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Abstract	A class of linear autonomous mixed type difference equations is considered, and some new results on the asymptotic behavior and the stability are given, via a positive root of the corresponding characteristic equation.

Keywords (separated by '-')	Mixed type difference equation - Asymptotic behavior - Stability - Characteristic equation - Solution
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On the behavior of the solutions for linear autonomous mixed type difference equation

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Abstract
A class of linear autonomous mixed type difference equations is considered, and some new 1
results on the asymptotic behavior and the stability are given, via a positive root of the 2
corresponding characteristic equation. 3

Keywords Mixed type difference equation · Asymptotic behavior · Stability · Characteristic equation · Solution

Mathematics Subject Classification 39A10 · 39A30

1 Introduction and preliminaries

The purpose of this paper is to investigate the stability behaviour of the solutions of the linear difference equation of mixed type

$$\triangle x(n) = \sum_{i=1}^{\ell} p_i x(n-i) + \sum_{j=1}^{m} q_j x(n+j), \quad n = 0, 1, 2, \ldots \tag{1.1}$$

where p_i for $i = 1, 2, \ldots, \ell$ and q_j for $j = 1, 2, \ldots, m$ are real numbers, and ℓ, m are positive integers and $\triangle x(n)$ is the forward operator defined by $\triangle x(n) = x(n+1) - x(n)$.

Qualitative theory of difference equations has drawn considerable attention in the past two decades. For the general background of difference equations, one can refer to the books by Agarwal and Wong [1], Agarwal et al. [3], Agarwal [4], Elaydi [7], Gyori and Ladas [11],

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Kelley and Peterson [13], and Lakshmikantham and Trigiante [15]. Our aim in this paper is to give some new results on the asymptotic behavior and the stability for a class of linear autonomous mixed type difference equations. The difference equations considered are the discrete versions of first order linear autonomous mixed differential equations with delays and advances [see, the book ([8], pp. 355–364) and [10]].

The Eq. (1.1) has been adequately introduced in [9] . In this reference, Ferreira and Pinelas have established the oscillatory criteria for the oscillatory mixed difference systems of form (1.1). The similar equation is studied in [2] [see also, ([3], Section 1.16)].

The reader can look at the references [6,14,16,18] and references therein of the equation

$$\triangle x(n) = \sum_{i=1}^{\ell} p_i x(n-i),$$

which is a special case of Eq. (1.1). In this references, some stability analysis and asymptotic behavior were obtained for this delay difference equation.

We remark that our results can be extended to nonautonomous linear mixed type difference equations of the form

$$\triangle x(n) = \sum_{i=1}^{\ell} p_i(n)x(n-i) + \sum_{j=1}^{m} q_j(n)x(n+j),$$

where $p_i(n)$ and $q_j(n)$ are sequences of real numbers (see [5,17]). Berezansky and Pinelas [5] have established oscillation properties for a scalar linear difference equation of mixed type. Finally, Pinelas [17] is obtained asymptotic behavior of a scalar linear difference equation of mixed equations. In this article, we have applied a different method for asymptotic behavior of (1.1).

In this paper, we are concerned with the behavior of the solutions of autonomous linear mixed type difference equations. The general case of mixed type difference equations is considered in Sect. 2. Our results will be obtained via an appropriate positive real root of the corresponding characteristic equation.

This paper deals with the asymptotic behavior and the stability for autonomous linear mixed type difference equations. A basic asymptotic criterion is established. Moreover, a useful estimate of the solutions is obtained and a stability criterion is derived. Our results are obtained by the use of a positive real root (with an appropriate property) of the corresponding characteristic equation. The techniques applied in obtaining our results are originated in a combination of the methods used in [14,16].

Throughout the paper, by Φ we will denote the set of all $\phi = (\phi(n))_{n=-\ell}^{m}$ with $\phi(n) \in \mathbf{R}$ for $n = -\ell, \ldots, 0, \ldots, m$; this set is a finite dimensional space with the usual sup-norm $\| \cdot \|$ defined by

$$\| \phi \| = \sup_{n=-\ell,\ldots,0,\ldots,m} |\phi(n)| \quad \text{for any} \quad \phi = (\phi(n))_{n=-\ell}^{m} \quad \text{in} \quad \Phi.$$

Along with the mixed type difference equation (1.1), we specify an initial condition of the form

$$x(n) = \phi(n) \quad \text{for} \quad n = -\ell, \ldots, 0, \ldots, m; \tag{1.2}$$

where the initial function ϕ is a given real-valued function for $n = -\ell, \ldots, 0, \ldots, m$ satisfying the "consistency condition"

$$\phi(1) - \phi(0) = \sum_{i=1}^{\ell} p_i \phi(-i) + \sum_{j=1}^{m} q_j \phi(j).$$

As usual, by a *solution* of the difference equation (1.1), we mean a sequence of real numbers $(x(n))_{n \geq -\ell}$ which satisfies (1.1) for all integers $n \geq 0$. In order to guarantee its existence and uniqueness for given initial values (1.2), we will assume throughout this paper that the numbers q_j for $j = 1, 2, \ldots, m$ are such that

$$q_1 \neq 1 \quad \text{if } m = 1$$

$$q_m \neq 0 \quad \text{if } m \geq 2$$

with no restrictions in other cases (see [11, Chapter 7] and [12]). The equations of (1.1) and (1.2) are an initial value problem (IVP, for short).

Together with the mixed type difference equation (1.1), we associate the following equation

$$1 = \lambda - \sum_{i=1}^{\ell} p_i \lambda^{-i} - \sum_{j=1}^{m} q_j \lambda^j \tag{1.3}$$

which will be called the *characteristic equation* of (1.1).

