A New Predictor-Corrector Method for Solving Nonlinear Fractional Differential Equations with Graded Meshes

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Abstract

In this dissertation we consider the numerical methods for solving non-linear fractional differential equations. We first review the predictor-corrector methods for solving the nonlinear fractional differential equation with uniform meshes and discussed in detail how to prove the error estimates. The convergence orders of the predictor-corrector methods for solving nonlinear fractional differential equations available in the literature are only $O(h^{1+\alpha})$, where $\alpha \in (0, 1)$ denotes the fractional order and h is the step size. It will take a long time to obtain the good approximate solutions by using such method. Therefore it is necessary to construct some higher order numerical methods to solve the nonlinear fractional differential equations. We construct a higher order numerical method with the convergence order $O(h^{1+2\alpha})$ by approximating the Riemann-Liouville fractional integral with the quadratic interpolation polynomials. The graded meshes can be used in the numerical methods to capture the singularity of the problem. Numerical examples are given to show that the numerical results are consistent with the theoretical results.

Keywords

- Adams-Bashforth-Moulton Method
- Fractional Adams-Bashforth-Moulton Method
- Higher-Order Fractional Adams-Bashforth-Moulton Method
- Uniform Mesh
- Graded Mesh
- Error Analysis

This work is original and has not been previously submitted for any academic purpose.

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Date: ....

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Contents

Chapter 1

Introduction

In this dissertation we will consider numerical methods for solving nonlinear fractional differential equation from [26],

$$\begin{cases} D_*^{\alpha} y(t) = f(t, y(t)), & 0 < t < T, \\ y^{(k)}(0) = y_0^{(k)}, & k = 0, 1, ..., \lceil \alpha \rceil - 1, \end{cases}$$

where the $y_0^{(k)}$ may be arbitrary real numbers and $\alpha > 0$.

From [25], by designing numerical methods to have optimal convergence orders when $D_*^{\alpha} y$ behaves as $t^{\lceil \alpha \rceil - \alpha}$, $\alpha > 0$. [5] used the graded meshes to recover the optimal convergence order for the approximation of the Hadamard finite-part integral. [22] and [21] used graded meshes to recover the convergence order of the finite difference method for solving a time-fractional diffusion equation when the solution is not sufficiently smooth. Other papers that solve fractional differential equations with non-uniform meshes can be found in, [13], [20], [27], [28].

By using specifically [7] and [25], we could understand the method and improve the method for a higher order.

In Chapter 2 we state a few definitions and theorems in fractional calculus. See [15], [16], [18], [3], [6], [29], [17], [12].

In Chapter 3 we consider the Adams-Bashforth-Moulton method to get an idea of how the method works and then consider the fractional Adams-Bashforth-Moulton method with uniform meshes. We can then find the error bounds of the method to find out when the method will converge.

In Chapter 4 we review [25] where they use the fractional Adams-Bashforth-Moulton method with graded meshes. We calculate how they got the weights and the method, then go on to find the error bounds for graded meshes.

In Chapter 5 and by using [26] we have introduced a new higher order fractional Adams-Bashforth-Moulton method with both uniform and graded meshes. Instead of separating the nodes into even and odd like they do in [26], we have separated the first integral from the rest of them using a quadratic interpolation to approximate the method.

In Chapter 6 we use the examples found in [25] and use MATLAB to simulate the method for a higher order fractional Adams-Bashforth-Moulton method and show our findings.

Finally, in the last chapter we give the conclusion and some possible further research.

Chapter 2

Fractional Calculus

1 Basics

Theorem 1.1 (Fundamental Theorem of Classical Calculus)**.** [4] *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function, and let* $F : [a, b] \to \mathbb{R}$ *be defined by,*

$$F(x) := \int_a^x f(t)dt.$$

Then, F *is differentiable and*

$$F' = f.$$

Definition 1.2 (Euler's Gamma Function)**.** [4] The function $\Gamma : (0, \infty) \to \mathbb{R}$, defined by

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt,$$

is called Euler's Gamma function.

Theorem 1.3. [4] *For* $n \in N$*, we have* $(n - 1)! = \Gamma(n)$*.*

Definition 1.4 (Function Spaces)**.** [4] Let $0 < \mu \leq 1$, $k \in N_0$ and $1 \leq p$.

$$\begin{aligned}
&L_p[a, b] := \{f : [a, b] \to \mathbb{R}; f \text{ is measurable on } [a, b] \text{ and } \int_a^b |f(x)|^P dx < \infty\},\\
&L_\infty[a, b] := \{f : [a, b] \to \mathbb{R}; f \text{ is measurable and essentially bounded on } [a, b]\},\\
&H_\mu[a, b] := \{f : [a, b] \to \mathbb{R}; \exists\, c > 0\, \forall\, x, y \in [a, b] : |f(x) - f(y)| \leq c|x - y|^\mu\},\\
&C^k[a, b] := \{f : [a, b] \to \mathbb{R}; f \text{ has a continuous } k\text{th derivative}\},\\
&C[a, b] := C^0[a, b],\\
&H_0[a, b] := C[a, b].
\end{aligned}$$

2 Riemann-Liouville Integrals

The Riemann-Liouville Integral is named after Bernhard Riemann and Joseph Liouville. We will start by defining the integral then show when it is well-defined and then show a few properties of the integral.

Definition 2.1. [4] Let $n \in \mathbb{R}_+$. The operator defined on $L_1[a, b]$ by,

$$J_a^n f(x) := \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t) dt \tag{2.1}$$

for $a \leq x \leq b$, is called the Riemann-Liouville fractional integral operator of order n.

Note: When $n = 0$, we set $J_a^0 = I$, the identity operator. The Riemann-Liouville integral can also be written like, $J_a^n = D_a^{-n}$.

Example 2.2. Let $f(x) = 1$, $n = \frac{1}{2}$ and $a = 0$. From the Riemann-Liouville Integral we will get,

$$J_0^{\frac{1}{2}} 1 = \frac{1}{\Gamma(\frac{1}{2})} \int_0^x (x-t)^{\frac{1}{2}-1} dt$$

We know $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, if we let $u = x - t$, then $du = -dt$, which will give,

$$\begin{aligned}
J_0^{\frac{1}{2}} 1 &= \frac{1}{\sqrt{\pi}} \int_x^0 (u)^{-\frac{1}{2}} - du \\
&= \frac{1}{\sqrt{\pi}} \left[- \frac{(u)^{\frac{1}{2}}}{\frac{1}{2}} \right]_x^0 \\
&= \frac{2}{\sqrt{\pi}} \left[- (0)^{-\frac{1}{2}} - -(x)^{\frac{1}{2}} \frac{1}{2} \right] \\
&= \frac{2}{\sqrt{\pi}} \sqrt{x}
\end{aligned}$$

Remark 2.3. [24] If $0 < n < 1$ and $|f(t)| \leq M$, then the operator $J_a^n f(x)$ is well-defined.

