Numerical Methods for Solving Nonlinear Fractional Differential Equations with Non-Uniform Meshes

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Abstract

In this dissertation, we consider numerical methods for solving fractional differential equations with non-uniform meshes. We first introduce some basic definitions and theories for fractional differential equations and then we consider the numerical methods fro solving fractional differential equation. In the literature, the popular numerical methods for solving fractional differential equation include the rectangle method, trapezoid method and predictor-corrector methods. We reviewed such methods and the ways to prove the stability and the error estimates for these methods. Since the fractional differential equation is a nonlocal problem, the computation cost is very long compared with the local problem. Therefore it is very important to design some higher order numerical methods for solving fractional differential equation. In this dissertation, we introduce a new higher order numerical method for solving fractional differential equation which is based on the quadratic interpolation polynomial approximation to the fractional integral. To capture the singularity near the origin we also introduce the non-uniform meshes. The numerical results show that the optimal convergence order can be recovered by using non-uniform meshes even if the data are not sufficiently smooth.

Key words

- Fractional differential equations
- Caputo Derivative
- Riemann-Liouville derivatives
- Hadamard finite integral
- Mittag-Leffler functions

- Trapezoidal method
- Rectangle method
- Predictor-Corrector method
- Stability estimate
- Error estimate

This work is original and has not been previously submitted for any academic purpose.

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Date: ....07/10/2017...

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Chapter 1

Introduction

Fractional differential equations are used for many practical purposes some of which can be found in [14][15][16][17][18]. Recently fractional differential equations have been used in many fields of science and engineering, more specifically in viscoelastic materials, fluid flow, rheology, diffusive transport and electromagnetic theory which can be found in [22][23][24][25]. This dissertation mainly looks at how to solve the following fractional differential equation [10]:

$$\begin{cases} {}_0^C D_t^{\alpha} y(t) = f(t, y), & T \geq t > 0, \\ y^{(k)}(0) = y_0^{(k)}, & k = 0, 1, \ldots, n-1. \end{cases} \tag{1.1}$$

Here the Caputo derivative is defined by [4]:

$${}_0^C D_t^{\alpha} y(t) = \frac{1}{\Gamma(\lceil \alpha \rceil - \alpha)} \int_0^t (t-s)^{\lceil \alpha \rceil - \alpha - 1} y^{\lceil \alpha \rceil}(s)\, ds, \tag{1.2}$$

and $\lceil \alpha \rceil$ is the smallest integer $\geq \alpha$.
Due to equation (1.2), equation (1.1) is equivalent to [4]

$$y(t) - y(0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))\, ds \tag{1.3}$$

For equation (1.1) to have a unique solution y we assume that $y_0^{(k)}$ is an arbitrary real number, f is a continuous function that fulfills a Lipschitz condition with respect to its second argument.

It is not possible to find an analytical soution of (1.1) for general f. We shall consider numerical methods which have better stability and higher convergence order when a non-uniform mesh is used compared to that of a uniform mesh. We will also use certain smoothness assumptions for the equations (1.1) and (1.3) to consider these numerical methods.

For solving equations (1.1)-(1.3) we will discuss the different numerical methods and consider their stability and convergence. This dissertation will include:

> In Chapter 2 we consider an introduction to basic fractional differential equations and fractional integral equations by looking at some basic definitions, theorems and relations for fractional calculus.
>
> In Chapters 3 we review a paper *Detailed error analysis for a fractional Adams method* [1] which used the fractional Adams method for solving (1.1). We show the theorems and definitions for the numerical methods and then look into the error analysis for these methods.
>
> In Chapter 4 we review a paper by *C.Li, Q. Yi and A. Chen* [10]. This paper looks into finding numerical methods for solving equation (1.1) with non-uniform meshes and then give theorems and definitions for their stability analysis and error analysis.
>
> In Chapter 5 we look at lecture notes by *Y. Yan* [4] which are an extension of the results found in Chapter 4. These lecture notes describe three different numerical methods for solving equation (1.1) and then goes into detail about finding the error analysis and stability analysis for all three numerical methods, and finally some numerical results are given for each numerical method.
>
> In Chapter 6 we will discuss three approximations for solving equation (1.1) two of which have been shown in previous chapters but greater detail on how they are calculated will be given and a new higher order method will be shown, we will also show the convergence orders of each numerical method. Numerical examples will be given to show that the theoretical results are consistent with experimental results.

In Chapter 7 we will summarize our findings and mention some possible future work on this topic.

Chapter 2

An Introduction to fractional differential equations

In this chapter we introduce fractional differential equations. We will look at the properties of fractional integrals, Riemann-Liouiville fractional derivatives and integrals, their relation to each other, the Hadamard finite part integral and briefly look at the Mittag-Leffler function. All of the theorems, definitions and lemmas found in this chapter can be found in the book entitled *The analysis of fractional differential equations* by Kia Diethelm [5].

2.1 Fractional Integrals

Definition 2.1. *[5]*

a) By D we denote the operator that maps a function onto its derivative, i.e.

$$Df(x) := f'(x)$$

b) By J_a, we denote the operator that maps a function f, assumed to be (Riemann) integrable on the compact interval [a,b], on its primitive centered at a, i.e.

$$D_x^{-1} = J_a f(x) = \int_a^x f(t)\,dt$$

for $a \leq x \leq b$

c) For $n \in \mathbb{N}$, we use the symbols D^n and J_a^n to denote the n-fold iterates of D and J_a, respectively, i.e. we set $D^1 := D$, $J_a^1 := J_a$, and $D^n = DD^{n-1}$ and $J_a^n = J_a J_a^{n-1}$ for $n \geq 2$

Note For our notation we have

$$DJ_a f = f$$

which implies that

$$D^n J_a^n f = f$$

for $n \in \mathbb{N}$. i.e. D^n is the left inverse of J_a^n in a suitable space of functions.

Now we will introduce two lemmas with proofs that again can be found in [5]

Lemma 2.2. *[5] Let f be Riemann integrable on $[a, b]$. Then, for $a \leq x \leq b$ and $n \in \mathbb{N}$, we have:*

$$J_a^n f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt$$

Proof:[19]

$$J_a^n f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt$$

When $n = 1$

$$\begin{aligned} J_a^1 f(x) &= \frac{1}{(1-1)!} \int_a^x (x-t)^{1-1} f(t)\, dt \\ &= \frac{1}{0!} \int_a^x (x-t)^0 f(t)\, dt \\ &= \int_a^x f(t)\, dt \end{aligned}$$

True when $n = 1$, now we assume true for when $n = k - 1$

$$J_a^{k-1} f(x) = \frac{1}{((k-1)-1)!} \int_a^x (x-t)^{(k-1)-1} f(t)\, dt$$

Prove true for when $n = k$ e.g

$$J_a^k f(x) = \frac{1}{(k-1)!} \int_a^x (x-t)^{k-1} f(t)\, dt$$

$$
\begin{aligned}
J_a^k f(x) &= J_a^1 J_a^{k-1} f(x)\\
&= J_a^1 \frac{1}{((k-1)-1)!}\int_a^x (x-t)^{(k-1)-1} f(t)\,dt\\
&= \frac{1}{1!}\frac{1}{((k-1)-1)!}\int_a^x (x-t)^{(k-1)-1} f(t)\,dt \cdot \int_a^x (x-t)\,dt\\
&= \frac{1}{(k-1)!}\int_a^x (x-t)^{k-1} f(t)\,dt
\end{aligned}
$$

Therefore true when $n = k$ hence true for all values of $n \in \mathbb{N}$ #

Lemma 2.3. *[5] Let $m, n \in \mathbb{N}$. Such that $m > n$, and let f be a function having a continuous n^{th} derivative on the interval $[a, b]$. Then*

$$D^n f = D^m J_a^{m-n} f$$

Proof:[19] Note that

$$f = D^{m-n} J_a^{m-n} f \qquad m, n \in \mathbb{N},\ m > n$$

Applying the operator D^n to both sides of the relation and using the fact that $D^n \cdot D^{m-n} = D^m$, we get

$$D^n f = D^n[D^{m-n} J_a^{m-n} f] = D^m J_a^{m-n} f$$

#

Now we will introduce Euler's Gamma function, followed by a theorem which can be used in order for us to be able to simplify the fractional differential equations.

Definition 2.4. *[5] The function $\Gamma : (0, \infty) \to \mathbb{R}$ defined by*

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\,dt, \qquad x > 0$$

is called Euler's Gamma Function.

Theorem 2.5. *[5] For $n \in \mathbb{N}$, we have*

$$\Gamma(n) = (n-1)!$$

The proof for this theorem can be found in [7]. Now we define a Hölder space which may be necessary for working through methods in later chapters.

Definition 2.6. *[5] Let* $0 < \mu \leq 1$, $k \in \mathbb{N}_0$ *and* $p \geq 1$

$$L_p[a,b] := \{f : [a,b] \to \mathbb{R};\, f\, is\ measurable\ on\, [a,b]\ and \int_a^b |f(x)|^p\, dx < \infty\},$$

$$L_\infty[a,b] := \{f : [a,b] \to \mathbb{R};\, f\, is\ measurable\ and\ essentially\ bounded\ on\, [a,b]\},$$

$$H_\mu[a,b] := \{f : [a,b] \to \mathbb{R}; \exists\, c > 0,\, \forall x, y \in [a,b] : |f(x) - f(y)| \leq c|x-y|^\mu,\, 0 < \mu < 1\},$$

$$C^k[a,b] := \{f : [a,b] \to \mathbb{R};\ f\, has\ a\ continuous\ kth\ derivative\},$$

$$C[a,b] := C^0[a,b],$$

$$H_0[a,b] := C[a,b],$$

where $L_p(a,b)$ $(1 \leq p \leq \infty)$ *is the usual Lebesgue space,* $H_\mu[a,b]$ *is a Hölder space of the Lipschitz space of order* μ*.*

Now we have a fundamental theorem in Lebesgue spaces which is presented here without proof but can be found in [6], followed by a definition of continuous derivatives.

Theorem 2.7. *[5] Let* $f \in L_1[a,b]$*. Then* $J_a f$ *id differentiable almost everywhere in* $[a,b]$*, and* $DJ_a f = f$ *also holds almost everywhere on* $[a,b]$

Definition 2.8. *[5] By* A^n *or* $A^n[a,b]$ *we denote the set of functions with an absolutely continuous* $(n-1)$*th derivative, i.e. the function* f *for which there exists (almost everywhere) a function* $g \in L_1(a,b)$ *such that*

$$f^{(n-1)}(x) = f^{(n-1)}(a) + \int_a^x g(t)\, dt$$

In this case we call g *the generalized nth derivative of* f*, we simply write* $g = f^{(n)}$*.*

$$f \in A^1 \Rightarrow f \in C[a,b]$$

$$f \in A^1 \Rightarrow f' \in L_1(a,b)$$

$$f' \in L_1(a,b) \nRightarrow f \in A^1$$

For more on Fractional differential equations see [26],[27],[28],[29],[30],[31],[32],[33].

2.2 Riemann - Liouville Integral operators

In this section we look at theorems and definitions for Riemann-Liouville integral operators.

Definition 2.9. *[5] Let $n \in \mathbb{R}_+$. The operator J_a^n, defined on $L_1(a,b)$ by*

$$J_a^n f(x) := \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)\, dt$$

for $a \leq x \leq b$, is called the Riemann - Liouville integral operator of order n

Note:[5] For $n = 0$, we set $J_a^0 := I$, the identity operator.
Remark: [5] If $0 < n < 1$, $|f(t)| \leq M$, then $J_a^n f(x)$ is well defined since

$$\begin{aligned} |J_a^n f(x)| &\leq \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} M\, dt \\ &\leq \frac{M}{\Gamma(n)} \cdot \frac{1}{n-1+1} (x-t)^n \Big|_{t=a}^{t=x} < \infty \end{aligned}$$

If $n \geq 1$, $f \in L^1(a,b)$, then $J_a^n f(x)$ exists for every $x \in [a,b]$ since

$$\begin{aligned} |J_a^n f(x)| &\leq \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} |f(x)|\, dt \\ &\leq \frac{1}{\Gamma(n)} (b-a)^{n-1} \cdot \int_a^x |f(t)|\, dt \leq M \end{aligned}$$

In general we now give two Theorems for $n > 0$, $f \in L^1(a,b)$, we have:

Theorem 2.10. *[5] Let $f \in L_1(a,b)$ and $n > 0$. Then the integral $J_a^n f(x)$ exists for almost every $x \in (a,b)$ moreover the function $J_a^n f$ itself is also an element of $L_1(a,b)$*

Proof[20] We will express the integral $J_a^n f(x)$ by using the convolution of two functions $\phi_1, \phi_2 \in L_1(\mathbb{R})$ then we use the following well-known Lebesgue theorem

Theorem 2.11. *[5] Let $\phi_1, \phi_2 \in L_1(\mathbb{R})$, then*

$$\phi_1 * \phi_2(x) = \int_{-\infty}^{\infty} \phi_1(x-t)\phi_2(t)\, dt$$

is well defined for almost every $x \in \mathbb{R}$ and $\phi_1 * \phi_2 \in L_1(\mathbb{R})$

We denote

$$J_a^n f(x) = \int_a^x (x-t)^{n-1} f(t)\,dt = \int_{-\infty}^{\infty} \phi_1(x-t)\phi_2(t)\,dt$$

Where

$$\phi_1(u) = \begin{cases} u^{n-1}, & \text{for } 0 < u \le b-a \\ 0, & \text{else} \end{cases}$$

and

$$\phi_2(u) = \begin{cases} f(u), & \text{for } a \le u \le b \\ 0, & \text{else} \end{cases}$$

By construction, we see that $\phi_1 \in L_1(\mathbb{R}), \phi_2 \in L_1(\mathbb{R})$ By the above theorem, we get

$$J_a^n f(x) = \int_{-\infty}^{\infty} \phi_1(x-t)\phi_2(t)\,dt = \phi_1 * \phi_2(x) \in L_1(\mathbb{R})$$

and $J_a^n f$ exists for almost every $x \in (a,b)$. #

Theorem 2.12. *[5] Let $m, n \ge 0$ and $\phi \in L_1(a,b)$ then,*

$$J^n J^m \phi = J_a^n J_a^m \phi = J^{n+m}\phi$$

holds almost everywhere on (a,b). If additionally $\phi \in C[a,b]$ or $m+n \ge 1$ then the identity holds everywhere on $[a,b]$

Proof:[20]

$$J_a^m J^n - a\phi = \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x (x-t)^{m-1} \left[\int_a^t (t-\tau)^{n-1}\phi(t)\,d\tau\right] dt$$

This integral exists and by Fubini's theorem [8], we may interchange the order of integration obtaining,

$$\begin{aligned} J_a^m J_a^n \phi(x) &= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x \int_a^t \left[(x-t)^{m-1}(t-\tau)^{n-1}\phi(\tau)\right] d\tau\,dt \\ &= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x \left[\int_\tau^x (x-t)^{m-1}(t-\tau)^{n-1}\phi(\tau)\,dt\right] d\tau \\ &= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x \phi(\tau) \left[\int_\tau^x (x-t)^{m-1}(t-\tau)^{n-1}\,dt\right] d\tau \end{aligned}$$

using the substitution $t = \tau + s(x-\tau)$

$$J_a^m J_a^n \phi(x) = \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x \phi(\tau) \left[\int_0^1 [(x-\tau)(1-s)]^{m-1} \times [s(x-\tau)]^{n-1} (x-\tau)\, ds \right] d\tau$$
$$= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x \phi(\tau)(x-\tau)^{m+n-1} \left[\int_0^1 (1-s)^{m-1} s^{n-1}\, ds \right] d\tau$$

Note that $\int_0^1 (1-s)^{m-1} s^{n-1} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(n+m)}$, thus

$$J_a^m J_a^n \phi(x) = \frac{1}{\Gamma(m+n)} \int_a^x \phi(\tau)(x-\tau)^{m+n-1}\, d\tau = J_a^{m+n}\phi(x)$$

Case 1: If $\phi \in C[a,b]$, then $J_a^n \phi \in C[a,b]$, therefore

$$J_a^m J_a^n \phi \in C[a,b] \text{ and } J_a^{m+n}\phi \in C[a,b] \text{too}$$

Thus since these two continuous functions coincide almost everywhere, they must coincide everywhere.
Case 2: If $\phi \in L_1[a,b]$ and $m+n \geq 1$ we have, by the above result

$$J_a^m J_a^n \phi = J_a^{m+n}\phi = J_a^{m+n-1} J_a^1 \phi$$

almost everywhere. Since $J_a^1 \phi$ is continuous, we also have that $J_a^{m+n}\phi = J_a^{m+n-1} J_a^1 \phi$ is continuous, and once again we may conclude that the two functions on either side of the equality almost everywhere are continuous; thus they must be identical everywhere. #

Now we present two theorems, the first concerns the Hölder function with the Riemann-Liouville integral this is a mapping property and relates to how fractional integration improves the smoothness properties of functions. The second does a similar thing but this time with the Lebesgue class.

Theorem 2.13. *[5] Let $\phi \in H_\mu[a,b]$ for some $\mu \in [0,1]$ then*

$$J_a^n \phi(x) = \frac{\phi(a)}{\Gamma(n+1)}(x-a)^n + \Phi(x),$$

with some function Φ. This function Φ satisfies

$$\Phi(x) = O\left((x-a)^{\mu+n}\right) \text{ as } x \to a$$

more over

$$\phi \in \begin{cases} H_{\mu+n}[a,b] & \text{if } \mu + n < 1 \\ H^*[a,b] & \text{if } \mu + n = 1 \\ H_1[a,b] & \text{if } \mu + n > 1 \end{cases}$$

The proof for this theorem can be found in [5] on pages 16-18.

Theorem 2.14. *[5] Let $n > 0, p > \max\{1, \frac{1}{n}\}$, and $\phi \in L_p(a,b)$ then*

$$J_a^n \phi(x) = O\left((x-a)^{n-\frac{1}{p}}\right) \text{ as } x \to a^+$$

If additionally $n - \frac{1}{p} \notin \mathbb{N}$, then $J_a^n \phi \in C^{[n-\frac{1}{p}]}[a,b]$ and $D^{[n-\frac{1}{p}]} J_a^n \phi \in H_{n-\frac{1}{p}-[n-\frac{1}{p}]}[a,b]$ here $[\beta]$ denote the integer part of β, $[\beta] \leq \beta$

The proof of this theorem can again be found in [5] but this time over pages 19-20. More on Riemann-Liouville can be found in [34], [35], [36] and [37].

2.3 Relation between Riemann-Liouville integral and derivative

We now look into the relation between the Riemann-Liouville integral and derivative. Beginning with a definition of the integral operator followed by a lemma which shows the derivative in continuous form.

Definition 2.15. *[5] Let $n \in \mathbb{R}_+$, $0 < n < 1$, $m = \lceil x \rceil = 1$, The operator D_a^n is defined by*

$$D_a^n f := D^1(D_a^{n-1} f)$$

Here $D_a^{n-1} f = J_a^{1-n} f$, the Riemann-Liouville integral operator

Lemma 2.16. *[5] Let $f \in A^1[a,b]$, it is absolutely continuous, and $0 < n < 1$. Then $D_a^n f$ exists almost everywhere in $[a,b]$*

Proof:[20] Note that $f \in A^1[a,b]$ implies that f' exists almost everywhere and

$$f(x) = f(a) + \int_a^x f'(t)\,dt, \qquad f' \in L_1[a,b]$$

Thus

$$
\begin{aligned}
D_a^n f(x) &:= D^1(D_a^{n-1} f(x)) \\
&= \frac{d}{dx}\frac{1}{\Gamma(1-n)}\int_a^x (x-t)^{-n}\underline{f(t)}\,dt \\
&\left\{= f(x) = +\int_a^x f'(t)\,dt, \qquad \text{since } f\in A^1[a,b]\right\} \\
&= \frac{d}{dx}\frac{1}{\Gamma(1-n)}\int_a^x (x-t)^{-n}\left(f(a)+\int_a^t f'(u)\,du\right)\,dt \\
&= \frac{1}{\Gamma(1-n)}\frac{d}{dx}\left[\underline{f(a)\int_a^x (x-t)^{-n}\,dt} + \int_a^x\int_a^t (x-t)^{-n} f'(u)\,du\,dt\right] \\
&= \frac{1}{\Gamma(1-n)}\left[\underline{\frac{f(a)}{(x-a)^n}} + \frac{d}{dx}\int_a^x\int_a^t (x-t)^{-n} f'(u)\,du\,dt\right]
\end{aligned}
$$

By Fubinis theorem we can change the order of integration of the double integral.

$$
D_a^n f(x) = \frac{1}{\Gamma(1-n)}\left[\frac{f(a)}{(x-a)^n} + \frac{d}{dx}\int_a^x f'(u)\frac{(x-u)^{1-n}}{1-n}\,du\right]
$$

By formula

$$
\begin{aligned}
\frac{d}{dx}\int_{\varphi_1(x)}^{\varphi_2(x)} f(u,x)\,du &= \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{d}{dx} f(u,x)\,du \\
&\quad + f(u,x)|_{u=\varphi_2(x)}\cdot\varphi_2'(x) - f(u,x)|_{u=\varphi_1(x)}\cdot\varphi_1'(x)
\end{aligned}
$$

Hence

$$
D_a^n f(x) = \frac{1}{\Gamma(1-n)}\left[\frac{f(a)}{(x-a)^n} + \int_a^x f'(u)(x-u)^{-n}\,du\right]
$$

From this we can see that $D_a^n f \in L^1[a,b]$, since,

$$
\begin{aligned}
\int_a^b |D_a^n f(x)|\,dx &\le C\int_a^b (x-a)^{-n}\,dx + \int_a^b\int_a^x |f'(u)||x-u|^{-n}\,du\,dx \\
&\le C\int_a^b (x-a)^{-n}\,dx + \int_a^b\left[\int_u^b |f'(u)||x-u|^{-n}\,dx\right]\,du \\
&\le C\int_a^b (x-a)^{-n}\,dx + \int_a^b |f'(u)|\left(\int_u^b |x-u|^{-n}\,dx\right)\,du
\end{aligned}
$$

$$\leq C\int_a^b (x-a)^{-n}\,dx + C\int_a^b |f'(u)|\cdot(b-u)^{-n+1}\,du$$

$$\leq C\int_a^b (x-a)^{-n}\,dx + C\int_a^b |f'(u)|\,du \leq \infty \qquad \text{since } 0<n<1\ \#$$

Now we have two theorems which shows the direct relation between the Riemann-Liouville integral and deriviative which can be found in [5].

