

THE MULTI-DIMENSIONAL STOCHASTIC STEFAN FINANCIAL MODEL FOR A PORTFOLIO OF ASSETS

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ABSTRACT. The financial model proposed in this work involves the liquidation process of a portfolio of n assets through sell or (and) buy orders placed, in a logarithmic scale, at a (vectorial) price $x \in \mathbb{R}^n$, with volatility. We present the rigorous mathematical formulation of this model in a financial setting resulting to an n -dimensional outer parabolic Stefan problem with noise. The moving boundary encloses the areas of zero trading, the so-called solid phase. We will focus on a case of financial interest when one or more markets are considered. In particular, our aim is to estimate for a short time period the areas of zero trading, and their diameter which approximates the minimum of the n spreads of the portfolio assets for orders from the n limit order books of each asset respectively.

In dimensions $n = 3$, and for zero volatility, this problem stands as a mean field model for Ostwald ripening, and has been proposed and analyzed by Niethammer in [25], and in [7] in a more general setting. There in, when the initial moving boundary consists of well separated spheres, a first order approximation system of odes had been rigorously derived for the dynamics of the interfaces and the asymptotic profile of the solution. In our financial case, we propose a spherical moving boundaries approach where the zero trading area consists of a union of spherical domains centered at portfolios various prices, while each sphere may correspond to a different market; the relevant radii represent the half of the minimum spread. We apply Itô calculus and provide second order formal asymptotics for the stochastic version dynamics, written as a system of stochastic differential equations for the radii evolution in time. A second order approximation seems to disconnect the financial model from the large diffusion assumption for the trading density. Moreover, we solve the approximating systems numerically.

1. INTRODUCTION

1.1. A Stefan problem for the liquidation of a portfolio. Decision making tools play an important role in quantifying the different sources of uncertainty in portfolio management (such as prices, market liquidation, etc.), and on deriving efficient portfolio strategies. Many studies are focused on the portfolio selection problem where the measuring of the performance of portfolios is based on various criteria such as the variance of expected returns, [22], risk minimization and utility maximization, [23].

Liquidation of a portfolio of n assets, is the process of transforming the aforementioned set of assets into cash, for example through sell and buy orders. A certain question of significant financial importance that naturally arises concerns the determination of a profitable price of trading at a specific time t . Moreover, the investor would like to predict an optimal time for liquidation and the dynamics of the spreads, even for short time periods.

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The proposed model in this paper is applicable to the next strategy of liquidation summarized as follows.

Sell and buy orders from the limit order book when one or more markets are considered:

- We observe the evolution of prices for a portfolio of n assets of analogous properties traded during the same financial day in one or different (but interacting) markets, for example currencies in European Union markets. An asset is defined as liquid when it is traded through sell or buy orders, and the prices of zero trading per asset define the relevant spreads. We aim to estimate (predict) through time an average spread at each market. The initial data of the problem will be taken from the limit order books, where the bid and ask prices, as well the volume of trading are included at discrete times in a financial day. The bid and ask prices at time $t = 0$ will induce the initial spreads, while the total liquidity will be estimated by using information from all the interacting markets.
- Under the assumption of infinitesimally small ‘tick size’, that is a minimum permitted price increment of the financial market tending to zero, we may consider continuous price models (continuous space-like coordinates). A Stefan problem for a Heat equation with stochastic volatility posed on the liquid phase will describe the diffusion of the sell or (and) buy orders in time.

We note that an alternative trading strategy would involve stop-loss orders, which consist standing orders to sell an asset when its price drops by a certain percentage. However, this approach is only temporarily effective, for example during a breakdown swing of the market, and may not be the optimal one for a long-term portfolio performance, [30]. On the other hand, there exists a behavioral finance characteristic called as ‘the disposition effect’, which describes the tendency of investors to sell assets when their price is increasing rather than decreasing; this effect is difficult to be predicted, see for example in [20], for a model of asset liquidation, where the investors realize utility over gains and losses, or in [14, 15] for various financial models estimating the prices dynamics related to the limit orders market.

1.2. Motivation for the proposed model. There exist so far some interesting and rigorous results on modeling and well posedness of financial Stefan problems for the Heat equation, even with noise, but up to now are restricted only in dimension one, where the price of one asset is considered; see the pioneering works of Ekström, Zhi Zheng and Müller in [16, 29, 24] for some 2-phases 1-dimensional stochastic Stefan systems, for sell and buy orders of one asset. X. Chen and Dai proposed and analyzed an optimal strategy for multiasset investment on correlated risky assets of a portfolio, [12], while Altarovici, Muhle-Karbez and Soner in [5], presented an optimal policy and leading order asymptotics related to multiple risky assets trading with small and fixed transaction cost.

A natural extension is to consider more than one assets consisting a portfolio and state analogous Stefan problems in dimensions $n \geq 2$ with stochastic volatility.

Various deterministic parabolic Stefan problems have been extensively used for describing the phase separation of alloys and a relevant mathematical theory is already well established. See for example the results of Niethammer, X. Chen and Reitich, Antonopoulou Karali and Yip in [25, 13, 7], or for the quasi-static problem in [1, 2, 3, 4, 26, 10, 9, 11]. Note that the quasi-static problem approximates the parabolic one when the diffusion tends to infinity as in the case of a very large trading activity.

We shall state a stochastic multi-dimensional moving boundary financial problem, and will enlighten in a financial setting for our parameters the existing theory for the boundary dynamics (which is only developed in the deterministic version), mainly motivated by the work of Niethammer in [25]. We provide a financial interpretation of the Gibbs Thomson condition involving the mean curvature of the interface, and propose a simplified model formulation for the approximation of the initial moving boundary by spheres; their radii may implement the price ranges of zero trading around the portfolio intrinsic values. In this case, the mathematical theory for zero volatility, [25], predicts the increasing of the large radii spheres at the expense of the smaller ones, as for example when the price strongly surpasses the assets intrinsic value (financial bubbles).

We use Itô calculus and derive second order formal asymptotics for the stochastic dynamics of the moving interface and the solution of the Heat equation of the Stefan problem. These are presented as a system of stochastic odes, which is solved numerically. Our numerical results indicate that in contrast to the deterministic problem, where static solutions (equilibrium of one sphere or of many equal spheres) evolve very slowly, when noise is present initial states of even one sphere may decrease their volume in relatively small times (or increase). The previous describes a faster liquidation process (or solidification process) for non zero volatility.

The investigation of well posedness and regularity for the fully stochastic version consists a work in progress and it is not considered in this paper. Moreover, the rigorous mathematical derivation of the stochastic dynamics for the moving boundary, which as we shall see involve the mean curvature of the surface, remains a challenging open problem.

2. THE n -DIMENSIONAL OUTER STOCHASTIC STEFAN PROBLEM FOR A PORTFOLIO

2.1. The mathematical statement. We consider a *portfolio* of $n \geq 2$ different assets, and define their trading prices through sell or (and) buy orders by x_1, x_2, \dots, x_n respectively in a logarithmic scale. So, $x := (x_1, x_2, \dots, x_n)$ in general belongs to \mathbb{R}^n (and not restricted as the usual asset prices in \mathbb{R}^{+n}), cf. in [29]. Each asset may consist of one only share, and thus $x_i = x_i(t)$ is the price of the specific share when traded at time t (enwritten, before the logarithmic rescaling, in the limit order book of this share).

Let $w = w(x, t)$ be the *fluctuating density*, cf. [24], or *volume*, cf. [29], of the portfolio placed at price $x = (x_1, \dots, x_n)$. We observe the evolution of the density w in time, for $t \in [0, T]$ and pose a stochastic heat equation on a *liquid phase* $:= \mathbb{R}^n - \mathcal{D}(t)$ with boundary $\Gamma(t)$; this phase is defined as the complement in \mathbb{R}^n of the areas of zero trading (i.e., the complement of the solid phase $\mathcal{D}(t)$).

The liquid phase domain for these n assets describes the set of prices $x \in \mathbb{R}^n$ on which trading is executed and is of course unknown, while it is one of the portfolio characteristics that we would like to determine through the moving boundary problem. The decision of trading, and thus the liquid phase domain, is induced by the distance of the vector

$$x := (x_1, x_2, \dots, x_n),$$

from the boundary $\Gamma = \Gamma(t)$ of $\mathcal{D}(t)$, and this distance is given by

$$\text{dist}(x, \Gamma) := \inf_{y \text{ on } \Gamma} \|x - y\|,$$

where Γ is a curve ($n = 2$), or surface ($n = 3$), or hyper-surface ($n \geq 4$). Here, $\|\cdot\|$ denotes the euclidean norm in \mathbb{R}^n .

The total price of the portfolio is defined by the euclidean norm of x

$$\|x\| := \left(x_1^2 + x_2^2 + \dots + x_n^2 \right)^{1/2}.$$

Remark 2.1. *The euclidean norm in \mathbb{R}^n was used in the mathematical analysis of the physical problem of phase separation of alloys and the asymptotics formulae derived in [25] (involving volumes and surface areas in \mathbb{R}^n measured with this norm) and seems to be proper for the multi-dimensional case; we will avoid thus to define the total price by different measures, like for example the average value of all assets, or the sum of the n prices, which may fit better to one-dimensional approaches.*

We shall consider the following stochastic outer Stefan problem for the Heat equation with noise

$$(2.1) \quad \left\{ \begin{array}{ll} \partial_t w = \alpha \Delta w + \sigma(\text{dist}(x, \Gamma)) \dot{W}(x, t), & x \in \mathbb{R}^n - \mathcal{D}(t) \text{ ('liquid' phase), } t > 0, \\ w = w_0 = 0, & x \in \mathcal{D}(t) \text{ ('solid' phase),} \\ w = -k + w_0 \text{ on } \Gamma(t) & \text{(Gibbs Thomson condition),} \\ V = -\nabla w \cdot \eta \text{ on } \Gamma(t) & \text{(change of liquidity} \\ & \text{driven by the strength of trade,} \\ & \text{also called Stefan condition),} \\ \Gamma(0) = \Gamma_0, & \end{array} \right.$$

where k is the mean curvature of Γ , V the velocity of Γ , $\dot{W}(x, t)$ a space-time noise, and σ is a noise diffusion. Moreover, $\alpha > 0$ is the positive constant coefficient of the Laplacian operator modeling the diffusion of the trading that stabilizes market's variations, cf. also in [30]. We also impose a condition at infinity (far-field value) of the form $\lim_{r \rightarrow \infty} w(r, t) = w_\infty(t)$.

The coefficient α reflects the liquidity of the market, and will be referred as *liquidity coefficient*. An increasing value for α implies that more intense active trading occurs, and thus, it is expected that the solid phase (for example the spreads domain in a case of interest) will become smaller and will reach at an equilibrium earlier in time. For simplicity, we assume that α remains constant for any $t \in [0, T]$ which is a reasonable assumption, when evolution is observed in short time intervals, as for example during a day.

The noise diffusion σ is a *volatility* that depends on the distance of the prices vector x from the liquidity boundary Γ . In dimension one, in [29, 24], the authors proposed a volatility of the form $\sigma = \sigma(|x - S^*(t)|)$ for $S^*(t) \in \mathbb{R}$ the mid price of one share from the limit order book, and $x \in \mathbb{R}$ its price in a logarithmic scale. This represents the dependence of the noise strength on the distance of the current price x from an average price S^* (mid price there), which is equal to $|x - S^*|$, when the spread is zero. The analogous argument for a model permitting non zero spreads, in dimensions n where the distance is measured by the euclidean norm in \mathbb{R}^n , leads to a volatility definition of the form

$$\sigma = \sigma(\text{dist}(x, \Gamma(t))).$$

The volatility of an asset is a measure of the dispersion of the prices of the asset as it evolves in time. If the price remains stable the volatility is low and the risk for holding (not trading) the asset is low. In the case of a portfolio the volatility can be generalized to be a measure of risk of the investment. Let us assume a price in the liquid phase; as the distance of the price vector from the boundary of the liquid phase increases the risk for not making a transaction increases as well, and the volatility achieves a higher level.