We say that the characteristic equation has the *Property A* if

$$\mu(\lambda_0) = \frac{1}{\lambda_0} \left\{ \sum_{i=1}^{\ell} i \, |p_i| \, \lambda_0^{-i} + \sum_{j=1}^{m} j \, |q_j| \, \lambda_0^{j} \right\} < 1. \tag{1.4}$$

The main result of this work establish conditions for the asymptotic behaviour of the Eq. (1.1).

Theorem 1.1 *Let λ_0 be a positive real root of the characteristic equation (1.3) with the Property A. Set*

$$\beta(\lambda_0) = \frac{1}{\lambda_0} \left(\sum_{i=1}^{\ell} i p_i \lambda_0^{-i} - \sum_{j=1}^{m} j q_j \lambda_0^{j} \right). \tag{1.5}$$

Then, for any $\phi = (\phi(n))_{n=-\ell}^{m}$ in Φ, the solution x of the IVP (1.1) and (1.2) satisfies

$$|x(n)| \leq N(\lambda_0) \parallel \phi \parallel \lambda_0^n, \qquad \textit{for all} \quad n \geq 0, \tag{1.6}$$

where

$$N(\lambda_0) = \frac{1 + \mu(\lambda_0)}{1 + \beta(\lambda_0)} + \mu(\lambda_0) k(\lambda_0) \left(1 + \frac{1 + \mu(\lambda_0)}{1 + \beta(\lambda_0)} \right) \tag{1.7}$$

and

$$k(\lambda_0) = max\{\lambda_0^{\ell}, \lambda_0^{-m}\}. \tag{1.8}$$

Moreover, the trivial solution of the mixed type difference equation (1.1) is:

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(i) *uniformly stable if* $\lambda_0 = 1$ *or, equivalently, if*

$$\sum_{i=1}^{\ell} p_i + \sum_{j=1}^{m} q_j = 0 \quad and \quad \sum_{i=1}^{\ell} i|p_i| + \sum_{j=1}^{m} j|q_j| < 1, \tag{1.9}$$

(ii) *uniformly asymptotically stable if* $\lambda_0 < 1$, *and*
(iii) *unstable if* $\lambda_0 > 1$.

2 Statement of the main results

Before to proof our main result, Theorem 1.1, we must to establish a the following theorem, which establishes a basic asymptotic property for the solutions of the mixed type difference equation (1.1).

Theorem 2.1 *Let* λ_0 *be a positive real root of the characteristic equation* (1.3) *with the Property A. Consider* $\beta(\lambda_0)$ *as in Theorem* 1.1. *Then for any* $\phi = (\phi(n))_{n=-\ell}^{m}$ *in* Φ, *it holds*

$$\lim_{n\to\infty}\left[\lambda_0^{-n}x(n)\right] = \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}, \tag{2.1}$$

where

$$L(\lambda_0;\phi) = \phi(0) + \frac{1}{\lambda_0}\left\{\sum_{i=1}^{\ell} p_i\lambda_0^{-i}\left(\sum_{s=-i}^{-1}\phi(s)\lambda_0^{-s}\right)\right.$$

$$\left. - \sum_{j=1}^{m} q_j\lambda_0^{j}\left(\sum_{s=0}^{j-1}\phi(s)\lambda_0^{-s}\right)\right\}. \tag{2.2}$$

Note: Property A guarantees that $1+\beta(\lambda_0) > 0$.

Note: It is a natural question if the characteristic equation (1.3) possesses a positive real root λ_0 with the property $\mu(\lambda_0)$ (introduced in Theorem 2.1). In Sect. 3, we will give sufficient conditions on the coefficients and the delays and advances of (1.1) for the characteristic equation (1.3) to have a positive real root λ_0 with the property $\mu(\lambda_0)$.

Proof The *Property A* implies $0 < \mu(\lambda_0) < 1$. From (1.5), we have

$$|\beta(\lambda_0)| = \left|\frac{1}{\lambda_0}\left(\sum_{i=1}^{\ell} ip_i\lambda_0^{-i} - \sum_{j=1}^{m} jq_j\lambda_0^{j}\right)\right| \le \frac{1}{\lambda_0}\left(\sum_{i=1}^{\ell} i|p_i|\lambda_0^{-i} + \sum_{j=1}^{m} j|q_j|\lambda_0^{j}\right).$$

In this case, $|\beta(\lambda_0)| \le \mu(\lambda_0)$ will be provided, so $|\beta(\lambda_0)| < 1$. Then $1+\beta(\lambda_0) > 0$ is come out.

Consider an arbitrary $\phi = (\phi(n))_{n=-\ell}^{m}$ in Φ and set

$$y(n) = \lambda_0^{-n}x(n) \quad \text{for} \quad n \ge -\ell.$$

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Then, for $n \geq 0$, we have

$$
\begin{aligned}
&\triangle x(n) - \sum_{i=1}^{\ell} p_i x(n-i) - \sum_{j=1}^{m} q_j x(n+j) \\
&\quad = \triangle\left[\lambda_0^n y(n)\right] - \sum_{i=1}^{\ell} p_i \lambda_0^{n-i} y(n-i) - \sum_{j=1}^{m} q_j \lambda_0^{n+j} y(n+j) \\
&\quad = \lambda_0^n \left\{ \lambda_0 y(n+1) - y(n) - \sum_{i=1}^{\ell} p_i \lambda_0^{-i} y(n-i) - \sum_{j=1}^{m} q_j \lambda_0^{j} y(n+j) \right\}.
\end{aligned}
$$