Theorem 2.17. *[5] Let $n \geq 0$. Then for every $f \in L_1[a,b]$,*

$$D_a^n D_a^{-n} f = f$$

almost everywhere, where $J_a^n = D_a^n$

Proof:[20] The proof for $n = 0$ is trivial, so we look at the proof for $0 < n < 1$

$$D_a^n D_a^{-n} f = D_a^1 D_a^{n-1} D_a^{-n} f = D_a^1(D_a^{-1} f) = f$$

now looking at $1 < n < 2$

$$D_a^n D_a^{-n} f = D_a^2 D_a^{n-2} D_a^{-n} f = D_a^2 D_a^{-2} f = D_a^1(D_a^1 D_a^{-1})(D_a^{-1} f) = D_a^1(D_a^{-1} f) = f$$

argument can similary be used to prove $n > 2$ #

Theorem 2.18. *[5] Assume that $n_1, n_2 > 0$. Let $\phi \in L_1[a,b]$ and $f = D_a^{-(n_1+n_2)}\phi$ then,*

$$D_a^{n_1} D_a^{n_2} f = D_a^{n_1+n_2} f$$

Proof: [20]

$$\begin{aligned}
\text{LHS} = D_a^{n_1} D_a^{n_2} f &= D_a^{n_1} D_a^{n_2} D_a^{-(n_1+n_2)}\phi \\
&= D_a^{\lceil n_1\rceil} D_a^{n_1-\lceil n_1\rceil}\cdot D_a^{\lceil n_2\rceil}\underbrace{D_a^{n_2-\lceil n_2\rceil}\cdot D_a^{-(n_1+n_2)}}_{\text{combine}}\phi \\
&= D_a^{\lceil n_1\rceil} D_a^{n_1-\lceil n_1\rceil}\cdot D_a^{\lceil n_2\rceil} D^{-\lceil n_2\rceil-n_1}\phi \\
&= D_a^{\lceil n_1\rceil} D_a^{n_1-\lceil n_1\rceil}\cdot \underbrace{D_a^{\lceil n_2\rceil} D^{-\lceil n_2\rceil}}_{\text{cancel}} D_a^{-n_1}\phi \\
&= D_a^{\lceil n_1\rceil}\underbrace{D_a^{n_1-\lceil n_1\rceil}\cdot D_a^{-n_1}}_{\text{combine}}\phi \\
&= \underbrace{D_a^{\lceil n_1\rceil} D_a^{-\lceil n_1\rceil}}_{\text{cancel}}\phi = \phi
\end{aligned}$$

$$
\begin{aligned}
\text{RHS} = D_a^{n_1+n_2} f &= D_a^{n_1+n_2} D_a^{-(n_1+n_2)} \phi \\
&= D_a^{\lceil n_1+n_2 \rceil} \underbrace{D_a^{(n_1+n_2)-\lceil n_1+n_2 \rceil} D_a^{-(n_1+n_2)}}_{\text{combine}} \phi \\
&= \underbrace{D_a^{\lceil n_1+n_2 \rceil} D_a^{-\lceil n_1+n_2 \rceil}}_{\text{cancel}} \phi = \phi \qquad \#
\end{aligned}
$$

More on this relation can be found in [38], [39], [40] and [41].

2.4 Relation between Hadamard integral and Riemann-Liouville derivative

Finally in this section we will look at the Hadamard integral specifically its relation to Riemann-Liouville derivative. We present the definition of the Hadamard finite-part integral, and move on to prove a lemma concerning its relation to the Riemann-Liouville derivative.

Definition 2.19. *[9] Assume that f is sufficiently smooth. Let $0 < n < 1$. We define the Hadamard finite-part integral*

$$
\oint_a^b f(x)(x-a)^{-1-n}\, dx = f(a)\left(\frac{1}{-n}\right)(b-a)^{-n} + \int_a^b (x-a)^{-1-n}\left[\int_a^x f'(s)\, ds\right]
$$

Lemma 2.20. *[9] Let $n > 0$, $n \notin \mathbb{N}$, and $m = \lceil n \rceil$. Assume that $f \in c^m[a,b]$, and $x \in [a,b]$. them*

$$
D_a^n f(x) = \frac{1}{\Gamma(-n)} \int_a^x (x-t)^{-n-1} f(t)\, dt
$$

Proof:[9] We will only consider the case of $0 < n < 1$

$$
\begin{aligned}
LHS = D_a^n f(x) &= D^1 \cdot D_a^{n-1} f(x) \\
&= \frac{d}{dx}\frac{1}{\Gamma(1-n)} \int_a^x (x-t)^{-n} f(t)\, dt \\
&= \frac{1}{\Gamma(1-n)} \oint_a^x \frac{\partial}{\partial x}(x-t)^{-n} f(t)\, dt \\
&= \frac{1}{\Gamma(1-n)} \oint_a^x -n(x-t)^{-n-1} f(t)\, dt
\end{aligned}
$$

$$= \frac{1}{\Gamma(1-n)} \left[f(a) + \int_a^x (x-t)^{-n} f'(t)\,dt \right]$$
$$= \frac{x^{-n}}{\Gamma(1-n)} f(a) + \frac{1}{\Gamma(1-n)} \int_a^x (x-t)^{-n} f'(t)\,dt$$
$$RHS = \frac{1}{\Gamma(-n)} \oint_a^x (x-t)^{-n-t} f(t)\,dt$$

let $x - t = xw$, Assume that $a = 0$ for simplicity

$$= \frac{1}{\Gamma(-n)} \oint_0^1 (xw)^{-n-1} f(x - xw).x\,dw$$
$$= \frac{x^{-n}}{\Gamma(-n)} \oint_0^1 w^{-n-1} f(x - xw)\,dw$$

= **Defintion 2.19**

$$= \frac{x^{-n}}{\Gamma(-n)} \left[f(0) \frac{1}{-n} (1-0)^{-n} + \int_0^1 \int_w^0 w^{-1-n} \frac{df}{dt}(x - xt)\,dt\,dw \right]$$

Changing the order of integration

$$= \frac{x^{-n}}{\Gamma(-n)} \left[f(0) \frac{1}{-n} (1-0)^{-n} + \int_0^1 \int_t^1 w^{-1-n} \frac{df}{dt}(x - xt)\,dw\,dt \right]$$
$$= \frac{x^{-n}}{\Gamma(-n)} \left[f(0) \frac{1}{-n} 1^{-n} + \int_0^1 \frac{df}{dt}(x - xt) \int_t^1 w^{-1-n}\,dw\,dt \right]$$
$$= \frac{x^{-n}}{\Gamma(-n)} \left[\frac{f(0)}{-n} + \int_0^1 \frac{df}{dt}(x - xt) \left[\frac{1}{-n} w^{-n} \right]_t^1 dt \right]$$
$$= \frac{x^{-n}}{\Gamma(-n)} \left[\frac{f(0)}{-n} + \frac{1}{-n} \int_0^1 \frac{df}{dt}(x - xt)(1 - t^{-n})\,dt \right]$$

Clearly this gives

$$= \frac{x^{-n}}{\Gamma(1-n)} f(0) + \frac{1}{\Gamma(1-n)} \int_a^x (x-t)^{-n} f'(t)\,dt$$
$$= LHS$$

#

For more on the Hadamard Finite integral see [42], [43], [44], [45], [46], [47], [48] and [49].

2.5 Mittag-Leffler Functions

We will now very briefly introduce the Mittag-Leffler function, we will do this by giving two definitions and two theorems.

Definition 2.21. *[21][1] Let $n > 0$ (This can be extended to $n \in \mathbb{C}$). The function E_n is defined by*

$$E_n(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(jn+1)}$$

Whenever the series converges this is called the Mittag-Leffler function of order n.
When $n = 1$, we get the exponential function

$$e^z = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j+1)} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

Which is convergent everywhere $|z| < \infty$

Definition 2.22. *[1][21] Let $n_1, n_2 > 0$ (This can be extended to $n_1, n_2 \in \mathbb{C}$). The function E_{n_1,n_2} defined by*

$$E_{n_1,n_2}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(jn_1+n_2)}$$

Whenever the series converges this is called the two-parameter Mittag-Leffler function with parameters n_1 and n_2.

Theorem 2.23. *[1][21] Consider the two-parameter Mittag-Leffler function E_{n_1,n_2} for some $n_1, n_2 > 0$. The power series defining $E_{n_1,n_2}(z)$ is convergent for all $z \in \mathbb{C}$. In other words E_{n_1,n_2} is an entire function*

Proof: can be found in [21].

Theorem 2.24. *[21][1] Let $n > 0$ and $\lambda \in \mathbb{R}$, Show that*

$${}_0^C D_x^n y(x) = \lambda y(x)$$

Has the solution

$$y(x) = E_n(\lambda x^n)$$

If $n = 1$ *we have* $y(t) = E_1(\lambda x) = e^{\lambda x}$

Proof: can be found in [21].
More on the Mittag-Leffler function can be found in [50], [51], [52], [53], [54], [55], [56] and [57].

Chapter 3

Detailed error analysis for a fractional Adams method

In this chapter we will review a paper written by K. Diethelm, N.J. Ford and A.D. Freed, called 'detailed error analysis for a fractional Adams method' [1]. All the Theorems, Corollaries, Lemmas and Conjunctures, included in this chapter have come directly from the paper [1]. The paper begins by introducing the general method for solving the fractional differential initial value problem [1]:

$$ {}_0^C D_t^{\alpha} y(t) = f(t, y(t)), \qquad y^{(k)}(0) = y_0^{(k)}, \qquad k = 0, 1, 2, \ldots, \lceil \alpha \rceil - 1, \quad (3.1) $$

to solve (3.1) the following results are necessary:

$$ {}_0^C D_t^{\alpha} y(t) = J^{n-\alpha} D^n z(t), $$

where

$$ J^{\mu} z(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-u)^{\mu-1} z(u)\, du. $$

in order for the above result to be true the following conditions must hold:

- $y_0^{(k)}$ are the arbitrary real numbers for $\alpha > 0$,
- ${}_0^C D_t^{\alpha} y(t)$ denotes the Caputo differential operator,
- $n := \lceil \alpha \rceil$,
- D^n is the usual differential operator of order n,
- where $\mu > 0$ j^{μ} is the Riemann-Liouville integral operator of order μ

The paper now goes on to explain all of the common uses for this method for example a few of these are: the diffusion process, modeling material and heat propagation. They then explain that they choose to utilizes the Caputo version as it allows for the specification inhomogeneous initial conditions which my be desired. The following equation is now introduced [1]:

$$y(t) = \sum_{v=0}^{\lceil\alpha\rceil-1} y_0^{(v)} \frac{t^v}{v!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\mu)^{\alpha-1} f(u, y(u))\, du \tag{3.2}$$

This Volterra integral equation is equivalent to equation (3.1), if and only if the continuous function that is a solution of (3.1) is also a solution of (3.2). Now the authors discuss the idea behind the classical one step Adams-Bashford-Moulton algorithm and its main problem,

$$y_{k+1} = y_k + \frac{h}{2}[f(t_k, y_k) + f(t_{k+1}, y_{k+1})] \tag{3.3}$$

This equation is implicit. Meaning the y_{k+1} term appears on both sides. Due to this we cannot solve for y_{k+1} directly and instead this paper uses an iterative process, by inserting a approximation called the predictor y_{k+1}^P for y_{k+1}, on the right hand side. The predictor is used in the same way as the classical one step Adams-Bashford-Moulton algorithm except that this time the rectangle rule,

$$\int_a^b g(z)\, dz \approx (b-a)g(a),$$

is utilised, instead of the trapeziodal quadrature formula,

$$\int_a^b g(z)\, dz \approx \frac{b-a}{2}(g(a) + g(b)),$$

this gives the equation:

$$y_{k+1} = y_k + \frac{h}{2}[f(t_k, y_k) + f(t_{k+1}, y_{k+1}^P)] \tag{3.4}$$

This is needed to be carried from the classical Adams method to the fractional Adams method. To construct this method the product trapezoidal quadrature formula is used and finds approximation for the fractional Adams-Moulton method to be [1],

$$y_{k+1} = \sum_{j=0}^{\lceil\alpha\rceil-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + a_{k+1,k+1} f(t_{k+1}, y_{k+1}^P) \right). \tag{3.5}$$

In order to solve this again a predictor was needed and by using the product rectangle rule the Authors found the Adams-Bashford method [1]:

$$y_{k+1}^P = \sum_{j=0}^{\lceil\alpha\rceil-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j). \tag{3.6}$$

The description of the numerical algorithm is now complete, the error analysis for the scheme is discussed. We will look into this paper using the same sections used in the original paper. Beginning with the axillary results in Section 3.1 which gives the theorems which are needed in order to describe the error of the method which be looked at in Section 3.2.

3.1 Auxiliary Results

This section now looks into how the numerical methods (3.5) and (3.6) are used to solve (3.1). It is stated that the function f must be continuous and fulfill a Lipschitz condition with respect to its second argument with Lipschitz constant L on a suitable set G. Then it can be proven by Theorems 3.1 and 3.2 that a uniquely determined solution y exists on some interval $[0, T]$. The authors also state that the error analysis requires additional information about smoothness, and note that α in the theorems corresponds to $\alpha - 1$ in the previous section. The same is true here.

Theorem 3.1. *[1]*

a) assume that $f \in C^2(G)$. Define $\hat{v} := \lceil 1/\alpha \rceil - 1$. Then there exists a function $\psi \in C^1[0, T]$ and some $c_1, \ldots, c_{\hat{v}} \in \mathbb{R}$ such that the solution y of (3.1) can be expressed in the form

$$y(t) = \psi(t) + \sum_{v=1}^{\hat{v}} c_v t^{v\alpha}$$

b) Assume that $f \in C^3(G)$. Define $\hat{v} := \lceil 2/\alpha \rceil - 1$ and $\tilde{v} := \lceil 1/\alpha \rceil - 1$. Then there exists a function $\psi \in C^2[0,T]$ and some $c_1, \ldots, c_{\hat{v}} \in \mathbb{R}$ and $d_1, \ldots, d_{\hat{v}} \in \mathbb{R}$ such that the solution y of (3.1) can be expressed in the form

$$y(t) = \psi(t) + \sum_{v=1}^{\hat{v}} c_v t^{v\alpha} + \sum_{v=1}^{\hat{v}} d_v t^{1+v\alpha}$$

The proof of Theorem 3.1 is not presented in the paper [1]. The next theorem is useful in relation to the smoothness properties of a function of the Caputo derivative.

Theorem 3.2. *[1] If $y \in C^m[0,T]$ for some $m \in \mathbb{N}$ and $0 < \alpha < m$ then*

$${}_0^C D_t^\alpha y(t) = \sum_{\ell=0}^{m-\lceil \alpha \rceil - 1} \frac{y^{(\ell+\lceil \alpha \rceil)}(0)}{\Gamma(\lceil \alpha \rceil - \alpha + \ell + 1)} t^{\lceil \alpha \rceil - \alpha + \ell} + g(t)$$

With some function $g \in C^{m-\lceil \alpha \rceil}[0,T]$. Moreover, the $(m - \lceil \alpha \rceil)$th derivative of g satisfies a Lipschitz condition of order $\lceil \alpha \rceil - \alpha$.

The Authors state a small proof for Theorem 3.2 however they say it is more directly proved in a different paper, written by S.G Samko, A.A Kilbas and O.I Marichev,[2].
Theroem 3.2 yields the following collorary:

Corollary 3.3. *[1] Let $y \in C^m[0,T]$ for some $m \in \mathbb{N}$ and assume that $0 < \alpha < m$. Then ${}_0^C D_t^\alpha y(t) \in C[0,T]$.*

It is stated that a simple counter example shows that this result does not hold for the Riemann-Liouvulle derivative, hence Colloray 3.3 is a indication of the practical usefulness of Caputo derivatives. the following theorem is shown for the product rectangle rule that was used for the predictor.

Theorem 3.4. *[1]*

a) Let $z \in C^1[0,T]$. Then

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} z(t)\, dt - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \le \frac{1}{\alpha} ||z'||_\infty t_{k+1}^\alpha h.$$

b) Let $z(t) = t^p$ for some $p \in (0, 1)$. Then

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} z(t)\, dt - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \leq C_{a,p}^{Re} t_{k+1}^{\alpha+p-1} h,$$

where $C_{a,p}^{Re}$ is a constant that depends only on α and p.

The proof of this Theorem 3.4 is shown in the paper [1] but is omitted here.
Now the corresponding result, this time for the product trapezoidal formula used for the corrector.

Theorem 3.5. *a) If $z \in C^2[0, T]$ then there is a constant C_α^{Tr} depending only on α such that*

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} z(t)\, dt - \sum_{j=0}^{k+1} a_{j,k+1} z(t_j) \right| \leq C_{a,p}^{Tr} ||z''||_\infty t_{k+1}^{\alpha} h^2.$$

b) Let $z \in C^1[0, T]$ and assume that z' fulfils a Lipschitz condition of order μ for some $\mu \in (0, 1)$. Then, there exists positive constants $B_{\alpha,\mu}^{Tr}$ (depending only on α and μ) and $M(z, \mu)$ (depending only on z and μ) such that

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} z(t)\, dt - \sum_{j=0}^{k+1} a_{j,k+1} z(t_j) \right| \leq B_{\alpha,\mu}^{Tr} M(z, \mu) t_{k+1}^{\alpha} h^{1+\mu}.$$

c) Let $z(t) = t^p$ for some $p \in (0, 2)$ and $Q := \min(2, p + 1)$. Then

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} z(t)\, dt - \sum_{j=0}^{k+1} a_{j,k+1} z(t_j) \right| \leq C_{a,p}^{Tr} t_{k+1}^{\alpha+p-Q} h^Q,$$

where $C_{a,p}^{Tr}$ is a constant that depends only on α and p.

As the proof for theorem 3.5 is similar to theorem 3.4 it is omitted from the paper [1].
The authors now make a final comment about theorem 3.5 about under which circumstances the error for parts b) and c) become smaller and larger. We do not include this statement as it not necessary to our understanding of the methods described.

3.2 Error Analysis for the Adams Method

The main results of the paper are presented in this section. In particular: the cases where either $\alpha < 1$ and $\alpha > 1$, the smoothness of the given function f and unknown solution y, and that if a solution is found and one is smooth, it implies that the other is non-smooth (except for some special cases). The error results under these special cases are shown.

Lemma 3.6. *[1] Assume that the solution y of the initial value problem is such that*

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} {}_0^C D_t^\alpha y(t)\, dt - \sum_{j=0}^{k} b_{j,k+1} {}_0^C D_t^\alpha y(t) \right| \leq C_1 t_{k+1}^{\gamma_1} h^{\delta_1}$$

and

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} {}_0^C D_t^\alpha y(t)\, dt - \sum_{j=0}^{k+1} a_{j,k+1} {}_0^C D_t^\alpha y(t) \right| \leq C_2 t_{k+1}^{\gamma_2} h^{\delta_2}$$

with some $\gamma_1, \gamma_2 \geq 0$ and $\delta_1, \delta_2 > 0$. Then, for some suitably chosen $T > 0$, we have

$$\max_{1 \leq j \leq N} |y(t_j) - y_j| = O(h^q),$$

where $q = min\{\delta_1 + \alpha, \delta_2\}$ and $N = \lfloor T/h \rfloor$.