In the proposed problem, by definition, in the solid phase zero trading occurs, and therefore, the density of trading therein is zero for any time t , i.e., $w_0(t) = 0$.

Remark 2.2. *Even if not analyzed in this paper, we point out that in various one-dimensional Stefan problems for limit orders (applied under the simplification assumption of a zero spread though)*

a usual choice for the density w_0 is again zero. However, in these approaches (due to the vanishing spread), $w_0 = 0$ only models the immediate execution of the sell or buy orders from the limit order book of one share (when decided), and seems feasible enough, due to the direct computerized interaction of the network of various trade markets, see in [29]. A positive constant $w_0 > 0$ could be also considered in dimensions one for these cases, when a certain delay of transactions is inserted in the model. Note that a simple change of variables of the form $w \rightarrow w - w_0$, as the spde and the Stefan condition for the velocity are linear, leads to a zero delay model.

The Stefan b.c. describes the velocity of the interface $\Gamma(t)$, which of course is given by the jump of the gradient of the density w along Γ . In our problem, in the solid phase $w = w_0 = \text{const} = 0$, so the jump involves only the gradient of the density in the liquid phase, since the other term in the difference is vanishing, cf. also [25]. A Stefan b.c. of this kind, where the velocity is given by the jump at a mid price (the moving boundary then consists of a moving point on a line), has been already proposed in [24, 30], for a system of 2 equations for sell and buy orders respectively in dimension one, and describes the change of the mid price driven by the strength of the ask price. In our case it is the liquidity area that changes and this change is driven by the strength of trading since the evolution of the (total) density w (volume of transactions in sell and buy orders) for all the n assets is given by one equation but posed in dimensions n .

A detailed motivation for the Gibbs-Thomson b.c. condition and its financial interpretation will be presented at a separate section, in the sequel.

Two-phases elliptic Stefan problems with analogous b.c. appear as the sharp interface limit of the Cahn-Hilliard equation, [1], or the stochastic limit of Cahn-Hilliard equation with noise, [6]. Moreover, considering Allen-Cahn or the stochastic Allen-Cahn equation, the law of motion on the sharp interface limit is described by a velocity given by the mean curvature or stochastic mean curvature respectively (and not by the jump); see for example the classical results of Evans, Soner, Souganidis in [17] for the deterministic equation, and this of Funaki for the stochastic case with mild noise, [18].

Let us describe the mathematical statement of (2.1) in terms of a moving boundary problem. It is a one-phase outer parabolic Stefan problem, since the parabolic type spde (Stochastic Heat equation) is posed only on one phase, the liquid phase, placed outside the solid one. At the initial time $t := 0$, the solid phase $\mathcal{D}_0 := \mathcal{D}(0)$ is considered already formed as a bounded domain in \mathbb{R}^n and thus, its boundary $\Gamma_0 := \Gamma(0)$ is given. Obviously, due to boundedness, Γ_0 is a closed hyper-surface of \mathbb{R}^{n-1} , embedded in \mathbb{R}^n . As it is usual to Stefan problems from phase separation, Γ_0 , in a more general setting, is a union of such surfaces, [26, 1]. So, at the initial time, (2.1) is fully determined by one Stochastic Heat equation posed on the unbounded domain $\mathbb{R}^n - \mathcal{D}_0$ with non-homogeneous Dirichlet b.c. on the boundary Γ_0 involving the mean curvature of Γ_0 . The solution w of the above, through ∇w , defines then the velocity V of the moving boundary $\Gamma(t)$, and therefore its evolution and shape at a next time, and this determines the new unbounded domain with boundary Γ where the spde is posed, and so on.

In [25], the initial boundary Γ_0 has been assumed to consist of a union of spherical surfaces; therein, in the deterministic setting and in dimensions $n = 3$, the same problem (2.1), for $\alpha > 0$ has been considered but for $\sigma = 0$, and for a different application from material science, the Ostwald Ripening of alloys. In particular, Niethammer in [25] analyzed a mean field approximation model where the solid phase is a union of spherical domains with fixed centers and evolving radii, and derived the dynamics of radii. Moreover, she proved well posedness for the static (elliptic) problem with undercooling, [26]. Antonopoulou, Karali and Yip in [7] proved well posedness for the full

parabolic problem with undercooling and obtained the modified dynamics of radii; cf. also the work of X. Chen and Reitich for the two-phases deterministic Stefan problem, [13], and in [2, 3, 4] for a quasi-static version.

Remark 2.3. *The existing rigorous mathematical literature on well posedness and dynamics of multi-dimensional two-phases Stefan type problems (cf. for example [1, 13] and the references therein), concerns so far the deterministic problem where the same exactly pde is posed on the two different phases, i.e. on $\mathbb{R}^n - \Gamma$. However, if the trading is to be classified in sell and buy orders corresponding to a liquid and a complementary solid phase respectively, the model would demand a system of 2 equations of the form appeared in (2.1) with different parameters, and it has been very recently analyzed and only in one dimension, [30, 24]. In higher dimensions, $n \geq 2$, there exist many open questions on existence, regularity and dynamics for Stefan problems posed as a system of two equations even in the absence of noise.*

2.2. The spherical boundaries stochastic Stefan model for $n = 3$. In the general stochastic Stefan problem (2.1) we set $n = 3$. So, we consider that the initial solid phase is in \mathbb{R}^3 . Moreover we assume that the initial solid phase is the union of I spherical domains, and, as in [25], that during evolution the centers remain constant.

So, we define

$$\mathcal{D}(t) := \cup_{i \in I} B_{R_i}(t),$$

for $B_{R_i}(t)$ a ball of radius $R_i(t)$ and fixed center $x_c^i \in \mathbb{R}^3$ for $i \in I$. The boundary $\Gamma(t)$ at time t is the union of the I spherical boundaries

$$\Gamma_i(t) := \partial B_{R_i}(t),$$

and so, given by

$$\Gamma(t) := \cup_{i \in I} \Gamma_i(t).$$

The problem (2.1) is transformed into I problems, given for each $i \in I$ by

$$\begin{aligned} \alpha^{-1} \partial_t v(x, t) &= \Delta v + \alpha^{-1} \sigma (\text{dist}(x, \Gamma)) \dot{W}(x, t), \quad x \in \mathbb{R}^n - \mathcal{D}(t) \quad (\text{'liquid' phase}), \quad t > 0, \\ v &= 0, \quad x \in \mathcal{D}(t) \quad (\text{'solid' phase}), \\ v &= \frac{1}{R_i(t)} \quad \text{on } \Gamma_i(t), \quad (\text{Gibbs Thomson condition}), \\ \dot{R}_i(t) &= \frac{1}{4\pi R_i^2(t)} \int_{\Gamma_i(t)} \nabla v \cdot \eta, \quad (\text{Stefan condition}), \\ \Gamma(0) &= \Gamma_0, \end{aligned} \tag{2.2}$$

where, as in [25], we applied the transformation

$$v := -w,$$

replaced the curvature of the sphere by the inverse of its radius, and integrated the last b.c. along the spherical boundary; for the general transformation see at the first lines of pg. 125 of [25], in particular, we took $H = K$, and all appearing constants equal to 1 except of T_0 taken as $T_0 := 0 = w_0$ and of C taken as $C := \alpha^{-1}$. Also see the statement of the transformed problem at pg. 127 of [25], for zero volatility σ . Note that \dot{R}_i denotes the time derivative of the i radius. The far-field value takes the form $\lim_{r \rightarrow \infty} v(r, t) = v_\infty(t)$ and $v_\infty(t)$ consists one of the unknowns of the problem.

When volatility is not a vanishing quantity, then the integral of the b.c. along $\Gamma_i(t)$ is formally taken, assuming that the sphere remains a sphere during evolution, but with stochastically fluctuating radius.

Remark 2.4. *A ball in \mathbb{R}^n , is the logarithmic image of a bounded simply connected domain in the initial coordinates in \mathbb{R}^{n+} , where the initial real values of the portfolio are set.*

2.3. Financial interpretation of the Gibbs Thomson condition. Ostwald in [27], first observed that during the late stages of phase separation also called as coarsening, the evolution favors the minimization of surface energy of the inner interfaces separating the phases. Considering the case of liquid/solid phase transitions, the previous is translated to the reduction of the surface area of the solid phase, where the diffusional mass (measured by the integral of our density solution in the liquid phase) is transferred from regions of high interfacial curvature to regions of low interfacial curvature, [25]. The Gibbs Thomson condition of problem (2.1) involving the curvature k is an effective approximation of the above growth law and is extensively used to the literature of multi-dimensional Stefan problems, where the geometric characteristic of the curvature of curves, $n = 2$, or surfaces $n \geq 3$, (which form the phase separation sharp interfaces), has a meaning.

When the solid phase at time t consists of 2 well separated spherical domains of radii $R_1(t) > R_2(t)$ and thus of curvatures $\frac{1}{R_1(t)} < \frac{1}{R_2(t)}$, in later times as separation evolves, the growth of the larger sphere is expected (here this of radius R_1) at the expense of the smaller. In fact this is rigorously proved for $n = 3$ and zero volatility in [25, 7] for the Stefan problem of type (2.2), and for a more general case where kinetic undercooling acts on the Gibbs Thomson condition.

Moreover in the case of solid phase of a more complex geometry, the aforementioned optimization constraint set by the growth law leads during evolution to minimizing area moving boundaries close to spheres, cf. [3, 4].

In the financial setting, let us consider that at our initial time, the solid phase consists of 2 well separated balls with centers two marginal estimations, or even real but different observations for the vectorial price x of our portfolio, as for example when two markets are participating by trading all the $n = 3$ assets during the same period of one financial day; $n = 3$ currencies in European Union is a basic case.

In accordance to (2.2) notation, we define the fixed centers by

$$x_c^1, \quad x_c^2,$$

and the initial radii by

$$R_1(0) < R_2(0),$$

small enough, which as we shall analyze in detail in a following section represent the half of the minimum of the n spreads respectively at the given initial time, and obtain the well separated balls condition

$$\|x_c^1 - x_c^2\| \gg R_1(0) + R_2(0).$$

Remark 2.5. *The scaling of the Stefan problem (2.2) is of significant importance, this being related to the mean field assumption of an initial solid phase consisting of I well separated spherical domains with relatively small radii, that do not touch during evolution; a result is a comparatively very large magnitude for the liquidity coefficient α .*

For example, see also in [25] for an analogous condition after rescaling, in dimensions $n = 3$ and for $\sigma = 0$, the condition (4.2), which we will analyze more extensively in a following section, must

hold, *i.e.*

$$I \max_{i=1, \dots, I} R_i(0) \ll \mathcal{O}(\alpha^{4/9}).$$

As time passes, for the problem (2.2), and when $\sigma := 0$, the theory predicts that the smaller ball of radius $R_1(0)$ will begin to shrink while the other will grow. This means that the smaller (minimum) spread will be reduced in the first market while the larger (minimum) spread will increase in the second market. The above is indeed expected since small spreads are observed to highly traded assets and tend thus to reduce. On the other hand a comparatively large spread is an index of low trading and of higher risk for the investor.

Remark 2.6. *Considering the problem (2.2) ($n = 3$), for $\sigma = 0$, for I initial balls, there exist the so-called vanishing times t_{vi} for the spherical domains constituting the solid phase, [25]. A case of interest is the equilibrium where only one of the I balls survives and stands as the final solid phase, while its diameter approximates a maximum spread value in the time interval $[0, T]$. Here, T is equal to the last vanishing ball time. However, there exist equilibria of more than one balls of equal radii.*

In the case of two balls for the initial solid phase ($I := 2$), with radii ordered as follows

$$R_1(0) < R_2(0),$$

since the smaller will eventually vanish, let us say at $t := t_v$, the maximum spread in $[0, T]$ coincides to

$$2R_2(t_v).$$

The above, gives a useful prediction for the future optimal investment of the portfolio.

The evolution of the I radii, and thus, their values and vanishing times are well estimated from the approximating dynamics for the radii which are given in a following section by the ODEs (4.4), (4.3).