Thus, by using the hypothesis that λ_0 is a positive real root of the characteristic equation (1.3), we obtain for $n \geq 0$

$$
\begin{aligned}
&\triangle x(n) - \sum_{i=1}^{\ell} p_i x(n-i) - \sum_{j=1}^{m} q_j x(n+j) \\
&\quad = \lambda_0^n \left\{ \lambda_0 y(n+1) - \left(\lambda_0 - \sum_{i=1}^{\ell} p_i \lambda_0^{-i} - \sum_{j=1}^{m} q_j \lambda_0^{j} \right) y(n) \right. \\
&\qquad \left. - \sum_{i=1}^{\ell} p_i \lambda_0^{-i} y(n-i) - \sum_{j=1}^{m} q_j \lambda_0^{j} y(n+j) \right\} \\
&\quad = \lambda_0^n \left\{ \lambda_0 y(n+1) - \lambda_0 y(n) \right. \\
&\qquad \left. + \sum_{i=1}^{\ell} p_i \lambda_0^{-i} \big(y(n) - y(n-i)\big) + \sum_{j=1}^{m} q_j \lambda_0^{j} \big(y(n) - y(n+j)\big) \right\}.
\end{aligned}
$$

So, $(x(n))_{n \geq -\ell}$ is a solution of (1.1) means exactly that $(y(n))_{n \geq -\ell}$ satisfies

$$
\begin{aligned}
\triangle y(n) = -\frac{1}{\lambda_0} &\left\{ \sum_{i=1}^{\ell} p_i \lambda_0^{-i} \big(y(n) - y(n-i)\big) \right. \\
&\left. + \sum_{j=1}^{m} q_j \lambda_0^{j} \big(y(n) - y(n+j)\big) \right\} \quad \text{for} \quad n \geq 0. \qquad (2.3)
\end{aligned}
$$

The initial condition (1.2) can be written in the following equivalent form:

$$
y(n) = \lambda_0^{-n} \phi(n) \quad \text{for} \quad n = -\ell, \ldots, 0, \ldots, m. \qquad (2.4)
$$

By using (2.4) and taking into account the definition of $L(\lambda_0; \phi)$, we can easily see that (2.3) is equivalent to

$$
\begin{aligned}
y(n) = -\frac{1}{\lambda_0} &\left\{ \sum_{i=1}^{\ell} p_i \lambda_0^{-i} \left(\sum_{s=n-i}^{n-1} y(s) \right) \right. \\
&\left. - \sum_{j=1}^{m} q_j \lambda_0^{j} \left(\sum_{s=n}^{n+j-1} y(s) \right) \right\} + L(\lambda_0; \phi) \quad \text{for} \quad n \geq 0. \qquad (2.5)
\end{aligned}
$$

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Now, we define

$$z(n) = y(n) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} \quad \text{for} \quad n \geq -\ell.$$

By the definition of $\beta(\lambda_0)$, one can immediately verify that (2.5) reduces to the following equivalent equation:

$$z(n) = -\frac{1}{\lambda_0}\left\{\sum_{i=1}^{\ell} p_i \lambda_0^{-i}\left(\sum_{s=n-i}^{n-1} z(s)\right) - \sum_{j=1}^{m} q_j \lambda_0^{j}\left(\sum_{s=n}^{n+j-1} z(s)\right)\right\} \quad \text{for} \quad n \geq 0. \tag{2.6}$$

Moreover, the initial condition (2.4) takes the following equivalent form:

$$z(n) = \lambda_0^{-n}\phi(n) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} \quad \text{for} \quad n = -\ell,\ldots,0,\ldots,m. \tag{2.7}$$

Because of the definitions of $(y(n))_{n\geq -\ell}$ and $(z(n))_{n\geq -\ell}$, it remains to show that

$$\lim_{n\to\infty} z(n) = 0. \tag{2.8}$$

In order to establish (2.8), we put

$$M(\lambda_0;\phi) = \sup_{n=-\ell,\ldots,0,\ldots,m}\left|\lambda_0^{-n}\phi(n) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}\right|. \tag{2.9}$$

Then (2.7) ensures that

$$|z(n)| \leq M(\lambda_0;\phi) \qquad \text{for} \quad n = -\ell,\ldots,0. \tag{2.10}$$

The constant $M(\lambda_0;\phi)$, is a bound of the sequence $(z(n))_{n\geq -\ell}$, i.e.

$$|z(n)| \leq M(\lambda_0;\phi) \qquad \text{for all} \quad n \geq -\ell. \tag{2.11}$$

In fact, let ϵ be an arbitrary positive number. We claim that

$$|z(n)| < M(\lambda_0;\phi) + \epsilon \qquad \text{for every} \quad n \geq -\ell. \tag{2.12}$$

Otherwise, in view of (2.10), there is an integer $n_0 > 0$ with

$$|z(n)| < M(\lambda_0;\phi) + \epsilon \quad \text{for } n = -\ell,\ldots,0,\ldots,n_0-1, n_0+1,\ldots n_0+m$$

and

$$|z(n_0)| = M(\lambda_0;\phi) + \epsilon$$

Then, by defining $\mu(\lambda_0)$, as in Theorem 2.1, from (2.6) we get

$$\begin{aligned} M(\lambda_0;\phi) + \epsilon = |z(n_0)| &\leq \frac{1}{\lambda_0}\left\{\sum_{i=1}^{\ell}|p_i|\lambda_0^{-i}\left(\sum_{s=n_0-i}^{n_0-1}|z(s)|\right)\right. \\ &\quad \left. + \sum_{j=1}^{m}|q_j|\lambda_0^{j}\left(\sum_{s=n_0}^{n_0+j-1}|z(s)|\right)\right\} \\ &\leq \frac{1}{\lambda_0}\left\{\sum_{i=1}^{\ell} i|p_i|\lambda_0^{-i} + \sum_{j=1}^{m} j|q_j|\lambda_0^{j}\right\}[M(\lambda_0;\phi)+\epsilon] \\ &= \mu(\lambda_0)\,[M(\lambda_0;\phi)+\epsilon] \\ &< M(\lambda_0;\phi)+\epsilon, \end{aligned}$$