The proof of this lemma can be found in [1] but is omitted here. Now the authors assume that the solution y is suitably differentiable which leads to the following theorem.

Theorem 3.7. *[1] Let $0 < \alpha$ and assume ${}_0^C D_t^\alpha y(t) \in C^2[0, T]$ for some suitable T. Then,*

$$\max_{1 \leq j \leq N} |y(t_j) - y_j| = \begin{cases} O(h^2) & \text{if } \alpha \geq 1, \\ O(h^{1+\alpha}) & \text{if } \alpha < 1. \end{cases}$$

The proof of this theorem can be found in [1] but again is omitted here. However before the proof of Theorem 3.7 is given a few comments are given.

It is pointed out that the order of convergence depends on a non-decreasing value of α, due to this fact that they discretized the integral operator in 3.2 so it behaves smoothly as α increases. Then another method used for proving convergence is briefly discussed, stating that the Adams scheme that is presented always converges for $\alpha > 0$. Theroem 3.7 therefore deals with the optimal solution. To find good error bounds the quadrature errors for this function must be made as small as possible. Looking at an extrapolation procedure that will allow a more accurate numerical approximations for the solution to be found. This leads to the following conjuncture.

Conjecture 3.8. *[1] Let $\alpha > 0$ and assume that ${}_0^C D_t^\alpha y(t) \in C^k[0,T]$ for some $k \geq 3$ and some suitable T. Then,*

$$y(T) - y_{T/h} = \sum_{j=2}^{k_1} c_j h^{2j} + \sum_{j=1}^{k_2} d_j h^{j+\alpha} + O(h^{k_3})$$

where k_1, k_2 and k_3 are certain constants depending only on k and satisfying $k_3 > \max(2k_1, k_2 + \alpha)$.

The belief is that Conjuncture 3.8 shows asymptotic expansions of this form, which hold for the fractional Adams-Moulton method and a similar expansion can be derived for the Adams-Bashford method. This however leads down an unrelated path. Theorem 3.2 allows the statement that smoothness of y implies non-smoothness of ${}_0^C D_t^\alpha y(t)$ which leads to:

Theorem 3.9. *[1] Let $\alpha > 1$ and assume that $y \in C^{1+\lceil\alpha\rceil}[0,T]$ for some suitable T. Then,*

$$\max_{1 \leq j \leq N} |y(t_j) - y_j| = O(h^{1+\lceil\alpha\rceil - \alpha})$$

The proof of this theorem is included in [1]. The speed of convergence of the method with respect to α is discussed, then a statement that it is possible to prove that the method converges for all $\alpha > 0$. is made.

Theorem 3.10. *[1] Let $0 < \alpha < 1$ and assume that $y \in C^2[0,T]$ for soe suitable T. Then, for $1 \leq j \leq N$ we have*

$$|y(t_j) - y_j| \leq C t_j^{\alpha-1} \times \begin{cases} h^{1+\alpha} & \text{if } 0 < \alpha < \frac{1}{2}, \\ h^{2-\alpha} & \text{if } \frac{1}{2} \leq \alpha < 1. \end{cases}$$

where C is a constant independent of j and h.

from this the following can be obtained

Corollary 3.11. *[1] Under the assumptions of theorem 3.10, we have*

$$\max_{0\leq j\leq N}|y(t_j)-y_j| = \begin{cases} O(h^{2\alpha}) & \text{if } 0<\alpha<\frac{1}{2}, \\ O(h) & \text{if } \frac{1}{2}\leq\alpha<1. \end{cases}$$

Moreover, for every $\epsilon \in (0,T)$ we have

$$\max_{t_j\in[\epsilon,T]}|y(t_j)-y_j| = \begin{cases} O(h^{1+\alpha}) & \text{if } 0<\alpha<\frac{1}{2}, \\ O(h^{2-\alpha}) & \text{if } \frac{1}{2}\leq\alpha<1. \end{cases}$$

The proof of Theorem 3.10 is now given in [1] again it is omitted here. To conclude this section the authors now state a final theorem in order to formulate a hypothesis for given data instead of in the terms of the unknown solution. For $\alpha > 1$ the following theorem is given.

Theorem 3.12. *[1] Let $\alpha > 1$. Then, if $f \in C^3(G)$,*

$$\max_{0\leq j\leq N}|y(t_j)-y_j| = O(h^2).$$

The proof of this theorem is given in [1] and then the authors say that the case when $\alpha < 1$ is less clear, however due to Theorem 3.1 the authors say a solution exists which leads to the final conjuncture.

Conjecture 3.13. *[1] Let $0 < \alpha < 1$. Then, if $f \in C^2(G)$, for every $\epsilon > 0$ we have*

$$\max_{t_j\in[\epsilon,T]}|y(t_j)-y_j| = O(h^{1+\alpha})$$

This is final result provided in the paper and numerical examples are look at to see if the theoretical results match experimental ones.

Chapter 4

Finite Difference Methods with Non-Uniform Meshes for Nonlinear Fractional Differential Equations

4.1 Introduction

This chapter looks at the paper written by C. Li, Q. Yi and A. Chen entitled *Finite difference methods with nonuniform meshes for nonlinear fractional differential equations* [10] which studies numerical solutions to the following problem [10]:

$$\begin{cases} {}_0^C D_t^\alpha y(t) = f(t, y), & T \geq t > 0, \\ y^{(k)}(0) = y_0^{(k)}, & k = 0, 1, \dots, n-1. \end{cases} \tag{4.1}$$

where $f(t, y)$ is a continuous function that is non-linear with respect to the unknown function y, the initial values $y_0^{(k)}$ are known and n is a positive integer such that $n - 1 < \alpha < n$. The Caputo derivative ${}_0^C D_t^\alpha y(t)$ is defined to be [10]:

$${}_0^C D_t^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s)\, ds, \qquad n - 1 < \alpha < n \ \in Z^+.$$

For equation (4.1) to have a unique solution, it is assumed that f satisfies the Lipschitz condition with respect to the second variable, defined by [10]

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

Where $L > 0$.
It is well known that the following equation is equivalent to equation (4.1) only if the continuous function is a solution of (4.1)[10]

$$y(t) = \sum_{j=0}^{n-1} \frac{t^j}{j!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))\, ds \tag{4.2}$$

is equivalent to equation (4.1) only if its solution is also a continuous function that is a solution of (4.1).
As most papers seem to focus on numerical methods having uniform meshes, the authors instead want to look into what happens when non-uniform meshes are used. This paper focuses on the stability and convergence analysis of the three numerical methods with non-uniform meshes. Firstly we give the numerical schemes, then look into their stability analysis and error analysis in separate sections [10].

4.2 Numerical Schemes on Non-uniform Meshes

This section establishes three separate numerical methods with non-uniform meshes. The following equations have been taken from [10].

Firstly the non-uniform mesh is introduced [10].
For an integer N and the given time T, we divide the interval $[0, T]$ into $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots < t_N = T$, with non-equidistant $\tau_i = t_{i+1} - t_i$, $o \leq i \leq N-1$, and denote $\tau_{\max} = \max\limits_{0 \leq i \leq N-1} \tau_i$, $\tau_{\min} = \min\limits_{0 \leq i \leq N-1} \tau_i$.
If the given question is singular at the origin, then the choice of the non-equidistant stepsizes obeys non-decreasing rule, i.e. $\tau_i \leq \tau_{i+1}$.

Presenting three approaches requires y_j to be the approximate solution of $y(t_j)$ $(j = 0, 1, \ldots, k)$ which should be determined. Now we need to calculate y_{k+1} to do this the following integral is considered [10]

$$I_{k+1} = \int_0^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} f(s, y(s))\, ds$$

$$= \sum_{j=0}^{k} \int_{t_j}^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} f(s, y(s))\, ds$$

where $k = 0, 1, \ldots, N-1$.
The integrals can be approximated by the following approach

$$I_{k+1} \approx \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1} - s)^{\alpha-1} \tilde{f}_j(s, y(s))\, ds,$$

where $\tilde{f}_j(s, y(s))$, $j = 0, 1, \ldots, k$ is the approximation of $f(s, y(s))$ on the interval $[t_j, t_{j+1})$.

The chosen three $\tilde{f}_j(s, y(s))$ used to derive the fractional rectangle, trapezoid and predictor-correct methods that can be found in [10] respectively are:

i) by choosing $\tilde{f}_j(s, y(s))$ as

$$\tilde{f}_j(s, y(s)) = f(t_j, y_j), \quad j = 0, 1, \ldots, k,$$

the fractional rectangle method is derived as [10]

$$y_{k+1} = \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \sum_{j=0}^{k} w_{j,k+1} f(t_j, y_j), \tag{4.3}$$

where

$$\begin{aligned} w_{j,k+1} &= \frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (t_{k+1} - s)^{\alpha-1}\, ds \\ &= \frac{(t_{k+1} - t_j)^\alpha - (t_{k+1} - t_{j+1})^\alpha}{\Gamma(\alpha+1)}, \qquad j = 0, 1, \ldots, k. \end{aligned} \tag{4.4}$$

ii) If $\tilde{f}_j(s, y(s))$ is selected as

$$\tilde{f}_j(s, y(s)) = \frac{s - t_{j+1}}{t_j - t_{j+1}} f(t_j, y_j) + \frac{s - t_j}{t_{j+1} - t_j} f(t_{j+1}, y_{j+1}), \qquad j = 0, 1, \ldots, k,$$

then the fractional trapeziod method is given by

$$y_{k+1} = \sum_{j=0}^{n-1} \frac{t_{k+1}^{j}}{j!} y_0^{(j)} + \sum_{j=0}^{k+1} \tilde{w}_{j,k+1} f(t_j, y_j) \tag{4.5}$$

in which [10]

$$\tilde{w}_{j,k+1} = \frac{1}{\Gamma(\alpha+2)} \begin{cases} \frac{1}{t_1} A_0, & \text{if } j = 0, \\ \frac{1}{t_{j+1}-t_j} A_j + \frac{1}{t_{j-1}-t_j} B_j, & \text{if } j = 1, 2, \ldots, k, \\ (t_{k+1} - t_k)^{\alpha}, & \text{if } j = k+1, \end{cases} \tag{4.6}$$

and

$$\begin{cases} A_0 = (t_{n+1} - t_1)^{\alpha+1} - t_{n+1}^{\alpha+1} + (\alpha+1) t_1 t_{n+1}^{\alpha}, \\ A_j = (t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1} + (\alpha+1)(t_{j+1} - t_j)(t_{n+1} - t_j)^{\alpha}, \\ B_j = (t_{n+1} - t_j)^{\alpha+1} - (t_{n+1} - t_{j-1})^{\alpha+1} + (\alpha+1)(t_j - t_{j-1})(t_{n+1} - t_j)^{\alpha}. \end{cases}$$

the following comment is now made in [10]. 'After the observation of (4.5) it is obvious to see that this is an implicit scheme. In order to decrease computational complexity, predictor-corrector method is naturally proposed.'

iii) The predictor-corrector method can be deduced with the following steps.
Firstly, take equation (4.3) as the predictor item y_{k+1}^{P}, then we replace y_{k+1} on the right-hand side of the scheme (4.5) with y_{k+1}^{P} to get the corrector item y_{k+1}. The predictor-correct method can be written as

[10] $$\begin{cases} y_{k+1}^{P} = \sum_{j=0}^{n-1} \frac{t_{k+1}^{j}}{j!} y_0^{(j)} + \sum_{j=0}^{k} w_{j,k+1} f(t_j, y_j), \\ y_{k+1} = \sum_{j=0}^{n-1} \frac{t_{k+1}^{j}}{j!} y_0^{(j)} + \{\sum_{j=0}^{k} \tilde{w}_{j,k+1} f(t_j, y_j) + \tilde{w}_{k+1,k+1} f(t_{k+1}, y_{k+1}^{P})\}, \end{cases} \tag{4.7}$$

where $w_{j,k+1}$, $\tilde{w}_{j,k+1}$ are respectively defined by (4.4) and (4.6).

4.3 Stability Analysis

This section looks into the stability analysis of the previously given numerical methods. The authors briefly explain what stability analysis is and then

give the following [10]:
'Suppose that y_k and z_k $(k = 1, 2, \ldots, N)$ are two solutions of the rectangle scheme (4.3) with different initial values $y_0^{(i)}$ and $z_0^{(i)}$ $(i = 0, 1, \ldots, n-1)$, respectively. If there exists a positive constant C independent of non-equidistant stepsizes τ_j $(j = 0, 1, \ldots, k)$ and k, such that

$$|y_k - z_k| \leq C \sum_{i=0}^{n-1} |y_0^{(i)} - z_0^{(i)}|, \qquad k = 1, 2, \ldots, N,$$

then we say that the rectangle scheme (4.3) is stable'.
It is similar to define the numerical stability for trapeziod scheme (4.5) and predictor-corrector scheme (4.7). Denote C as a generic positive constant that does not depend on meshes or k but on T, α and the smoothness of f if no ambiguousness occurs.
This idea is used throughout this section, the following lemmas are introduced in order for the stability to be described properly. These lemmas are not only important to stability analysis but also to the error analysis.

Lemma 4.1. *[10] If $\alpha > 0$, k is a nonnegative integer, $\tau_j \leq \tau_{j+1}$ $(j = 0, 1, \ldots, k-1)$, then $w_{j,k+1}$ and $\tilde{w}_{j,k+1}$ defined by equations (4.4) and (4.6) respectively have the following estimates*

$$w_{j,k+1} \leq C_\alpha \tau_j (t_{k+1} - t_j)^{\alpha-1}, \qquad j = 0, 1, \ldots, k, \tag{4.8}$$

and

$$\tilde{w}_{j,k+1} \leq C_\alpha \begin{cases} \tau_0 t_{k+1}^{\alpha-1} & \text{if } j = 0, \\ [\tau_j (t_{k+1} - t_j)^{\alpha-1} + \tau_{j-1} (t_{k+1} - t_{j-1})^{\alpha-1}] & \text{if } j = 1, 2, \ldots, k+1, \end{cases} \tag{4.9}$$

where $C_\alpha = \dfrac{max[2, \alpha]}{\Gamma(\alpha + 1)}$.

The proof of this lemma can be found in the paper [10].

Lemma 4.2. *[10] (Gronwall inequality) Assume that $\{k_n\}$ and $\{p_n\}$ are nonnegative sequences, $g_0 \geq 0$, and the sequence $\{\phi_n\}$ satisfies*

$$\begin{cases} \phi_0 \leq g_0, \\ \phi_n \leq g_0 + \sum_{j=0}^{n-1} p_j + \sum_{j=0}^{n-1} k_j \phi_j, \qquad n \geq 1. \end{cases}$$

Then

$$\phi_n \leq (g_0 + \sum_{j=0}^{n-1} p_j) exp(\sum_{j=0}^{n-1} k_j), \qquad n \geq 1.$$

There is no proof given in [10] for Lemma 4.2.

Lemma 4.3. *[10] Assume that α, C_0, $T > 0$ and $b_{j,k} = C_0\tau_j(t_k - t_j)^{\alpha-1}$, $(j = 0, 1, \ldots, k-1)$ for $0 = t_0 < t_1 < \cdots < t_N = T$, $k = 1, 2, \ldots, N$ where N is a positive integer and $\tau_j = t_{j+1} - t_j$. Let g_0 be positive and the sequence $\{\psi_k\}$ meet*

$$\begin{cases} \psi_0 \leq g_0, \\ \psi_k \leq \sum_{j=0}^{k-1} b_{j,k}\psi_j + g_0, \end{cases} \tag{4.10}$$

then

$$\psi_k \leq Cg_0, \qquad k = 1, 2, \ldots, N. \tag{4.11}$$

Lemma 4.3 is describing the Gronwall Inequality, the proof omitted here but can be found in [10]. Now theorems for the stability of the rectangle scheme, trapeziodal scheme and the predictor-corrector scheme respectively are presented in [10] and are as follows.

Theorem 4.4. *[10] Suppose that y_j $(j = 1, 2, \ldots, k)$ are the solutions of the rectangle scheme (4.3) where the non-equidistant stepsize is non-decreasing, $f(t, y)$ satisfies the Lipschitz condition with respect to the second argument y with a Lipschitz constant L on the existed interval of its unique solution. Then the rectangle scheme(4.3) is stable.*

Theorem 4.5. *[10] Suppose that y_j $(j = 1, 2, \ldots, k)$ are the solutions of the trapezoid scheme (4.5) where the non-equidistant stepsize is non-decreasing, $f(t, y)$ satisfies the Lipschitz condition with respect to the second argument y with a lipschitz constant L on the existed interval of its unique solution. Then the trapezoid scheme(4.5) is stable.*

Theorem 4.6. *[10] Suppose that y_j $(j = 1, 2, \ldots, k)$ are the solutions of the predictor-corrector scheme (4.7) where the non-equidistant stepsize is non-decreasing, $f(t, y)$ satisfies the Lipschitz condition with respect to the second argument y with a Lipschitz constant L on the existed interval of its unique solution. Then the predictor-corrector scheme (4.7) is stable.*

The proofs for Theorems 4.4 and 4.6 is given in the paper [10] but are omitted here and the proof for Theorem 4.5 had also been omitted in the paper as it is similar to the proof of Theorem 4.4.

4.4 Error Analysis

This section explores the error analysis of the rectangle scheme, trapezoid scheme and the predictor-corrector scheme the same way that [10] does. The authors produce some lemmas needed in order to prove the theorems that are presented in this section.

Lemma 4.7. *[10] If $g(t) \in C^1[0, T]$, then*

$$\left| \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} g(s)\, ds - \sum_{j=0}^{k} w_{j,k+1} g(t_j) \right| \leq \frac{||g^{'}||_\infty}{\Gamma(\alpha - 1)} T^\alpha \tau_{max}.$$

Lemma 4.8. *[10] If $g(t) \in C^2[0, T]$, then*

$$\left| \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} g(s)\, ds - \sum_{j=0}^{k} \tilde{w}_{j,k+1} g(t_j) \right| \leq \frac{||g^{''}||_\infty}{2\Gamma(\alpha - 1)} T^\alpha \tau_{max}^2.$$

Lemma 4.9. *[10] Let $g(t) = t^\sigma$ $(0 < \sigma < 1)$ which is not smooth at the origin $t = 0$. Set non-uniform meshes with variable stepsizes $\tau_j = t_{j+1} - t_j = (j + 1)\mu$, $0 \leq j \leq N - 1$, where $\mu = \frac{2T}{N(N+1)}$. Then for $0 < \alpha < 1$, we have*

$$\left| \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} g(s)\, ds - \sum_{j=0}^{k+1} \tilde{w}_{j,k+1} g(t_j) \right| \leq C_{\sigma,\alpha} N^{-2(\sigma+\alpha)} (k + 1)^{2(\sigma+\alpha-1)}, \quad (4.12)$$

where $k = 0, 1, \ldots, N - 1$, and $C_{\sigma,\alpha}$ is a constant depending only on T, α, σ.

In the paper [10] the Lemmas 4.7, 4.8 and 4.9 have proofs. It is acknowledged that the smooth conditions in the integrals are somewhat strong in Lemmas 4.7 and 4.8. For non-uniform meshes and uniform meshes it is highly possible that the convergence rates are the same for smooth functions under these conditions,. When the function is non-smooth the non-uniform meshes are far better suited, this is why Lemma 4.9 is included. Before the theorems for the error analysis are introduced the following remark is given:

Remark 4.10. *[10] For uniform meshes with a uniform stepsize $h = \frac{T}{N}$ and $g(t) = t^\sigma$ $(0 < \sigma < 1)$, it has been proved in [1] that*

$$\left| \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} g(s)\, ds - \sum_{j=0}^{k+1} \tilde{w}_{j,k+1} g(t_j) \right| \leq C_{\sigma,\alpha} t_{k+1}^{\alpha-1} h^{\sigma+1},$$

i.e.,

$$\left| \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} g(s)\, ds - \sum_{j=0}^{k+1} \tilde{w}_{j,k+1} g(t_j) \right| \leq C_{\sigma,\alpha} N^{-(\sigma+\alpha)} (k+1)^{\alpha-1}. \tag{4.13}$$

in view of equations (4.12), (4.13), we find that the order of convergence with non-uniform meshes are higher than that with uniform meshes

From the above lemmas and the previous remark the following theorems for the error estimate of each of the three schemes can be found.