Remark 2.7. *The quasi-static version of the parabolic problem (2.2), assumes a diffusion coefficient $\alpha \rightarrow \infty$. For this model and for $\sigma := 0$, volume conservation holds for the solid phase, see Lemma 2 at pg. 135 of [25]. Thus, the diameter of the last surviving ball is given by*

$$(2.3) \quad 2 \left(\sum_{i \in I} R_i(0)^3 \right)^{1/3},$$

*which is the largest observed spread and depends strongly on the initial definition of the solid phase, in particular on the initial radii, *i.e.* the initial spreads.*

2.4. A liquidation strategy. Efficient strategies for portfolios management are based on the quantification of the uncertainty of prices and of market liquidity. The portfolio optimal performance is restrained by the control of the variance of the expected returns under minimum risk investment policies where a certain utility function is maximized, [22, 23].

In [22], an investor wants to allocate his initial amount among a given number of assets where the expectations of returns are taken as known. The criterion for determining the set of optimal portfolios, that is, the optimal weights (proportions of total wealth) assigned to the assets, is to minimize the variance of the expected returns. However, the limitation of this approach is that the expected returns (which are estimated by financial data) are assumed to be constant in time; this is also called the static optimization problem.

Merton, in [23], performs a continuous-time analysis for the problem of optimal portfolio selection where the rates of return of the individual assets are generated by a Wiener Brownian motion

process. The optimal proportions of total wealth (or weights) that are invested in each asset for any given time is derived by maximizing its expected utility as a function of wealth.

As we shall describe, the solution of the Stefan problem (2.1) with properly defined parameters can contribute as a recommendation tool to an investor who already holds a portfolio of assets and wishes to liquidate to cash some fraction of each one of them during a time interval $(0, T)$.

We consider a financial market of n risky assets with prices per share

$$p_i(t) \in \mathbb{R}^+, \quad i = 1, \dots, n,$$

at time t . An investor holds a portfolio of these assets with allocations

$$s(t) = (s_1(t), \dots, s_n(t));$$

here, $s_i(t)$ is the number of shares of the i asset at t . So, the value of the portfolio at time t is given by

$$(2.4) \quad \mathcal{V}(t) = \sum_{j=1}^n s_j(t)p_j(t).$$

When the portfolio is liquidated, $p_i(t)$ can be specified on real time by the limit order book of asset i , or be predicted in advance. In general, s_i varies in time.

Let $f_i(t)$ denote the fraction of the initial amount of asset i that the investor wants to sell at time t ($f_i \in [0, 1]$). The allocation of asset i at time t can be modeled by

$$(2.5) \quad s_i(t) := s_i(0) - f_i(t)s_i(0) = (1 - f_i(t))s_i(0),$$

for $s_i(0)$ some initial given allocation of asset i at the initial time $t = 0$. Here, we consider that, for any i , f_i are defined to satisfy $f_i(0) = 0$. This implies that at the initial time the investor will never choose to sell any share of his portfolio. Therefore, by replacing s_i , the investor's portfolio allocation is given by the vector

$$(2.6) \quad s(t) = \left(s_1(0) - f_1(t)s_1(0), \dots, s_n(0) - f_n(t)s_n(0) \right).$$

We consider as time $t \in [0, T]$ the first instant that the investor sells parts of his portfolio, and thus, no transaction has been performed in the interval $[0, t)$. Time t is a part of investor's strategy that will be derived based on information offered by the evolution of the whole market as shown in the next section. As soon as a transaction is performed at time t , the model is initiated and the time is set to 0 again. So, a new period $[0, T]$ starts for the investor for future transactions.

This trading activity will affect the portfolio performance in terms of returns and risk. Liquidation strategies are developed and applied in order to ensure that the remaining portfolio will have a high rate of return.

We fix a time t , and define

$$(2.7) \quad C(t) := \sum_{j=1}^n f_j(t)s_j(0)p_j(t),$$

as the amount of consumption resulted by the liquidation of the portfolio.

At the asset i , for any $i = 1, \dots, n$, we assign at time t the non-negative weight $z_i = z_i(t)$, defined as

$$(2.8) \quad z_i(t) = \frac{s_i(t)}{\sum_{j=1}^n s_j(t)},$$

so that

$$\sum_{i=1}^n z_i(t) = 1, \text{ and } z_i(t) \in [0, 1].$$

Let $\mathcal{R}(t)$ be the rate of return of the remaining portfolio at time t ; $\mathcal{R}(t)$ is defined as

$$\mathcal{R}(t) := \sum_{i=1}^n z_i(t) \frac{p_i(t)}{p_i(0)} = \sum_{i=1}^n \frac{(1 - f_i(t))s_i(0)}{\sum_{j=1}^n (1 - f_j(t))s_j(0)} \frac{p_i(t)}{p_i(0)},$$

where we replaced the weights z_i by using (2.8) and (2.5).

We define as utility U of the investor a measure that captures the satisfaction he obtains when involved in trading activities concerning his portfolio. More precisely,

$$(2.9) \quad U = U(\mathcal{V}(t), C(t)),$$

is assumed to be a strictly concave function of the value \mathcal{V} of his portfolio, and of the consumption level C that is liquidated at time t .

Definition 2.8. *We define the liquidation strategy at time t to be the vector of fractions*

$$f(t) = (f_1(t), \dots, f_n(t)),$$

for f the solution of the following maximization problem with constraint:

$$(2.10) \quad \begin{aligned} & \max_f U(\mathcal{V}(t), C(t)), \\ \text{s.t.} \quad & \sum_{j=1}^n s_j(0) - \sum_{j=1}^n s_j(t) \leq w^*, \end{aligned}$$

for w^* the level of the available total volume of all the shares of the assets in the portfolio. Here, remind that the fractions f_i appear in the formulae of \mathcal{V} and C , cf. (2.4), (2.6), (2.7), while U is the utility function given at (2.9).

In order to solve the optimization problem (2.10) (which is not in the aims of the current work), one has first to estimate the prices vector $p(t) = (p_1(t), \dots, p_n(t))$ of the shares which appear in the definition of $\mathcal{V}(t)$ and $C(t)$, and w^* . The solution of the Stefan problem (2.2), in particular the moving boundary, provides a prediction in a logarithmic scale (for example the spreads) for the price vector $p(t)$ at a given time t . This information is crucial for maximizing the utility function.

Evidently, the liquidity parameter α and the choice of initial solid phase, even if defined mathematically, when replaced should be related to a specific financial application since they concern the market's characteristics.

Our aim in the following section is to address a main financial application of the Stefan problem as stated in (2.2), which is posed in a logarithmic scale for the spatial coordinates x , where the trading (i.e., the diffusion of the density) is observed to one or more financial markets.

The spatial coordinates defining $x = x(t) = (x_1, \dots, x_n)$ will correspond to the prices of trading (sell/buy) at time t of n different shares, while the solid phase diameter will approximate the minimum of the n spreads for orders from the limit order book.

We shall properly define the prices x_i , $i = 1, \dots, n$ and the initial data of (2.1) in the version (2.2): α , $\mathcal{D}(0)$ (the initial solid phase which is related to zero trading areas), and $\Gamma(0)$ (the initial solid phase boundary). In addition, we shall present carefully the financial interpretation of all these parameters.

3. SELL/BUY ORDERS AND SPREADS FROM THE LIMIT ORDER BOOK

3.1. Preliminaries. In portfolio selection, the investor uses financial data such as expected prices, rates of return, market liquidity and many other; these parameters are often estimated by historical data provided by the limit order books of the assets of interest.

The evolution of market sell or buy limit orders for a particular asset placed by investors in a financial market is described in the limit order book, [19]. At any time $t \in [0, T]$, the limit order book contains a list of sell and buy limit orders for an asset, and it is continuously updated in $[0, T]$. The information contained in an order book is significant for discovering the price of an asset, and affects substantially the investors' decisions on choosing optimal trading strategies.

A trading strategy (or limit order) is characterized by three components: the time to place the order, the quantity of shares that is for trade and the limit price per share.

In particular, a sell (buy) order is placed in the i limit order book (which is denoted by LOB_i), when an investor wants to sell (buy) a specified number of shares of asset i at or over (below) a specified price; this price is called *limit price*.

Let $A_i(t)$ be the *ask price* which is the lowest sell order (i.e., the minimum price at which the investor is willing to receive), and let $B_i(t)$ be the *bid price* which is the highest buy order (i.e., the maximum price at which the investor is willing to pay), both contained in the order book. The ask price is always higher than the bid price. Thus, a sell order that arrives at time t is executed, more specifically the asset is sold, if the associated price being set by the investor is lower than the current bid price at time t . Otherwise, the sell order is sorted in the list of the order book.

The average of the the ask and bid prices of the asset i at time t

$$(3.1) \quad \bar{p}_i(t) := \frac{A_i(t) + B_i(t)}{2},$$

is called *mid price*, while the difference

$$(3.2) \quad spr_i(t) := A_i(t) - B_i(t),$$

between the ask and bid prices at time t defines the *spread* for the order book of the asset i .

Remark 3.1. *The spread reflects the liquidity of the asset. Liquidity is a measure that describes how quickly the asset is traded. For example, a high liquidity asset is cash or currency, while a low liquidity asset is art or real estate. An overview of indicators that can be used to measure liquidity can be found in [21]. The dependence of the spread of an asset with its liquidity indicates an inverse relation: a wide spread implies a low liquidity asset.*

For a portfolio of n different shares with prices $p_i \in \mathbb{R}^+$, $i = 1, \dots, n$, let

$$x = (x_1, \dots, x_n) := (\ln(p_1), \dots, \ln(p_n)) \in \mathbb{R}^n.$$

Let $w = w(x, t)$ (in accordance to the notation used in (2.1)) be the *fluctuating density*, cf. [24], or *volume*, cf. [29] of the *limit sell and buy orders* of all n assets placed at price x (i.e., corresponding to $p := (p_1, \dots, p_n)$ before the logarithmic scaling).

Referring to the initial coordinates p_i , the zero trading domain, contains all the possible prices in an n -dimensional open rectangular domain \mathcal{S} induced by the ask and bid prices of each of the n shares. More specifically, for all prices p_i , $i = 1, \dots, n$, being lower than the respective ask prices, and higher than the respective bid prices contained in the limit order books at time t , no trading is possible. The edges of the rectangle have lengths equal to the spreads $spr_1(t), \dots, spr_n(t)$, since for each $p(t) \in \mathcal{S}$ the coordinate $p_i(t)$ is within the interval $[B_i(t), A_i(t)]$ of the respective order book. Obviously, the mid price $\bar{p}_i(t)$ of each asset is the midpoint of $[B_i(t), A_i(t)]$, and the ‘center’ of \mathcal{S} at the same time t is given by the coordinates of the mid prices vector $\bar{p}(t) := (\bar{p}_1(t), \dots, \bar{p}_n(t))$.

Note that when more (interacting) markets are considered at the same time t , the zero trading area consists of more than one domains, defined by the corresponding spreads and mid prices for the same shares taken from the limit order books of the different markets.

At any price vector p outside \mathcal{S} there is a possibility of trading (either sell or buy), and the volume of the portfolio at this price indicates the total number of shares that may be sold or bought. So, we may have one of the following trading activities depending on the position of this p :

- (1) Sell opportunities for all or some assets.
- (2) Buy opportunities for all or some assets.
- (3) Sell opportunities for some assets and buy opportunities for other.

Of course since the evaluation of spreads is observed in discrete times, the aforementioned definition of zero trading areas is an idealized one. In practice the boundary of \mathcal{S} , as defined, will include all price vectors that are the most favorable for obtaining the available shares of the assets, if the respective sell/buy orders are executed. Roughly speaking, in $\bar{\mathcal{S}}$ (i.e., the boundary included) the volume of trading is minimized; we mention that on the boundary, the prices optimize trading: i.e. for a sell order the price on the boundary results in higher profits (though the probability of trade is decreased).