which in view of *Property A*, leads to a contradiction. So, our claim is true, i.e. (2.12) holds. We have thus proved that (2.12) is fulfilled for all numbers $\epsilon > 0$. Hence, (2.11) is satisfied. Now, by virtue of (2.11), from (2.6) we derive for $n \geq 0$

$$\begin{aligned} |z(n)| &\leq \frac{1}{\lambda_0}\left\{\sum_{i=1}^{\ell}|p_i|\lambda_0^{-i}\left(\sum_{s=n-i}^{n-1}|z(s)|\right)+\sum_{j=1}^{m}|q_j|\lambda_0^{j}\left(\sum_{s=n}^{n+j-1}|z(s)|\right)\right\} \\ &\leq \frac{1}{\lambda_0}\left\{\sum_{i=1}^{\ell} i|p_i|\lambda_0^{-i}+\sum_{j=1}^{m} j|q_j|\lambda_0^{j}\right\} M(\lambda_0;\phi). \end{aligned}$$

Consequently, by the definition of $\mu(\lambda_0)$, we have

$$|z(n)| \leq \mu(\lambda_0)M(\lambda_0;\phi), \qquad \text{for all} \quad n \geq 0. \tag{2.13}$$

By (2.11) and (2.13), an easy induction leads to the conclusion that $(z(n))_{n\geq-\ell}$ satisfies

$$|z(n)| \leq (\mu(\lambda_0))^{\upsilon}\, M(\lambda_0;\phi), \qquad \text{for all} \quad n \geq \upsilon\ell-\ell, \tag{2.14}$$

where $\upsilon = 0, 1, \ldots$. Because of *Property A*, we have $\lim_{n\to\infty}(\mu(\lambda_0))^{\upsilon} = 0$. Thus, from (2.14) it follows that $\lim_{n\to\infty} z(n) = 0$, i.e. (2.8) holds.

The proof of Theorem 2.1 is complete. □

Note: By applying Theorem 2.1 with $\lambda_0 = 1$, we immediately obtain the following result: Let (1.9) be satisfied. Then, for any $\phi = (\phi(n))_{n=-\ell}^{m}$ in Φ, the solution x of the IVP (1.1) and (1.2) satisfies

$$\lim_{n\to\infty} x(n) = \frac{\phi(0)+\sum_{i=1}^{\ell} p_i\left(\sum_{s=-i}^{-1}\phi(s)\right)-\sum_{j=1}^{m} q_j\left(\sum_{s=0}^{j-1}\phi(s)\right)}{1+\sum_{i=1}^{\ell} ip_i-\sum_{j=1}^{m} jq_j}.$$

Let us now to prove the Theorem 1.1

Proof Consider an arbitrary $\phi = (\phi(n))_{n=-\ell}^{m}$ in Φ. From (2.2), we obtain

$$\begin{aligned} |L(\lambda_0;\phi)| &\leq |\phi(0)|+\frac{1}{\lambda_0}\left\{\sum_{i=1}^{\ell}|p_i|\lambda_0^{-i}\left(\sum_{s=-i}^{-1}|\phi(s)|\lambda_0^{-s}\right)+\sum_{j=1}^{m}|q_j|\lambda_0^{j}\left(\sum_{s=0}^{j-1}|\phi(s)|\lambda_0^{-s}\right)\right\} \\ &\leq \parallel\phi\parallel\left(1+\frac{1}{\lambda_0}\left\{\sum_{i=1}^{\ell} i|p_i|\lambda_0^{-i} max\{\lambda_0,\lambda_0^{i}\}+\sum_{j=1}^{m} j|q_j|\lambda_0^{j} max\{1,\lambda_0^{-j+1}\}\right\}\right). \end{aligned}$$

Since

$$\begin{aligned} max\{\lambda_0,\lambda_0^{i}\} &\leq max\{1,\lambda_0^{\ell}\}, \quad i = 1,\ldots,\ell, \\ max\{1,\lambda_0^{-j+1}\} &\leq max\{1,\lambda_0^{-m}\}, \quad j = 1,\ldots,m \end{aligned}$$

and using (1.8)

$$max\{1,\lambda_0^{\ell}\} \leq k(\lambda_0), \quad max\{1,\lambda_0^{-m}\} \leq k(\lambda_0),$$

we have

$$|L(\lambda_0;\phi)| \leq \parallel\phi\parallel\left(1+\frac{k(\lambda_0)}{\lambda_0}\left\{\sum_{i=1}^{\ell} i|p_i|\lambda_0^{-i}+\sum_{j=1}^{m} j|q_j|\lambda_0^{j}\right\}\right).$$

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So, because of the definition of $\mu(\lambda_0)$ by *Property A*

$$|L(\lambda_0;\phi)| \leq \parallel \phi \parallel \Big(1+\mu(\lambda_0)k(\lambda_0)\Big). \tag{2.15}$$