Theorem 4.11. *[10] If ${}_0^C D_t^\alpha y(t) \in C^1[0,T]$ and the non-equidistant stepsizes is non-decreasing, then the rectangle scheme (4.3) for equation (4.2) has the following estimate,*

$$|y_{k+1} - y(t_{k+1})| \leq C\tau_{\max}, \qquad k = 0, 1, \ldots, N-1. \tag{4.14}$$

Theorem 4.12. *[10] If ${}_0^C D_t^\alpha y(t) \in C^2[0,T]$ and the non-equidistant stepsizes is non-decreasing, then the trapeziod scheme (4.4) for equation (4.2) has the following estimate,*

$$|y_{k+1} - y(t_{k+1})| \leq C\tau_{\max}^2, \qquad k = 0, 1, \ldots, N-1. \tag{4.15}$$

Theorem 4.13. *[10] If ${}_0^C D_t^\alpha y(t) \in C^2[0,T]$ and the non-equidistant stepsizes is non-decreasing, then the predictor-corrector scheme (4.5) has the estimate below,*

$$|y_{k+1} - y(t_{k+1})| \leq C\tau_{\max}^q, \qquad k = 0, 1, \ldots, N-1, \tag{4.16}$$

where $q = \min\{2, 1+\alpha\}$.

The proofs for Theorems 4.11 and 4.12 are not included [10] however the proof for Theorem 4.13 is included, but is ommited here.

Chapter 5

Finite Difference Methods for Nonlinear Fractional Differential Equations with Non-Uniform Meshes

5.1 Introduction

In this chapter we will be reviewing lecture notes written by Y. Yan entitled *finite difference methods for nonlinear fractional differential equations with non-uniform meshes* [4].
The lecture notes begin with an initial non-linear fractional differential equation to which the author will be applying numerical methods in order to find a solution. This equation is:

$$ {}_0^C D_t^\alpha y(t) = f(t, y(t)), \quad t > 0, \quad y^{(k)}(0) = y_0^{(k)}, \qquad k = 0, 1, \ldots, \lceil \alpha \rceil - 1, \tag{5.1} $$

It can be shown that equation (5.1) is equivalent to:

$$ y(t) = \sum_{v=0}^{\lceil \alpha \rceil - 1} y_0^{(v)} \frac{t^v}{v!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) \, ds \tag{5.2} $$

In order for equation (5.1) to have a unique solution y on some interval $[0, T]$ then; let $y_0^{(k)}$ can be an arbitrary real number, the function f must be continuous and fulfill a Lipschitz condition with respect to its second argument with Lipschitz constant L, where the Caputo derivative ${}_0^C D_t^\alpha y(t)$

defined by

$$ {}_{0}^{C}D_t^{\alpha} y(t) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^t (t-s)^{\lceil\alpha\rceil-\alpha-1} y^{\lceil\alpha\rceil}(s)\, ds, \tag{5.3} $$

and $\lceil\alpha\rceil$ is the smallest integer $\geq \alpha$.
As it is not possible to find an analytical solution of (5.1) for general f, numerical methods need to be applied to solving equation (5.1). The stability and convergence of these numerical methods is then analyzed under certain smoothness assumptions for equation (5.1). That is what will be shown in this review of the lecture notes [4].
As most numerical methods are under the assumption that the meshes are uniform, these lecture notes look into non-uniform meshes, meaning that the solutions of 5.1 no longer need to be sufficiently smooth. The lecture notes introduce three numerical methods:

- The rectangular formula,
- The trapezoid formula,
- The predictor-corrector method.

Some well know smoothness properties of ${}_{0}^{C}D_t^{\alpha} y(t), \alpha > 0$ under some assumptions of y are recalled before the methods are discussed.

Theorem 5.1. *[4] If $y \in C^m[0,T]$ for some $m \in \mathbb{N}$ and $0 < \alpha < m$, then*

$$ {}_{0}^{C}D_t^{\alpha} y(t) = \varphi(t) + \sum_{l=0}^{m-\lceil\alpha\rceil-1} \frac{y^{(l+\lceil\alpha\rceil)}(0)}{\Gamma(\lceil\alpha\rceil - \alpha + l + 1)} t^{\lceil\alpha\rceil-\alpha+l} $$

with some function $\varphi \in C^{m-\lceil\alpha\rceil}[0,T]$. Moreover, the $(m-\lceil\alpha\rceil)$th derivative of φ satisfies a Lipschitz condition of order $\lceil\alpha\rceil - \alpha$.

A brief description of this theorem is then given, then the following Assumption 5.2 for simplicity the only case considered is $0 < \alpha < 2$.

Assumption 5.2. *[4] Let $0 < \alpha < 2$ and let $g(t) := {}_{0}^{C}D_t^{\alpha} y(t)$. There exists a constant $c > 0$ such that*

$$ |g(t)| \leq ct^{\sigma}, \qquad |g'(t)| \leq ct^{\sigma-1}, \qquad |g''(t)| \leq ct^{\sigma-2}. $$

Let N be a positive integer and let $0 = t_0 < t_1 < \cdots < t_N = T$ be the non-uniform meshes on $[0, T]$. For simplicity, we assume that $T = 1$. Let $\mu = \frac{2T}{N(N+1)}$. Assume that the non-uniform satisfy

$$\tau_j = t_{j+1} - t_j = (j+1)\mu, \quad j = 0, 1, 2, \ldots, N-1, \tag{5.4}$$

lets consider the following integral, with $n = 0, 1, 2, \ldots, N-1$

$$I_{n+1} = \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} f(s, y(s))\, ds = \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} f(s, y(s))\, ds.$$

it can be approximated by the approach

$$I_{n+1} \approx \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} \tilde{f}_j(s, y(s))\, ds,$$

where $\tilde{f}_j(s, y(s))$, $j = 0, 1, 2, \ldots, n$ is the approximation of $f(s, y(s))$ on the interval $[t_j, t_{j+1}]$.
It will lead to a different scheme by choosing different $\tilde{f}_j(s, y(s))$ to derive the fractional rectangle, trapezoid, and predict-correct methods receptively.

i) ***Fractional rectangle method***
By choosing $\tilde{f}_j(s, y(s))$ as,

$$\tilde{f}_j(s, y(s)) = f(t_j, y_j), \quad \text{on } [t_j, t_{j+1}],$$

where $j = 0, 1, 2, \ldots, n$.
The fractional rectangle method is derived where $0 < \alpha < 2$

$$y_{n+1} = y(0) + y'(0)t_{n+1} + \sum_{j=0}^{n} w_{j,n+1} f(t_j, y_j), \tag{5.5}$$

where , with $j = 0, 1, 2, \ldots, n$,

$$w_{j,n+1} = \frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1}\, ds = \frac{(t_{n+1} - t_{j+1})^{\alpha} - (t_{n+1} - t_j)^{\alpha}}{\Gamma(\alpha+1)}. \tag{5.6}$$

We remark that when $0 < \alpha < 1$, there is no term $y'(0)t_{n+1}$ at the right hand of (5.5). We will not repeat this remark below for $1 < \alpha < 2$

ii) ***Fractional trapezoid method***
If $\tilde{f}_j(s, y(s))$ *is selected as, with* $j = 0, 1, 2, \ldots, n$,

$$\tilde{f}_j(s, y(s)) = \frac{s - t_{j+1}}{t_j - t_{j+1}} f(t_j, y_j) + \frac{s - t_j}{t_{j+1} - t_j} f(t_{j+1}, y_{j+1}) \text{ on } [t_j, t_{j+1}],$$

the fractional trapezoid method is derived where $0 < \alpha < 2$

$$y_{n+1} = y(0) + y'(0)t_{n+1} + \sum_{j=0}^{n+1} \tilde{w}_{j,n+1} f(t_j, y_j), \tag{5.7}$$

where

$$\tilde{w}_{j,n+1} = \frac{1}{\Gamma(\alpha+2)} \begin{cases} \frac{1}{t_1} A_0, & \text{if } j = 0, \\ \frac{1}{t_{j+1}-t_j} A_j + \frac{1}{t_{j-1}-t_j} B_j, & \text{if } j = 1, 2, \ldots, n, \\ (t_{n+1} - t_n)^{\alpha}, & \text{if } j = n+1 \end{cases} \tag{5.8}$$

and
$A_0 = (t_{n+1} - t_1)^{\alpha+1} - t_{n+1}^{\alpha+1} + (\alpha+1)t_1 t_{n+1}^{\alpha}$,
$A_j = (t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1} + (\alpha+1)(t_{j+1} - t_j)(t_{n+1} - t_j)^{\alpha}$,
$B_j = (t_{n+1} - t_j)^{\alpha+1} - (t_{n+1} - t_{j-1})^{\alpha+1} + (\alpha+1)(t_j - t_{j-1})(t_{n+1} - t_j)^{\alpha}$.
Note that (5.7) is an implicit scheme. In order to decrease computational complexity, it is natural to introduce the following predictor correct method.

iii) ***Predictor-corrector method***

$$y_{n+1}^{P} = y(0) + y'(0)t_{n+1} + \sum_{j=0}^{n} w_{j,n+1} f(t_j, y_j), \tag{5.9}$$

$$y_{n+1} = y(0) + y'(0)t_{n+1} + \left(\sum_{j=0}^{n} \tilde{w}_{j,n+1} f(t_j, y_j) + \tilde{w}_{n+1,n+1} f(t_{n+1}, y_{n+1}^{P}) \right), \tag{5.10}$$

where $w_{j,n+1}$ *and* $\tilde{w}_{j,n+1}$ *are defined in (5.6) and (5.8), respectively.*

Now each of these methods will be looked into in separate sections; the rectangle formula, the trapezoid formula and the predictor-corrector method respectively, then some numerical results are used to show that they are consistent with the theoretical results.

5.2 Rectangle Formula

We will now look into the error estimate of the rectangle method for solving equation (5.2), as it is done in [4] the following two theorems are introduced.

Theorem 5.3. *[4] Let $\alpha > 0$. If ${}_0^C D_t^\alpha y \in C^1[0,T]$ and the non-equidistant stepsize is non-decreasing, then the rectangle scheme (5.5) for equation (5.2) has the following error estimates*

$$\underset{0\leq n\leq N}{max} |y_{n+1} - y(t_{n+1})| \leq C\tau_{\max}, \qquad k = 0, 1, 2, \ldots, N-1,$$

where $\tau_{\max}$ denotes the maximum stepsizes, i.e.,

$$\tau_{\max} = \max_{0\leq j\leq N-1}(t_{j+1} - t_j). \tag{5.11}$$

Theorem 5.4. *[4] Let $0 < \alpha < 2$ and assume that ${}_0^C D_t^\alpha$ satisfies Assumption 5.2. Let τ_j, $j = 0, 1, 2, \dot{;} N-1$ be the non-uniform meshes defined in (5.4) Assume that $y(t_j)$ and y_j are the solutions of equation (5.2) and (5.5) respectively.*

If $0 < \alpha \leq 1$, we have

$$\max_{0\leq j\leq N} |y(t_j) - y_j| \leq \begin{cases} CN^{-2\sigma}, & \text{if } \sigma < \frac{1}{2}, \\ CN^{-1}\ln(N), & \text{if } \sigma = \frac{1}{2}, \\ CN^{-1}, & \text{if } \sigma > \frac{1}{2}. \end{cases}$$

If $1 < \alpha < 2$, we have

$$\max_{0\leq j\leq N} |y(t_j) - y_j| \leq CN^{-1}.$$

No proof is given to Theorem 5.3, however the following lemmas are required in order to prove Theorem 5.4.

Lemma 5.5. *[4] Let $0 < \alpha < 2$ and assume that $g := {}_0^C D_t^\alpha y$ satisfies Assumption 5.2. Let τ_j, $j = 0, 1, 2, \ldots, N-1$ be the non-uniform meshes defined in (5.4). If $0 < \alpha < 1$, then we have*

$$\left| \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} g(s)\, ds - \sum_{j=0}^{n} w_{j,n+1} g(t_j) \right| \leq \begin{cases} CN^{-2\sigma}, & \text{if } \sigma < \frac{1}{2}, \\ CN^{-1}\ln(N), & \text{if } \sigma = \frac{1}{2}, \\ CN^{-1}, & \text{if } \sigma > \frac{1}{2}. \end{cases}$$

If $1 < \alpha < 2$, *then we have*

$$\left| \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} g(s)\, ds - \sum_{j=0}^{n} w_{j,n+1} g(t_j) \right| \leq CN^{-1}.$$

In [4] Lemma 5.5 has a proof, it is however omitted here.

Lemma 5.6. *[4] Assume that* $\alpha, C_0, T > 0$ *and* $b_{j,n} = C_0 \tau_j (t_n - t_j)^{\alpha-1}$, $(j = 0, 1, 2, \ldots, n-1)$ *for* $0 = t_0 < t_1 < \cdots < t_n < \cdots < t_N = T$, $n = 1, 2, \ldots, N$ *where* N *is a positive integer and* $\tau_j = t_{j+1} - t_j$. *Let* g_0 *be positive and the sequence* $\{\psi_k\}$ *meet*

$$\begin{cases} \psi_0 \leq g_0 \\ \psi_n \leq \sum_{j=1}^{n-1} b_{j,n} \psi_j + g_0 \end{cases}$$

then

$$\psi_n \leq C g_0, \qquad n = 1, 2, \ldots, N.$$

Lemma 5.7. *[4] Let* $\Delta t, H$ *and* a_n, b_n, c_n, r_n, *for integers* $n \geq 0$, *be nonnegative numbers such that*

$$a_m + \Delta t \sum_{n=0}^{m} \leq \Delta t \sum_{n=0}^{m} r_n a_n + \Delta t \sum_{n=0}^{m} c_n + H, \quad m \geq 0.$$

Suppose that $r_n \Delta t < 1$ *for all* n, *set* $\sigma_n = (1 - r_n \Delta t)^{-1}$. *Then we have*

$$a_m + \Delta t \sum_{n=0}^{m} b_n \leq exp\left(\Delta t \sum_{n=0}^{m} \sigma_n r_n \right) \left(\Delta t \sum_{n=0}^{m} c_n + H \right), \quad m \geq 0.$$

Lemma 5.8. *[4] Let* $\alpha > 0$. *We have*

1) $w_{j,n+1} > 0$, $j = 0, 1, 2, \ldots, n$, *where* $w_{j,n+1}$ *are the weights defined in* (5.6).

2) $\tilde{w}_{j,n+1} > 0$, $j = 0, 1, 2, \ldots, n+1$ *where* $\tilde{w}_{j,n+1}$ *are the weights defined in* (5.8).

Lemma 5.9. *[4] If $\alpha > 0$, k is a nonnegative integer, $\tau_j \leq \tau_{j+1}$, $(j = 0, 1, \ldots, n-1)$, then $w_{j,n+1}$ and $\tilde{w}_{j,n+1}$ defined by (5.6) and (5.7) respectively have the following estimates*

$$w_{j,n+1} \leq C_\alpha \tau_j (t_{n+1} - t_j)^{\alpha-1}, \qquad j = 0, 1, \ldots, n,$$

and

$$\tilde{w}_{j,n+1} \leq C_\alpha \begin{cases} \tau_0 t_{n+1}^{\alpha-1}, & j = 0 \\ \tau_j (t_{n+1} - t_j)^{\alpha-1} + \tau_{j-1}(t_{n+1} - t_{j-1})^{\alpha-1}, & j = 1, 2, \ldots, n+1, \end{cases}$$

where $C_\alpha = \dfrac{\max\{2, \alpha\}}{\Gamma(\alpha+1)}$.

Lemma 5.8 has a proof which can be found in [4]. The Lemmas 5.6, 5.7 and 5.9 do not have proofs [4]. The proof of Theorem 5.4 is now given in the lecture notes [4] This proof is omitted here.

5.3 Trapezium Formula

The lecture notes [4] now look into the error estimates for the trapezium formula to solve equation (5.2). To do this the follwing theorems are introduced. Theorem 5.10 has no proof in these lecture notes.

Theorem 5.10. *[4] If ${}_0^C D_t^\alpha y \in C^2[0, T]$ and the non-equidistant stepsize in non-decreasing, then the trapeziod scheme (5.7) for solving equation (5.2) has the following error estimates*

$$\max_{0 \leq n \leq N-1} |y_{n+1} - y(t_{n+1})| \leq C\tau_{\max}^2, \qquad k = 0, 1, 2, \ldots, N-1,$$

where $\tau_{\max}$ is defined by (5.11).

Theorem 5.11. *[4] Let $0 < \alpha < 2$ and assume that $g := {}_0^C D_t^\alpha y$ satisfies Assumption 5.2. Let τ_j, $j = 0, 1, 2, \ldots, N-1$ be the non-uniform meshes defined in (5.4). Assume that $y(t_j)$ and y_j are the solutions of equation (5.2), respectively.*

If $0 < \alpha < 1$, *then we have*

$$\max_{0\le j\le N}|y(t_j)-y_j| \le \begin{cases} CN^{-2(\alpha+\sigma)}, & \text{if } 0<\alpha+\sigma<1,\\ CN^{-2}ln(N), & \text{if } \alpha+\sigma=1,\\ CN^{-2}, & \text{if } 1<\alpha+\sigma<2. \end{cases}$$

If $1 < \alpha < 2$, *then we have*

$$\max_{0\le j\le N}|y(t_j)-y_j| \le CN^{-2}.$$

In order to prove Theorem 5.11 the following lemmas are introduced.

Lemma 5.12. *[4] Let* $0 < \alpha < 2$. *Assume that* g *satisfies Assumption 5.2.*

1) If $0 < \alpha \le 1$, *then*

$$\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}}(t_{n+1}^r - s)^{\alpha-1}g(s)\,ds - \sum_{j=0}^{n+1}\tilde{w}_{j,n+1}g(t_j)\right|$$
$$\le \begin{cases} CN^{-2(\alpha+\sigma)}, & \text{if } 0<\alpha+\sigma<1,\\ CN^{-2}ln(N), & \text{if } \alpha+\sigma=1,\\ CN^{-2}, & \text{if } 1<\alpha+\sigma<2. \end{cases}$$

2) If $1 \le \alpha < 2$, *then*

$$\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}}(t_{n+1}^r - s)^{\alpha-1}g(s)\,ds - \sum_{j=0}^{n+1}\tilde{w}_{j,n+1}g(t_j)\right| \le CN^{-2}$$

The proof Lemma 5.12 can be found in [4]. Lemma 5.12 leads on to the proof of Theorem 5.11 which can be found in [4], but will not be shown here.

5.4 Predictor-Correct Method

Finally the Author looks into error estimates of the predictor correct method for solving equation (5.2). This leads to the following theorems. Theorem 5.13 does not have an accompanying proof in [4].

Theorem 5.13. *[4] If ${}_0^C D_t^\alpha \in c^2[0,T]$ and the non-equidistant stepsize is non-decreasing, then the predictor-corrector scheme as described in (5.9) and (5.10) for solving equation (5.2) has the following error estimates*

$$\max_{0\le n\le N-1} |y_{n+1} - y(t_{n+1})| \le C_{\tau^q_{\max}}, \qquad k = 0, 1, 2, \ldots, N-1,$$

where $q = \min\{2, 1+\alpha\}$ and $\tau_{\max}$ is defined by (5.11)

Theorem 5.14. *[4] Let $0 < \alpha < 2$ and assume that $g := {}_0^C D_t^\alpha y$ satisfies Assumption 5.2. Let τ_j $j = 0, 1, 2, \ldots, N-1$ be the non-uniform meshes defined in (5.4). Assume that $y(t_j)$ and y_j are the solutions of equation (5.2), (5.9) and (5.10), respectively.*

If $0 < \alpha < 1$, then we have

$$\max_{0\le j\le N} |y(t_j) - y_j| \le \begin{cases} CN^{-2(\alpha-\sigma)}, & \text{if } \alpha + 2\sigma < 1, \\ CN^{-2(\alpha-\sigma)} ln(N), & \text{if } \alpha + 2\sigma = 1, \\ CN^{-1-\alpha}, & \text{if } \alpha + 2\sigma > 1, \end{cases}$$

If $1 < \alpha < 2$, then we have

$$\max_{0\le j\le N} |y(t_j) - y_j| \le CN^{-2}.$$

In order to prove Theorem 5.14 the following Lemma is required

Lemma 5.15. *[4] Let $0 < \alpha < 2$. Assume that $g(t)$ satisfies Assumption 5.2.*

1) If $0 < \alpha < 1$, them we have

$$\left| \tilde{w}_{n+1,n+1} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} (g(s) - g(t_j))\, ds \right|$$
$$\le \begin{cases} CN^{-2(\alpha-\sigma)}, & \text{if } \alpha + 2\sigma < 1, \\ CN^{-2(\alpha-\sigma)} ln(N), & \text{if } \alpha + 2\sigma = 1, \\ CN^{-1-\alpha}, & \text{if } \alpha + 2\sigma > 1. \end{cases}$$

2) If $1 < \alpha < 2$, then we have

$$\left| \tilde{w}_{n+1,n+1} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} (g(s) - g(t_j))\, ds \right| \le CN^{-1-\alpha}.$$

The proof of this Lemma can be found in [4] but will not be introduced here, neither will the proof of Theorem 5.14. The lecture notes begin to look at the numerical results of the the schemes in this paper in the next section.