Proof. Since,

$$J_a^n f(x) := \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t) dt.$$

We can apply moduli to both sides of the equation to get,

$$\begin{aligned}
|J_a^n f(x)| &= \left| \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t) dt \right| \\
&\leq \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} M dt \\
&= \frac{M}{\Gamma(n)} \int_a^x (x-t)^{n-1} dt \\
&= \frac{M}{\Gamma(n)} \left[\frac{1}{n-1+1} (x-t)^{n-1+1} \right]_a^x \\
&= \frac{M}{\Gamma(n)} \left[\frac{1}{n} (x-t)^n \right]_a^x \\
&< \infty
\end{aligned}$$

If $n \geq 1$ and $f \in L^1(a,b)$ then $J_a^n f(x)$ exists for every $x \in [a,b]$, since,

$$\begin{aligned}
|J_a^n f(x)| &\leq \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} |f(t)| dt \\
&\leq \frac{1}{\Gamma(n)} (b-a)^{n-1} \int_a^x |f(t)| dt \\
&\leq \mathrm{M}
\end{aligned}$$

□

In general for $n > 0$ and $f \in L^1(a,b)$, we have the following theorem,

Theorem 2.4 (Existence Theorem)**.** [24] *Let $f \in L_1(a,b)$ and $n > 0$. Then the integral $J_a^n f(x)$ exists for almost every $x \in (a,b)$. Moreover the function $J_a^n f$ itself is also an element of $f \in L_1(a,b)$.*

Proof. By expressing the integral, $J_a^n f(x)$, as the convolution of two functions ϕ_1, $\phi_2 \in L_1(\mathbb{R})$, we can use the following well-known Lebesgue Theorem,

> **Lebesgue Theorem** [24]
> Let ϕ_1, $\phi_2 \in L_1(\mathbb{R})$, then
>
> $$\phi_1 * \phi_2(x) = \textstyle\int_{-\infty}^{\infty} \phi_1(x-t)\phi_2(t) dt,$$
>
> is well-defined for almost every $x \in \mathbb{R}$ and $\phi_1 * \phi_2 \in L_1(\mathbb{R})$.

We can write the integral as,

$$\int_a^x (x-t)^{n-1} f(t) dt = \int_{-\infty}^{\infty} \phi_1(x-t)\phi_2(t) dt$$

where,

$$\phi_1(u) = \begin{cases} u^{n-1}, & \text{for } 0 < u \leq b-a, \\ 0, & \text{else,} \end{cases} \tag{2.2}$$

and,

$$\phi_2(u) = \begin{cases} f(u), & \text{for } a \leq u \leq b, \\ 0, & \text{else.} \end{cases} \tag{2.3}$$

By construction, we see that $\phi_1 \in L_1(\mathbb{R})$, $\phi_2 \in L_1(\mathbb{R})$. Thus from the Lebesgue Theorem above, we get,

$$\int_{-\infty}^{\infty} \phi_1(x-t)\phi_2(t) dt = \phi_1 * \phi_2(x) \in L_1(\mathbb{R})$$

and therefore exists for almost every $x \in (a,b)$. □

2.1 Properties

Theorem 2.5. [24] *Let $m, n \geq 0$ and $\phi \in L_1(a,b)$. Then,*

$$J_a^m J_a^n \phi = J_a^{m+n} \phi$$

holds almost everywhere on (a,b).

Proof. So by definition, we have,

$$\begin{aligned} J_a^m J_a^n \phi &= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x (x-t)^{m-1} \left[\int_a^t (t-\tau)^{n-1} \phi(\tau) d\tau \right] dt \\ &= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x \int_a^t (x-t)^{m-1} (t-\tau)^{n-1} \phi(\tau) d\tau dt \end{aligned}$$

By the existence Theorem we know that the integral exists. We can prove this Theorem by interchanging the order of integration. We can do this by producing diagrams on the integral limits like so,

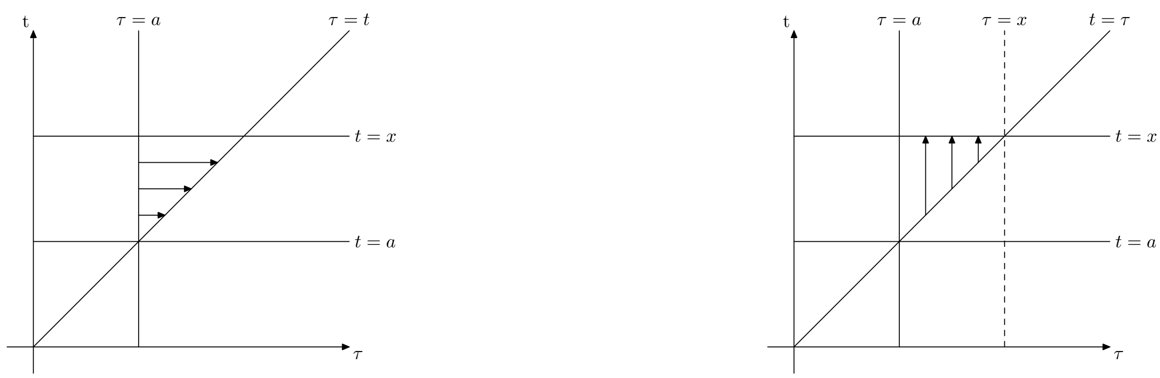


Figure 2.1: Changing the Order of Integration

From these diagrams we can see how to change the order of integration. The one on the left shows $\int_a^x \int_a^t d\tau dt$ and the right diagram is the change of order so you can use it to find $\int_a^x \int_\tau^x dt d\tau$. Therefore the above equation becomes,

$$\begin{aligned} J_a^m J_a^n \phi &= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x \int_a^t (x-t)^{m-1} (t-\tau)^{n-1} \phi(\tau) d\tau dt \\ &= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x \int_\tau^x (x-t)^{m-1} (t-\tau)^{n-1} \phi(\tau) dt d\tau \\ &= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x \phi(\tau) \left[\int_\tau^x (x-t)^{m-1} (t-\tau)^{n-1} dt \right] d\tau \end{aligned}$$

Then by substituting $t = \tau + s(x-\tau)$, we will get,

$$
\begin{aligned}
J_a^m J_a^n \phi &= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x \phi(\tau) \left[\int_0^1 \left[(x-\tau)(1-s)\right]^{m-1} \left[s(x-\tau)\right]^{n-1} (x-\tau) ds \right] d\tau \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x \phi(\tau)(x-\tau)^{m+n-1} \Bigg[\underbrace{\int_0^1 (1-s)^{m-1} s^{n-1} ds}_{=\frac{\Gamma(m)\Gamma(n)}{\Gamma(n+m)}} \Bigg] d\tau \\
&= \frac{1}{\Gamma(m+n)} \int_a^x \phi(\tau)(x-\tau)^{m+n-1} d\tau \\
&= J_a^{m+n}\phi
\end{aligned}
$$

Case 1
If $\phi \in C[a,b]$, then $J_a^n \phi \in C[a,b]$, therefore $J_a^m J_a^n \phi \in C[a,b]$ and $J_a^{m+n}\phi \in C[a,b]$ too. Thus since these two continuous functions coincide almost everywhere, they must coincide everywhere.