Let us consider the constant $M(\lambda_0;\phi)$ defined by (2.9). Then using (1.8) and (2.15)

$$\begin{aligned} M(\lambda_0;\phi) &\leq \parallel \phi \parallel k(\lambda_0) + \frac{|L(\lambda_0;\phi)|}{1+\beta(\lambda_0)} \leq \parallel \phi \parallel k(\lambda_0) + \frac{\parallel \phi \parallel \Big(1+\mu(\lambda_0)k(\lambda_0)\Big)}{1+\beta(\lambda_0)} \\ &= \left(k(\lambda_0) + \frac{\Big(1+\mu(\lambda_0)k(\lambda_0)\Big)}{1+\beta(\lambda_0)}\right) \parallel \phi \parallel . \end{aligned}$$

Next, define $(y(n))_{n\geq-\ell}$ and $(z(n))_{n\geq-\ell}$ as well as $M(\lambda_0;\phi)$ as in the proof of Theorem 2.1. Then, as it has been shown in the proof of Theorem 2.1, the sequence $(z(n))_{n\geq-\ell}$ satisfies (2.13). By the definition of $(z(n))_{n\geq-\ell}$ and using (1.7), from (2.13) we get

$$\begin{aligned} |y(n)| &\leq M(\lambda_0;\phi)\mu(\lambda_0) + \frac{|L(\lambda_0;\phi)|}{1+\beta(\lambda_0)} \\ &\leq \mu(\lambda_0)\left(k(\lambda_0) + \frac{\Big(1+\mu(\lambda_0)k(\lambda_0)\Big)}{1+\beta(\lambda_0)}\right) \parallel \phi \parallel + \frac{\Big(1+\mu(\lambda_0)k(\lambda_0)\Big)}{1+\beta(\lambda_0)} \parallel \phi \parallel \\ &= \left\{\mu(\lambda_0)k(\lambda_0) + \big(1+\mu(\lambda_0)\big)\frac{\Big(1+\mu(\lambda_0)k(\lambda_0)\Big)}{1+\beta(\lambda_0)}\right\} \parallel \phi \parallel \\ &= N(\lambda_0) \parallel \phi \parallel . \end{aligned}$$

Hence, we have

$$|y(n)| \leq N(\lambda_0) \parallel \phi \parallel \quad \text{for all} \quad n \geq 0,$$

which, by the definition of $(y(n))_{n\geq-\ell}$, yields

$$|x(n)| \leq N(\lambda_0) \parallel \phi \parallel \lambda_0^n \quad \text{for all} \quad n \geq 0.$$

This completes the proof of the first part of the theorem. It remains to show the stability criterion contained in the theorem.

Let us suppose that $\lambda_0 \leq 1$. Let $\phi = (\phi(n))_{n=-\ell}^{m}$ be an arbitrary initial function in Φ and let x be the solution of the IVP (1.1) and (1.2). Then (1.6) holds and hence

$$|x(n)| \leq N(\lambda_0) \parallel \phi \parallel \quad \text{for all} \quad n \geq 0.$$

Since $|\beta(\lambda_0)| \leq \mu(\lambda_0)$, we obviously have $N(\lambda_0) > 1$, and so, we have

$$|x(n)| \leq N(\lambda_0) \parallel \phi \parallel \quad \text{for all} \quad n \geq -\ell.$$

Using this inequality, we can immediately verify that the trivial solution of (1.1) is uniformly stable. Moreover, if $\lambda_0 < 1$. Then (1.6) guarantees that

$$\lim_{n\to\infty} x(n) = 0.$$

Thus, for $\lambda_0 < 1$ the trivial solution of (1.1) is uniformly asymptotically stable.

Finally, we assume that $\lambda_0 > 1$ and we will show that the trivial solution of (1.1) is unstable. Suppose, for the sake of contradiction, that the trivial solution of (1.1) is stable.

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Then we can choose a number $\delta > 0$ such that, for each $\phi = (\phi(n))_{n=-\ell}^{m}$ in Φ with $\| \phi \| \leq \delta$, it holds

$$|x(n)| < 1 \qquad \text{for all} \quad n \geq -\ell. \tag{2.16}$$

Define $\phi_0 = (\phi_0(n))_{n=-\ell}^{m}$ in Φ, where

$$\phi_0(n) = \lambda_0^n \quad \text{for} \quad n = -\ell, \ldots, 0, \ldots, m.$$

From (2.2), we can verify that

$$\begin{aligned} L(\lambda_0; \phi_0) &= 1 + \frac{1}{\lambda_0} \left\{ \sum_{i=1}^{\ell} p_i \lambda_0^{-i} \left(\sum_{s=-i}^{-1} 1 \right) - \sum_{j=1}^{m} q_j \lambda_0^{j} \left(\sum_{s=0}^{j-1} 1 \right) \right\} \\ &= 1 + \frac{1}{\lambda_0} \left\{ \sum_{i=1}^{\ell} i p_i \lambda_0^{-i} - \sum_{j=1}^{m} j q_j \lambda_0^{j} \right\} \\ &= 1 + \beta(\lambda_0) > 0. \end{aligned} \tag{2.17}$$

Furthermore, we consider a number $\delta_0 > 0$ with $0 < \delta_0 < \delta$ and we put

$$\phi(n) = \frac{\delta_0}{\| \phi_0 \|} \phi_0(n) \quad \text{for} \quad n = -\ell, \ldots, 0, \ldots, m.$$

Then, by setting $\phi = (\phi(n))_{n=-\ell}^{m}$, we have $\| \phi \| = \delta_0 < \delta$. So (2.16) is satisfied for the initial function ϕ in Φ. Thus, since $\lambda_0 > 1$, we have

$$\lim_{n \to \infty} \left[\lambda_0^{-n} x(n) \right] = \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)} = \frac{\delta_0}{\| \phi_0 \|} \cdot \frac{L(\lambda_0; \phi_0)}{1 + \beta(\lambda_0)} = \frac{\delta_0}{\| \phi_0 \|} > 0.$$

But, since $\lambda_0 > 1$, from (2.16) it follows that

$$\lim_{n \to \infty} \left[\lambda_0^{-n} x(n) \right] = 0.$$

We have thus arrived at a contradiction. The proof of Theorem 1.1 is now complete. □

3 Additional lemmas

In this section, we give some conditions, under which the characteristic equation (1.3) has a positive real root λ_0 with the *Property A*. Here, we will give two lemmas about the positive real roots of (1.3).