5.5 Numerical Examples

We now look at some numerical results for the numerical methods found previously.
Where $N \geq 1$ is a positive integer, let $0 = t_0 < t_1 < \cdots < t_N = T$ is the non uniform mesh defined as [4]:

$$\tau_j = t_{j+1} - t_j = (j+1)\mu, \qquad j = 0, 1, 2, \ldots, N-1 \tag{5.4}$$

Where $\mu = \dfrac{2T}{N(N+1)}$.

Example 5.16. *In this example we Look into the rectangular formula defined in equation (5.5) and the trapezoidal formula found by equation (5.7). Consider the following*

$$^{C}_{0}D^{\alpha}_{t}y(t) = \frac{120}{\Gamma(6-\alpha)}t^{5-\alpha} - y^2 + (t^5)^2, \tag{5.12}$$

with the initial value

$$y(0) = 0, \qquad\qquad if\, 0 < \alpha < 1.$$

We can see that the exact solution of this equation is $y(t) = t^5$, and

$$^{C}_{0}D^{\alpha}_{t}y(t) = \frac{120}{\Gamma(6-\alpha)}t^{5-\alpha}$$

For this example we only want to look at values for α between $0 < \alpha < 1$. We can see that $^{C}_{0}D^{\alpha}_{t}y(t) \notin C^2[0,T]$.
In table 1 we use different values for N to obtain the maximum nodal errors, from Theorem 5.11 we define this to be [4]:
for the rectangle method

$$||e_N||_\infty := \max_{0\leq j\leq N}|y(t_j) - y_j| \leq CN^{-1}$$

for the trapezoidal method

$$||e_N||_\infty := \max_{0\le j\le N} |y(t_j) - y_j| \le CN^{-2}\ln(N)$$

In order to find the experimental order of convergence (EOC) we want to find the following $log_2\left(\frac{||e_N||_\infty}{||e_{2N}||_\infty}\right)$ *should approximately be equal to 1 to be consistent with the theoretical results for the rectangle method and for the trapezoidal method we have:*
$log_2\left(\frac{||e_N||_\infty}{||e_{2N}||_\infty}\right) \approx 2$ *for the EOC this can be found in Theorem 5.11.*

Meshes	N	$\alpha = 0.25$	EOC	$\alpha = 0.55$	EOC	$\alpha = 0.7$	EOC
Uniform	40	4.10E-02		4.38E-02		4.66E-02	
	80	1.91E-02	1.096	2.09E-02	1.07	2.26E-02	1.04
	160	9.12E-03	1.072	1.02E-02	1.04	1.11E-02	1.03
	320	4.38E-03	1.058	4.98E-03	1.03	5.49E-03	1.02
	640	2.12E-03	1.048	2.46E-03	1.02	2.73E-03	1.01
Non-Uniform	40	8.93E-02		9.02E-02		9.16E-02	
	80	4.06E-02	1.140	4.19E-02	1.10	4.36E-02	1.07
	160	1.89E-02	1.096	2.00E-02	1.07	2.12E-02	1.04
	320	9.01E-03	1.073	9.71E-03	1.04	1.04E-02	1.03
	640	4.33E-03	1.059	4.75E-03	1.03	5.14E-03	1.02

Table 5.1: Maximum nodal errors for T=1, for example 5.16 using rectangle method

Meshes	N	$\alpha = 0.25$	EOC	$\alpha = 0.55$	EOC	$\alpha = 0.7$	EOC
Uniform	40	3.31E-04		5.89E-04		6.88E-04	
	80	8.79E-05	1.91	1.50E-04	1.98	1.73E-04	1.99
	160	2.31E-05	1.93	3.78E-05	1.98	4.35E-05	1.99
	320	5.99E-06	1.94	9.53E-06	1.99	1.09E-05	2.00
	640	1.55E-06	1.96	2.39E-06	1.99	2.73E-06	2.00
Non-Uniform	40	1.12E-03		2.01E-03		2.29E-03	
	80	3.07E-04	1.87	5.19E-04	1.95	5.82E-04	1.97
	160	8.23E-05	1.90	1.32E-04	1.97	1.47E-04	1.99
	320	2.17E-05	1.92	3.35E-05	1.98	3.69E-05	1.99
	640	5.65E-06	1.94	8.44E-06	1.99	9.25E-06	2.00

Table 5.2: Maximum nodal errors for T=1, for example 5.16 using trapezium method

In both of these cases we can see that the EOC found for uniform meshes are almost the same as the EOC results for non-uniform meshes. This is because the exact solution is a smooth function.

Example 5.17. *For this example we consider the following fractional differential equation and solve using the predictor-corrector method [4]:*

$$ {}_0^C D_t^{\alpha} y(t) + y(t) = 0, \qquad y(0) = 1, \qquad 0 < \alpha < 1. \tag{5.13}$$

We know that the analytical solution is

$$y(t) = E_{\alpha,1}(-t^{\alpha}),$$

where $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha, \beta > 0. \tag{5.14}$$

we know that equation (5.14) is the Mittag Leffler function [21]. This means that for ${}_C D_{0,t}^{\alpha} y(t)$, where $y(0) = 1$ we have,

$$ {}_0^C D_t^{\alpha} y(t) = -1 - \frac{-t^{\alpha}}{\Gamma(\alpha+1)} - \frac{-t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{-t^{3\alpha}}{\Gamma(3\alpha+1)} - \dots$$

now we know that if ${}_0^C D_t^{\alpha} y(t)$ has regularity then the function behaves as $a + at^{\alpha}$ where $\alpha \in [0, T]$. The maximum nodal error found in Theorem 5.14 can be obtained for this as such [4]:

$$||e_N||_{\infty} := \max_{0 \le j \le N} |y(t_j) - y_j| \le \begin{cases} CN^{-4\alpha}, & \text{if } \alpha < \frac{1}{3}, \\ CN^{-4\alpha}\ln(N), & \text{if } \alpha = \frac{1}{3}, \\ CN^{-(1+\alpha)}, & \text{if } \alpha > \frac{1}{3} \end{cases} \tag{5.15}$$

Meshes	N	$\alpha = 0.25$	EOC	$\alpha = 0.5$	EOC	$\alpha = 0.7$	EOC
Uniform	40	1.35E-03		1.09E-03		3.39E-04	
	80	5.54E-04	1.28	9.12E-04	0.26	1.56E-04	1.12
	160	4.65E-04	0.25	5.92E-04	0.62	6.54E-05	1.21
	320	3.02E-04	0.62	3.46E-04	0.78	2.63E-05	1.31
	640	1.77E-04	0.77	1.91E-04	0.86	1.03E-05	1.35
Non- Uniform	40	2.55E-02		5.31E-04		3.34E-04	
	80	1.38E-02	0.89	1.77E-04	1.59	1.01E-04	1.73
	160	6.96E-03	0.98	5.99E-05	1.56	3.08E-05	1.71
	320	3.15E-03	1.14	2.06E-05	1.54	9.48E-06	1.70
	640	1.11E-03	1.50	7.14E-06	1.53	2.92E-06	1.69

Table 5.3: Maximum nodal errors for T=1, for example 5.17 using predictor-corrector method

From table 5.3 we can see that the experimental order of convergence, where $\alpha = 0.25$ which is $< \frac{1}{3}$thats means that the EOC should be $4\alpha = 1$, we can see that the uniform results are all slightly lower and the non-uniform results are slightly higher than the theoretical results. For $\alpha = 0.5$ and $\alpha = 0.7$ both of which are bigger than $\frac{1}{3}$ we know the EOC should be $1 + \alpha$ and in the non-uniform mesh size case the theoretical results hold true.

Example 5.18. *We now consider an example for the predictor-corrector method where* $0 < \alpha < 1$, $0 < \delta < 1$ *and* $\alpha < \delta$.

$$ {}_0^C D_t^\alpha y(t) = \frac{\Gamma(1+\delta)}{\Gamma(1+\delta-\alpha)} t^{\delta-\alpha} + t^{2\delta} - y^2, \qquad t \in (0, T], \tag{5.16} $$

Where $y(0) = 0$.

The exact solution is $y(t) = t^\delta$ *and where* ${}_0^C D_t^\alpha y(t) = \dfrac{\Gamma(1+\delta)}{\Gamma(1+\delta-\alpha)} t^{\delta-\alpha}$, *this implies that* ${}_0^C D_t^\alpha y(t)$ *has regularity and behaves as* $t^{\delta-\alpha}$. *In this case by Theorem 5.14 the maximum nodal erros can be found by [4]:*

$$ ||e_N||_\infty := \max_{0\le j\le N} |y(t_j) - y_j| \le CN^{-2}\ln(N) \begin{cases} CN^{-2\delta}, & if\, 2\delta < 1+\alpha, \\ CN^{-2\delta}\ln(N), & if\, 2\delta = 1+\alpha, \\ CN^{-(1+\alpha)}, & if\, 2\delta > 1+\alpha \end{cases} \tag{5.17} $$

In table 4 we have $\delta = 0.75$ and with different values of N we find the corresponding EOC values.

Meshes	N	$\alpha = 0.2$	EOC	$\alpha = 0.5$	EOC	$\alpha = 0.7$	EOC
Uniform	40	2.26E-02		1.44E-02		2.43e-02	
	80	6.39E-03	1.82	8.75E-03	0.72	1.45e-02	0.742
	160	1.99E-03	1.67	5.24E-03	0.74	8.63E-03	0.747
	320	7.64E-04	1.38	3.13E-03	0.75	5.14e-03	0.749
	640	4.68E-04	0.71	1.86E-03	0.75	3.06e-03	0.750
Non- Uniform	40	6.37E-02		1.96E-03		1.54E-03	
	80	1.92E-02	1.73	5.97E-04	1.72	5.43E-04	1.50
	160	5.56E-03	1.79	1.91E-04	1.65	1.92E-04	1.50
	320	1.74E-03	1.68	6.29E-05	1.60	6.79E-05	1.50
	640	5.91E-04	0.56	2.12e-05	1.57	2.40E-05	1.50

Table 5.4: Maximum nodal errors for T=1, for example 5.18 using predictor-corrector method where $\delta = 0.75$

We can see that from table 4 the that EOC of the non-uniform meshes the EOC is as we expect it to be according to the theoretical results. The same can be said for the EOC of the non-uniform mesh values in table 5 however this time $\delta = 0.99$.

Meshes	N	$\alpha = 0.25$	EOC	$\alpha = 0.5$	EOC	$\alpha = 0.9$	EOC
Uniform	40	1.29E-02		3.93E-03		1.11E-02	
	80	4.06E-03	1.67	2.00E-03	0.97	5.61E-03	0.9887
	160	1.36E-03	1.58	1.01E-03	0.98	2.82E-03	0.9897
	320	4.82E-04	1.50	5.11E-04	0.99	1.42E-03	0.9899
	640	1.77E-04	1.44	2.57E-04	0.99	7.16E-04	0.9900
Non- Uniform	40	3.63E-02		3.86E-03		2.89E-04	
	80	1.13E-02	1.68	1.18E-03	1.71	7.33E-05	1.98
	160	3.60E-03	1.66	3.77E-04	1.65	1.86E-05	1.98
	320	1.21E-03	1.57	1.24E-04	1.60	4.71E-05	1.98
	640	4.27E-04	1.50	4.18E-05	1.57	1.19E-06	1.98

Table 5.5: Maximum nodal errors for T=1, for example 5.18 using predictor-corrector method where $\delta = 0.99$

This is where the lecture notes end, there is no formal conclusion to sum up or explain future work.

Chapter 6

Higher Order Numerical Methods for Solving Fractional Differential Equations

This chapter considers different methods used to solve the equation [10]:

$$\begin{cases} {}_0^C D_t^\alpha y(t) = f(t, y), & T \geq t > 0, \\ y^{(k)}(0) = y_0^{(k)}, & k = 0, 1, \ldots, n-1. \end{cases} \tag{6.1}$$

this equation (6.1) is equivalent to [10]

$$y(t) - y(0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))\, ds \tag{6.2}$$

Now we look into more detail on how to solve equation (6.2) with different schemes for $f(s, y(s))$. We will be looking at solving this equation over an interval of $[0, T]$ where $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ and where Δt is the step size of the non-uniform mesh.

We will look into 2 previous schemes, the rectangle rule and the trapezium rule, then we will be looking into a new method, the quadratic polynomial scheme. Firstly we look into solving equation (6.2) for each method and at the end of each method we will briefly discuss the error of the method.

6.1 Rectangle scheme

Firstly we work with equation (6.2).

$$y(t) - y(0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))\, ds \tag{6.2}$$

We want to find a value of y_n where $t = t_n$ so the equation becomes:

$$y_n - y_0 = \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t_n - s)^{\alpha-1} f(s, y(s))\, ds \tag{6.3}$$

Here the integral can be split into:

$$\int_0^{t_1} (t_n - s)^{\alpha-1} f(s, y(s))\, ds + \int_{t_1}^{t_2} (t_n - s)^{\alpha-1} f(s, y(s))\, ds + \int_{t_1}^{t^3} (t_n - s)^{\alpha-1} f(s, y(s))\, ds$$

$$+ \cdots + \int_{t_{n-3}}^{t_{n-2}} (t_n - s)^{\alpha-1} f(s, y(s))\, ds + \int_{t_{n-2}}^{t_{n-1}} (t_n - s)^{\alpha-1} f(s, y(s))\, ds$$

which equates to

$$\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} f(s, y(s))\, ds$$

this gives the result for y_n

$$y_n - y_0 = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} f(s, y(s))\, ds \tag{6.4}$$

In order to find the rectangle scheme we use the rectange rule to approximate $f(s, y(s))$ on the interval $[t_j, t_{j+1}]$ by $P_0(s)$ where

$$P_0(s) = f(t_j, y(t_j)) \qquad \in [t_j, t_{j+1}]$$

Now if we refer back to equation (6.4)

$$y_n - y_0 = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} f(s, y(s))\, ds \tag{6.4}$$

substituting in the rectangle approximation for $f(s, y(s))$ the equation becomes

$$y_n - y_0 \approx \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} f(t_j, y(t_j))\, ds \tag{6.5}$$

Now we wish to solve equation (6.5)

$$\begin{aligned} y_n - y_0 &\approx \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} f(t_j, y(t_j))\, ds \\ &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} f(t_j, y(t_j)) \int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1}\, ds \\ &= \sum_{j=0}^{n} w_{j,n} f(t_j, y(t_j)) \end{aligned}$$

Where

$$w_{j,n} = \frac{1}{\Gamma(\alpha+1)}[(t_{n+1} - t_{j+1})^{\alpha} - (t_{n+1} - t_j)^{\alpha}] \qquad j = 0, 1, 2, \ldots, n$$

$w_{j,n}$ is found by solving the integral $\frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (t - s)^{\alpha-1}\, ds$ which is done as follows:

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1}\, ds &= \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\alpha} \left[(t_n - s)^{\alpha}\right]_{t_j}^{t_{j+1}} \\ &= \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\alpha}[(t_n - t_{j+1})^{\alpha} - (t_n - t_j)^{\alpha}] \\ &= \frac{1}{\Gamma(\alpha+1)}[(t_n - t_{j+1})^{\alpha} - (t_n - t_j)^{\alpha}] \end{aligned}$$

Finally we have the **Rectangle scheme** to be:

$$y_n = y_0 + \sum_{j=0}^{n} w_{j,n} f(t_j, y(t_j))$$

where

$$w_{j,n} = \frac{1}{\Gamma(\alpha+1)}[(t_n - t_{j+1})^{\alpha} - (t_n - t_j)^{\alpha}] \qquad j = 0, 1, 2, \ldots, n \tag{6.6}$$

As we are only looking at values for α to be $0 < \alpha < 1$, we only need to be given the initial value y_0 so solving equation (6.6) we have:

$$\begin{aligned}
y_0 &\to \text{ given} \\
y_1 &= y_0 + w_{0,1} f(t_0, y(t_0)) \\
y_2 &= y_0 + w_{0,2} f(t_0, y(t_0)) + w_{1,2} f(t_1, y(t_1)) \\
y_3 &= y_0 + w_{0,3} f(t_0, y(t_0)) + w_{1,3} f(t_1, y(t_1)) + w_{2,3} f(t_2, y(t_2)) \\
&\vdots \\
y_{n-1} &= y_0 + w_{0,n-1} f(t_0, y(t_0)) + w_{1,n-1} f(t_1, y(t_1)) + \dots + w_{n-2,n-1} f(t_{n-2}, y(t_{n-2})) \\
y_n &= y_0 + w_{0,n} f(t_0, y(t_0)) + w_{1,n} f(t_1, y(t_1)) + \dots + w_{n-1,n} f(t_{n-1}, y(t_{n-1}))
\end{aligned} \tag{6.7}$$

This is an explicit method. Which is fairly easy to solve, now we want to find the error of this method. This is found by taking the modulus of the exact value minus the approximate value for each n giving

$$|y_n - y(t_n)| = O(\Delta t^1)$$

6.2 Trapezoidal Method

In this section we look at how to find the trapezoidal method to solve equation (6.2) Again we begin by working with:

$$y(t) - y(0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))\, ds \tag{6.2}$$

We want to find a value of y_{n+1} where $t = t_{n+1}$ so the equation becomes:

$$y_{n+1} - y_0 = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} f(s, y(s))\, ds \tag{6.8}$$

The integral is again split into:

$$\int_0^{t_1} (t_{n+1} - s)^{\alpha-1} f(s, y(s))\, ds + \int_{t_1}^{t_2} (t_{n+1} - s)^{\alpha-1} f(s, y(s))\, ds + \dots$$
$$+ \int_{t_{n-1}}^{t_n} (t_{n+1} - s)^{\alpha-1} f(s, y(s))\, ds + \int_{t_n}^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} f(s, y(s))\, ds$$

this equates to

$$\sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} f(s, y(s))\, ds$$

which gives us the result for y_{n+1} to be

$$y_{n+1} - y_0 = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} f(s, y(s))\, ds \tag{6.9}$$

In order to find the rectangle scheme we need to use the rectangle rule to approximate $f(s, y(s))$ on the interval $[t_j, t_{j+1}]$ by $P_1(s)$ where

$$P_1(s) = \frac{s - t_{j+1}}{t_j - t_{j+1}} f(t_j, y(t_j)) + \frac{s - t_j}{t_{j+1} - t_j} f(t_{j+1}, y(t_{j+1})) \qquad s \in [t_j, t_{j+1}]$$

thus we get,

$$\begin{aligned} y_{n+1} - y_0 &\approx \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} \left[P_1(s)\right] \\ &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} \Bigg[\frac{s - t_{j+1}}{t_j - t_{j+1}} f(t_j, y(t_j)) \\ &\qquad\qquad + \frac{s - t_j}{t_{j+1} - t_j} f(t_{j+1}, y(t_{j+1}))\Bigg]\, ds \\ &= \sum_{j=0}^{n+1} \tilde{w}_{j,n+1} f(t_j, y(t_j)) \end{aligned} \tag{6.10}$$

where

$$\tilde{w}_{j,n+1} = \frac{1}{\Gamma(\alpha+2)} \begin{cases} \frac{1}{t_1} A_0, & \text{if } j = 0, \\ \frac{1}{t_{j+1} - t_j} A_j + \frac{1}{t_{j-1} - t_j} B_j, & \text{if } j = 1, 2, \ldots, k, \\ (t_{k+1} - t_k)^{\alpha}, & \text{if } j = k+1, \end{cases}$$

and

$$\begin{cases} A_0 = (t_{n+1} - t_1)^{\alpha+1} - t_{n+1}^{\alpha+1} + (\alpha+1) t_1 t_{n+1}^{\alpha}, \\ A_j = (t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1} + (\alpha+1)(t_{j+1} - t_j)(t_{n+1} - t_j)^{\alpha}, \\ B_j = (t_{n+1} - t_j)^{\alpha+1} - (t_{n+1} - t_{j-1})^{\alpha+1} + (\alpha+1)(t_j - t_{j-1})(t_{n+1} - t_j)^{\alpha}. \end{cases}$$