Case 2
If $\phi \in L_1[a,b]$ and $m+n \geq 1$ we have from the equation above,

$$
J_a^m J_a^n \phi = J_a^{m+n}\phi = J_a^{m+n-1} J_a^1 \phi
$$

Since $J_a^1\phi$ is continuous, we also know that $J_a^{m+n}\phi = J_a^{m+n-1}J_a^1\phi$ which is also continuous, we can now conclude that the two functions on either side are continuous almost everywhere, then they must be identical everywhere. □

Corollary 2.6. [4] Under the assumptions of Theorem 2.5,

$$
J_a^m J_a^n \phi = J_a^n J_a^m \phi
$$

Proof. We know from Theorem 2.5, that

$$
\begin{aligned}
J_a^m J_a^n \phi &= J_a^{m+n}\phi \\
&= \frac{1}{\Gamma(m+n)} \int_a^x \phi(\tau)(x-\tau)^{m+n-1} d\tau,
\end{aligned}
$$

and we know that addition is commutative for all real numbers. We can rearrange the above equation to find $J_a^n J_a^m \phi$. Hence,

$$
\begin{aligned}
\frac{1}{\Gamma(m+n)} \int_a^x \phi(\tau)(x-\tau)^{m+n-1} d\tau &= \frac{1}{\Gamma(n+m)} \int_a^x \phi(\tau)(x-\tau)^{n+m-1} d\tau \\
&= J_a^n J_a^m \phi
\end{aligned}
$$

□

Example 2.7. [24] If $f(x) = (x-a)^\beta$, $\beta > -1$ and $n > 0$. Then,

$$
\begin{aligned}
J_a^n f(x) &= \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t) dt, \\
&= \frac{1}{\Gamma(n)} \int_a^x (t-a)^\beta (x-t)^{n-1} dt
\end{aligned}
$$

Let $t = a + s(x-a)$, then $dt = (x-a)ds$, we will obtain,

$$J_a^n f(x) = \frac{1}{\Gamma(n)}(x-a)^{n+\beta}\Bigg[\underbrace{\int_0^1 s^\beta (1-s)^{n-1}}_{=\frac{\Gamma(n)\Gamma(\beta+1)}{\Gamma(n+\beta+1)}}\Bigg]ds$$
$$= \frac{\Gamma(\beta+1)}{\Gamma(n+\beta+1)}(x-a)^{n+\beta}$$

3 Riemann-Liouville Derivative

Definition 3.1. [4] Let $n \in \mathbb{R}_+$ and let $m = \lceil n \rceil$. The operator D_a^n, defined by

$$D_a^n f = D^m J_a^{m-n} f$$

is called the Riemann-Liouville fractional differential operator of order n.

Note: When $n = 0$, we set $D_a^0 = I$, the identity operator.

Remark 3.2. [24] Let $f \in A^1[a,b]$ and $0 < n < 1$. Then $D_a^n f$ exists almost everywhere in $[a,b]$.

Proof. We have the following theorem,

> **Theorem**
> [1]By A^n or $A^n[a,b]$ we denote the set of functions with an absolutely continuous $(n-1)$st derivative, i.e. the functions f for which there exists (almost everywhere) a function $g_1[a,b]$ such that
>
> $$f^{(n-1)}(x) = f^{(n-1)}(a) + \int_a^x g(t)dt.$$
>
> In this case we call g the (generalized) nth derivative of f, and we simply write $g = f^{(n)}$.

Since $f \in A^1[a,b]$, and by the definition above, we see that $f \in A^1[a,b]$ implies that f' exists almost everywhere and $f(x) = f(a) + \int_a^x f'(t)dt$, $f' \in L_1[a,b]$. Thus,

$$\begin{aligned}
D_a^n f(x) &= D^1(D_a^{n-1} f(x)) \\
&= \frac{d}{dx}\underbrace{\frac{1}{\Gamma(1-n)}\int_a^x (x-t)^{-n} f(t)dt}_{=D_a^{-(1-n)}f(x)} \\
&= \left(f(x) = f(a) + \int_a^x f'(t)dt\right) \\
&= \frac{d}{dx}\frac{1}{\Gamma(1-n)}\int_a^x (x-t)^{-n} \underbrace{\left(f(a) + \int_a^t f'(u)du\right)}_{=f(t)} dt
\end{aligned}$$

$$= \frac{1}{\Gamma(1-n)} \frac{d}{dx} \left[f(a) \underbrace{\int_a^x (x-t)^{-n} dt}_{= \frac{1}{1-n}(x-a)^{-n+1}} + \int_a^x \int_a^t (x-t)^{-n} f'(u) du dt \right]$$

$$= \frac{1}{\Gamma(1-n)} \left(\frac{f(a)}{(x-a)^n} + \frac{d}{dx} \int_a^x \int_a^t (x-t)^{-n} f'(u) du dt \right)$$

Now we can change the order of integration on the second part of the equation. We will obtain graphs similar to Figure (2.1), except $u = \tau$. So you will obtain,

$$D_a^n f(x) = \frac{1}{\Gamma(1-n)} \left(\frac{f(a)}{(x-a)^n} + \frac{d}{dx} \int_a^x f'(u) \underbrace{\left[\int_u^x (x-t)^{-n} dt \right]}_{= \frac{1}{1-n}(x-u)^{-n+1}} du \right)$$

$$= \frac{1}{\Gamma(1-n)} \left(\frac{f(a)}{(x-a)^n} + \frac{d}{dx} \int_a^x f'(u) \frac{(x-u)^{1-n}}{1-n} du \right)$$

To bring the differential into the integral we need the following theorem.

Theorem
By formula,
$$\frac{d}{dx} \int_{\phi_1(x)}^{\phi_2(x)} f(u,x) du = \int_{\phi_1(x)}^{\phi_2(x)} \frac{d}{dx} f(u,x) du + f(u,x) \Big|_{u=\phi_2(x)} .\phi_2'(x) - f(u,x) \Big|_{u=\phi_1(x)} .\phi_1'(x)$$

Hence,

$$D_a^n f(x) = \frac{1}{\Gamma(1-n)} \left(\frac{f(a)}{(x-a)^n} + \int_a^x f'(u)(x-u)^{-n} du \right).$$

It's easy to see $D_a^n f^1[a,b]$, since

$$\int_a^b |D_a^n f(x)| dx \leq C \int_a^b (x-a)^{-n} dx + \int_a^b \int_a^x |f'(u)|.|(x-u)|^{-n} du dx$$

Then by using the change of order again, we will find the graphs are similar to Figure (2.1), where $t = x$, $x = b$, $d\tau = dx$ and $dt = du$. This implies,

$$\begin{aligned} \int_a^b |D_a^n f(x)| dx &\leq C \int_a^b (x-a)^{-n} dx + \int_a^b \int_u^b |f'(u)|.|(x-u)|^{-n} dx du \\ &= C \int_a^b (x-a)^{-n} dx + \int_a^b |f'(u)| \left[\int_u^b (x-u)^{-n} dx \right] du \\ &= C \int_a^b (x-a)^{-n} dx + C_1 \int_a^b |f'(u)|.(b-u)^{-n+1} du \\ &= C \int_a^b (x-a)^{-n} dx + C_2 \int_a^b |f'(u)| du \\ &< \infty \end{aligned}$$

Since $0 < n < 1$. $\square$

Theorem 3.3. [24] *Let $n \geq 0$. Then for every $f_1[a, b]$,*

$$D_a^n D_a^{-n} f = f,$$

almost everywhere.

Example 3.4. [24] For $n = 0$, as D_a^0 and J_a^0 are the identity operator's, it's obvious that $D_a^n D_a^{-n} f = f$.

For $0 < n < 1$,

$$D_a^n D_a^{-n} f = D_a^1 D_a^{n-1} D_a^{-n} f = D_a^1 D_a^{-1} f = f.$$

For $1 < n < 2$,

$$D_a^n D_a^{-n} f = D_a^2 D_a^{n-2} D_a^{-n} f = D_a^2 D_a^{-2} f = D_a^1 (D_a^{-1} D_a^1)(D_a^{-1} f) = D_a^1 (D_a^{-1} f) = f.$$

Theorem 3.5. [4] *Let $n > 0$. If there exists some $\phi \in L_1[a, b]$ such that $f = J_a^n \phi$ then,*

$$J_a^n D_a^n f = f$$

almost everywhere.

Proof. From the previous Theorem and the definition of f, we have

$$J_a^n D_a^n f = J_a^n [D_a^n J_a^n \phi] = J_a^n \phi = f.$$

□

3.1 Properties

Lemma 3.6. From [9], (cf. [11]), let $n \in \mathbb{N}$, then

$$D^n J^n = I, \tag{2.4}$$

but $J^n D^n \neq I$, where I is the identity operator. In fact,

$$J^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, \tag{2.5}$$

when $t > 0$.

