpha}} - zk}{k}$$

$$= \frac{(zk)(1 + \frac{c_2}{\alpha}(zk)^{2-\alpha} + \cdots) - zk}{k} = O(k^{2-\alpha}z^{3-\alpha}), \quad \text{as } kz \to 0.$$

Thus we have, following the proof [63, (4.6)] and noting $||\hat{k}'(z)|| \leq C|z|^{-2}$ in [63, (3.12)]

$$||\hat{k}(z_k) - \hat{k}(z)|| \leq C|z|^{-2}k^{2-\alpha}|z|^{3-\alpha} = Ck^{2-\alpha}|z|^{1-\alpha}$$

Finally we show (8.3.28). Following the same proof as in the proof of [63, (4.3)], we have

$$||\mu_2(\zeta)\hat{k}(z_k) - \hat{k}(z)|| \leq ||(\mu_2(\zeta) - 1)\hat{k}(z_k)|| + ||\hat{k}(z_k) - \hat{k}(z)||$$
$$\leq |zk|^{2-\alpha}C|z|^{-1} + k^{2-\alpha}|z|^{1-\alpha} \leq Ck^{2-\alpha}|z|^{1-\alpha}.$$

Together these estimates complete the proof of Lemma 8.3.9. $\qquad\square$

Remark 10. We remark that assuming that $u_0 \in \mathcal{D}(A)$ rather than $u_0 \in L^2(\Omega)$ reduces the singular behavior of the error bound at $t = 0$. We can prove the convergence order $O(k^{2-\alpha}), 0 < \alpha < 1$ similarly, see Lubich et al. [63].

8.4 Numerical Simulations

In this section, we will consider the convergence rates of the numerical methods with both smooth and nonsmooth data for the following homogeneous problem. Consider

$$_0^C D_t^\alpha u(x,t) - u_{xx} = 0, \quad 0 < x \leq 1, \ t > 0, \tag{8.4.1}$$

$$u(0,t) = u(1,t) = 0, \tag{8.4.2}$$

$$u(x,0) = u_0(x), \tag{8.4.3}$$

where $u_0(x) = x(1 - x)$ or $u_0(x) = \chi_{(0,1/2)}$.

Let $0 < t_0 < t_1 < t_2 < \cdots < t_N = T$ be the time partition and k the time step size. Let N_h be a positive integer. Let $0 = x_0 < x_1 < x_2 < \cdots < x_{N_h} = 1$ be the space partition and h the space step size. We will use finite element method to consider the spatial discretization.

We first consider the scheme (8.1.13)-(8.1.14) and the convergence rate was proved to be $O(k)$ for both smooth and nonsmooth data in [47].

To observe this convergence order, we first calculate the reference solution $u_{ref}(t)$ at $T = 1$ with $h_{ref} = 2^{-6}$ and $k_{ref} = 2^{-10}$. We then use $h = 2^{-6}$ and $k = kappa * k_{ref}$ with $kappa = [2^2, 2^3, 2^4, 2^5, 2^6]$ to obtain the approximate solutions $u(t)$ at $t = 1$. We choose

α		$k = 2^{-8}$	$k = 2^{-7}$	$k = 2^{-6}$	$k = 2^{-5}$	$k = 2^{-4}$	Rate
0.1	(a)	0.02212e-4	0.0496e-4	0.1067e-4	0.2218e-4	0.4564e-4	1.1063
	(b)	0.0055e-3	0.0127e-3	0.0274e-3	0.0570e-3	0.1172e-3	1.1063
0.3	(a)	0.0056e-3	0.0130e-3	0.0280e-3	0.0585e-3	0.1209e-3	1.1100
	(b)	0.0143e-3	0.0333e-3	9 0.0718e-3	0.1479e-3	0.3094e-3	1.1099
0.8	(a)	0.0078e-3	0.0185e-3	0.0403e-3	0.0857e-3	0.1824e-3	1.1359
	(b)	0.0198e-3	0.0466e-3	0.1017e-3	0.2160e-3	0.4595e-3	1.1350
0.9	(a)	0.0054e-3	0.0128e-3	0.0284e-3	0.0621e-3	0.1404e-3	1.1766
	(b)	0.0134e-3	0.0320e-3	0.0708e-3	0.1546e-3	0.3490e-3	1.1757

Table 8.4.1: Time convergence orders with the different α for the L1 scheme

the smooth and nonsmooth initial data (a) $u_0(x) = x(1-x)$ and the nonsmooth data (b) $u_0(x) = \chi_{(0,1/2)}$ we obtain the following results in Table 8.4.1, which are consistent with the Table 8.4.1 in [47]. The convergence order indeed is almost $O(k)$ for the different $\alpha \in (0,1)$ in both smooth and nonsmooth data cases.

We next consider the modified L1 scheme (8.1.19)-(8.1.20) and the convergence rate is $O(k^{2-\alpha})$ for both smooth and nonsmooth data.