It is expected that the higher the distance of prices vector p from the spreads area is, the higher the trading is. Note that the rate of change of the trading volume of an asset may vary significantly in the time interval $[0, T]$; for example, as frequently observed, there is a decrease of trading during lunch time. In addition, this rate may be influenced by the impact of trading activities involving the same asset in other financial markets, or when new information arrives about the asset.

Remark 3.2. *In our approach, we shall consider one differential equation for both sell and buy orders assuming that the demand and the supply of the assets in the portfolio evolve according to a single parameter α that is related to the total liquidity of the markets.*

3.2. Solid phase of spherical domains with varying radii and constant centers. As already mentioned, $p(t)$ is the price vector of the portfolio at time t . Remind that the portfolio consists of n different shares.

We apply the change of variables

$$x = (\ln(p_i), \dots, \ln(p_n)),$$

for the new coordinate system in space. At time t the solution of the Stefan problem defined in (2.1) provides the total density $w(x, t)$ and the solid phase $\mathcal{D}(t)$ at which $w = 0$.

Motivated by the model described by Niethammer in [25] for zero volatility, where a union of spherical domains constitutes the initial solid phase, while these domains remain spherical, we approximate the initial solid phase at time $t = 0$ with a spherical one centered at the rescaled coordinates of the initial mid price vector

$$(3.3) \quad \bar{p}(0) = \left(\frac{A_1(0) + B_1(0)}{2}, \dots, \frac{A_n(0) + B_n(0)}{2} \right) =: (\bar{p}_1(0), \dots, \bar{p}_n(0)),$$

i.e. for $I = 1$ the center is given by

$$(3.4) \quad x_c = (x_{c1}, \dots, x_{cn}) := (\ln(\bar{p}_1(0)), \dots, \ln(\bar{p}_n(0))).$$

The analogous approach can be applied for $I \geq 1$, by using the data of the limit order books of each $i = 1, \dots, I$ market.

The radius of this initial spherical solid phase $\mathcal{D}(0)$ is defined by

$$(3.5) \quad R(0) := \min_{i=1, \dots, n} \frac{lspr_i}{2},$$

for

$$(3.6) \quad lspr_i := \ln(A_i(0)) - \ln(B_i(0)).$$

So, we exclude the larger spread values; this is a reasonable strategy when in our portfolio assets of analogous spreads are considered as for example currencies. Moreover, for small spreads of order $\mathcal{O}(10^{-1}) - \mathcal{O}(10^{-4})$, which is the usual case for assets of high liquidity, even in the logarithmic scale, the minimum spread enclosed area is a good approximation of the solid phase; see for example the following data for the British Pound versus US Dollar currency taken in March 2019, [8], 19 March 2019 British Pound v US Dollar Data Latest GBP/USD: Exchange Rate: 1.3275, Bid: 1.3275, Ask: 1.3276, Market Status: Live, Percent Change: +0.0939, Today's Open (00:01 GMT): 1.3262, Today's High: 1.3309, Today's Low: 1.3241, Previous day's Close (23:59 GMT): 1.3263, Current Week High: 1.3309, Current Week Low: 1.31841, Current Month High: 1.33786.

The center of the spherical domain represents the mid vectorial price of the n assets and the radius represents the range of the minimum spread of all n assets around the mid price.

We also assume that the center of the spherical domain remains constant in time, that is the mid price of each order book does not change in a small time horizon (e.g. within a day).

We can extend our model of one financial market to the scenario where the investor is interested in taking part in more than one markets for the same portfolio. This is translated to considering more than one domains of different radii $R_i(t)$, one for each financial market, and as solid phase the union of them. When significantly different mid prices per market and relatively small spreads occur, the initial spherical domains can be assumed well separated and placed far enough one from the other so that during evolution they do not touch. This is in accordance to the necessary assumption for the deterministic Stefan problem model of [25] where the theory predicts the increase of the larger spherical domain at the expense of the smaller, at least when the volatility is zero. In such a case, the coefficient α of the SPDE of (2.2) will be related to the total liquidity of the different markets.

In the above, our aim was to approximate the initial solid phase by a spherical domain of diameter the minimum spread value that theoretically can be taken at the specific initial time t_0 from the limit order books. However, in realistic cases, the provided data are discrete. We define instead an

average value of historical data very close to the initial time t . This is implemented by using the data of the following Definition 3.3.

Definition 3.3. *Let t_1, \dots, t_m be m time instants prior to time interval $[t_0, t_0 + T]$ (where t_0 is the initial time). We will use the following data for each order book LOB_i , $i = 1, \dots, n$ at each time t_j , $j = 1, \dots, m$: the ask price $A_i(t_j)$, the bid price $B_i(t_j)$, the spread $spr_i(t_j) = A_i(t_j) - B_i(t_j)$ and the total volume w_i of sell orders for each asset i that have been executed additively during the m time instants.*

More precisely, in order to estimate the mid price and the spreads of the n assets at t_0 (which will be set to 0), we define the average version of (3.3), where an average initial mid price vector is used, given by

$$(3.7) \quad \bar{p}_a := \left(\sum_{j=1}^m \frac{A_1(t_j) + B_1(t_j)}{2m}, \dots, \sum_{j=1}^m \frac{A_n(t_j) + B_n(t_j)}{2m} \right) =: (\bar{p}_{a1}, \dots, \bar{p}_{an}).$$

This will define the center of the solid phase at $t_0 := 0$ by the logarithm of its coordinates, i.e., the definition (3.4) is replaced by

$$(3.8) \quad x_c := (\ln(\bar{p}_{a1}), \dots, \ln(\bar{p}_{an})).$$

In this case the initial radius at $t_0 := 0$ is given by the following averaged version of (3.5)

$$(3.9) \quad R(0) := \min_{i=1, \dots, n} \frac{lspra_i}{2},$$

for

$$(3.10) \quad \begin{aligned} lspra_i &:= \ln \left(\sum_{j=1}^m A_i(t_j)/m \right) - \ln \left(\sum_{j=1}^m B_i(t_j)/m \right) \\ &= \ln \left(\sum_{j=1}^m A_i(t_j) \right) - \ln \left(\sum_{j=1}^m B_i(t_j) \right). \end{aligned}$$

We also assume that the fixed cost of liquidation per share is equal to a fixed rate of general transaction costs ($H = K$ for the physical problem, which appears in the Stefan condition for the velocity).

3.3. The liquidity coefficient α . Estimations of various characteristics of the market, such as the level of liquidity, the assets evaluation, and the volume of trading activity per temporal period (for example during a financial year) consist the context of the limit order book as we discussed in the previous section; see for example the relevant survey presented in [28]. The limit order represents the trade of a specified amount of an asset at a predetermined price.

The Laplacian coefficient $\alpha > 0$ of the SPDE of Stefan problem (2.1) (appearing also in (2.2)), measures the diffusion strength of sell and buy orders during trading; remind that for the financial application considered in this section, the total trading is observed, and so, in liquidation sell and as well buy orders participate.

A large value $\alpha \gg 0$ models a high-volume market with intense trading activity.

We shall assume that α is constant in a short period of one day, this implying that the tendency of the demand of the market concerning the assets of interest will not change its pattern. Since α reflects the liquidity of the market, we expect that it will increase as the number of the shares of

the assets that are traded (sold and/or bought) in the recent past is increased. Also a low spread shows a tendency of the market to face a high liquidity.

Taking these remarks into account we define α to be a weighted average of the liquidity measures of each asset separately which will be denoted by the symbol α_i .

We shall use the following historical data from the order book of each of n shares of Definition 3.3.

Let

$$(3.11) \quad w_{\text{tot}} := \sum_{i=1}^n w_i,$$

be the total number of sell and buy orders for all assets that have been executed during the m time instants prior to $[t_0, t_0 + T]$ (we will set $t_0 := 0$).

Let also

$$(3.12) \quad s\bar{p}r_i := \sum_{j=1}^m \frac{A_i(t_j) - B_i(t_j)}{m},$$

be the average spread of i share; we define

$$(3.13) \quad \alpha_i = \frac{w_i}{s\bar{p}r_i},$$

as the measure of liquidation of asset i .

The liquidity coefficient α_{in} is defined then as follows

$$(3.14) \quad \alpha_{\text{in}} := \sum_{i=1}^n \alpha_i \frac{w_i}{w_{\text{tot}}} = \sum_{i=1}^n \frac{w_i^2}{s\bar{p}r_i w_{\text{tot}}}.$$

However, since we will use a logarithmic scale for the space variables, we shall define in the rescaled problem (2.1) or (2.2) α as follows

$$(3.15) \quad \alpha := \sum_{i=1}^n \frac{w_i^2}{(l\text{spra}_i) w_{\text{tot}}}.$$

Remark 3.4. *The formulae (3.11)-(3.15) were implemented for the case of one market participating and so for only one ball for the initial solid phase of zero trading. In the general case of I balls (see (2.2)), we apply the same formulae for the limit order books of each $i = 1, \dots, I$ market for the same n assets, and compute each respective liquidity coefficient; let this be denoted by α^i . We may then define as α the average, i.e.*

$$\alpha := \frac{1}{I} \sum_{i=1}^I \alpha^i.$$

3.3.1. *An example.* We consider a portfolio of three assets ($n = 3$) and historical data for $m = 5$ time instances. Tables 1, 2, 3 show the data of the respective order books. Table 4 shows the number of shares of the assets sold and bought in the $m = 5$ periods, or in the logarithmic scale, for defining α we use the Table 5. Based on the above data, we derive the following parameters: the initial center of the spherical domain at time 0 is given by

$$(3.16) \quad x_c = (x_{c1}, x_{c2}, x_{c3}) = (3.416906675, 2.714694744, 3.022860941),$$

TABLE 1. A sample of 5 quotes for asset 1

Time t_j	$A_1(t_j)$	$B_1(t_j)$	$spr_1(t_j)$	$\frac{A_1(t_j)+B_1(t_j)}{2}$
9:00	30.25	29.75	0.5	30
9:02	30.75	29.50	1.25	30.125
9:04	31.00	29.25	1.75	30.125
9:06	31.50	29.00	2.50	30.25
9:08	35.00	28.75	6.25	31.875
Sum	158.5	146.25	12.25	152.375
\bar{spr}_1	$12.25/5 = 2.45$			
$lspra_1$	$\ln(158.5) - \ln(146.25) = 0.080437107$			
x_{c1}	$\ln(152.375/5) = 3.416906675$			

TABLE 2. A sample of 5 quotes for asset 2

Time t_j	$A_2(t_j)$	$B_2(t_j)$	$spr_2(t_j)$	$\frac{A_2(t_j)+B_2(t_j)}{2}$
9:00	15.00	14.25	0.75	14.625
9:02	15.25	14.25	1.00	14.75
9:04	15.25	15.00	0.25	15.125
9:06	15.50	15.25	0.25	15.375
9:08	15.75	15.50	0.25	15.625
Sum	76.75	74.25	2.50	75.50
\bar{spr}_2	$2.50/5 = 0.5$			
$lspra_2$	$\ln(76.75) - \ln(74.25) = 0.033115609$			
x_{c2}	$\ln(75.50/5) = 2.714694744$			

TABLE 3. A sample of 5 quotes for asset 3

Time t_j	$A_3(t_j)$	$B_3(t_j)$	$spr_3(t_j)$	$\frac{A_3(t_j)+B_3(t_j)}{2}$
9:00	20.75	19.50	1.25	20.125
9:02	21.00	19.50	1.50	20.25
9:04	21.25	19.25	2.00	20.25
9:06	22.00	18.25	3.75	20.125
9:08	25.50	18.50	7.00	22.00
Sum	110.5	95	15.50	102.75
\bar{spr}_3	$15.50/5 = 3.1$			
$lspra_3$	$\ln(110.5) - \ln(95) = 0.151138629$			
x_{c3}	$\ln(102.75/5) = 3.022860941$			

the radius of the spherical domain at time 0 by

$$(3.17) \quad R(0) = \frac{1}{2} \min\{0.080437107, 0.033115609, 0.151138629\} = 0.016557805,$$