Lemma 3.1 *Let $r = max\{\ell, m\}$ and assume that*

$$\frac{1}{r+1} + \sum_{i=1}^{\ell} p_i \left(\frac{r+1}{r} \right)^i + \sum_{j=1}^{m} q_j \left(\frac{r}{r+1} \right)^j > 0 \tag{3.1}$$

$$\frac{1}{r} - \sum_{i=1}^{\ell} p_i \left(\frac{r}{r+1} \right)^i - \sum_{j=1}^{m} q_j \left(\frac{r+1}{r} \right)^j > 0 \tag{3.2}$$

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and

$$\sum_{i=1}^{\ell} i|p_i|\left(\frac{r+1}{r}\right)^{i+1} + \sum_{j=1}^{m} j|q_j|\left(\frac{r+1}{r}\right)^{j-1} \leq 1. \tag{3.3}$$

Then, in the interval $\left(\frac{r}{r+1}, \frac{r+1}{r}\right)$, *the characteristic equation* (1.3) *has a unique positive root* λ_0, *and this root satisfies the Property A.*

Proof Define

$$F(\lambda) = \lambda - 1 - \sum_{i=1}^{\ell} p_i\lambda^{-i} - \sum_{j=1}^{m} q_j\lambda^{j} \quad \text{for } \lambda \in \left[\frac{r}{r+1}, \frac{r+1}{r}\right]. \tag{3.4}$$

It follows from (3.1) that

$$\begin{aligned} F\left(\frac{r}{r+1}\right) &= \frac{r}{r+1} - 1 - \sum_{i=1}^{\ell} p_i\left(\frac{r}{r+1}\right)^{-i} - \sum_{j=1}^{m} q_j\left(\frac{r}{r+1}\right)^{j} \\ &= -\frac{1}{r+1} - \sum_{i=1}^{\ell} p_i\left(\frac{r+1}{r}\right)^{i} - \sum_{j=1}^{m} q_j\left(\frac{r}{r+1}\right)^{j} \end{aligned}$$

and so, we get $F(\frac{r}{r+1}) < 0$. Moreover,

$$\begin{aligned} F\left(\frac{r+1}{r}\right) &= \frac{r+1}{r} - 1 - \sum_{i=1}^{\ell} p_i\left(\frac{r+1}{r}\right)^{-i} - \sum_{j=1}^{m} q_j\left(\frac{r+1}{r}\right)^{j} \\ &= \frac{1}{r} - \sum_{i=1}^{\ell} p_i\left(\frac{r}{r+1}\right)^{i} - \sum_{j=1}^{m} q_j\left(\frac{r+1}{r}\right)^{j} \end{aligned}$$

and hence from (3.2) it follows that $F(\frac{r+1}{r}) > 0$. Furthermore, by taking into account (3.3), for $\lambda \in \left(\frac{r}{r+1}, \frac{r+1}{r}\right)$, we obtain

$$\begin{aligned} F'(\lambda) &= 1 + \sum_{i=1}^{\ell} ip_i\lambda^{-i-1} - \sum_{j=1}^{m} jq_j\lambda^{j-1} \\ &\geq 1 - \sum_{i=1}^{\ell} i|p_i|\lambda^{-i-1} - \sum_{j=1}^{m} j|q_j|\lambda^{j-1} \\ &> 1 - \sum_{i=1}^{\ell} i|p_i|\left(\frac{r}{r+1}\right)^{-i-1} - \sum_{j=1}^{m} j|q_j|\left(\frac{r+1}{r}\right)^{j-1} \geq 0. \end{aligned}$$

Therefore, F is strictly increasing on the interval $\left(\frac{r}{r+1}, \frac{r+1}{r}\right)$. So, as $F(\frac{r}{r+1}) < 0$, $F(\frac{r+1}{r}) > 0$ and $F'(\lambda) > 0$ on $\left(\frac{r}{r+1}, \frac{r+1}{r}\right)$, the equation $F(\lambda) = 0$ has a unique root λ_0 in the interval $\left(\frac{r}{r+1}, \frac{r+1}{r}\right)$. This root satisfies *Property A*. Indeed, by using again (3.3), we have

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$$\frac{1}{\lambda_0}\left\{\sum_{i=1}^{\ell} i\,|p_i|\,\lambda_0^{-i} + \sum_{j=1}^{m} j\,|q_j|\,\lambda_0^{j}\right\}$$
$$= \sum_{i=1}^{\ell} i\,|p_i|\,\lambda_0^{-i-1} + \sum_{j=1}^{m} j\,|q_j|\,\lambda_0^{j-1}$$
$$< \sum_{i=1}^{\ell} i|p_i|\left(\frac{r}{r+1}\right)^{-i-1} + \sum_{j=1}^{m} j|q_j|\left(\frac{r+1}{r}\right)^{j-1} \leq 1.$$

This completes the proof. □

Note: We can use Lemma 3.1 and the stability criterion contained in Theorem 1.1 to derive the following corollary.