To observe this convergence order, we first calculate the reference solution $u_{ref}(t)$ at $T = 1$ with $h_{ref} = 2^{-6}$ and $k_{ref} = 2^{-10}$, We then use $h = 2^{-6}$ and $kappa = [2^2, 2^3, 2^4, 2^5, 2^6]$ to obtain the approximate solutions $u(t)$ at $t = 1$. We choose the smooth and nonsmooth initial data (a) $u_0 = x(1-x)$ and the nonsmooth data (b) $u_0 = \chi_{(0,1/2)}$ we obtain the following results in Table 8.4.2.

We found that the modified L1 scheme has the better accuracy than L1 scheme and the errors are about $1e-05$ or $1e-04$ for all $\alpha \in (0,1)$. The error of the L1 scheme are only $1e-03$.

α		$k = 2^{-8}$	$k = 2^{-7}$	$k = 2^{-6}$	$k = 2^{-5}$	$k = 2^{-4}$	Rate
0.1	(a)	0.0007e-5	0.0030e-5	0.0125e-5	0.0517e-5	0.2197e-5	2.0674
	(b)	0.0018e-5	0.0078e-5	0.0322e-5	0.1333e-5	0.5658e-5	2.0668
0.3	(a)	0.0013e-5	0.00064e-5	0.0291e-5	0.1302e-5	0.5891e-5	2.1914
	(b)	0.0004e-4	0.00017e-4	9 0.0076e-4	0.0339e-4	0.1527e-4	2.1839
0.8	(a)	0.0079e-4	0.0201e-4	0.0462e-4	0.0981e-4	0.1782e-4	1.1223
	(b)	0.0196e-4	0.0496e-4	0.1140e-4	0.2421e-4	0.4407e-4	1.1230
0.9	(a)	0.0141e-4	0.0345e- 4	0.0778e-4	0.1687e-4	0.3848e-4	1.1573
	(b)	0.0347e-4	0.0851e-4	0.1920e-4	0.4162e-4	0.8597e-4	1.1572

Table 8.4.2: Time convergence orders with the different α for the L1 scheme

Chapter 9

Conclusions and Forthcoming Research

This thesis mainly consists of four papers which have been published in the peer refereed international journals. In Chapter 5, we consider the Fourier spectral methods for solving some linear stochastic space fractional partial differential equations perturbed by space-time white noises. The space fractional derivative is defined by using the eigenvalues and eigenfunctions of Laplacian subject to some boundary conditions. We approximate the space-time white noise by using piecewise constant functions and obtain the approximated stochastic space fractional partial differential equations. The approximated stochastic space fractional partial differential equations are then solved by using Fourier spectral methods.

In Chapter 6, we consider the Fourier spectral methods for solving stochastic space fractional partial differential equation driven by special additive noises. The space fractional derivative is defined by using the eigenvalues and eigenfunctions of Laplacian subject to some boundary conditions. The space-time noise is approximated by the piecewise constant functions in the time direction and by appropriate approximations in the space direction. The approximated stochastic space fractional partial differential equations is then solved by using Fourier spectral methods.

In Chapter 7, we consider the discontinuous Galerkin time stepping methods for solving the linear space fractional partial differential equations. The space fractional derivatives are defined by using Riesz fractional derivative. The space variable is discretized by means of a Galerkin finite element method and the time variable is discretized by the discontinous Galerkin method. The approximate solution will be sought as a piecewise polynomial function in t of degree at most $q-1, q \geq 1$, which is not necessarily continuous at the nodes of the defining partition. The error estimates in the fully discrete case are obtained and the numerical examples are given.

Finally, in Chapter 8, we consider error estimates for the modified L1 scheme for solving time fractional partial differential equation. Jin et al. (2016, An analysis of the L1 scheme for the subdiffifusion equation with nonsmooth data, IMA J. of Number. Anal., 36, 197-221) established the $O(k)$ convergence rate for the L1 scheme for booth smooth and nonsmooth initial data. We introduce a modified L1 scheme and prove that the convergence rate is $O(k^{2-\alpha}), 0 < \alpha < 1$ for both smooth and nonsmooth initial data. We first write the time fractional partial differential equations as a Volterra integral equation which is then approximated by using the convolution quadrature with some special generating functions. A Laplace transform method is used to prove the error estimates for the homogeneous time fractional partial differential equation for both smooth and nonsmooth data. Numerical examples are given to show that the numerical results are consistent with the theoretical results.

The importance of research into stochastic space fractional differential equations, time fractional differential equation and their significance to future applications warrants the continued study. We propose some possible research topics in this active research area:

- Spectral method for solving stochastic space fractional PDEs with some special space-time noise.

- Finite element method for solving stochastic space fractional PDEs.

- Stochastic space fractional PDEs with multiplication noise.

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