TABLE 4. Number of shares sold, and liquidity coefficient

Asset	w_i	$a_i = w_i / \bar{s} \bar{p} r_i$	w_i / w_{tot}	$a_i w_i / w_{\text{tot}}$
1	550	$550 / 2.45 = 224.4897959$	$550 / 1600 = 0.34375$	77.16836735
2	750	$750 / 0.5 = 1500$	$750 / 1600 = 0.46875$	703.125
3	300	$300 / 3.1 = 96.77419355$	$300 / 1600 = 0.1875$	18.14516129
Sum	1600			$\alpha_{\text{in}} = 798.4385286$

TABLE 5. Number of shares sold, and liquidity coefficient in logarithmic scale

Asset	w_i	$w_i / \bar{l} \bar{s} \bar{p} r_i$	w_i / w_{tot}	$(w_i / \bar{l} \bar{s} \bar{p} r_i)(w_i / w_{\text{tot}})$
1	550	$550 / 0.080437107 = 6837.640227$	$550 / 1600 = 0.34375$	2350.438828
2	750	$750 / 0.033115609 = 22647.93031$	$750 / 1600 = 0.46875$	10616.21733
3	300	$300 / 0.151138629 = 1984.932649$	$300 / 1600 = 0.1875$	372.1748718
Sum	1600			$\alpha = 13338.83103$

and the coefficient α_{in} is

$$\alpha_{\text{in}} = 798.4385286,$$

while at the logarithmic scale

$$(3.18) \quad \alpha = 13338.83103.$$

Remark 3.5. *At a next section we will use this computed value of α , given by (3.18), and the specific data presented as above (together with the values in (3.16), (3.17)) in a simulation where the stochastic Stefan problem (2.2) with one initial ball in the zero trading area will be solved numerically for the corresponding time of 8 minutes in the financial day; the data used refer to the number of shares traded (sold and bought) in 8 minutes and the liquidity coefficient numerator uses this number which is highly increasing during the day, while the denominator involves the spread that tends to be less varying.*

4. ASYMPTOTIC EXPANSIONS AND APPROXIMATING DYNAMICS IN DIMENSIONS $n = 3$

4.1. Preliminaries. The deterministic version of the general stochastic Stefan problem (2.1) in the union of balls solid phase statement (2.2), i.e. when $n = 3$ and $\sigma = 0$, has been fully analyzed in [25]. Our aim is to derive through asymptotic expansions the approximating dynamics of the moving boundary of (2.2) in the presence of noise (stochastic volatility) as a system of stochastic differential equations. This will provide a useful tool for the prediction of the spreads of 3 shares participating in I markets since the system can be solved numerically. In particular, for various cases of financial interest we will present the numerical results of a number of simulations. We note that the analysis here is restricted to $n = 3$ (as in [25]) but can be easily extended for $n \neq 3$, once careful calculations are applied in the derivation of the statement of (2.2) in dimensions $n \neq 3$; the surface area of a ball is present at the Stefan condition and will involve n , while the n -dependent euclidean norm in \mathbb{R}^n will appear and may modify many other formulae.

4.2. Zero volatility. First we analyze briefly the known results of [25] in the absence of noise, and then we solve the approximating ODEs system numerically; this numerical part appears for first time in the literature.

We present first some existing results for the problem (2.2), for $\sigma = 0$.

The deterministic Stefan problem (2.2), takes the form

$$\begin{aligned}
(4.1) \quad & \alpha^{-1} \partial_t v = \Delta v, \quad x \in \mathbb{R}^3 - \mathcal{D}(t) \text{ ('liquid' phase)}, \quad t > 0, \\
& v = 0, \quad x \in \mathcal{D}(t) \text{ ('solid' phase)}, \\
& v = \frac{1}{R_i(t)} \text{ on } \Gamma_i(t), \\
& \dot{R}_i(t) = \frac{1}{4\pi R_i^2(t)} \int_{\Gamma_i(t)} \nabla v \cdot \eta, \\
& \Gamma(0) = \Gamma_0, \\
& v_\infty(t) = \lim_{r \rightarrow \infty} v(r, t),
\end{aligned}$$

where the initial Γ_0 is given.

The so-called mean-field variable v_∞ describes the limiting behaviour of the density v away from the phase transitions interfaces. A scaling on the space variables of the form $x \in [0, 1] \rightarrow \delta^{-4}x \in [0, \delta^{-4}]$, where $\alpha^{-1} = \delta^9$, cf. [25, 7], approximates as $\alpha \rightarrow \infty$, the background domain \mathbb{R}^3 of the moving boundary problem by some domain Ω of very large volume $|\Omega| = (\delta^{-4})^3 = \delta^{-12} = \alpha^{4/3}$. The assumption

$$\text{Vol}(\text{solid phase}) \ll \text{Vol}(\Omega) =: |\Omega|,$$

leads to

$$\sum_{i=1}^I R_i(0) \leq I \max_i R_i(0) \ll \text{diameter}(\Omega) = \mathcal{O}(\delta^{-4}) = \mathcal{O}(\alpha^{4/9}).$$

So, we impose the next condition for our initial data

$$(4.2) \quad I \max_{i=1, \dots, I} R_i(0) \ll \mathcal{O}(\alpha^{4/9}),$$

which is a condition for the proper scaling of the problem (2.2).

In our approach we will not use a δ for rescaling the equation (as done in [25, 7], where $\delta \ll 1$ is used also in relation with a very large number of radii, in a macroscopic level, not needed here), but we will consider instead the α -dependent Stefan problem for α and $R_i(0)$ satisfying (4.2).

It is known that as $\alpha^{-1} \rightarrow 0$, the exact solution

$$v_\infty(t) + \sum_i \frac{1 - R_i(t)v_\infty(t)}{|x - x_c^i|},$$

of the quasi-static elliptic problem (replace 0 at the left-hand side of the pde of (4.1)) for $x_c^i \in \mathbb{R}^3$ the center of the ball B_{R_i} , approximates the solution v of (4.1); see also the comments at pg. 4683 of [7].

For $\alpha > 0$ very large, which describes here a strong diffusion of the sell/buy orders, v_∞ satisfies approximately the i.v.p.

$$(4.3) \quad \partial_t v_\infty(t) = 4\pi\alpha^{-1/3} \sum_{i \in I} (1 - R_i(t)v_\infty(t)), \quad v_\infty(0) = v_{\infty 0},$$

for example for $v_{\infty 0} \approx \frac{I}{\sum_{i \in I} R_i(0)}$. Here, $\alpha^{-1/3}$ corresponds to

$$\frac{1}{\alpha^{-1}|\Omega|} = \frac{1}{\alpha^{-1}a^{4/3}} = \alpha^{-1/3},$$

see at pg. 4684 of [7], and in the sequel of this section.

Also, in the weak sense, an approximating formula for the dynamics of the radii as $\alpha \rightarrow \infty$ is given by

$$(4.4) \quad \dot{R}_i(t) = \frac{v_{\infty}(t)}{R_i(t)} - \frac{1}{R_i^2(t)},$$

for $v_{\infty}(t)$ the solution of (4.3). See for example the approximation estimate in $W^{1,1}(0, T)$ of [25] at pg. 175 of [25] (for $\|z\|_{W^{1,1}(0, T)} := \int_0^T (|z| + |z_t|) dt$), or at pg. 4712 of [7] for $\beta = g_i := 0$ in the formula (87) therein, derived for the rescaled problem.

Also, the density solution v is approximated by the quasi-static one

$$(4.5) \quad v(x, t) \approx v_{\infty}(t) + \sum_i \frac{1 - R_i(t)v_{\infty}(t)}{|x - x_c^i|}.$$

The ODEs system (4.3), (4.4) for the dynamics of the I radii consists of $I + 1$ equations with unknowns

$$v_{\infty}(t), R_1(t), R_2(t), \dots, R_I(t),$$

and initial values

$$(4.6) \quad v_{\infty}(0) := I \left(\sum_{i=1}^I R_i(0) \right)^{-1}, R_1(0), R_2(0), \dots, R_I(0).$$

We rewrite the system in the equivalent form

$$(4.7) \quad \begin{aligned} \partial_t v_{\infty}(t) &= 4\pi\alpha^{-1/3} \sum_{i \in I} (1 - R_i(t)v_{\infty}(t)), & v_{\infty}(0) &= I \left(\sum_{i=1}^I R_i(0) \right)^{-1}, \\ \partial_t (R_i^3)(t) &= 3v_{\infty}(t)R_i(t) - 3, & R_i(0) &:= R_{i0}, \quad i = 1, \dots, I, \end{aligned}$$

or by setting

$$z_i(t) := R_i^3(t),$$

we obtain the equivalent system

$$(4.8) \quad \begin{aligned} \partial_t v_{\infty}(t) &= 4\pi\alpha^{-1/3} \sum_{i \in I} (1 - z_i(t)^{1/3}v_{\infty}(t)), & v_{\infty}(0) &= I \left(\sum_{i=1}^I R_i(0) \right)^{-1}, \\ \partial_t z_i(t) &= 3v_{\infty}(t)z_i(t)^{1/3} - 3, & z_i(0) &:= R_{i0}^3, \quad i = 1, \dots, I. \end{aligned}$$

Here, each equation for z_i holds until the vanishing time of the i ball.

The solution $z_i(t)$, $v_{\infty}(t)$ of the above system of ODEs is then used to specify $R_i(t)$ and $v(x, t)$ by

$$(4.9) \quad \begin{aligned} R_i(t) &= z_i(t)^{1/3}, \quad i = 1, \dots, I, \\ v(x, t) &= v_{\infty}(t) + \sum_i \frac{1 - R_i(t)v_{\infty}(t)}{|x - x_c^i|}, \quad i = 1, \dots, I, \end{aligned}$$

where for the equation for v we used the approximation formula given by (4.5).

Remark 4.1. *The system (4.8) will be solved numerically, and its solution will be used in the direct formulae (4.9). We propose (4.8), (4.9) (and their numerical solution) as a financial tool for the estimation at time t of the spreads $2R_i(t)$ and the density $v(x, t)$ of the Stefan problem (2.2) when $\sigma = 0$.*

4.2.1. *Numerical experiments.* We constructed a double precision Matlab code for the numerical solution of the system (4.8), (4.9); there, we used the ODE45 routine.

We applied our code for a number of numerical experiments with initial data satisfying the proper scaling condition (4.2).

(1) 4 radii:

For the first experiment, we took $I := 4$ balls for the initial solid phase, and $R_i(0) := 1, 2, 3, 8$, and $\alpha = 10000$. For the graphs of the radii as functions of t , and their vanishing times at the horizontal t - axis, see Fig. 4.1. Obviously, the expected dominance of the larger ball at the expense of the smaller ones is observed.

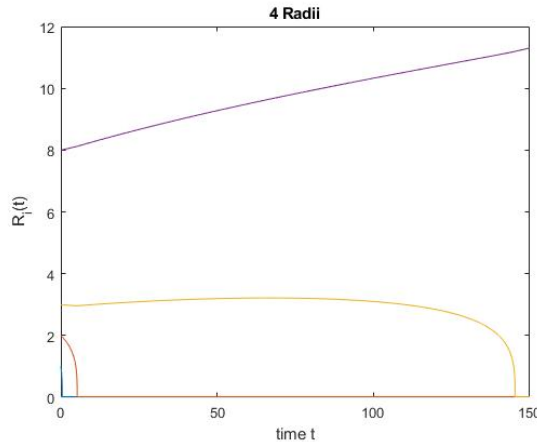


FIGURE 4.1. Radii dynamics of 4 balls at the solid phase.

(2) 100 radii:

We checked our code for a very large number $I = 100$ of initial balls, with centers a small perturbation of $x_{\text{intr}} \in \mathbb{R}^3$ where an intrinsic value $\|x_{\text{intr}}\| = 15$ is assigned. The initial radii are defined in a comparative way through x_{intr} by $R_i(0) := \delta \% v_i$, for $v_i := \|p_i x_{\text{intr}}\|$, $p_i \in [0, 1]$ (randomly evaluated), and $\delta := 25$. We took $\alpha = 10000000$. In this run the initial data are given by a random perturbation of a historical values set of data; here, $I = 100$ does not represent I different markets (as in the main financial application we presented) and R_i are not related to spreads. The next figure, Fig. 4.2 presents the evolution of the radii.