4 Caputo Derivative

Definition 4.1. [4] Let $\alpha \geq 0$ and $n = \lceil \alpha \rceil$. Then, D_*^α denotes the Caputo differential operator, defined by,

$$D_*^\alpha f = J^{n-\alpha} D^n f, \tag{2.6}$$

whenever $D^n f \in L_1[a, b]$, (cf. [4], [7]).

Note: $D_*^\alpha f \neq D^\alpha f$, so the Caputo derivative is not equal to the Riemann-Louiville derivative.

Theorem 4.2. [4] *If $f \in C^\mu[a,b]$ for some $\mu \in \mathbb{N}$ and $0 < n < \mu$ then*

$$D^n_{*a} f(x) = \sum_{\ell=0}^{\mu-\lceil n\rceil-1} \frac{f^{(\ell+\lceil n\rceil)}(a)}{\Gamma(\lceil n\rceil - n + \ell + 1)}(x-a)^{(\lceil n\rceil - n+\ell)} + g(x) \tag{2.7}$$

with some function $g \in C^{\mu-\lceil n\rceil}[a,b]$. Moreover, the $(\mu - \lceil n\rceil)$th derivative of g satisfies a Lipschitz condition of order $\lceil n\rceil - n$.

5 Hadamard Finite-Part Integral

Definition 5.1. (cf. [24], [4]) We can express the Riemann-Liouville derivative, $D^n_a f(x)$, $n > 0$, by the Hadamard finite-part Integral. It is defined by a Taylor expansion of a function, f, at $x = a$, which is defined by,

$$\oint_a^b (x-a)^{-\mu} dx = \frac{1}{1-\mu}(b-a)^{1-\mu}, (\mu > 1).$$

Hadamard suggested to simply ignore the unbounded contribution and assign a value of the remaining finite expression.

Example 5.2. Let us consider the integral, with $0 < n < 1$,

$$\int_a^b (x-a)^{-n-1} dx.$$

Which is infinity since,

$$\begin{aligned}
\int_a^b (x-a)^{-n-1} dx &= -\frac{1}{n}(x-a)^{-n}\Big|_a^b \\
&= -\frac{1}{n}\left[(b-a)^{-n} - (a-a)^{-n}\right] \\
&= -\frac{1}{n}(b-a)^{-n} + \frac{1}{n}(a-a)^{-n} \\
&= \underbrace{-\frac{1}{n}(b-a)^{-n}}_{\text{finite part}} + \underbrace{\infty}_{\text{infinite part}}
\end{aligned}$$

Hadamard denotes $-\frac{1}{n}(b-a)^{-n}$ as the finite part of the integral, such that,

$$\oint_a^b (x-a)^{-n-1} dx = -\frac{1}{n}(b-a)^{-n}$$

For sufficiently smooth function, $f(x)$, we may define the Hadamard finite-part integral, for $0 < n < 1$,

$$\int_a^b f(x)(x-a)^{-n-1} dx,$$

By the Taylor Formula,

$$f(x) = f(a) + \underbrace{\int_a^x f'(t)dt}_{\text{remainder term}}$$

Thus,

$$\begin{aligned}\int_a^b f(x)(x-a)^{-n-1}dx &= \int_a^b \left[f(a) + \int_a^x f'(t)dt\right](x-a)^{-n-1}dx \\ &= \int_a^b f(a)(x-a)^{-n-1}dx + \int_a^b (x-a)^{-n-1}\left[\int_a^x f'(t)dt\right]dx\end{aligned}$$

The first part of the integral $\int_a^b f(a)(x-a)^{-n-1}dx$ is infinity and the finite part is $\frac{f(a)}{-n}(b-a)^{-n}$. The second integral $\int_a^b (x-a)^{-n-1}\left[\int_a^x f'(t)dt\right]dx$ is well-defined for a sufficiently smooth $f(x)$, for example, if $f \in C^1[0,T]$, we have

$$\begin{aligned}\left|\int_a^b (x-a)^{-n-1}\left[\int_a^x f'(t)dt\right]dx\right| &\leq \int_a^b (x-a)^{-n-1}M(x-a)dx \\ &\leq M\int_a^b (x-a)^{-n}dx \\ &< \infty\end{aligned}$$

Note: So, the Hadamard finite-part integral is,

$$\oint_a^b f(x)(x-a)^{-n-1}dx = \frac{f(a)}{-n}(b-a)^{-n} + \int_a^b (x-a)^{-n-1}\left[\int_a^x f'(t)dt\right]dx.$$

Chapter 3

Detailed Error Analysis for a Fractional Adams Method with Uniform Meshes

1 Introduction

We discuss a numerical method for the fractional initial value problem,

$$\begin{cases} D_*^\alpha y(t) = f(t, y(t)), \\ y^{(k)}(0) = y_0^{(k)}, \\ k = 0, 1, ..., \lceil \alpha \rceil - 1, \end{cases} \tag{3.1}$$

where $y_0^{(k)}$ may be arbitrary real numbers and $\alpha > 0$.

Lemma 1.1. The fractional initial value problem (3.1), is equivalent to the Volterra integral equation ([7]),

$$y(t) = \sum_{k=0}^{\lceil \alpha \rceil - 1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u, y(u)) du, \tag{3.2}$$

in the sense that a continuous function is a solution of (3.1), if and only if it is a solution of (3.2).

Proof. Let $n = \lceil \alpha \rceil$, if we apply the integral operator from (2.1) to equation (3.1), we will get,

$$\begin{aligned} J^\alpha D_*^\alpha y(t) &= J^\alpha f(t, y(t)), \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u, y(u)) du. \end{aligned}$$

By replacing $J^\alpha D_*^\alpha y(t)$ by (2.5), we will find,

$$y(t) - \sum_{k=0}^{\lceil \alpha \rceil - 1} y_0^{(k)} \frac{t^k}{k!} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u, y(u)) du.$$

After rearranging the equation you will obtain the desired equation. □

2 Adams-Bashforth-Moulton Method

We will first start with the one-step Adams-Bashforth-Moulton algorithm for a first-order equation to understand how to obtain the method for this particular case. We will start with the well-known first-order IVP,

$$\begin{cases} Dy(t) = f(t, y(t)), \\ y(0) = y_0. \end{cases} \tag{3.3}$$

We assume the function f has a unique solution on some interval [0,T]. One way to solve this nonlinear equation, (3.3), is by using the predictor-corrector method found in [10], p.360. We can assume that we are working on a uniform grid, [7], such that $\{t_j = jh : j = 0, 1, ..., N\}$ and $h = T/N$, when $N \in \mathbb{Z}$. By integrating (3.3) on the interval $[t_k, t_{k+1}]$ and assuming we have already calculated approximations $y_j \approx y(t_j)$, for $j = 1, 2, ..., k$. This gives the equation,

$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} f(z, y(z))dz. \tag{3.4}$$

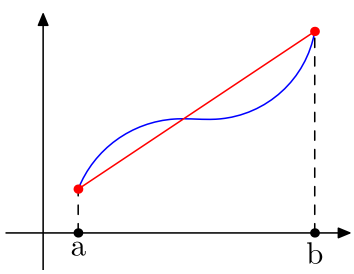


Figure 3.1: Trapezoidal rule

Both $y(t_k)$ and the integral expression are unknown quantities, but we have an approximation for $y(t_k)$, called y_k which we can use instead. The integral expression can be be modified using the two-point trapezoidal quadrature formula, shown in Figure 3.1. This method is used to approximate the area under the graph with a straight line instead of the a curve. Thus, given us the equation for the trapezoidal rule,

$$\int_a^b g(z)dz \approx \frac{b-a}{2}\left(g(a) + g(b)\right). \tag{3.5}$$

By substituting (3.5) and the approximations, $y(t_k) \approx y_k$ and $y(t_{k+1}) \approx y_{k+1}$, into (3.4), we will find,

$$y_{k+1} = y_k + \frac{h}{2}[f(t_k, y(t_k)) + f(t_{k+1}, y(t_{k+1}))]. \tag{3.6}$$

Note: It's obvious to see that $t_{k+1} - t_k = h$.