In order to find $\tilde{w}_{j,n+1}$ we need to solve the integral

$$\frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}\left[\frac{s-t_{j+1}}{t_j-t_{j+1}}+\frac{s-t_j}{t_{j+1}-t_j}\right]$$

$$\Rightarrow \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}\frac{s-t_{j+1}}{t_j-t_{j+1}}+\frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}\frac{s-t_j}{t_{j+1}-t_j}$$

we shall solve each of these integrals separately starting with

$$\frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}\frac{s-t_{j+1}}{t_j-t_{j+1}} \tag{6.11}$$

and then solve:

$$\frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}\frac{s-t_j}{t_{j+1}-t_j} \tag{6.12}$$

In order to solve these we need to use the following identity from [11],[12]

$$J_{n,j}^{k}=\frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_n}(t_{n+1}-s)^{\alpha-1}(s-t_j)^k\,ds=\frac{(t_{n+1}-t_j)^{\alpha+k}}{\Gamma(\alpha+1+k)} \tag{6.13}$$

Firstly we begin by solving equation (6.11) we need to also use $\frac{1}{t_{j+1}-t_j}\left[J_{n,j}^{(1)}+(t_{j+1}-t_j)J_{n,j}^{(0)}-J_{n,j+1}^{(1)}\right]$ which again can be found in [12]. Hence we have

$$\frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}\frac{s-t_{j+1}}{t_j-t_{j+1}}=\frac{1}{t_{j+1}-t_j}\left[J_{n,j}^{(1)}+(t_{j+1}-t_j)J_{n,j}^{(0)}-J_{n,j+1}^{(1)}\right]$$

$$\begin{aligned}J_{n,j}^{(1)}&=\frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_n}(t_{n+1}-s)^{\alpha-1}(s-t_j)\,ds\\&=\frac{(t_{n+1}-t_j)^{\alpha+1}}{\Gamma(\alpha+2)}\\J_{n,j}^{(0)}&=\frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_n}(t_{n+1}-s)^{\alpha-1}(s-t_j)^0\,ds\end{aligned}$$

$$
\begin{aligned}
&= \frac{(t_{n+1}-t_j)^{\alpha}}{\Gamma(\alpha+1)} \\
J^{(1)}_{n,j+1} &= \frac{1}{\Gamma(\alpha)}\int_{t_{j+1}}^{t_n} (t_{n+1}-s)^{\alpha-1}(s-t_{j+1})\ ds \\
&= \frac{(t_{n+1}-t_{j+1})^{\alpha+1}}{\Gamma(\alpha+2)}
\end{aligned}
$$

$$
\frac{1}{t_{j+1}-t_j}\left[J^{(1)}_{n,j} + (t_{j+1}-t_j)J^{(0)}_{n,j} - J^{(1)}_{n,j+1}\right]
$$

$$
\begin{aligned}
&\Rightarrow \frac{1}{t_{j+1}-t_j}\left[\frac{(t_{n+1}-t_j)^{\alpha+1}}{\Gamma(\alpha+2)} + (t_{j+1}-t_j)\frac{(t_{n+1}-t_j)^{\alpha}}{\Gamma(\alpha+1)} - \frac{(t_{n+1}-t_{j+1})^{\alpha+1}}{\Gamma(\alpha+2)}\right] \\
&= \frac{1}{\Gamma(\alpha+2)}\frac{1}{t_{j+1}-t_j}\Bigg[(t_{n+1}-t_j)^{\alpha+1} + (\alpha+1)(t_{j+1}-t_j)(t_{n+1}-t_j)^{\alpha} \\
&\qquad\qquad - (t_{n+1}-t_{j+1})^{\alpha+1}\Bigg]
\end{aligned}
$$

Now we wish to solve equation (6.12) this is done by using a similar function to the previous example this time it is: $\frac{1}{t_{j-1}-t_j}\left[J^{(1)}_{n,j} + (t_{j-1}-t_j)J^{(0)}_{n,j} - J^{(1)}_{n,j-1}\right]$ which again can be found in [12]. For equation (6.12) the solution is found as follows:

$$
\frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha-1}\frac{s-t_j}{t_{j-1}-t_j} = \frac{1}{t_{j-1}-t_j}\left[J^{(1)}_{n,j} + (t_{j-1}-t_j)J^{(0)}_{n,j} - J^{(1)}_{n,j-1}\right]
$$

$$
\begin{aligned}
J^{(1)}_{n,j} &= \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_n} (t_{n+1}-s)^{\alpha-1}(s-t_j)\ ds \\
&= \frac{(t_{n+1}-t_j)^{\alpha+1}}{\Gamma(\alpha+2)} \\
J^{(0)}_{n,j} &= \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_n} (t_{n+1}-s)^{\alpha-1}(s-t_j)^0\ ds \\
&= \frac{(t_{n+1}-t_j)^{\alpha}}{\Gamma(\alpha+1)}
\end{aligned}
$$

$$
\begin{aligned}
J^{(1)}_{n,j-1} &= \frac{1}{\Gamma(\alpha)} \int_{t_{j-1}}^{t_n} (t_{n+1} - s)^{\alpha-1}(s - t_{j-1})\, ds \\
&= \frac{(t_{n+1} - t_{j-1})^{\alpha+1}}{\Gamma(\alpha+2)}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{t_{j-1} - t_j} \left[J^{(1)}_{n,j} + (t_{j-1} - t_j) J^{(0)}_{n,j} - J^{(1)}_{n,j-1} \right] \\
&\qquad \Rightarrow \frac{1}{t_{j-1} - t_j} \left[\frac{(t_{n+1} - t_j)^{\alpha+1}}{\Gamma(\alpha+2)} + (t_{j-1} - t_j) \frac{(t_{n+1} - t_j)^{\alpha}}{\Gamma(\alpha+1)} - \frac{(t_{n+1} - t_{j-1})^{\alpha+1}}{\Gamma(\alpha+2)} \right] \\
&\qquad = \frac{1}{\Gamma(\alpha+2)} \frac{1}{t_{j-1} - t_j} \Bigg[(t_{n+1} - t_j)^{\alpha+1} + (\alpha+1)(t_{j-1} - t_j)(t_{n+1} - t_j)^{\alpha} \\
&\qquad\qquad - (t_{n+1} - t_{j-1})^{\alpha+1} \Bigg]
\end{aligned}
$$

this makes $\tilde{w}_{j,n+1}$ equivalent to

$$
\begin{aligned}
\tilde{w}_{j,n+1} = &\frac{1}{\Gamma(\alpha+2)} \frac{1}{t_{j+1} - t_j} \Big[(t_{n+1} - t_j)^{\alpha+1} + (\alpha+1)(t_{j+1} - t_j)(t_{n+1} - t_j)^{\alpha} \\
&\qquad - (t_{n+1} - t_{j+1})^{\alpha+1} \Big] \\
&+ \frac{1}{\Gamma(\alpha+2)} \frac{1}{t_{j-1} - t_j} \Big[(t_{n+1} - t_j)^{\alpha+1} + (\alpha+1)(t_{j-1} - t_j)(t_{n+1} - t_j)^{\alpha} \\
&\qquad - (t_{n+1} - t_{j-1})^{\alpha+1} \Big]
\end{aligned}
$$

Obviously the $\frac{1}{\Gamma(\alpha+2)}$ is common to both terms and can be taken out then differing values of j give,

- When $j = 0$ the second equation tends towards ∞ and using the fact that $t_0 = 0$ we are left with

  $$
  \frac{1}{\Gamma(\alpha+1)} \frac{1}{t_1} \left[(t_{n+1})^{\alpha+1} + (\alpha+1)(t_1)(t_{n+1})^{\alpha} - (t_{n+1} - t_1)^{\alpha+1} \right]
  $$

  this leads directly to $A_0 \Rightarrow (t_{n+1} - t_1)^{\alpha+2} - t_{n+1}^{\alpha+1} + (\alpha+1) t_1 t_{n+1}^{\alpha}$

- When $j = 1, 2, 3, \dots, n$ it is easy to see that:
  A_j is equivalent to

$$(t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1} + (\alpha + 1)(t_{j+1} - t_j)(t_{n+1} - t_j)^{\alpha},$$

  and B_j is equivalent to

$$(t_{n+1} - t_j)^{\alpha+1} - (t_{n+1} - t_{j-1})^{\alpha+1} + (\alpha + 1)(t_j - t_{j-1})(t_{n+1} - t_j)^{\alpha}$$

- When $j = n + 1$ then we are left with the second equation giving:

$$\frac{1}{t_n - t_{n+1}} \left[(t_{n+1} - t_{n+1})^{\alpha+1} + (\alpha + 1)(t_n - t_{n+1})(t_{n+1} - t_{n+1})^{\alpha} - (t_{n+1} - t_n)^{\alpha+1} \right]$$

$$\frac{1}{t_n - t_{n+1}} \times -(t_{n+1} - t_n)^{\alpha+1} \Rightarrow \frac{-(t_{n+1} - t_n)^{\alpha+1}}{-(t_{n+1} - t_n)} \Rightarrow (t_{n+1} - t_n)^{\alpha}$$

Hence we have shown that:

$$\tilde{w}_{j,n+1} = \frac{1}{\Gamma(\alpha + 2)} \begin{cases} \frac{1}{t_1} A_0, & \text{if } j = 0, \\ \frac{1}{t_{j+1} - t_j} A_j + \frac{1}{t_{j-1} - t_j} B_j, & \text{if } j = 1, 2, \dots, k, \\ (t_{k+1} - t_k)^{\alpha}, & \text{if } j = k + 1, \end{cases} \tag{6.14}$$

and

$$\begin{cases} A_0 = (t_{n+1} - t_1)^{\alpha+1} - t_{n+1}^{\alpha+1} + (\alpha + 1) t_1 t_{n+1}^{\alpha}, \\ A_j = (t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1} + (\alpha + 1)(t_{j+1} - t_j)(t_{n+1} - t_j)^{\alpha}, \\ B_j = (t_{n+1} - t_j)^{\alpha+1} - (t_{n+1} - t_{j-1})^{\alpha+1} + (\alpha + 1)(t_j - t_{j-1})(t_{n+1} - t_j)^{\alpha}. \end{cases} \tag{6.15}$$

$$y_{n+1} = y_0 + \sum_{j=0}^{n+1} \tilde{w}_{j,n+1} f(t_j, y(t_j)) \tag{6.16}$$

where $\tilde{w}_{j,n+1}$ is as shown in equations (6.14) and (6.15) respectively.
Now we can begin to find the algorithm to solve y_{n+1} again we are only interested in values of $0 < \alpha < 1$, we only need to be given the initial value

y_0 so solving equation (6.16) we have:

$$
\begin{aligned}
y_0 &\rightarrow \text{ given}\\
y_1 &= y_0 + w_{0,1}f(t_0, y(t_0)) + w_{1,1}f(t_1, y(t_1))\\
y_2 &= y_0 + w_{0,2}f(t_0, y(t_0)) + w_{1,2}f(t_1, y(t_1)) + w_{2,2}f(t_2, y(t_2))\\
y_3 &= y_0 + w_{0,3}f(t_0, y(t_0)) + w_{1,3}f(t_1, y(t_1)) + w_{2,3}f(t_2, y(t_2)) + w_{3,3}f(t_3, y(t_3))\\
&\vdots\\
y_n &= y_0 + w_{0,n}f(t_0, y(t_0)) + w_{1,n}f(t_1, y(t_1)) + \cdots + w_{n,n}f(t_n, y(t_n))\\
y_{n+1} &= y_0 + w_{0,n+1}f(t_0, y(t_0)) + w_{1,n+1}f(t_1, y(t_1)) + \cdots + w_{n+1,n+1}f(t_{n+1}, y(t_{n+1}))
\end{aligned}
\tag{6.17}
$$

This is an implicit algorithm and as $y_1 = y_0 + w_{0,1}f(t_0, y(t_0)) + w_{1,1}f(t_1, y(t_1))$ is a non-linear equation we use newtons method to solve $g(g) = 0$. In our case newtons iteration method is, with a given y_0:

Theorem 6.1. *[13]*

$$y_{j+1} = y_j - \frac{f(t_j, y(t_j))}{f'(t_j, y(t_j))} \qquad \textit{for } j = 1, 2, 3, \ldots, n$$

We will now look briefly into the error of this algorithm, again we need to find the modulus of the exact value minus the approximated value for each y_n value giving:

$$|y_n - y(t_n)| = O(\Delta t^{\gamma})$$

where $\gamma = \min(1 + \alpha, 2)$.

6.3 Quadratic method

In this section we look at how to find a solution to (6.2) by using an approximation for $f(s, y(s))$ to be the quadratic polynomial.
Again we begin by working with:

$$y(t) - y(0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, y(s))\, ds \tag{6.2}$$

We want to find a value of y_{n+1} where $t = t_{n+1}$ so the equation becomes:

$$y_{n+1} - y_0 = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} f(s, y(s))\, ds$$

The integral is again split into:

$$\int_0^{t_1} (t_{n+1} - s)^{\alpha-1} f(s, y(s))\, ds + \int_{t_1}^{t_2} (t_{n+1} - s)^{\alpha-1} f(s, y(s))\, ds + \dots$$
$$+ \int_{t_{n-1}}^{t_n} (t_{n+1} - s)^{\alpha-1} f(s, y(s))\, ds + \int_{t_n}^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} f(s, y(s))\, ds$$

this equates to

$$\sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} f(s, y(s))\, ds$$

which gives us the result for y_{n+1} to be

$$y_{n+1} - y_0 = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} f(s, y(s))\, ds$$

In order to find the quadratic scheme we need to use the quadratic polynomial rule to approximate $f(s, y(s))$ on the interval $[t_j, t_{j+1}]$ by $P_2(s)$ where

$$P_2(s) = \frac{(s - t_j)(s - t_{j+1})}{(t_{j-1} - t_j)(t_{j-1} - t_{j+1})} f(t_{j-1}, y(t_{j-1})) + \frac{(s - t_{j-1})(s - t_{j+1})}{(t_j - t_{j-1})(t_j - t_{j+1})} f(t_j, y(t_j))$$
$$+ \frac{(s - t_{j-1})(s - t_j)}{(t_{j+1} - t_{j-1})(t_{j+1} - t_j)} f(t_{j+1}, y(t_{j+1}))$$

giving

$$y_{n+1} - y_0 = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n+1} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} \left[\frac{(s - t_j)(s - t_{j+1})}{(t_{j-1} - t_j)(t_{j-1} - t_{j+1})} f(t_{j-1}, y(t_{j-1})) + \right.$$
$$\left. \frac{(s - t_{j-1})(s - t_{j+1})}{(t_j - t_{j-1})(t_j - t_{j+1})} f(t_j, y(t_j)) + \frac{(s - t_{j-1})(s - t_j)}{(t_{j+1} - t_{j-1})(t_{j+1} - t_j)} f(t_{j+1}, y(t_{j+1})) \right] ds$$
$$= \sum_{j=0}^{n+1} \tilde{\tilde{w}}_{j,n+1} f(t_j, y(t_j))$$

Hence

$$y_{n+1} - y_0 = \sum_{j=0}^{n+1} \tilde{\tilde{w}}_{j,n+1} f(t_j, y(t_j)) \tag{6.18}$$

to find $\tilde{\tilde{w}}_{j,n+1}$ we must solve

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} \left[\frac{(s - t_j)(s - t_{j+1})}{(t_{j-1} - t_j)(t_{j-1} - t_{j+1})} f(t_{j-1}, y(t_{j-1})) \right. \\
&\qquad \left. + \frac{(s - t_{j-1})(s - t_{j+1})}{(t_j - t_{j-1})(t_j - t_{j+1})} f(t_j, y(t_j)) + \frac{(s - t_{j-1})(s - t_j)}{(t_{j+1} - t_{j-1})(t_{j+1} - t_j)} f(t_{j+1}, y(t_{j+1})) \right] ds \\
&\qquad = A_{j-1} f(t_{j-1}, y(t_{j-1})) + B_j f(t_j, y(t_j)) + C_{j+1} f(t_{j+1}, y(t_{j+1}))
\end{aligned}$$

this means we have 3 integrals to solve

1) $\frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} \frac{(s - t_j)(s - t_{j+1})}{(t_{j-1} - t_j)(t_{j-1} - t_{j+1})} ds$

2) $\frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} \frac{(s - t_{j-1})(s - t_{j+1})}{(t_j - t_{j-1})(t_j - t_{j+1})} ds$

3) $\frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} \frac{(s - t_{j-1})(s - t_j)}{(t_{j+1} - t_{j-1})(t_{j+1} - t_j)} ds$

solving each of these separately gives

1)

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} \frac{(s - t_j)(s - t_{j+1})}{(t_{j-1} - t_j)(t_{j-1} - t_{j+1})} \\
&\qquad = \frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} \frac{(s - t_{n+1} + t_{n+1} - t_j)(s - t_{n+1} + t_{n+1} - t_{j+1})}{(t_{j-1} - t_j)(t_{j-1} - t_{j+1})} ds \\
&\qquad = \frac{1}{\Gamma(\alpha)} \frac{1}{(t_{j-1} - t_j)(t_{j-1} - t_{j+1})} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} \big((s - t_{n+1}) \\
&\qquad\qquad + (t_{n+1} - t_j)\big)\big((s - t_{n+1}) + (t_{n+1} - t_{j+1})\big) ds \\
&\qquad = \frac{1}{\Gamma(\alpha)} \frac{1}{(t_{j-1} - t_j)(t_{j-1} - t_{j+1})} \int_{t_j}^{t_{j+1}} \Big[(t_{n+1} - s)^{\alpha-1} (s - t_{n+1})^2 \\
&\qquad\qquad + (2t_{n+1} - t_j - t_{j+1})(s - t_{n+1})(t_{n+1} - s)^{\alpha-1} \\
&\qquad\qquad + (t_{n+1} - t_j)(t_{n+1} - t_{j+1})(t_{n+1} - s)^{\alpha-1} \Big] ds
\end{aligned}$$

$$= \frac{1}{\Gamma(\alpha)} \frac{1}{(t_{j-1}-t_j)(t_{j-1}-t_{j+1})} \bigg[\int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha+1} ds$$

$$- \int_{t_j}^{t_{j+1}} (2t_{n+1}-t_j-t_{j+1})(t_{n+1}-s)^{\alpha} ds$$

$$+ \int_{t_j}^{t_{j+1}} (t_{n+1}-t_j)(t_{n+1}-t_{j+1})(t_{n+1}-s)^{\alpha-1} ds \bigg]$$

$$= \frac{1}{\Gamma(\alpha)} \frac{1}{(t_{j-1}-t_j)(t_{j-1}-t_{j+1})} \bigg[\frac{1}{\alpha+2} (t_{n+1}-s)^{\alpha+2} \bigg|_{t_{j+1}}^{t_j}$$

$$- \frac{(2t_{n+1}-t_j-t_{j+1})}{\alpha+1} (t_{n+1}-s)^{\alpha+1} \bigg|_{t_{j+1}}^{t_j}$$

$$+ \frac{(t_{n+1}-t_j)(t_{n+1}-t_{j+1})}{\alpha} (t_{n+1}-s)^{\alpha} \bigg|_{t_{j+1}}^{t_j} \bigg]$$

$$= \frac{1}{\Gamma(\alpha+3)} \frac{1}{(t_{j-1}-t_j)(t_{j-1}-t_{j+1})} \bigg[\alpha(\alpha+1) \big[(t_{n+1}-t_j)^{\alpha+2} - (t_{n+1}-t_{j+1})^{\alpha+2} \big]$$

$$- \alpha(\alpha+2)(2t_{n+1}-t_j-t_{j+1}) \big[(t_{n+1}-t_j)^{\alpha+1} - (t_{n+1}-t_{j+1})^{\alpha+1} \big]$$

$$+ (\alpha+1)(\alpha+2)(t_{n+1}-t_j)(t_{n+1}-t_{j+1}) \big[(t_{n+1}-t_j)^{\alpha} - (t_{n+1}-t_{j+1})^{\alpha} \big] \bigg]$$

$$= \frac{1}{\Gamma(\alpha+3)} \frac{1}{(t_{j-1}-t_j)(t_{j-1}-t_{j+1})} A_{j-1}$$