(3) 2 radii:

We took $\alpha = 1000$, $R_1(0) := 2.5$, $R_2(0) := 1.5$. We present the dynamics of the 2 radii at the next figure, Fig. 4.3.

(4) 1 ball at the solid phase with very large radius (large spread case):

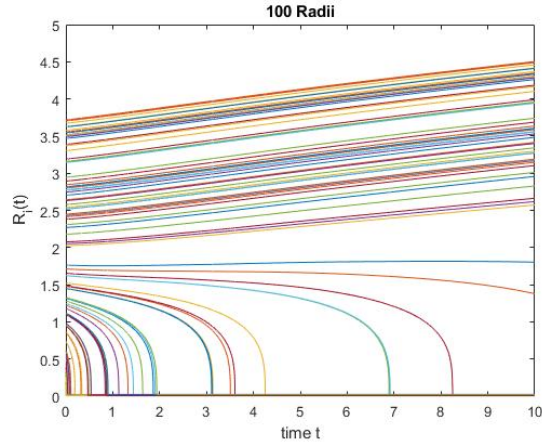


FIGURE 4.2. Radii dynamics of 100 balls at the solid phase.

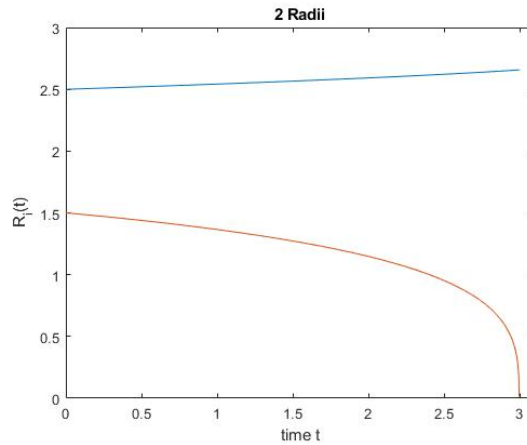


FIGURE 4.3. Radii dynamics of 2 balls at the solid phase.

We took one sphere of center $x_1 = (3/1000, 7/1000, 15/1000)$ and initial radius

$$R(0) = \|x_1\|250/100 = 4.205650960315179e - 01,$$

and defined $\alpha = 100$; this case exceeds severely a normal percentage between the spread and the value of the asset measured by $\|x_1\|$. Recall that in the financial application analyzed in the previous sections the diameter $2R(0)$ stands as a measure of the minimum spread of the 3 shares at the initial time. Our run demonstrated a sudden drop of the radius, see Figure 4.4. Here, we remind that the scaling of initial data satisfied (4.2). However, one ball is a static solution and theoretically it is expected its radius to change very slowly, as seen at the next experiment.

- (5) 1 ball at the solid phase with small radius (small spread case):

Finally, we took

$$R(0) := \|x_1\|25/100 = 4.205650960315179e - 02,$$

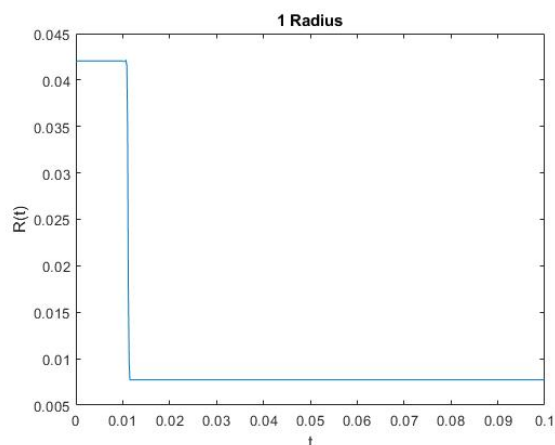


FIGURE 4.4. Radius dynamics of one ball at the solid phase with relatively large spread.

(more normal range between value and spread), and we kept the same other data as in the previous experiment; we derived numerically the expected quasi-static solution approximate profile $R(t) \approx R(0)$ for all t (remind that one ball is an equilibrium of the quasistatic case), see Figure 4.5.

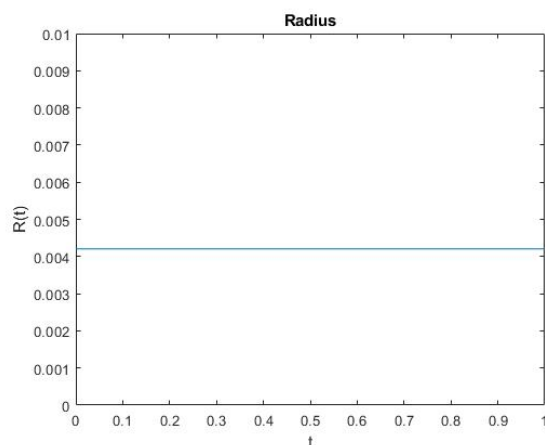


FIGURE 4.5. Radius dynamics of one ball at the solid phase with relatively small spread.

4.3. Formal asymptotics for the stochastic Stefan problem with time noise. We proceed to the formal calculations analogous to those presented in [7] (by defining the parameters β , g_i of the Stefan problem of [7] as $\beta = g_i = 0$) and additionally, we insert the extra noise term in the parabolic equation.

First we present the result of Lemma 5.2 for the formula of differentiation in time of integrals on domains of stochastic time dependent spherical boundary; its proof involves integrals defined on stochastic on time spherical surfaces embedded in \mathbb{R}^3 (case of stochastic radius); (5.4) there is used in the sequel for the second order asymptotics of the problem's stochastic dynamics, in case of time

noise given as the formal derivative of a Wiener process; see at the Appendix for the analytical proof, and for the version where $\partial_t \int_{B(R(t))} u(x, t) dx$ is computed (Lemma 5.1) that we included for completeness of the text.

Let $x \in \mathbb{R}^3$, $t \in \mathbb{R}$, and $u = u(x, t)$, $R = R(t)$, be real stochastic processes compatible with Itô calculus in time, and let u be smooth in space. If $B_{R(t)} =: B(R(t))$ is a ball in \mathbb{R}^3 of radius $R(t)$, the (5.4) holds i.e.

$$\begin{aligned} \partial_t \int_{\mathbb{R}^3 - B(R(t))} u(x, t) dx &= \int_{\mathbb{R}^3 - B(R(t))} u_t(x, t) dx - \dot{R}(t) \left[1 + \frac{\dot{R}(t)}{R(t)} \right] \int_{\partial B(R(t))} u(s, t) ds \\ &\quad - \frac{(\dot{R}(t))^2}{2} \int_{\partial B(R(t))} \nabla u(s, t) \cdot \eta(s) ds - \dot{R}(t) \int_{\partial B(R(t))} u_t(s, t) ds, \end{aligned}$$

if the appearing integrals are well defined. Here, $\dot{R} := R_t = dR(t)$, and η is the outward normal vector to $\partial B(R)$.

Remark 4.2. *In the deterministic case, due to the usual chain rule, in dimensions $n = 3$, and for general u , the result of Lemma 5.1 takes the form*

$$(4.10) \quad \partial_t \int_{B(R(t))} u(x, t) dx = \int_{B(R(t))} u_t(x, t) dx + \dot{R}(t) \int_{\partial B(R(t))} u(s, t) ds,$$

which is a well known formula.

Moreover in the deterministic case again, the result of Lemma (5.2) takes the form

$$(4.11) \quad \partial_t \int_{\mathbb{R}^3 - B(R(t))} u(x, t) dx = \int_{\mathbb{R}^3 - B(R(t))} u_t(x, t) dx - \dot{R}(t) \int_{\partial B(R(t))} u(s, t) ds,$$

for $\dot{R} := R_t$.

Let us consider the problem (2.2) posed in \mathbb{R}^3 , with non-smooth noise $\dot{W} := \dot{W}(t)$, depending only on time given as the formal derivative of a time dependent one dimensional, one parameter Wiener process (for example $W(t) = \beta(t)$ a brownian process). This problem for one only (open) ball B_R with radius R , has the following statement

$$(4.12) \quad \begin{aligned} \alpha^{-1} \partial_t v &= \Delta v + \alpha^{-1} \sigma(\text{dist}(x, \partial B_R(t))) \dot{W}(t), \quad x \in \mathbb{R}^3 - B_R(t), \quad t > 0, \\ v &= 0, \quad x \in B_R(t), \\ v &= \frac{1}{R(t)} \quad \text{on } \partial B_R(t), \\ \dot{R}(t) &= \frac{1}{4\pi R^2(t)} \int_{\partial B_R(t)} \nabla v \cdot \eta, \end{aligned}$$

and $\lim_{r \rightarrow \infty} v(r, t) = v_\infty(t)$, for r the distance of $x \in \mathbb{R}^3$ from the origin.

The formal construction of an approximate solution for the multiple spheres problem (2.2), is based on the following argument. Near one of the spherical domains of the solid phase of (2.2), the solution of (2.2), should look approximately like the solution of the single spherical domain solid phase problem (4.12); see the analogous argument in [7].

The quasi-static version of (4.12) is given as $\alpha^{-1} \rightarrow 0$ by

$$(4.13) \quad \begin{aligned} \Delta v &= 0, \quad x \in \mathbb{R}^3 - B_R(t), \quad t > 0, \\ v &= 0, \quad x \in B_R(t), \\ v &= \frac{1}{R(t)} \quad \text{on } \partial B_R(t), \\ \dot{R}(t) &= \frac{1}{4\pi R^2(t)} \int_{\partial B_R(t)} \nabla v \cdot \eta, \end{aligned}$$

with $\lim_{r \rightarrow \infty} v(r, t) = v_\infty(t)$; observe that α^{-1} acts also to the noise term of (4.12), which thus, vanishes in the quasi-static case.

The exact solution of (4.13) is given by

$$(4.14) \quad v(r, t) = v_\infty(t) + \frac{1 - R(t)v_\infty(t)}{r},$$

and

$$(4.15) \quad \dot{R}(t) = \frac{v_\infty(t)}{R(t)} - \frac{1}{R^2(t)}.$$

Since $\alpha^{-1} \ll 1$, the solution $v(x, t)$ of (2.2) for time noise $\dot{W} := \dot{W}(t)$, is approximated by a linear combination of individual (single sphere) solutions of the quasi-static problem (4.13), as follows

$$(4.16) \quad v(x, t) \approx v_\infty(t) + \sum_i \frac{1 - R_i(t)v_\infty(t)}{|x - x_c^i|},$$

for x_c^i the center of the ball B_{R_i} with radius R_i .

As $\alpha^{-1} \rightarrow 0$ the background domain \mathbb{R}^3 of the moving boundary problem is approximated by some domain Ω of very large volume $|\Omega| = \alpha^{4/3}$. Moreover, the liquid phase $\mathbb{R}^3 - \mathcal{D}$ is very close to Ω .