We can now replace $y(t_k)$ and $y(t_{k+1})$ by the approximations too. This provides us with the equation for the implicit one-step Adams-Moulton method,

$$y_{k+1} = y_k + \frac{h}{2}[f(t_k, y_k) + f(t_{k+1}, y_{k+1})]. \tag{3.7}$$

We can see that y_{k+1} is on both sides of the equation, this causes a problem as we have a nonlinear function f so we can't solve for y_{k+1}, directly. To overcome this issue, we can use (3.7) as an iterative process, a preliminary approximation, to obtain a better approximation for the the right-hand side.

We will call this preliminary approximation, y_{k+1}^P, the predictor. To formulate the predictor, we can use a similar approach like before, but instead of the trapezoidal quadrature formula we will use the rectangular formula. The integral expression for the rectangle formula, shown in Figure 3.2, is given by,

$$\int_a^b g(z)dz \approx (b-a)g(a). \tag{3.8}$$

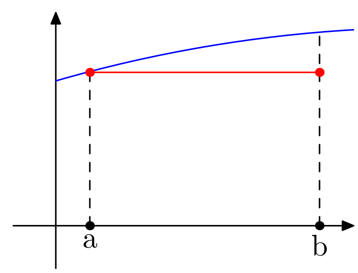


Figure 3.2: Rectangle rule

Then by applying this and the approximations to (3.4) like before, we get,

$$y_{k+1}^P = y_k + hf(t_k, y_k). \tag{3.9}$$

This is commonly known as the explicit (forward) Euler method or one-step Adams-Bashforth method. We can now combine the Adams-Moutlon method and Adams-Bashforth method to get the Adams-Bashforth-Moulton method, by replacing y_{k+1} in (3.7), by the the predictor y_{k+1}^P, from (3.9). Giving us the equation,

$$y_{k+1} = y_k + \frac{h}{2}(f(t_k, y_k) + f(t_{k+1}, y_{k+1}^P)). \tag{3.10}$$

We now have the one-step Adams-Bashforth-Moutlon method,

$$\begin{cases} y_{k+1}^P = y_k + hf(t_k, y_k), \\ y_{k+1} = y_k + \frac{h}{2}(f(t_k, y_k) + f(t_{k+1}, y_{k+1}^P)), \end{cases} \tag{3.11}$$

which is also known as Heun's or trapezoidal rule method. Hence, this equation has the following coefficient tableau, [?trapmeth], p.94 (see RK21).

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Table 3.1: Trapezoidal Rule Method

Table 3.1 is a second stage method and corresponds to the second tableaux when $\theta = 1$, [?trapmeth] p.93-4, so the method converges with order 2. Hence,

$$\max_{j=0,1,\ldots,N} |y(t_j) - y_j| = O(h^2). \tag{3.12}$$

This method is said to be a PECE (Predict, Evaluate, Correct, Evaluate) type. To calculate y_{k+1}, we would calculate and evaluate in this particular order,

Calculate Predictor (y_{k+1}^P) → Evaluate $f(t_{k+1}, y_{k+1}^P)$ → Calculate Corrector (y_{k+1}) → Evaluate $f(t_{k+1}, y_{k+1})$

The final output, $f(t_{k+1}, y_{k+1})$, is then stored and used in the next integration step.

3 Fractional Adams-Bashforth-Moulton Method with Uniform Meshes

Now we have introduced the method for a basic IVP, we can use a similar idea for a fractional-order problem. We are aiming to obtain an equation similar to (3.4). Equation (3.2) resembles (3.4) except the range of integration starts from 0 instead of t_k. This is from the non-local structure of the fractional-order differential operators, see [?nonlocal], p.1.

To construct the formula, we can use the product trapezoidal quadrature formula and use the weight function $(t_{k+1} - \cdot)^{\alpha-1}$ to replace the integral, with nodes t_j $(j = 0, 1, ..., k+1)$ used like before. We apply the approximation,

$$\int_0^{t_{k+1}} (t_{k+1} - z)^{\alpha-1} g(z) dz \approx \int_0^{t_{k+1}} (t_{k+1} - z)^{\alpha-1} \tilde{g}_{k+1}(z) dz, \tag{3.13}$$

where $\tilde{g}_{k+1}$ is the piecewise linear interpolant for g, which is obtained from the trapezoidal rule shown in Figure 3.1. We can write the integral on the right-hand side of (3.13) as,

$$\int_0^{t_{k+1}} (t_{k+1} - z)^{\alpha-1} \tilde{g}_{k+1}(z) dz = \sum_{j=0}^{k+1} a_{j,k+1} g(t_j), \tag{3.14}$$

where,

$$a_{j,k+1} = \frac{h^\alpha}{\alpha(\alpha+1)} \times \begin{cases} (k^{\alpha+1} - (k-\alpha)(k+1)^\alpha), & \text{if } j = 0, \\ ((k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} \\ \quad -2(k-j+1)^{\alpha+1}), & \text{if } 1 \le j \le k, \\ 1, & \text{if } j = k+1. \end{cases} \tag{3.15}$$

We can obtain this by the following computations,

$$\begin{aligned} \int_0^{t_{k+1}} (t_{k+1} - z)^{\alpha-1} \tilde{g}_{k+1}(z) dz &= \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{\alpha-1} \tilde{g}_{k+1}(z) dz, \\ &= \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{\alpha-1} \Bigg[g(t_j) \frac{z - t_{j+1}}{t_j - t_{j+1}} \\ &\quad + g(t_{j+1}) \frac{z - t_j}{t_{j+1} - t_j} \Bigg] dz, \\ &= \sum_{j=0}^{k} \Bigg[\frac{g(t_j)}{t_j - t_{j+1}} \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{\alpha-1} (z - t_{j+1}) dz \\ &\quad + \frac{g(t_{j+1})}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{\alpha-1} (z - t_j) dz \Bigg], \end{aligned}$$

$$= \sum_{j=0}^{k} \Bigg[\underbrace{\frac{g(t_j)}{t_j - t_{j+1}} \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{\alpha-1}(z - t_{j+1})dz}_{①}$$

$$+ \underbrace{\frac{g(t_{j+1})}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{\alpha-1}(z - t_j)dz}_{②} \Bigg].$$