Corollary 3.2 *Assume that* (3.1)–(3.3) *are satisfied. Then the trivial solution of* (1.1) *is uniformly asymptotically stable if*

$$\sum_{i=1}^{\ell} p_i + \sum_{j=1}^{m} q_j < 0$$

and it is unstable if

$$\sum_{i=1}^{\ell} p_i + \sum_{j=1}^{m} q_j > 0.$$

Proof The Lemma 3.1 guarantees that, in the interval $\left(\frac{r}{r+1}, \frac{r+1}{r}\right)$, the characteristic equation (3.1) has a unique root λ_0; this root satisfies the *Property A*. Let F be defined by (3.4). For this function, as in the proof of Lemma 3.1, we have

$$F\left(\frac{r}{r+1}\right) < 0 \quad \text{and} \quad F\left(\frac{r+1}{r}\right) > 0.$$

Clearly, $\lambda_0 < 1$ if $F(1) > 0$, and $\lambda_0 > 1$ if $F(1) < 0$. On the other hand,

$$F(1) = -\sum_{i=1}^{\ell} p_i - \sum_{j=1}^{m} q_j.$$

So, the proof of Corollary 3.2 is now complete.

□

Lemma 3.3 *Let r is defined as in Lemma 3.1. Suppose that*

$$p_i < 0 \quad for\ i = 1, \ldots, \ell, \quad and \quad q_j \leq 0 \quad for\ j = 1, \ldots, m.$$

Assume that (3.1) *holds. Then,*

(i) *in the interval* $\left(0, \frac{r}{r+1}\right)$, *the characteristic equation* (1.3) *has a unique root,*
(ii) *in the interval* $\left(\frac{r}{r+1}, 1\right)$, *the characteristic equation* (1.3) *has a unique root and*
(iii) *in the interval* $[1, \infty)$, *the characteristic equation* (1.3) *has no roots.*

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Proof We introduce the function F defined by

$$F(\lambda) = \lambda - 1 - \sum_{i=1}^{\ell} p_i \lambda^{-i} - \sum_{j=1}^{m} q_j \lambda^j \quad \text{for } \lambda > 0.$$

We immediately obtain

$$F''(\lambda) = -\sum_{i=1}^{\ell} i(i+1) p_i \lambda^{-i-2} - \sum_{j=1}^{m} j(j-1) q_j \lambda^{j-2} \quad \text{for } \lambda > 0.$$

So, using the hypothesis that $p_i < 0$ and $q_j \leq 0$, we conclude that

$$F''(\lambda) > 0 \quad \text{for all } \lambda > 0,$$

and consequently F is convex for $\lambda > 0$. Next, we observe that, as in the proof of Lemma 3.1, assumption (3.1) means that $F(\frac{r}{r+1}) < 0$ holds true. Furthermore, it is not difficult to show that $F(0+) = \infty$. Therefore, as $F\left(\frac{r}{r+1}\right) < 0$, $F(0+) = \infty$ and $F''(\lambda) > 0$ for $\lambda > 0$, the characteristic equation (1.3) has a unique root in the interval $\left(0, \frac{r}{r+1}\right)$, and so part (i) has been proved. Furthermore, we obtain

$$F(1) = -\sum_{i=1}^{\ell} p_i - \sum_{j=1}^{m} q_j > 0.$$

So, as $F\left(\frac{r}{r+1}\right) < 0$, $F(1) > 0$ and $F''(\lambda) > 0$ for $\lambda > 0$, the characteristic equation (1.3) has a unique root in the interval $\left(\frac{r}{r+1}, 1\right)$, and so, part (ii) has been shown. It remains to establish parts (iii). Because there are two roots in the interval $(0, 1)$ and since F is convex, there is no root in the interval $[1, \infty)$.

The proof of the lemma is complete. □

4 Examples

In the following examples, we will apply the stability criteria of the Theorem 1.1.

Example 4.1 Consider

$$\begin{aligned} \triangle x(n) &= \sum_{i=1}^{2} \left(-\frac{1}{4}\right)^i x(n-i) + \sum_{j=1}^{2} \left(-\frac{1}{4}\right)^{j+1} x(n+j), \quad n = 0, 1, 2, \ldots \\ x(n) &= \phi(n) \quad \text{for} \quad n = -2, -1, 0, 1, 2, \end{aligned} \tag{4.1}$$

where $\phi(n) \in \mathbf{R}$.

The characteristic equation of (4.1) is

$$1 = \lambda - \sum_{i=1}^{2} \left(-\frac{1}{4}\right)^i \lambda^{-i} - \sum_{j=1}^{2} \left(-\frac{1}{4}\right)^{j+1} \lambda^j$$

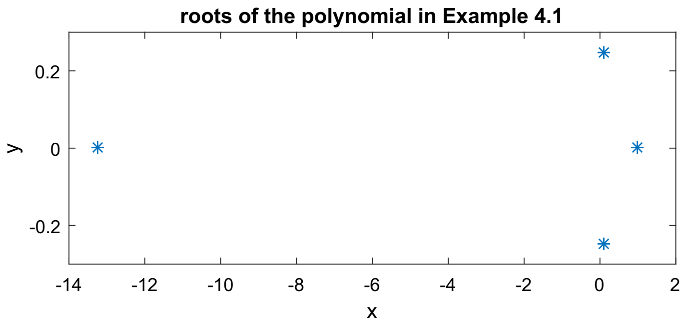

Fig. 1 Locations of the roots of the polynomial in Example 4.1

or

$$\lambda^4 + 12\lambda^3 - 16\lambda^2 + 4\lambda - 1 = 0. \tag{4.2}$$

Here are the roots of (4.2) we obtain by the MATLAB function "roots.m":

$$\lambda_1 = 1,\ \lambda_2 = -13.2324,\ \lambda_3 = 0.1162 + 0.2491i,\ \lambda_4 = 0.1162 - 0.2491i.$$

In Fig. 1, we give the locations of the roots of the polynomial in Example 4.1. We have $\lambda = 1$ is unique positive real root of (4.2). Then, for $\lambda_0 = 1$, from *Property A* we get

$$\mu(1) = \sum_{i=1}^{2} i \left| \left(-\frac{1}{4}\right)^i \right| + \sum_{j=1}^{2} j \left| \left(-\frac{1}{4}\right)^{j+1} \right| = \frac{3}{4} < 1.$$

So, the trivial solution of (4.1) is uniformly stable by Theorem 1.1.