Hence, we consider that

$$|\Omega|v_\infty = \int_\Omega v_\infty dx \approx \int_{\mathbb{R}^3 - \mathcal{D}} v dx,$$

which yields that

$$|\Omega|v_\infty \approx \int_{\mathbb{R}^3 - \mathcal{D}} v dx.$$

We integrate in the liquid phase both sides of the stochastic equation of (2.2), use the above approximation, and the b.c. of (2.2), and derive for

$$m(t) := \sum_i \frac{(dR_i)^2}{2} \int_{\partial B_{R_i}} \nabla v \cdot \eta ds + \sum_i dR_i \int_{\partial B_{R_i}} v_t ds = 2\pi \sum_i (dR_i)^3 R_i^2 - 4\pi \sum_i R_i \dot{R}_i^2 \frac{1}{R_i + \dot{R}_i},$$

where we used Itô calculus to differentiate $v = 1/R_i$ on the spheres,

$$v_t = -\dot{R}_i / (R_i(R_i + \dot{R}_i)), \quad \text{on } \partial B_i,$$

(4.17)

$$\begin{aligned}
|\Omega|\partial_t v_\infty &\approx \partial_t \int_{\mathbb{R}^3-\mathcal{D}} v dx = \int_{\mathbb{R}^3-\mathcal{D}} v_t dx - \sum_i dR_i [1 + dR_i R_i^{-1}] \int_{\partial B_{R_i}} v ds - m(t) \\
&= \int_{\mathbb{R}^3-\mathcal{D}} \alpha \Delta v dx + \dot{W}(t) \int_{\mathbb{R}^3-\mathcal{D}} \sigma(\text{dist}(x, \Gamma(t))) dx - \sum_i \int_{\partial B_{R_i}} \dot{R}_i [1 + \dot{R}_i R_i^{-1}] v ds - m(t) \\
&= - \int_{\cup \partial B_{R_i}} \alpha \nabla v \cdot \eta ds + \dot{W}(t) \int_{\mathbb{R}^3-\mathcal{D}} \sigma(\text{dist}(x, \Gamma(t))) dx - \sum_i \int_{\partial B_{R_i}} \dot{R}_i [1 + \dot{R}_i R_i^{-1}] v ds - m(t) \\
&= - \alpha 4\pi \sum_i R_i^2 \dot{R}_i + \dot{W}(t) \int_{\mathbb{R}^3-\mathcal{D}} \sigma(\text{dist}(x, \Gamma(t))) dx - \sum_i \int_{\partial B_{R_i}} \dot{R}_i [1 + \dot{R}_i R_i^{-1}] v ds - m(t),
\end{aligned}$$

where we used Lemma 5.2 (formula (5.4)). So, we arrive at

$$\begin{aligned}
\partial_t v_\infty &\approx - \frac{\alpha}{|\Omega|} 4\pi \sum_i R_i^2 \dot{R}_i + \frac{1}{|\Omega|} \dot{W}(t) \int_{\mathbb{R}^3-\mathcal{D}} \sigma(\text{dist}(x, \Gamma(t))) dx \\
(4.18) \quad &- \frac{1}{|\Omega|} \left[\sum_i \int_{\partial B_{R_i}} \dot{R}_i [1 + \dot{R}_i R_i^{-1}] v ds + m(t) \right].
\end{aligned}$$

Using in the above that $\alpha \gg 1$, we ignore the last term. However the same argument is avoided for the noise term (being non smooth and not comparable). Replacing (4.15) for each sphere, and using that $|\Omega| = \alpha^{4/3}$, we derive the next system of stochastic differential equations for the approximating dynamics of (2.2)

$$(4.19) \quad \dot{R}_i(t) \approx \frac{v_\infty(t)}{R_i(t)} - \frac{1}{R_i^2(t)}, \quad i = 1, \dots, I,$$

$$(4.20) \quad \partial_t v_\infty(t) \approx 4\pi \alpha^{-1/3} \sum_{i=1}^I (1 - R_i(t) v_\infty(t)) + \alpha^{-4/3} \dot{W}(t) \int_{\mathbb{R}^3 - \cup B_{R_i}(t)} \sigma(\text{dist}(x, \cup \partial B_{R_i}(t))) dx,$$

for $B_{R_i}(t)$ the balls of constant centers x_c^i and radii $R_i(t)$ respectively. Remind that the solution v of the stochastic Stefan (the density of the sell and buy orders in the financial setting) is approximated by (4.16).

Note that for $\sigma = 0$ (4.19), (4.20) coincide to the rigorous first order asymptotics given by (4.8), (4.9).

Remark 4.3. For more general data, we do not replace $|\Omega|$, and also keep the second order approximation term in (4.18), i.e. we do not ignore

$$\begin{aligned}
& \frac{1}{|\Omega|} \left[\sum_i \int_{\partial B_{R_i}} \dot{R}_i [1 + \dot{R}_i R_i^{-1}] v ds + m(t) \right] \\
&= \frac{1}{|\Omega|} \left[\sum_i \int_{\partial B_{R_i}} \dot{R}_i [1 + \dot{R}_i R_i^{-1}] \frac{1}{R_i} ds + 2\pi \sum_i (dR_i)^3 R_i^2 - 4\pi \sum_i R_i \dot{R}_i^2 \frac{1}{R_i + \dot{R}_i} \right] \\
&= \frac{1}{|\Omega|} \left[4\pi \sum_i R_i^2 \dot{R}_i [1 + \dot{R}_i R_i^{-1}] \frac{1}{R_i} + 2\pi \sum_i (dR_i)^3 R_i^2 - 4\pi \sum_i R_i \dot{R}_i^2 \frac{1}{R_i + \dot{R}_i} \right] \\
&= 4\pi \frac{1}{|\Omega|} \sum_i (R_i \dot{R}_i [1 + \dot{R}_i R_i^{-1}] + (dR_i)^3 R_i^2 / 2 - R_i \dot{R}_i^2 \frac{1}{R_i + \dot{R}_i}) \\
(4.21) \quad &\approx 4\pi \frac{1}{|\Omega|} \sum_i \left(R_i [1 + \dot{R}_i R_i^{-1}] \left[\frac{v_\infty(t)}{R_i(t)} - \frac{1}{R_i^2(t)} \right] + \frac{1}{2} R_i^2 \left[\frac{v_\infty(t)}{R_i(t)} - \frac{1}{R_i^2(t)} \right]^3 \right) \\
&\quad - 4\pi \frac{1}{|\Omega|} \sum_i \frac{1}{R_i(R_i + \dot{R}_i)} \left(v_\infty(t) - \frac{1}{R_i(t)} \right)^2 \\
&= 4\pi \frac{1}{|\Omega|} \sum_i [1 + \dot{R}_i R_i^{-1}] \left(v_\infty(t) - \frac{1}{R_i(t)} \right) + 2\pi \frac{1}{|\Omega|} \sum_i \frac{1}{R_i} \left(v_\infty(t) - \frac{1}{R_i(t)} \right)^3 \\
&\quad - 4\pi \frac{1}{|\Omega|} \sum_i \frac{1}{R_i(R_i + \dot{R}_i)} \left(v_\infty(t) - \frac{1}{R_i(t)} \right)^2,
\end{aligned}$$

and derive the next formula (by replacing once again \dot{R}_i by (4.19) in the above)

$$\begin{aligned}
\partial_t v_\infty(t) &\approx 4\pi \frac{\alpha}{|\Omega|} \sum_{i=1}^I (1 - R_i(t) v_\infty(t)) \\
&\quad - 4\pi \frac{1}{|\Omega|} \sum_i \left(v_\infty(t) - \frac{1}{R_i(t)} \right) - 4\pi \frac{1}{|\Omega|} \sum_i \left(v_\infty(t) - \frac{1}{R_i(t)} \right)^2 \\
(4.22) \quad &\quad - 2\pi \frac{1}{|\Omega|} \sum_i \frac{1}{R_i} \left(v_\infty(t) - \frac{1}{R_i(t)} \right)^3 \\
&\quad + 4\pi \frac{1}{|\Omega|} \sum_i \frac{1}{v_\infty(t) - \frac{1}{R_i} + R_i^2} \left(v_\infty(t) - \frac{1}{R_i(t)} \right)^2 \\
&\quad + \frac{1}{|\Omega|} \dot{W}(t) \int_{\mathbb{R}^3 - \cup B_{R_i}(t)} \sigma(\text{dist}(x, \cup \partial B_{R_i}(t))) dx,
\end{aligned}$$

in place of (4.20).

For this case we may use a general scaling for defining our domain $|\Omega|$ approximating \mathbb{R}^3 , of the form $x \in [0, 1] \rightarrow c_s x \in [0, c_s]$, and $|\Omega| = c_s^3$ for $c_s \gg 1$. However, we also consider α relatively large (since the main argument was to approximate with the static problem formula for the derivatives of the radii).

Thus, (4.22) takes the form

$$\begin{aligned}
(4.23) \quad \partial_t v_\infty(t) &\approx 4\pi \frac{\alpha}{c_s^3} \sum_{i=1}^I (1 - R_i(t) v_\infty(t)) \\
&\quad - 4\pi \frac{1}{c_s^3} \sum_i \left(v_\infty(t) - \frac{1}{R_i(t)} \right) - 4\pi \frac{1}{c_s^3} \sum_i \left(v_\infty(t) - \frac{1}{R_i(t)} \right)^2 \\
&\quad - 2\pi \frac{1}{c_s^3} \sum_i \frac{1}{R_i} \left(v_\infty(t) - \frac{1}{R_i(t)} \right)^3 \\
&\quad + 4\pi \frac{1}{c_s^3} \sum_i \frac{1}{v_\infty(t) - \frac{1}{R_i} + R_i^2} \left(v_\infty(t) - \frac{1}{R_i(t)} \right)^2 \\
&\quad + \frac{1}{c_s^3} \dot{W}(t) \int_{\mathbb{R}^3 - \cup B_{R_i}(t)} \sigma(\text{dist}(x, \cup \partial B_{R_i}(t))) dx.
\end{aligned}$$

We may chose for example $c_s \gg 1$, and treat the problem of more general diffusion constant α , independent from the initial radii order $\mathcal{O}(R_i(0))$ or the centers; the c_s does not depend on α , and can take care of the initial radii order and of the prices-centers (the initial price vector must belong to Ω and the diameter of Ω is equal to $\mathcal{O}(c_s)$).

Once the Stefan problem is used in a financial setting, we would like to consider diffusion coefficients α with magnitude not depending from the initial radii, or the placement of the centers. Hence, we propose the second order approximation formula (4.23) instead of the first order one (4.20).

Remark 4.4. *Since the liquid phase approximates the background domain of the Stefan problem, we may consider*

$$\int_{\mathbb{R}^3 - \cup B_{R_i}(t)} \sigma(\text{dist}(x, \cup \partial B_{R_i}(t))) dx \approx \int_{\mathbb{R}^+} \sigma(r) dr,$$

for $\sigma(r)$ a sufficiently decaying function as $r \rightarrow \infty$, or of compact support (satisfying for example $\sigma(r) \sim \mathcal{O}(r^{-(1+a)})$, for some $a > 0$, as $r \rightarrow \infty$); see also in [24] the discussion of analogous properties for σ for a Stefan problem posed in dimension 1.

4.4. Numerical experiments. The first set of numerical experiments considers one initial ball for the solid phase and implements numerically the first and second order approximation stochastic differential systems proposed. For all cases we used a double precision Matlab code and the ODE45 routine. All initial data satisfy the proper scaling condition (4.2).

4.4.1. 1 radius, first order versus second order asymptotics with stochastic volatility. We took $\alpha = 100$ and $R(0) = 1$.

We solved numerically the first order approximation stochastic dynamics system (4.19), (4.20), with initial condition given by (4.6), for $c_0 = \int_{\mathbb{R}^+} \sigma(r) dr = 1$, and $W(t) := \beta(t)$ the brownian motion following the normal distribution $N(0, t)$. The noise $dW(t)$ was approximated by using the brownian increments as follows (finite differences)

$$(4.24) \quad dW(t) \simeq \frac{\beta(t_j) - \beta(t_{j-1})}{t_j - t_{j-1}},$$

for $\mathcal{O}(t_j - t_{j-1}) = 10^{-6}$, $j = 1, \dots, J$, where $0 = t_0 < t_1 < \dots < t_J := 15$, was the time discretization of our numerical scheme. We applied a Monte Carlo simulation for 100 realizations (100 runs) and computed, for each realization, the radius $R(t)$ for $t \in [0, 15]$, see figure 4.6. Moreover, we plot the value $R(t)$ for $t = 15$, for each realization, see figure 4.7. Remind that as predicted by the theory of the deterministic parabolic Stefan problem, the case of one spherical initial solid phase boundary has an almost constant radius profile in time $R(t) \simeq R(0)(= 1$ there), since the constant sphere is a solution of the quasi-static deterministic problem. The main observation of our experiment for the stochastic case is in contrast to the previous property. There existed quite a few realizations where the radius vanished at finite time $t < 15$, while in other the profile was oscillating. The computed experimental mean value of $R(t)$, for $t = 15$, was equal to 0.6096590298805101, and thus, significantly smaller than the initial radius $R(0) = 1$.