We can work out each part separately like so,

$$\begin{aligned}
① &= \frac{g(t_j)}{t_j - t_{j+1}} \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{\alpha-1}(z - t_{j+1})dz, \\
&= \frac{g(t_j)}{t_j - t_{j+1}} \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{\alpha-1}[(z - t_{k+1}) + (t_{k+1} - t_{j+1})]dz, \\
&= \frac{g(t_j)}{-h} \Bigg[- \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{\alpha} dz + (t_{k+1} - t_{j+1}) \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{\alpha-1} dz \Bigg], \\
&= \frac{g(t_j)}{-h} \Bigg[\frac{(t_{k+1} - z)^{\alpha+1}}{\alpha+1} \Bigg|_{t_j}^{t_{j+1}} - (t_{k+1} - t_{j+1}) \frac{(t_{k+1} - z)^{\alpha}}{\alpha} \Bigg|_{t_j}^{t_{j+1}} \Bigg], \\
&= \frac{g(t_j)}{-h} \Bigg[\frac{(t_{k+1} - t_{j+1})^{\alpha+1}}{\alpha+1} - \frac{(t_{k+1} - t_j)^{\alpha+1}}{\alpha+1} - \frac{(t_{k+1} - t_{j+1})(t_{k+1} - t_{j+1})^{\alpha}}{\alpha} \\
&\quad + \frac{(t_{k+1} - t_{j+1})(t_{k+1} - t_j)^{\alpha}}{\alpha} \Bigg], \\
&= \frac{g(t_j)}{h\alpha(\alpha+1)} \Bigg[\alpha(t_{k+1} - t_j)^{\alpha+1} + (t_{k+1} - t_{j+1})^{\alpha+1} \\
&\quad - (\alpha+1)(t_{k+1} - t_{j+1})(t_{k+1} - t_j)^{\alpha} \Bigg].
\end{aligned}$$

$$\begin{aligned}
② &= \frac{g(t_{j+1})}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{\alpha-1}(z - t_j)dz, \\
&= \frac{g(t_{j+1})}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{\alpha-1}[(z - t_{k+1}) + (t_{k+1} - t_j)]dz, \\
&= \frac{g(t_{j+1})}{h} \Bigg[- \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{\alpha} dz + (t_{k+1} - t_j) \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{\alpha-1} dz \Bigg], \\
&= \frac{g(t_{j+1})}{h} \Bigg[\frac{(t_{k+1} - z)^{\alpha+1}}{\alpha+1} \Bigg|_{t_j}^{t_{j+1}} - (t_{k+1} - t_j) \frac{(t_{k+1} - z)^{\alpha}}{\alpha} \Bigg|_{t_j}^{t_{j+1}} \Bigg], \\
&= \frac{g(t_{j+1})}{h} \Bigg[\frac{(t_{k+1} - t_{j+1})^{\alpha+1}}{\alpha+1} - \frac{(t_{k+1} - t_j)^{\alpha+1}}{\alpha+1} - \frac{(t_{k+1} - t_j)(t_{k+1} - t_{j+1})^{\alpha}}{\alpha} \\
&\quad + \frac{(t_{k+1} - t_j)(t_{k+1} - t_j)^{\alpha}}{\alpha} \Bigg], \\
&= \frac{g(t_{j+1})}{h\alpha(\alpha+1)} \Bigg[\alpha(t_{k+1} - t_{j+1})^{\alpha+1} + (t_{k+1} - t_j)^{\alpha+1} \\
&\quad - (\alpha+1)(t_{k+1} - t_j)(t_{k+1} - t_{j+1})^{\alpha} \Bigg].
\end{aligned}$$

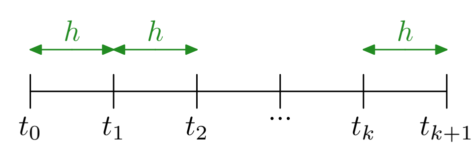
Giving us the following,

$$\sum_{j=0}^{k}\left[\textcircled{1}+\textcircled{2}\right]=\sum_{j=0}^{k}\left[\frac{g(t_j)}{h\alpha(\alpha+1)}\Big[\alpha(t_{k+1}-t_j)^{\alpha+1}+(t_{k+1}-t_{j+1})^{\alpha+1}\right.$$
$$-(\alpha+1)(t_{k+1}-t_{j+1})(t_{k+1}-t_j)^{\alpha}\Big]+\frac{g(t_{j+1})}{h\alpha(\alpha+1)}\Big[\alpha(t_{k+1}-t_{j+1})^{\alpha+1}$$
$$\left.+(t_{k+1}-t_j)^{\alpha+1}-(\alpha+1)(t_{k+1}-t_j)(t_{k+1}-t_{j+1})^{\alpha}\Big]\right],$$

We will compute the summation like so,

$$=\frac{g(t_0)}{h\alpha(\alpha+1)}\Big[\alpha(t_{k+1}-t_0)^{\alpha+1}+(t_{k+1}-t_1)^{\alpha+1}$$
$$-(\alpha+1)(t_{k+1}-t_1)(t_{k+1}-t_0)^{\alpha}\Big]+\frac{g(t_1)}{h\alpha(\alpha+1)}\Big[\alpha(t_{k+1}-t_1)^{\alpha+1}$$
$$+(t_{k+1}-t_0)^{\alpha+1}-(\alpha+1)(t_{k+1}-t_0)(t_{k+1}-t_1)^{\alpha}\Big]$$
$$+\frac{g(t_1)}{h\alpha(\alpha+1)}\Big[\alpha(t_{k+1}-t_1)^{\alpha+1}+(t_{k+1}-t_2)^{\alpha+1}$$
$$-(\alpha+1)(t_{k+1}-t_2)(t_{k+1}-t_1)^{\alpha}\Big]+\frac{g(t_2)}{h\alpha(\alpha+1)}\Big[\alpha(t_{k+1}-t_2)^{\alpha+1}$$
$$+(t_{k+1}-t_1)^{\alpha+1}-(\alpha+1)(t_{k+1}-t_1)(t_{k+1}-t_2)^{\alpha}\Big]+\dots$$
$$+\frac{g(t_k)}{h\alpha(\alpha+1)}\Big[\alpha(t_{k+1}-t_k)^{\alpha+1}+(t_{k+1}-t_{k+1})^{\alpha+1}$$
$$-(\alpha+1)(t_{k+1}-t_{k+1})(t_{k+1}-t_k)^{\alpha}\Big]+\frac{g(t_{k+1})}{h\alpha(\alpha+1)}\Big[\alpha(t_{k+1}-t_{k+1})^{\alpha+1}$$
$$+(t_{k+1}-t_k)^{\alpha+1}-(\alpha+1)(t_{k+1}-t_k)(t_{k+1}-t_{k+1})^{\alpha}\Big].$$

We can now replace $(t_{k+1}-\cdot)$ with the length, h, which you can see from the chart opposite. We know that $(t_{j+1}-t_j)=h$ and hence, $(t_{k+1}-t_j)=h(k+1-j)$. Using basic algebra and collecting the $g(t_j)$ terms together we will get,



$$=\frac{g(t_0)h^{\alpha}}{\alpha(\alpha+1)}\Big[k^{\alpha+1}+(\alpha-k)(k+1)^{\alpha}\Big]+\frac{g(t_1)h^{\alpha}}{\alpha(\alpha+1)}\Big[-2k^{\alpha+1}$$
$$+(k+1)^{\alpha+1}+(k-1)^{\alpha+1}\Big]+\frac{g(t_2)h^{\alpha}}{\alpha(\alpha+1)}\Big[-2(k-1)^{\alpha+1}+k^{\alpha+1}$$
$$+(k-2)^{\alpha+1}\Big]+\dots+\frac{g(t_{k+1})h^{\alpha}}{\alpha(\alpha+1)}.$$

Which is the exact same as (3.14) with the weights, (3.15). Giving us the corrector formula,

$$y_{k+1} = \sum_{j=0}^{\lceil\alpha\rceil-1} y_0^{(j)} \frac{t_{k+1}^j}{j!} + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + a_{k+1,k+1} f(t_{k+1}, y_{k+1}^P) \right). \tag{3.16}$$

Which can be found from substituting (3.14) into the Volterra integral equation, (3.2). Then separate the summation sign to get the $k+1$ term on its own.