Example 4.2 Consider

$$\begin{aligned} \triangle x(n) &= \sum_{i=1}^{3} \left(\frac{1}{2}\right)^i x(n-i) + \left(\frac{1}{3^4}\right) x(n+j), \quad n = 0, 1, 2, \cdots \\ x(n) &= \phi(n) \quad \text{for} \quad n = -3, -2, -1, 0, 1, \end{aligned} \tag{4.3}$$

where $\phi(n) \in \mathbf{R}$.

The characteristic equation of (4.3) is

$$1 = \lambda - \sum_{i=1}^{3} \left(\frac{1}{2}\right)^i \lambda^{-i} - \frac{\lambda}{3^4}$$

or

$$640\lambda^4 - 648\lambda^3 - 324\lambda^2 - 162\lambda - 81 = 0. \tag{4.4}$$

Here are the roots of (4.4) we obtain by the MATLAB function "roots.m":

$$\lambda_1 = 3/2,\ \lambda_2 = -0.425,\ \lambda_3 = 0.0312 + 0.4444i,\ \lambda_4 = 0.0312 - 0.4444i.$$

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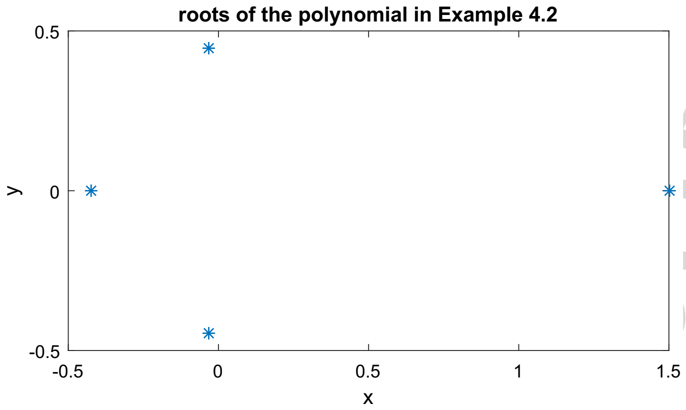

Fig. 2 Locations of the roots of the polynomial in Example4.2

In Fig. 2, we give the locations of the roots of the polynomial in Example 4.2. We have $\lambda = \frac{3}{2}$ is unique positive real root of (4.4). Then, for $\lambda_0 = \frac{3}{2}$, from *Property A* we get

$$\mu\left(\frac{3}{2}\right) = \frac{2}{3}\left\{\sum_{i=1}^{3} i\left|\left(\frac{1}{2}\right)^{i}\right|\left(\frac{3}{2}\right)^{-i} + \left|\left(\frac{1}{3^4}\right)\right|\frac{3}{2}\right\} = \frac{37}{81} < 1.$$

So, the trivial solution of (4.3) is unstable by Theorem 1.1.

Example 4.3 Consider

$$\begin{aligned} \triangle x(n) &= -\frac{1}{9}x(n-1) - \frac{1}{4}x(n+1), \quad n = 0, 1, 2, \ldots \\ x(n) &= \phi(n) \quad \text{for} \quad n = -1, 0, 1, \end{aligned} \tag{4.5}$$

where $\phi(n) \in \mathbf{R}$.

The characteristic equation of (4.5) is

$$45\lambda^2 - 36\lambda + 4 = 0. \tag{4.6}$$

We have $\lambda_1 = \frac{2}{15}$ and $\lambda_2 = \frac{2}{3}$ are roots of (4.6). Let $\lambda_0 = \frac{2}{15}$. Then

$$\mu\left(\frac{2}{15}\right) = \frac{15}{2}\left\{\frac{1}{9}\,\frac{15}{2} + \frac{1}{4}\,\frac{2}{15}\right\} = \frac{13}{2} > 1.$$

Therefore, Theorem 1.1 cannot be applied to equation (4.5). But, for $\lambda_0 = \frac{2}{3}$, we get

$$\mu\left(\frac{2}{3}\right) = \frac{3}{2}\left\{\frac{1}{9}\,\frac{3}{2} + \frac{1}{4}\,\frac{2}{3}\right\} = \frac{1}{2} < 1.$$

Then, the trivial solution of (4.5) is uniformly asymptotically stable by Theorem 1.1.

In this example, stability analysis could be performed using Corollary 3.2. Indeed, if the conditions of (3.1), (3.2) and (3.3) are calculated respectively, since $r = 1$, we obtain

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$$\frac{1}{2}-\frac{2}{9}-\frac{1}{8}=\frac{11}{72}>0,$$

$$1+\frac{1}{18}+\frac{1}{2}=\frac{14}{9}>0,$$

$$\frac{4}{9}+\frac{1}{4}=\frac{25}{36}<1.$$

Therefore, since $\sum_{i=1}^{\ell} p_i+\sum_{j=1}^{m} q_j=-\frac{1}{9}-\frac{1}{4}<0$, the trivial solution of (4.5) is uniformly asymptotically stable by Corollary 3.2.

Moreover, it is easily seen that in this example provides the Lemmas 3.1and 3.3.

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