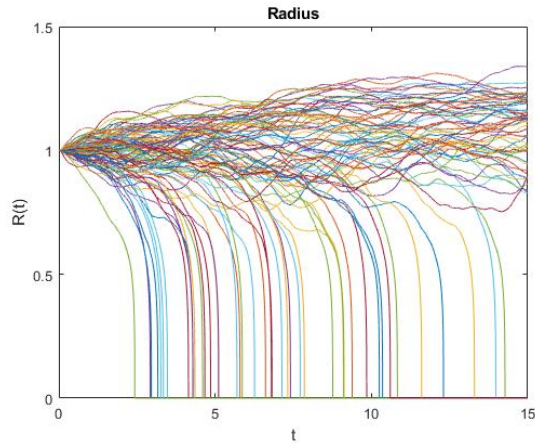


FIGURE 4.6. 100 realizations of $R(t)$, for $t \in [0, 15]$, with first order approximation.

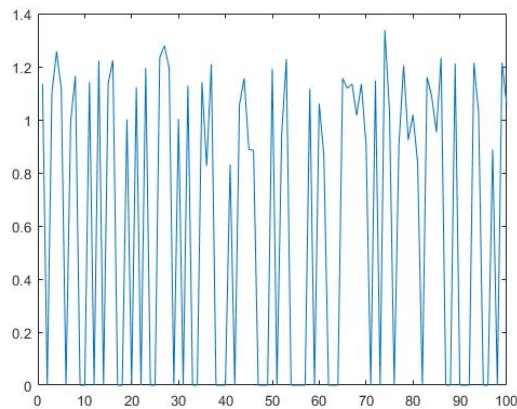


FIGURE 4.7. 100 realizations of $R(t)$, for $t = 15$ (first order approximation).

We repeated the same experiment by using the second order approximation for the stochastic dynamics system, (4.19), (4.23), and (4.6), for $c_s^3 = \alpha^{4/3}$. We computed, for each realization, the

radius $R(t)$ for $t \in [0, 15]$, see now figure 4.8. We also plot the value $R(t)$ for $t = 15$, for each realization, see figure 4.9. Again there existed quite a few realizations where the radius vanished at finite time $t < 15$. The computed experimental mean value of $R(t)$, for $t = 15$, was equal to 0.69573807862059131, again, smaller than the initial radius $R(0) = 1$. However, through the 100 realizations, the profile of $R(t)$ for $t = 15$ was less oscillating than this of the first order approximation, see fig. 4.7, 4.9.

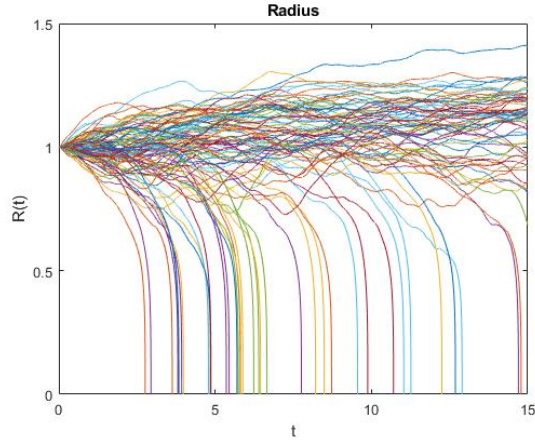


FIGURE 4.8. 100 realizations of $R(t)$, for $t \in [0, 15]$, with second order approximation.

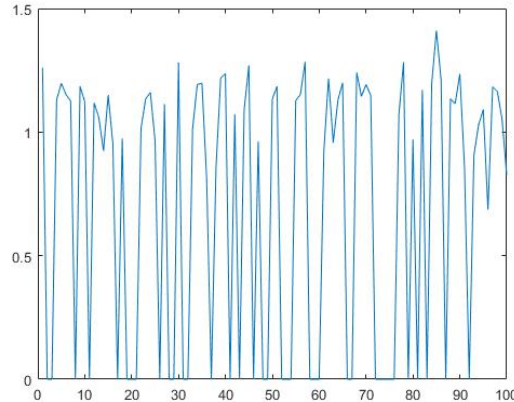


FIGURE 4.9. 100 realizations of $R(t)$, for $t = 15$ (second order approximation).

4.4.2. *Financial data experiment.* The next set of runs was devoted to the financial application presented in Section 3.3.1 and the tables therein; we also used the computed values given by (3.16), (3.17), (3.18). We considered one initial ball of center $(3.416906675, 2.714694744, 3.022860941)$, and radius $R(0) = 0.016557805$, while the liquidity coefficient was given by $\alpha = 13338.83103$. Note that the above financial data happen to satisfy the scaling condition (4.2). We applied our

double precision Matlab code and implemented numerically the second order approximation given by (4.19), (4.23), with (4.6). For all runs the time length used for the experiments was crucial and related to the computed value of α by data given during 8 consecutive minutes in a financial day, cf. the tables in Section 3.3.1.

The max ask price appeared in tables was equal to 35.00, i.e. equal to

$$\ln(35.00) = 3.555348061489414e + 00$$

in the logarithmic scale. We took c_s such that

$$\begin{aligned} c_s^3 &= |\Omega| \gg (4/3)\pi \ln(35.00)^3 \\ &= \text{the volume of the ball of radius } \ln(35.00) \\ &= 1.882499980769812e + 02, \end{aligned}$$

i.e. $c_s \gg 5.731192468848986e + 00$ (we note that in this experiment the radius is very small while the max ask price in logarithmic scale is very larger, so since the vectorial price is in $|\Omega|$ for the financial example, the order of the measure of the vector price should be used instead of the radius for the scaling). We took again $c_0 = \int_{\mathbb{R}^+} \sigma(r)dr = 1$, and $dW(t)$ was approximated by (4.24).

We used $c_s = 5 \times 10^4 \times 5.731192468848986e + 00$, and run our Monte Carlo simulation for 300 realizations in a time period less or equal to 8 minutes. In the first 2 minutes the spread (radius) had a very small increase while at the end of the 8 minutes period the value (oscillating) was $1.932132649058731e - 02$, as shown in the next table.

1.655780500000000e-02	0.000000000000000e+00	minutes
1.655780500000000e-02	2.000000000000000e+00	minutes
2.028801437719118e-02	7.079646017699115e+00	minutes
1.932132649058731e-02	8.000000000000000e+00	minutes

When we used 2 initial balls of different radii (but near the radius of the previous example) under the same other data as above, we observed the fast decrease of the smaller one, while the larger was increasing.

5. APPENDIX

In this Appendix we present some important results of Itô calculus for space integrals on domains of stochastic boundary.

Lemma 5.1. *Let $x \in \mathbb{R}^3$, $t \in \mathbb{R}$, and $u = u(x, t)$, $R = R(t)$, be real stochastic processes compatible with Itô calculus in time, and let u be smooth in space. Then for $B_{R(t)} =: B(R(t))$ a ball in \mathbb{R}^3 of radius $R(t)$, it holds that*

$$(5.1) \quad \begin{aligned} \partial_t \int_{B(R(t))} u(x, t) dx &= \int_{B(R(t))} u_t(x, t) dx + \dot{R}(t) \left[1 + \frac{\dot{R}(t)}{R(t)} \right] \int_{\partial B(R(t))} u(s, t) ds \\ &+ \frac{(\dot{R}(t))^2}{2} \int_{\partial B(R(t))} \nabla u(s, t) \cdot \eta(s) ds + \dot{R}(t) \int_{\partial B(R(t))} u_t(s, t) ds, \end{aligned}$$

for $\dot{R} := R_t = dR$ and η the outward normal vector to $\partial B(R)$.

Proof. Set

$$g(y, t) := \int_{B(y)} u(x, t) dx.$$

We aim to compute $\partial_t(g(R(t), t))$, i.e.

$$\partial_t \int_{B(R(t))} u(x, t) dx.$$

Itô formula (2 variables Taylor) when y, t depend stochastically, while z, t do not depend stochastically yields

$$(5.2) \quad \partial_t g(y, z) = g_y \frac{y_t}{1!} + g_{yy} \frac{(y_t)^2}{2!} + g_z \frac{z_t}{1!} + g_{yz} \frac{y_t z_t}{1!1!}.$$

In our case, $R(t)$ is stochastic, and $t, R(t)$ stochastically dependent, while t, t are not depending stochastically. So, by applying (5.2), for $y := R(t)$, $z = t$ and $y_t = \dot{R}$, $z_t = 1$, we obtain

$$(5.3) \quad \partial_t(g(R(t), t)) = \dot{R}(t)g_y(R(t), t) + \frac{(\dot{R}(t))^2}{2}g_{yy}(R(t), t) + g_t(R(t), t) + g_{ty}(R(t), t)\dot{R}(t),$$

i.e. for d denoting the differentiation in t

$$d(g(R(t), t)) = dR(t)g_y(R(t), t) + \frac{(dR(t))^2}{2}g_{yy}(R(t), t) + dg(R(t), t) + dg_y(R(t), t)dR(t).$$

Considering the functional formula of g as a function $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, we will compute g_y, g_{yy}, g_t and g_{ty} .

Moreover, we remind that our processes are smooth in space variables and differentiation in space follows the usual calculus (not Itô).

By using spherical coordinates, we have

$$g(y, t) = \int_0^y \int_0^{2\pi} \int_0^\pi \tau^2 \hat{u}(\tau, \theta, \phi, t) \sin(\theta) d\theta d\phi d\tau.$$

So, we obtain

$$g_y(y, t) = \int_0^{2\pi} \int_0^\pi y^2 \hat{u}(y, \theta, \phi, t) \sin(\theta) d\theta d\phi = \int_{\partial B(y)} u(s, t) ds,$$

while

$$\begin{aligned} g_{yy}(y, t) &= 2y \int_0^{2\pi} \int_0^\pi \hat{u}(y, \theta, \phi, t) \sin(\theta) d\theta d\phi + \int_0^{2\pi} \int_0^\pi y^2 \partial_y [\hat{u}(y, \theta, \phi, t)] \sin(\theta) d\theta d\phi \\ &= \frac{2}{y} \int_{\partial B(y)} u(s, t) ds + \int_{\partial B(y)} \nabla u(s, t) \cdot \eta(s) ds. \end{aligned}$$

Moreover, we have

$$g_t(y, t) = \int_{B(y)} u_t(x, t) dx,$$

and differentiation in t of g_y gives

$$g_{yt}(y, t) = \int_{\partial B(y)} u_t(s, t) ds.$$

Replacing in (5.3) we derive the result. □

The next Lemma is a direct result.

Lemma 5.2. *Let $x \in \mathbb{R}^3$, $t \in \mathbb{R}$, and $u = u(x, t)$, $R = R(t)$, be real stochastic processes compatible with Itô calculus in time, and let u be smooth in space. If $B_{R(t)} =: B(R(t))$ is a ball in \mathbb{R}^3 of radius $R(t)$, it holds that*

$$(5.4) \quad \begin{aligned} \partial_t \int_{\mathbb{R}^3 - B(R(t))} u(x, t) dx &= \int_{\mathbb{R}^3 - B(R(t))} u_t(x, t) dx - \dot{R}(t) \left[1 + \frac{\dot{R}(t)}{R(t)} \right] \int_{\partial B(R(t))} u(s, t) ds \\ &\quad - \frac{(\dot{R}(t))^2}{2} \int_{\partial B(R(t))} \nabla u(s, t) \cdot \eta(s) ds - \dot{R}(t) \int_{\partial B(R(t))} u_t(s, t) ds, \end{aligned}$$

if the appearing integrals are well defined. Here, $\dot{R} := R_t = dR(t)$, and η is the outward normal vector to $\partial B(R)$.

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