2)

$$\frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha-1} \frac{(s-t_{j-1})(s-t_{j+1})}{(t_j-t_{j-1})(t_j-t_{j+1})} ds$$

$$= \frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha-1} \frac{(s-t_{n+1}+t_{n+1}-t_{j-1})(s-t_{n+1}+t_{n+1}-t_{j+1})}{(t_j-t_{j-1})(t_j-t_{j+1})} ds$$

$$= \frac{1}{\Gamma(\alpha)} \frac{1}{(t_j-t_{j-1})(t_j-t_{j+1})} \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha-1} \big((s-t_{n+1})$$

$$+ (t_{n+1}-t_{j-1}))((s-t_{n+1}) + (t_{n+1}-t_{j+1}) \big) ds$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \frac{1}{(t_j - t_{j-1})(t_j - t_{j+1})} \int_{t_j}^{t_{j+1}} \Big[(t_{n+1} - s)^{\alpha-1} (s - t_{n+1})^2 \\
&\quad + (2t_{n+1} - t_{j-1} - t_{j+1})(s - t_{n+1})(t_{n+1} - s)^{\alpha-1} \\
&\quad\quad + (t_{n+1} - t_{j-1})(t_{n+1} - t_{j+1})(t_{n+1} - s)^{\alpha-1} \Big] ds \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{(t_j - t_{j-1})(t_j - t_{j+1})} \Bigg[\int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha+1} ds \\
&\quad - \int_{t_j}^{t_{j+1}} (2t_{n+1} - t_{j-1} - t_{j+1})(t_{n+1} - s)^{\alpha} ds \\
&\quad\quad + \int_{t_j}^{t_{j+1}} (t_{n+1} - t_{j-1})(t_{n+1} - t_{j+1})(t_{n+1} - s)^{\alpha-1} ds \Bigg] \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{(t_j - t_{j-1})(t_j - t_{j+1})} \Bigg[\frac{1}{\alpha+2} (t_{n+1} - s)^{\alpha+2} \Bigg|_{t_{j+1}}^{t_j} \\
&\quad - \frac{(2t_{n+1} - t_{j-1} - t_{j+1})}{\alpha+1} (t_{n+1} - s)^{\alpha+1} \Bigg|_{t_{j+1}}^{t_j} \\
&\quad\quad + \frac{(t_{n+1} - t_{j-1})(t_{n+1} - t_{j+1})}{\alpha} (t_{n+1} - s)^{\alpha} \Bigg|_{t_{j+1}}^{t_j} \Bigg] \\
&= \frac{1}{\Gamma(\alpha+3)} \frac{1}{(t_j - t_{j-1})(t_j - t_{j+1})} \Bigg[\alpha(\alpha+1) \big[(t_{n+1} - t_j)^{\alpha+2} - (t_{n+1} - t_{j+1})^{\alpha+2} \big] \\
&\quad - \alpha(\alpha+2)(2t_{n+1} - t_{j-1} - t_{j+1}) \big[(t_{n+1} - t_j)^{\alpha+1} - (t_{n+1} - t_{j+1})^{\alpha+1} \big] \\
&\quad\quad + (\alpha+1)(\alpha+2)(t_{n+1} - t_{j-1})(t_{n+1} - t_{j+1}) \big[(t_{n+1} - t_j)^{\alpha} - (t_{n+1} - t_{j+1})^{\alpha} \big] \Bigg] \\
&= \frac{1}{\Gamma(\alpha+3)} \frac{1}{(t_j - t_{j-1})(t_j - t_{j+1})} B_j
\end{aligned}
$$

3)

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}\frac{(s-t_{j-1})(s-t_j)}{(t_{j+1}-t_{j-1})(t_{j+1}-t_j)}ds\\
&\quad=\frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}\frac{(s-t_{n+1}+t_{n+1}-t_{j-1})(s-t_{n+1}+t_{n+1}-t_j)}{(t_{j+1}-t_{j-1})(t_{j+1}-t_j)}ds\\
&\quad=\frac{1}{\Gamma(\alpha)}\frac{1}{(t_{j+1}-t_{j-1})(t_{j+1}-t_j)}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}\big((s-t_{n+1})\\
&\qquad\quad+(t_{n+1}-t_{j-1})\big)\big((s-t_{n+1})+(t_{n+1}-t_j)\big)ds\\
&\quad=\frac{1}{\Gamma(\alpha)}\frac{1}{(t_{j+1}-t_{j-1})(t_{j+1}-t_j)}\int_{t_j}^{t_{j+1}}\Big[(t_{n+1}-s)^{\alpha-1}(s-t_{n+1})^2\\
&\qquad+(2t_{n+1}-t_{j-1}-t_j)(s-t_{n+1})(t_{n+1}-s)^{\alpha-1}\\
&\qquad\quad+(t_{n+1}-t_{j-1})(t_{n+1}-t_j)(t_{n+1}-s)^{\alpha-1}\Big]ds\\
&\quad=\frac{1}{\Gamma(\alpha)}\frac{1}{(t_{j+1}-t_{j-1})(t_{j+1}-t_j)}\Bigg[\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha+1}ds\\
&\qquad-\int_{t_j}^{t_{j+1}}(2t_{n+1}-t_{j-1}-t_j)(t_{n+1}-s)^{\alpha}ds\\
&\qquad\quad+\int_{t_j}^{t_{j+1}}(t_{n+1}-t_{j-1})(t_{n+1}-t_j)(t_{n+1}-s)^{\alpha-1}ds\Bigg]\\
&\quad=\frac{1}{\Gamma(\alpha)}\frac{1}{(t_{j+1}-t_{j-1})(t_{j+1}-t_j)}\Bigg[\frac{1}{\alpha+2}(t_{n+1}-s)^{\alpha+2}\Bigg|_{t_{j+1}}^{t_j}\\
&\qquad-\frac{(2t_{n+1}-t_{j-1}-t_j)}{\alpha+1}(t_{n+1}-s)^{\alpha+1}\Bigg|_{t_{j+1}}^{t_j}\\
&\qquad\quad+\frac{(t_{n+1}-t_{j-1})(t_{n+1}-t_j)}{\alpha}(t_{n+1}-s)^{\alpha}\Bigg|_{t_{j+1}}^{t_j}\Bigg]
\end{aligned}
$$

$$= \frac{1}{\Gamma(\alpha+3)} \frac{1}{(t_{j+1}-t_{j-1})(t_{j+1}-t_j)} \Bigg[\alpha(\alpha+1)\big[(t_{n+1}-t_j)^{\alpha+2} - (t_{n+1}-t_{j+1})^{\alpha+2}\big]$$

$$- \alpha(\alpha+2)(2t_{n+1}-t_{j-1}-t_j)\big[(t_{n+1}-t_j)^{\alpha+1} - (t_{n+1}-t_{j+1})^{\alpha+1}\big]$$

$$+ (\alpha+1)(\alpha+2)(t_{n+1}-t_{j-1})(t_{n+1}-t_j)\big[(t_{n+1}-t_j)^{\alpha} - (t_{n+1}-t_{j+1})^{\alpha}\big] \Bigg]$$

$$= \frac{1}{\Gamma(\alpha+3)} \frac{1}{(t_{j+1}-t_{j-1})(t_{j+1}-t_j)} C_{j+1}$$

For the first integral between $[t_0, t_1]$ we need to repeat the integration as we are borrowing a point from the right, instead of the left like we will with all other points. Hence the integration becomes:

1) $\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha-1} \frac{(s-t_1)(s-t_2)}{(t_0-t_1)(t_0-t_2)} ds$

2) $\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha-1} \frac{(s-t_0)(s-t_2)}{(t_1-t_0)(t_1-t_2)} ds$

3) $\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha-1} \frac{(s-t_0)(s-t_1)}{(t_2-t_0)(t_2-t_1)} ds$

solving each of these separately gives

1)

$$\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha-1} \frac{(s-t_1)(s-t_2)}{(t_0-t_1)(t_0-t_2)}$$

$$= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha-1} \frac{(s-t_{n+1}+t_{n+1}-t_1)(s-t_{n+1}+t_{n+1}-t_2)}{(t_0-t_1)(t_0-t_2)} ds$$

$$= \frac{1}{\Gamma(\alpha)} \frac{1}{(t_0-t_1)(t_0-t_2)} \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha-1} \big((s-t_{n+1}) + (t_{n+1}-t_1))((s-t_{n+1}) + (t_{n+1}-t_2)\big) ds$$

$$= \frac{1}{\Gamma(\alpha)} \frac{1}{(t_0-t_1)(t_0-t_2)} \int_{t_0}^{t_1} \Big[(t_{n+1}-s)^{\alpha-1}(s-t_{n+1})^2 + (2t_{n+1}-t_1-t_2)(s-t_{n+1})(t_{n+1}-s)^{\alpha-1} + (t_{n+1}-t_1)(t_{n+1}-t_2)(t_{n+1}-s)^{\alpha-1} \Big] ds$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \frac{1}{(t_0-t_1)(t_0-t_2)} \Bigg[\int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha+1} ds \\
&\quad - \int_{t_0}^{t_1} (2t_{n+1}-t_1-t_2)(t_{n+1}-s)^{\alpha} ds \\
&\quad + \int_{t_0}^{t_1} (t_{n+1}-t_1)(t_{n+1}-t_2)(t_{n+1}-s)^{\alpha-1} ds \Bigg] \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{(t_0-t_1)(t_0-t_2)} \Bigg[\frac{1}{\alpha+2} (t_{n+1}-s)^{\alpha+2} \Bigg|_{t_1}^{t_0} \\
&\quad - \frac{(2t_{n+1}-t_1-t_2)}{\alpha+1} (t_{n+1}-s)^{\alpha+1} \Bigg|_{t_1}^{t_0} + \frac{(t_{n+1}-t_1)(t_{n+1}-t_2)}{\alpha} (t_{n+1}-s)^{\alpha} \Bigg|_{t_1}^{t_0} \Bigg] \\
&= \frac{1}{\Gamma(\alpha+3)} \frac{1}{(t_0-t_1)(t_0-t_2)} \Bigg[\alpha(\alpha+1) \big[(t_{n+1}-t_0)^{\alpha+2} - (t_{n+1}-t_1)^{\alpha+2} \big] \\
&\quad - \alpha(\alpha+2)(2t_{n+1}-t_1-t_2) \big[(t_{n+1}-t_0)^{\alpha+1} - (t_{n+1}-t_1)^{\alpha+1} \big] \\
&\quad + (\alpha+1)(\alpha+2)(t_{n+1}-t_1)(t_{n+1}-t_2) \big[(t_{n+1}-t_0)^{\alpha} - (t_{n+1}-t_1)^{\alpha} \big] \Bigg] \\
&= A
\end{aligned}
$$

2)

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha-1} \frac{(s-t_0)(s-t_2)}{(t_1-t_0)(t_1-t_2)} ds \\
&\quad = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha-1} \frac{(s-t_{n+1}+t_{n+1}-t_0)(s-t_{n+1}+t_{n+1}-t_2)}{(t_1-t_0)(t_1-t_2)} ds \\
&\quad = \frac{1}{\Gamma(\alpha)} \frac{1}{(t_1-t_0)(t_1-t_2)} \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha-1} \big((s-t_{n+1}) \\
&\qquad + (t_{n+1}-t_0) \big) \big((s-t_{n+1}) + (t_{n+1}-t_2) \big) ds
\end{aligned}
$$

$$= \frac{1}{\Gamma(\alpha)} \frac{1}{(t_1 - t_0)(t_1 - t_2)} \int_{t_0}^{t_1} \Big[(t_{n+1} - s)^{\alpha-1} (s - t_{n+1})^2$$
$$+ (2t_{n+1} - t_0 - t_2)(s - t_{n+1})(t_{n+1} - s)^{\alpha-1}$$
$$+ (t_{n+1} - t_0)(t_{n+1} - t_2)(t_{n+1} - s)^{\alpha-1} \Big] ds$$
$$= \frac{1}{\Gamma(\alpha)} \frac{1}{(t_1 - t_0)(t_1 - t_2)} \Bigg[\int_{t_0}^{t_1} (t_{n+1} - s)^{\alpha+1} ds$$
$$- \int_{t_0}^{t_1} (2t_{n+1} - t_0 - t_2)(t_{n+1} - s)^{\alpha} ds$$
$$+ \int_{t_0}^{t_1} (t_{n+1} - t_0)(t_{n+1} - t_2)(t_{n+1} - s)^{\alpha-1} ds \Bigg]$$
$$= \frac{1}{\Gamma(\alpha)} \frac{1}{(t_1 - t_0)(t_1 - t_2)} \Bigg[\frac{1}{\alpha+2} (t_{n+1} - s)^{\alpha+2} \Bigg|_{t_1}^{t_0}$$
$$- \frac{(2t_{n+1} - t_0 - t_2)}{\alpha+1} (t_{n+1} - s)^{\alpha+1} \Bigg|_{t_1}^{t_0} + \frac{(t_{n+1} - t_0)(t_{n+1} - t_2)}{\alpha} (t_{n+1} - s)^{\alpha} \Bigg|_{t_1}^{t_0} \Bigg]$$
$$= \frac{1}{\Gamma(\alpha+3)} \frac{1}{(t_1 - t_0)(t_1 - t_2)} \Bigg[\alpha(\alpha+1) \big[(t_{n+1} - t_0)^{\alpha+2} - (t_{n+1} - t_1)^{\alpha+2} \big]$$
$$- \alpha(\alpha+2)(2t_{n+1} - t_0 - t_2) \big[(t_{n+1} - t_0)^{\alpha+1} - (t_{n+1} - t_1)^{\alpha+1} \big]$$
$$+ (\alpha+1)(\alpha+2)(t_{n+1} - t_0)(t_{n+1} - t_2) \big[(t_{n+1} - t_0)^{\alpha} - (t_{n+1} - t_1)^{\alpha} \big] \Bigg]$$
$$= B$$

3)

$$\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_{n+1} - s)^{\alpha-1} \frac{(s - t_0)(s - t_1)}{(t_2 - t_0)(t_2 - t_1)} ds$$
$$= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_{n+1} - s)^{\alpha-1} \frac{(s - t_{n+1} + t_{n+1} - t_0)(s - t_{n+1} + t_{n+1} - t_1)}{(t_2 - t_0)(t_2 - t_1)} ds$$
$$= \frac{1}{\Gamma(\alpha)} \frac{1}{(t_2 - t_0)(t_2 - t_1)} \int_{t_0}^{t_1} (t_{n+1} - s)^{\alpha-1} \big((s - t_{n+1})$$
$$+ (t_{n+1} - t_0) \big) \big((s - t_{n+1}) + (t_{n+1} - t_1) \big) ds$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \frac{1}{(t_2-t_0)(t_2-t_1)} \int_{t_0}^{t_1} \Big[(t_{n+1}-s)^{\alpha-1}(s-t_{n+1})^2 \\
&\quad + (2t_{n+1}-t_0-t_1)(s-t_{n+1})(t_{n+1}-s)^{\alpha-1} \\
&\quad + (t_{n+1}-t_0)(t_{n+1}-t_1)(t_{n+1}-s)^{\alpha-1}\Big] ds \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{(t_2-t_0)(t_2-t_1)} \Bigg[\int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha+1} ds \\
&\quad - \int_{t_0}^{t_1} (2t_{n+1}-t_0-t_1)(t_{n+1}-s)^{\alpha} ds \\
&\quad + \int_{t_0}^{t_1} (t_{n+1}-t_0)(t_{n+1}-t_1)(t_{n+1}-s)^{\alpha-1} ds\Bigg] \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{(t_2-t_0)(t_2-t_1)} \Bigg[\frac{1}{\alpha+2}(t_{n+1}-s)^{\alpha+2}\Bigg|_{t_1}^{t_0} \\
&\quad - \frac{(2t_{n+1}-t_0-t_1)}{\alpha+1}(t_{n+1}-s)^{\alpha+1}\Bigg|_{t_1}^{t_0} + \frac{(t_{n+1}-t_0)(t_{n+1}-t_1)}{\alpha}(t_{n+1}-s)^{\alpha}\Bigg|_{t_1}^{t_0}\Bigg] \\
&= \frac{1}{\Gamma(\alpha+3)} \frac{1}{(t_2-t_0)(t_2-t_1)} \Bigg[\alpha(\alpha+1)\big[(t_{n+1}-t_0)^{\alpha+2} - (t_{n+1}-t_1)^{\alpha+2}\big] \\
&\quad - \alpha(\alpha+2)(2t_{n+1}-t_0-t_1)\big[(t_{n+1}-t_0)^{\alpha+1} - (t_{n+1}-t_1)^{\alpha+1}\big] \\
&\quad + (\alpha+1)(\alpha+2)(t_{n+1}-t_0)(t_{n+1}-t_1)\big[(t_{n+1}-t_0)^{\alpha} - (t_{n+1}-t_1)^{\alpha}\big]\Bigg] \\
&= C
\end{aligned}
$$

in order to solve for y_{n+1} we solve:

$$
\begin{aligned}
y_{n+1} - y_0 &= \frac{1}{\Gamma(\alpha)} \underbrace{\int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha-1} P_2(s)\, ds}_{Af(t_0,y_0)+Bf(t_1,y_1)+Cf(t_2,y_2)} \\
&\quad + \frac{1}{(t_0-t_1)(t_0-t_2)} A_0 f(t_0,y_0) + \frac{1}{(t_1-t_0)(t_1-t_2)} B_1 f(t_1,y_1) \\
&\quad + \frac{1}{(t_2-t_0)(t_2-t_1)} C_2 f(t_2,y_2)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{(t_1 - t_2)(t_1 - t_3)} A_1 f(t_1, y_1) + \frac{1}{(t_2 - t_1)(t_2 - t_3)} B_2 f(t_2, y_2) \\
&+ \frac{1}{(t_3 - t_1)(t_3 - t_2)} C_3 f(t_3, y_3) \\
&+ \frac{1}{(t_2 - t_3)(t_2 - t_4)} A_2 f(t_2, y_2) + \frac{1}{(t_3 - t_2)(t_3 - t_4)} B_3 f(t_3, y_3) \\
&+ \frac{1}{(t_4 - t_2)(t_4 - t_3)} C_4 f(t_4, y_4) \\
&\qquad \vdots \\
&+ \frac{1}{(t_{n-2} - t_{n-1})(t_{n-2} - t_n)} A_{n-2} f(t_{n-2}, y_{n-2}) \\
&+ \frac{1}{(t_{n-1} - t_{n-2})(t_{n-1} - t_n)} B_{n-1} f(t_{n-1}, y_{n-1}) + \frac{1}{(t_n - t_{n-2})(t_n - t_{n-1})} C_n f(t_n, y_n) \\
&+ \frac{1}{(t_{n-1} - t_n)(t_{n-1} - t_{n+1})} A_{n-1} f(t_{n-1}, y_{n-1}) + \frac{1}{(t_n - t_{n-1})(t_n - t_{n+1})} B_n f(t_n, y_n) \\
&+ \frac{1}{(t_{n+1} - t_{n-1})(t_{n+1} - t_n)} C_{n+1} f(t_{n+1}, y_{n+1}) \\
&= \sum_{j=0}^{n+1} \tilde{\tilde{w}}_{j,n+1} f(t_j, y(t_j))
\end{aligned}
$$

Where A,B and C have been found. Hence for differing values of $j = 0, 1, 2, \ldots, n, n+1$ for $\tilde{\tilde{w}}_{j,n+1}$ we have

$$\tilde{\tilde{w}}_{0,n+1} = A + \frac{1}{(t_0 - t_1)(t_0 - t_1)} A_0$$

$$\tilde{\tilde{w}}_{1,n+1} = B + \frac{1}{(t_1 - t_0)(t_1 - t_2)} B_1 + \frac{1}{(t_1 - t_2)(t_1 - t_3)} A_1$$

$$\tilde{\tilde{w}}_{2,n+1} = C + \frac{1}{(t_2 - t_0)(t_2 - t_1)} C_2 + \frac{1}{(t_2 - t_1)(t_2 - t_3)} B_2 + \frac{1}{(t_2 - t_3)(t_2 - t_4)} A_2$$

$$\tilde{\tilde{w}}_{3,n+1} = \frac{1}{(t_3 - t_1)(t_3 - t_2)} C_3 + \frac{1}{(t_3 - t_2)(t_3 - t_4)} B_3 + \frac{1}{(t_3 - t_4)(t_3 - t_5)} C_3$$

$$\vdots$$

$$\tilde{\tilde{w}}_{j,n+1} = \frac{1}{(t_j - t_{j-2})(t_j - t_{j-1})} C_j + \frac{1}{(t_j - t_{j-1})(t_j - t_{j+1})} B_j + \frac{1}{(t_j - t_{j+1})(t_j - t_{j+2})} A_j$$

$$\vdots$$

$$\tilde{\tilde{w}}_{n-1,n+1} = \frac{1}{(t_{n-1} - t_{n-3})(t_{n-1} - t_{n-2})} C_{n-1} + \frac{1}{(t_{n-1} - t_{n-2})(t_{n-1} - t_n)} B_{n-1} + \frac{1}{(t_{n-1} - t_n)(t_{n-1} - t_{n+1})} A_{n-1}$$

$$\tilde{\tilde{w}}_{n,n+1} = \frac{1}{(t_n - t_{n-2})(t_n - t_{n-1})} C_n + \frac{1}{(t_n - t_{n-1})(t_n - t_{n+1})} B_n$$

$$\tilde{\tilde{w}}_{n+1,n+1} = \frac{1}{(t_{n+1} - t_{n-1})(t_{n+1} - t_n)} C_{n+1}$$