Like before we next have to determine the predictor formula, y_{k+1}^P. We can find this by applying the product rectangle rule to the last term of the right hand side in a similar manner to the corrector formula, to get,

$$\int_0^{t_{k+1}} (t_{k+1} - z)^{\alpha-1} \tilde{g}_{k+1}(z) dz \approx \sum_{j=0}^{k} b_{j,k+1} g(t_j), \tag{3.17}$$

where,

$$b_{j,k+1} = \frac{h^\alpha}{\alpha}((k+1-j)^\alpha - (k-j)^\alpha), \tag{3.18}$$

see [8]. This weight can be formulated by using the same idea as before but we will have $g(t_j) = g(t_{j+1})$ as it is using the rectangle formula shown in Figure 3.2.

$$\begin{aligned} \int_0^{t_{k+1}} (t_{k+1} - z)^{\alpha-1} \tilde{g}_{k+1}(z) dz = \sum_{j=0}^{k} \Bigg[& \frac{g(t_j)}{t_j - t_{j+1}} \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{\alpha-1} (z - t_{j+1}) dz \\ & + \frac{g(t_j)}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{\alpha-1} (z - t_j) dz \Bigg]. \end{aligned}$$

Giving us the predictor formula,

$$y_{k+1}^P = \sum_{j=0}^{\lceil\alpha\rceil-1} y_0^{(j)} \frac{t_{k+1}^j}{j!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j). \tag{3.19}$$

We will finally get the fractional Adams-Bashforth-Moutlon method,

$$\begin{cases} y_{k+1}^P = \sum_{j=0}^{\lceil\alpha\rceil-1} y_0^{(j)} \frac{t_{k+1}^j}{j!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j), \\ y_{k+1} = \sum_{j=0}^{\lceil\alpha\rceil-1} y_0^{(j)} \frac{t_{k+1}^j}{j!} + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + a_{k+1,k+1} f(t_{k+1}, y_{k+1}^P) \right) \end{cases} \tag{3.20}$$

with the following weights,

$$a_{j,k+1} = \frac{h^\alpha}{\alpha(\alpha+1)} \times \begin{cases} (k^{\alpha+1} - (k-\alpha)(k+1)^\alpha), & \text{if } j = 0, \\ ((k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} & \\ \quad -2(k-j+1)^{\alpha+1}), & \text{if } 1 \le j \le k, \\ 1, & \text{if } j = k+1. \end{cases} \tag{3.21}$$

and,

$$b_{j,k+1} = \frac{h^\alpha}{\alpha}((k+1-j)^\alpha - (k-j)^\alpha). \tag{3.22}$$

4 Error Analysis for the Adams-Bashforth-Moulton Method

In this section we will look at the error analysis for the method. Firstly, we will state some results that we will need later on to prove the errors. These results can be found in [7], (cf. [4], p.210, p.58). For the rest of the paper we will assume that the fractional Adams method given by (3.20) is used to solve the initial value problem (3.1). We demand that the function f is continuous and fulfils a Lipschitz condition with respect to its second argument with Lipschitz constant L on a suitable set G.

4.1 Auxiliary Results

By [4], a uniquely determined solution y of the problem exists on some interval $[0, T]$, we are aiming to approximate this solution. We will need some additional properties, such as the smoothness, which can be found in [14], p.88-90. Giving us the following results.

Theorem 4.1. *(a) Assume that $f \in C^2(G)$. Define $\hat{v} := \lceil 1/\alpha \rceil - 1$. Then there exist a function $\psi \in C^1[0, T]$ and some $c_1, ..., c_{\hat{v}} \in \mathbb{R}$ such that the solution y of (3.1) can be expressed in the form,*

$$y(t) = \psi(t) + \sum_{v=1}^{\hat{v}} c_v t^{v\alpha}.$$

(b) Assume that $f \in C^3(G)$. Define $\hat{v} := \lceil 2/\alpha \rceil - 1$ and $\tilde{v} := \lceil 1/\alpha \rceil - 1$. Then there exist a function $\psi \in C^2[0, T]$ and some $c_1, ..., c_{\hat{v}} \in \mathbb{R}$ and $d_1, ..., d_{\tilde{v}} \in \mathbb{R}$ such that the solution y of (3.1) can be expressed in the form,

$$y(t) = \psi(t) + \sum_{v=1}^{\hat{v}} c_v t^{v\alpha} + \sum_{v=1}^{\tilde{v}} d_v t^{1+v\alpha}.$$

Corollary 4.2. Let $y \in C^m[0, T]$ for some $m \in \mathbb{N}$ and assume that $0 < \alpha < m$. Then $D_*^\alpha y \in C[0, T]$.

To find the errors of the quadrature formulas we have used to find the predictor and the corrector, we can use the following theorems.

Theorem 4.3. *(a) Let $z \in C^1[0, T]$. Then,*

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} z(t) dt - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \leq \frac{1}{\alpha} ||z'||_\infty t_{k+1}^\alpha h.$$

(b) Let $z(t) = t^p$ for some $p \in (0, 1)$. Then,

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} z(t) dt - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \leq C_{\alpha,p}^{Re} t_{k+1}^{\alpha+p-1} h,$$

where $C_{\alpha,p}^{Re}$ is a constant that depends only on α and p.

Proof. We know that,

$$\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} z(t)dt - \sum_{j=0}^{k} b_{j,k+1} z(t_j) = \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1}-t)^{\alpha-1} z(t)dt$$

$$- \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1}-t)^{\alpha-1} z(t_j)dt,$$

$$= \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1}-t)^{\alpha-1} (z(t)-z(t_j))dt. \tag{3.23}$$

To prove (a), we can apply the following theorem,

> **Mean Value Theorem**
> From [1], suppose that f is continuous on a closed interval $I := [a,b]$, and that f has a derivative in the open interval (a,b). Then there exists at least one point c in (a,b) such that,
>
> $$f(b) - f(a) = f'(c)(b-a).$$

By taking the modulus of the Mean Value Theorem we know that $|f'(c)(b-a)| \leq ||f'||_\infty.|b-a|$. In our case $|b-a|$ is always positive, hence,

$$\left| \int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} z(t)dt - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \leq ||z'||_\infty \sum_{j=0}^{k} \underbrace{\int_{t_j}^{t_{j+1}} (t_{k+1}-t)^{\alpha-1}(t-t_j)dt}_{①},$$

By computing the integral, we will obtain,

$$① = \int_{t_j}^{t_{j+1}} (t_{k+1}-t)^{\alpha-1} [(t-t_{k+1}) + (t_{k+1}-t_j)]dt,$$

$$= \int_{t_j}^{t_{j+1}} (t_{k+1}-t)^{\alpha-1}(t-t_{k+1})dt + \int_{t_j}^{t_{j+1}} (t_{k+1}-t)^{\alpha-1}(t_{k+1}-t_j)dt,$$

$$= -\int_{t_j}^{t_{j+1}} (t_{k+1}-t)^{\alpha}dt + (t_{k+1}-t_j)\int_{t_j}^{t_{j+1}} (t_{k+1}-t)^{\alpha-1}dt,$$

$$= \frac{(t_{k+1}-t)^{\alpha+1}}{\alpha+1}\bigg|_{t_j}^{t_{j+1}} - (t_{k+1}-t_j)\frac{(t_{k+1}-t)^{\alpha}}{\alpha}\bigg|_{t_j}^{t_{j+1}},$$

$$= \frac{1}{\alpha+1}\Big[(t_{k+1}-t_{j+1})^{\alpha+1} - (t_{k+1}-t_j)^{\alpha+1}\Big] - \frac{1}{\alpha}\Big[(t_{k+1}-t_{j+1})^{\alpha}$$