Hence

$$\tilde{\tilde{w}}_{j,n+1} = \frac{1}{\Gamma(\alpha+3)}$$

$$\begin{cases} \frac{2}{(t_0-t_1)(t_0-t_1)}C_0 & j=0, \\ \frac{2}{(t_1-t_0)(t_1-t_2)}C_1 + \frac{1}{(t_1-t_2)(t_1-t_3)}D_1 & j=1, \\ \frac{2}{(t_2-t_0)(t_2-t_1)}C_2 + \frac{1}{(t_2-t_1)(t_2-t_3)}D_2 + \frac{1}{(t_2-t_3)(t_2-t_4)}E_2 & j=2, \\ \frac{1}{(t_j-t_{j-2})(t_j-t_{j-1})}C_j + \frac{1}{(t_j-t_{j-1})(t_j-t_{j+1})}D_j + \frac{1}{(t_j-t_{j+1})(t_j-t_{j+2})}E_j & j=3,4,\ldots,n-1, \\ \frac{1}{(t_n-t_{n-2})(t_n-t_{n-1})}D_n + \frac{1}{(t_n-t_{n-1})(t_n-t_{n+1})}E_n & j=n, \\ \frac{1}{(t_{n+1}-t_{n-1})(t_{n+1}-t_n)}E_{n+1} & j=n+1 \end{cases} \quad (6.19)$$

Where

$$\begin{aligned} C_0 = \alpha(\alpha+1)\big[(t_{n+1}-t_0)^{\alpha+2} - (t_{n+1}-t_1)^{\alpha+2}\big] \\ - \alpha(\alpha+2)(2t_{n+1}-t_1-t_2)\big[(t_{n+1}-t_0)^{\alpha+1} - (t_{n+1}-t_1)^{\alpha+1}\big] \\ + (\alpha+1)(\alpha+2)(t_{n+1}-t_1)(t_{n+1}-t_2)\big[(t_{n+1}-t_0)^{\alpha} - (t_{n+1}-t_1)^{\alpha}\big] \end{aligned}$$

$$\begin{aligned} C_1 = \alpha(\alpha+1)\big[(t_{n+1}-t_0)^{\alpha+2} - (t_{n+1}-t_1)^{\alpha+2}\big] \\ - \alpha(\alpha+2)(2t_{n+1}-t_0-t_2)\big[(t_{n+1}-t_0)^{\alpha+1} - (t_{n+1}-t_1)^{\alpha+1}\big] \\ + (\alpha+1)(\alpha+2)(t_{n+1}-t_0)(t_{n+1}-t_2)\big[(t_{n+1}-t_0)^{\alpha} - (t_{n+1}-t_1)^{\alpha}\big] \end{aligned}$$

$$\begin{aligned} D_1 = \alpha(\alpha+1)\big[(t_{n+1}-t_2)^{\alpha+2} - (t_{n+1}-t_3)^{\alpha+2}\big] \\ - \alpha(\alpha+2)(2t_{n+1}-t_2-t_3)\big[(t_{n+1}-t_2)^{\alpha+1} - (t_{n+1}-t_3)^{\alpha+1}\big] \\ + (\alpha+1)(\alpha+2)(t_{n+1}-t_2)(t_{n+1}-t_3)\big[(t_{n+1}-t_2)^{\alpha} - (t_{n+1}-t_3)^{\alpha}\big] \end{aligned}$$

$$\begin{aligned} C_2 = \alpha(\alpha+1)\big[(t_{n+1}-t_0)^{\alpha+2} - (t_{n+1}-t_1)^{\alpha+2}\big] \\ - \alpha(\alpha+2)(2t_{n+1}-t_0-t_1)\big[(t_{n+1}-t_0)^{\alpha+1} - (t_{n+1}-t_1)^{\alpha+1}\big] \\ + (\alpha+1)(\alpha+2)(t_{n+1}-t_0)(t_{n+1}-t_1)\big[(t_{n+1}-t_0)^{\alpha} - (t_{n+1}-t_1)^{\alpha}\big] \end{aligned}$$

$$
\begin{aligned}
D_2 = \alpha(\alpha+1)\big[(t_{n+1}-t_2)^{\alpha+2} - (t_{n+1}-t_3)^{\alpha+2}\big] \\
- \alpha(\alpha+2)(2t_{n+1}-t_1-t_3)\big[(t_{n+1}-t_2)^{\alpha+1} - (t_{n+1}-t_3)^{\alpha+1}\big] \\
+ (\alpha+1)(\alpha+2)(t_{n+1}-t_1)(t_{n+1}-t_3)\big[(t_{n+1}-t_2)^{\alpha} - (t_{n+1}-t_3)^{\alpha}\big]
\end{aligned}
$$

$$
\begin{aligned}
E_2 = \alpha(\alpha+1)\big[(t_{n+1}-t_3)^{\alpha+2} - (t_{n+1}-t_4)^{\alpha+2}\big] \\
- \alpha(\alpha+2)(2t_{n+1}-t_3-t_4)\big[(t_{n+1}-t_3)^{\alpha+1} - (t_{n+1}-t_4)^{\alpha+1}\big] \\
+ (\alpha+1)(\alpha+2)(t_{n+1}-t_3)(t_{n+1}-t_4)\big[(t_{n+1}-t_3)^{\alpha}(t_{n+1}-t_4)^{\alpha}\big]
\end{aligned}
$$

$$
\begin{aligned}
C_j = \alpha(\alpha+1)\big[(t_{n+1}-t_j)^{\alpha+2} - (t_{n+1}-t_{j+1})^{\alpha+2}\big] \\
- \alpha(\alpha+2)(2t_{n+1}-t_{j-2}-t_{j-1})\big[(t_{n+1}-t_j)^{\alpha+1} - (t_{n+1}-t_{j+1})^{\alpha+1}\big] \\
+ (\alpha+1)(\alpha+2)(t_{n+1}-t_{j-2})(t_{n+1}-t_{j-1})\big[(t_{n+1}-t_j)^{\alpha} - (t_{n+1}-t_{j+1})^{\alpha}\big]
\end{aligned}
$$

$$
\begin{aligned}
D_j = \alpha(\alpha+1)\big[(t_{n+1}-t_j)^{\alpha+2} - (t_{n+1}-t_{j+1})^{\alpha+2}\big] \\
- \alpha(\alpha+2)(2t_{n+1}-t_{j-1}-t_{j+1})\big[(t_{n+1}-t_j)^{\alpha+1} - (t_{n+1}-t_{j+1})^{\alpha+1}\big] \\
+ (\alpha+1)(\alpha+2)(t_{n+1}-t_{j-1})(t_{n+1}-t_{j+1})\big[(t_{n+1}-t_j)^{\alpha} - (t_{n+1}-t_{j+1})^{\alpha}\big]
\end{aligned}
$$

$$
\begin{aligned}
E_j = \alpha(\alpha+1)\big[(t_{n+1}-t_j)^{\alpha+2} - (t_{n+1}-t_{j+1})^{\alpha+2}\big] \\
- \alpha(\alpha+2)(2t_{n+1}-t_{j+1}-t_{j+2})\big[(t_{n+1}-t_j)^{\alpha+1} - (t_{n+1}-t_{j+1})^{\alpha+1}\big] \\
+ (\alpha+1)(\alpha+2)(t_{n+1}-t_{j+1})(t_{n+1}-t_{j+2})\big[(t_{n+1}-t_j)^{\alpha} - (t_{n+1}-t_{j+1})^{\alpha}\big]
\end{aligned}
$$

$$
\begin{aligned}
D_n = \alpha(\alpha+1)\big[(t_{n+1}-t_{n-1})^{\alpha+2} - (t_{n+1}-t_n)^{\alpha+2}\big] \\
- \alpha(\alpha+2)(2t_{n+1}-t_{n-2}-t_{n-1})\big[(t_{n+1}-t_{n-1})^{\alpha+1} - (t_{n+1}-t_n)^{\alpha+1}\big] \\
+ (\alpha+1)(\alpha+2)(t_{n+1}-t_{n-2})(t_{n+1}-t_{n-1})\big[(t_{n+1}-t_{n-1})^{\alpha} - (t_{n+1}-t_n)^{\alpha}\big]
\end{aligned}
$$

$$
\begin{aligned}
E_n = \alpha(\alpha+1)(t_{n+1}-t_n)^{\alpha+2} - \alpha(\alpha+2)(t_{n+1}-t_{n-1})(t_{n+1}-t_n)^{\alpha+1} \\
+ (\alpha+1)(\alpha+2)(t_{n+1}-t_{n-1})(t_{n+1}-t_n)^{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
E_{n+1} = \alpha(\alpha+1)(t_{n+1}-t_n)^{\alpha+2} - \alpha(\alpha+2)(2t_{n+1}-t_{n-1}-t_n)(t_{n+1}-t_n)^{\alpha+1} \\
+ (\alpha+1)(\alpha+2)(t_{n+1}-t_{n-1})(t_{n+1}-t_n)(t_{n+1}-t_n)^{\alpha}
\end{aligned}
$$

Now we can begin to find the algorithm to solve y_{n+1} as we are only interested in values of $0 < \alpha < 1$, we only need to be given the initial value y_0 however having y_1 would make the calculation easier, so solving equation (6.19) with (6.19) we have:

$$\begin{aligned} y_0 &\rightarrow \text{ given} \\ y_1 &= y_0 + w_{0,1}f(t_0, y(t_0)) + w_{1,1}f(t_1, y(t_1)) \\ y_2 &= y_0 + w_{0,2}f(t_0, y(t_0)) + w_{1,2}f(t_1, y(t_1)) + w_{2,2}f(t_2, y(t_2)) \\ y_3 &= y_0 + w_{0,3}f(t_0, y(t_0)) + w_{1,3}f(t_1, y(t_1)) + w_{2,3}f(t_2, y(t_2)) + w_{3,3}f(t_3, y(t_3)) \\ &\vdots \\ y_n &= y_0 + w_{0,n}f(t_0, y(t_0)) + w_{1,n}f(t_1, y(t_1)) + \cdots + w_{n,n}f(t_n, y(t_n)) \\ y_{n+1} &= y_0 + w_{0,n+1}f(t_0, y(t_0)) + w_{1,n+1}f(t_1, y(t_1)) + \cdots + w_{n+1,n+1}f(t_{n+1}, y(t_{n+1})) \end{aligned} \tag{6.20}$$

This is an implicit algorithm and as $y_1 = y_0 + w_{0,1}f(t_0, y(t_0)) + w_{1,1}f(t_1, y(t_1))$ is a non-linear equation we use newtons method to solve $g(g) = 0$. In our case newtons iteration method is, with a given y_0:

Theorem 6.2. *[13]*

$$y_{j+1} = y_j - \frac{f(t_j, y(t_j))}{f'(t_j, y(t_j))} \qquad \textit{for } j = 1, 2, 3, \ldots, n$$

We will now look briefly into the error of this higher order algorithm, again we need to find the modulus of the exact value minus the approximated value for each y_n value giving in this case:

$$|y_n - y(t_n)| = O(\Delta t^{\gamma})$$

where $\gamma = \min(1 + 2\alpha, 3)$.

6.4 Numerical Results

We will now look at some numerical results for the previous numerical methods [4].

$$\tau_j = t_{j+1} - t_j = (j+1)\mu, \qquad j = 0, 1, 2, \ldots, N-1 \tag{6.21}$$

Where $\mu = \dfrac{2T}{N(N+1)}$. Where $N \geq 1$ is a positive integer, let $0 = t_0 < t_1 < \cdots < t_N = T$ is the non uniform mesh defined above.

Example 6.3. *In this example we Look into the rectangular formula defined in equation (6.6). Consider the following*

$$ {}_0^C D_t^\alpha y(t) = \frac{720}{\Gamma(7-\alpha)} t^{6-\alpha} + \frac{18}{\Gamma(5-\alpha)} t^{4-\alpha} - \frac{2}{\Gamma(2+\alpha)} t^{1-\alpha} - y^2 + (t^6 + \frac{3}{4} t^4 - 2t)^2, \tag{6.22} $$

with the initial value $y(0) = 0$, if $0 < \alpha < 1$
We can see that the exact solution of this equation is $y(t) = t^6 + \frac{3}{4} t^4 - 2t$, and

$$ {}_0^C D_t^\alpha y(t) = \frac{\Gamma(7)}{\Gamma(7-\alpha)} t^{6-\alpha} + \frac{24}{\Gamma(5-\alpha)} t^{4-\alpha} - \frac{2}{\Gamma(2+\alpha)} t^{1-\alpha} $$

For this example we only want to look at values for α between $0 < \alpha < 1$. We can see that ${}_0^C D_t^\alpha y(t) \notin C^2[0,T]$ and the non uniform mesh (6.21). In table 6.1 we use different values for N to obtain the maximum nodal errors which form Theorem 5.11 [4] and we have

$$ ||e_N||_\infty := \max_{0 \leq j \leq N} |y(t_j) - y_j| \leq C N^{-1} $$

In order to find the experimental order of convergence (EOC) we want to find the following $log_2 \left(\dfrac{||e_N||_\infty}{||e_{2N}||_\infty} \right)$ this should approximately be equal to 1 to be consistent with the theoretical results.

Meshes	N	$\alpha = 0.25$	EOC	$\alpha = 0.5$	EOC	$\alpha = 0.8$	EOC
Uniform	40	7.66E-01		1.55E-01		6.76E-02	
	80	7.53E-01	0.024	9.65E-02	0.680	3.14E-02	0.99
	160	7.25E-01	0.055	5.53E-02	0.803	1.68E-02	1.02
	320	6.87E-01	0.079	2.98e-02	0.892	8.14E-03	1.04
	640	6.40E-01	0.102	1.55E-02	0.946	3.93E-03	1.05
Non- Uniform	40	6.67E-01		2.10E-01		2.57E-01	
	80	6.98E-01	-0.066	1.01E-01	1.06	1.31E-01	0.97
	160	6.91E-01	0.015	4.66E-02	1.12	6.54E-02	0.99
	320	6.64E-01	0.058	2.16E-02	1.11	3.26E-02	1.00
	640	6.23E-01	0.090	1.02E-02	1.09	1.63E-02	1.00

Table 6.1: Maximum nodal errors for T=1, for example 6.3 using rectanguar method

In this case we can see that the EOC found for uniform meshes are almost the same as the EOC results for non-uniform meshes, in the case that the rectangular method is used.

Example 6.4. *For this example we again look at equation (6.22), however this time we will use the trapezium method found by equation (6.16) worked with (6.14),(6.15) and the maximum nodal errors can be found by solving:*

$$||e_N||_\infty := \max_{0\leq j\leq N} |y(t_j) - y_j| \leq CN^{-2}\ln(N)$$

For this example the EOC should be $log_2\left(\frac{||e_N||_\infty}{||e_{2N}||_\infty}\right) \approx 2$ *this can be found in Theorem 5.11 in chapter 4 [4]. In table 2 we show the results for different values of N and for different values of α to the previous example where* $0 < \alpha < 1$. *Again the non uniform mesh is described by equation (6.21).*

Meshes	N	$\alpha = 0.25$	EOC	$\alpha = 0.55$	EOC	$\alpha = 0.7$	EOC
Uniform	40	5.43E-01		3.97E-02		4.53E-02	
	80	4.12E-01	0.399	1.43E-02	1.47	1.82E-02	1.32
	160	2.74E-01	0.591	5.14E-03	1.48	7.28E-03	1.32
	320	1.47E-01	0.893	1.84E-03	1.48	2.93E-03	1.32
	640	6.13E-02	1.26	6.65E-04	1.47	1.18E-03	1.31
Non-Uniform	40	5.04E-01		1.86e-02		1.43E-02	
	80	3.51E-01	0.523	4.85E-03	1.94	3.65E-03	1.97
	160	1.98E-01	0.827	1.24E-03	1.97	9.21E-04	1.99
	320	8.29E-02	1.25	3.14E-04	1.98	2.31E-04	1.99
	640	2.65E-02	1.65	7.93E-05	1.99	5.80E-05	2.00

Table 6.2: Maximum nodal errors for T=1, for example 6.4 using trapezoidal method

As we can see this time that the experimental order of convergence (EOC) for the trapezoidal method for uniform meshes is not what we expected in fact it is closer to $O(N^{-1})$ whereas the non-uniform mesh sizes have an EOC closer to $O(N^{-2})$ which is consistent with the theoretical results.

Example 6.5. *In this example we will work through the problem*

$$f(t,y) = \frac{\Gamma(3)}{\Gamma(3-\alpha)}t^{2-\alpha} + (t^2)^2 - y^2, \tag{6.23}$$

where the exact solution is $y(t) = t^2$. *We will run this example with all three of the methods discussed in the chapter, the rectangle method, the trapezoidal method and the quadratic method. We present these results in three separate tables.*

Meshes	N	$\alpha = 0.4$	EOC	$\alpha = 0.6$	EOC	$\alpha = 0.8$	EOC
Uniform	40	1.34E-02		1.36E-02		1.47E-02	
	80	6.42E-03	1.06	6.66E-03	1.04	7.24E-03	1.02
	160	3.11E-03	1.04	3.28E-03	1.02	3.59E-03	1.01
	320	1.52E-03	1.03	1.62E-03	1.01	1.79E-03	1.01
	640	7.49E-04	1.02	8.07E-04	1.01	8.93E-04	1.00
Non-Uniform	40	2.41E-02		2.30E-02		2.28E-02	
	80	1.14E-02	1.08	1.10E-02	1.06	1.11E-02	1.04
	160	5.46E-03	1.06	5.38E-03	1.04	5.47E-03	1.02
	320	2.65E-03	1.04	2.65E-03	1.02	2.72E-03	1.01
	640	1.30E-03	1.03	1.31E-03	1.01	1.35E-03	1.01

Table 6.3: Maximum nodal errors for T=1, for example 6.5 using rectangle method

Meshes	N	$\alpha = 0.4$	EOC	$\alpha = 0.6$	EOC	$\alpha = 0.8$	EOC
Uniform	40	8.98E-05		9.09E-05		6.96E-05	
	80	2.32E-05	1.95	2.34E-05	1.95	1.84E-05	1.92
	160	5.96E-06	1.96	6.01E-06	1.97	4.83E-06	1.93
	320	1.52E-06	1.97	1.53E-06	1.97	1.26E-06	1.94
	640	3.86E-07	1.98	3.88E-07	1.98	3.25E-07	1.95
Non-Uniform	40	1.01E-04		8.25E-05		4.72E-05	
	80	2.59E-05	1.95	2.09E-05	1.98	1.18E-05	2.00
	160	6.64E-06	1.97	5.25E-06	1.99	2.96E-06	2.00
	320	1.69E-06	1.98	1.32E-06	1.99	7.41E-07	2.00
	640	4.28E-07	1.98	3.31E-07	2.00	1.85E-07	2.00

Table 6.4: Maximum nodal errors for T=1, for example 6.5 using trapezoidal method

Meshes	N	$\alpha = 0.4$	EOC	$\alpha = 0.6$	EOC	$\alpha = 0.8$	EOC
Uniform	40	2.61E-06		6.85EE-07		3.70E-06	
	80	6.56E-07	1.99	1.75E-07	1.97	9.26E-07	1.99
	160	1.64E-07	1.99	4.39E-08	1.99	2.32E-07	1.99
	320	4.10E-08	1.99	1.10E-08	1.99	5.79E-08	2.00
	640	1.03E-08	2.00	2.74E-09	2.00	1.45E-08	2.00
Non-Uniform	40	1.58E-06		2.09E-06		1.80E-06	
	80	2.01E-07	2.98	2.59E-07	3.00	2.21E-07	3.02
	160	2.54E-08	2.98	3.23E-08	3.00	2.73E-08	3.01
	320	3.21E-09	2.99	4.03E-09	3.00	3.39E-09	3.00
	640	4.04E-10	2.99	5.04E-10	3.00	4.24E-10	3.00

Table 6.5: Maximum nodal errors for T=1, for example 6.4 using quadrature method

In Tables 6.3, 6.4 and 6.5 we can see that the experimental results match up with the theoretical results.

Chapter 7

Conclusion and Future work

During this dissertation we have had an introduction to basic fractional differential equations and fractional calculus, and considered the numerical methods for solving such fractional differential equations by approximating values for $f(s, y(s))$. We considered some basic definitions, theorems and relations for fractional calculus, reviewed three papers with different numerical methods for solving (1.1). Discussed some of the methods found in the papers in more detail and worked out a higher order method for solving (1.1) with a non-uniform mesh size and finally we presented some numerical results. Future work may go to proving the error estimates and the stability of the higher order numerical method in chapter 6 and extend the idea to solve fractional partial differential equations.

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