$$- (t_{k+1}-t_j)^{\alpha}\Big],$$

$$
\begin{aligned}
&= \frac{1}{\alpha(\alpha+1)}\Big[\alpha(t_{k+1}-t_{j+1})^{\alpha+1} - \alpha(t_{k+1}-t_j)^{\alpha+1} - (\alpha+1)(t_{k+1}-t_{j+1})^{\alpha} \\
&\quad + (\alpha+1)(t_{k+1}-t_j)^{\alpha}\Big], \\
&= \frac{1}{\alpha(\alpha+1)}\Big[(t_{k+1}-t_{j+1})^{\alpha}\Big[\alpha(t_{k+1}-t_{j+1}) - (\alpha+1)(t_{k+1}-t_j)\Big] \\
&\quad + (t_{k+1}-t_j)^{\alpha+1}\Big], \\
&= \frac{1}{\alpha(\alpha+1)}\Big[(h(k+1-(j+1)))^{\alpha}\Big[\alpha(h(k+1-(j+1))) \\
&\quad - (\alpha+1)(h(k+1-j))\Big] + (h(k+1-j))^{\alpha+1}\Big], \\
&= \frac{h^{\alpha+1}}{\alpha(\alpha+1)}\Big[(k-j)^{\alpha}\Big[\alpha(k-j) - (\alpha+1)(k+1-j)\Big] + (k+1-j)^{\alpha+1}\Big], \\
&= \frac{h^{\alpha+1}}{\alpha(\alpha+1)}\Big[(k-j)^{\alpha}(j-k-\alpha-1) + (k+1-j)^{\alpha+1}\Big], \\
&= \frac{h^{\alpha+1}}{\alpha(\alpha+1)}\Big[(k-j)^{\alpha}(j-k) - (k-j)^{\alpha}(\alpha+1) + (k+1-j)^{\alpha+1}\Big], \\
&= \frac{h^{\alpha+1}}{\alpha(\alpha+1)}\Big[-(k-j)^{\alpha+1} - (k-j)^{\alpha}(\alpha+1) + (k+1-j)^{\alpha+1}\Big].
\end{aligned}
$$

Giving us the results to (3.23) as follows,

$$
= ||z'||_{\infty}\frac{h^{\alpha+1}}{\alpha}\sum_{j=0}^{k}\left(\frac{1}{\alpha+1}[(k+1-j)^{\alpha+1} - (k-j)^{\alpha+1}] - (k-j)^{\alpha}\right).
$$

By computing the summation and gathering terms we will get,

$$
\begin{aligned}
&= ||z'||_{\infty}\frac{h^{\alpha+1}}{\alpha}\Bigg[\frac{1}{\alpha+1}\Big((k+1)^{\alpha+1} - k^{\alpha+1}\Big) - k^{\alpha} + \frac{1}{\alpha+1}\Big(k^{\alpha+1} - (k-1)^{\alpha+1}\Big) \\
&\quad - (k-1)^{\alpha} + ... + \frac{1}{\alpha+1}\Big(2^{\alpha+1} - 1^{\alpha+1}\Big) - 1^{\alpha} + \frac{1}{\alpha+1}\Big(1^{\alpha+1} - 0^{\alpha+1}\Big) - 0^{\alpha}\Bigg], \\
&= ||z'||_{\infty}\frac{h^{\alpha+1}}{\alpha}\left(\frac{(k+1)^{\alpha+1}}{\alpha+1} - \sum_{j=0}^{k} j^{\alpha}\right), \\
&= ||z'||_{\infty}\frac{h^{\alpha+1}}{\alpha}\left(\int_0^{k+1} t^{\alpha}dt - \sum_{j=0}^{k} j^{\alpha}\right).
\end{aligned}
$$

We know that the term in the brackets will give you the error after using the rectangle rule. From [7], (cf. [2], theorem 97), we can see that the integrand is monotonic so the term is bounded by the total variation by the quantity $(k+1)^{\alpha}$. Thus,

$$
\left|\int_0^{t_{k+1}}(t_{k+1}-t)^{\alpha-1}z(t)dt - \sum_{j=0}^{k} b_{j,k+1}z(t_j)\right| \leq ||z'||_{\infty}\frac{h^{\alpha+1}}{\alpha}(k+1)^{\alpha}.
$$

We can use a similar idea to prove (b), but the monotonicity of z in (3.23) gives us,

$$
\begin{aligned}
&\Big| \int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} z(t)dt - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \Big| \\
&\leq \sum_{j=0}^{k} |z(t_{j+1}) - z(t_j)| \int_{t_j}^{t_{j+1}} (t_{k+1}-t)^{\alpha-1} dt, \\
&= \sum_{j=0}^{k} |t_{j+1}^p - t_j^p| \cdot - \frac{(t_{k+1}-t)^{\alpha}}{\alpha} \Big|_{t_j}^{t_{j+1}}, \\
&= \sum_{j=0}^{k} \Big(h^p (j+1)^p - h^p j^p \Big) - \Big(\frac{(t_{k+1}-t_{j+1})^{\alpha}}{\alpha} - \frac{(t_{k+1}-t_j)^{\alpha}}{\alpha} \Big), \\
&= \frac{h^p}{\alpha} \sum_{j=0}^{k} \Big((j+1)^p - j^p \Big) h^{\alpha} \Big(-(k+1-j-1)^{\alpha} + (k+1-j)^{\alpha} \Big), \\
&= \frac{h^{p+\alpha}}{\alpha} \sum_{j=0}^{k} \Big((j+1)^p - j^p \Big) \Big(-(k-j)^{\alpha} + (k+1-j)^{\alpha} \Big).
\end{aligned}
$$

By taking $j = 0$ and $j = k$ out of the sum, we will get,

$$
\begin{aligned}
= \frac{h^{p+\alpha}}{\alpha} \Big[&(k+1)^{\alpha} - k^{\alpha} + (k+1)^p - k^p + \sum_{j=1}^{k-1} \Big((j+1)^p - j^p \Big) \Big(-(k-j)^{\alpha} \\
&+ (k+1-j)^{\alpha} \Big) \Big].
\end{aligned}
$$

Then by using the Mean Value Theorem, such that $(j+1)^p - j^p = pc^{p-1}$ for $c \in (j, j+1)$ and $(k+1-j)^{\alpha} - (k-j)^{\alpha} = \alpha d^{\alpha-1}$ for $d \in (k-j, k+1-j)$,

$$
\begin{aligned}
= \frac{h^{p+\alpha}}{\alpha} \Big[&(k+1)^{\alpha} - k^{\alpha} + (k+1)^p - k^p + \sum_{j=1}^{k-1} \Big((j+1)^p - j^p \Big) \Big(-(k-j)^{\alpha} \\
&+ (k+1-j)^{\alpha} \Big) \Big], \\
\leq \frac{h^{p+\alpha}}{\alpha} \Big[&(k+1)^{\alpha} - k^{\alpha} + (k+1)^p - k^p + p\alpha \sum_{j=1}^{k-1} j^{p-1} (k-j+q)^{\alpha-1} \Big],
\end{aligned}
$$

where,

$$
\begin{cases} q = 0, & \text{if } \alpha \leq 1, \\ q = 1, & \text{otherwise.} \end{cases}
$$

Then by using the Mean Value Theorem again,

$$\leq \frac{h^{p+\alpha}}{\alpha}\left[\alpha(k+q)^{\alpha-1} + pk^{p-1} + p\alpha\sum_{j=1}^{k-1} j^{p-1}(k-j+q)^{\alpha-1}\right],$$

$$\leq \frac{h^{p+\alpha}}{\alpha}\left[2\alpha(k+q)^{\alpha-1} + pk^{p-1} + p\alpha\sum_{j=1}^{k-1} j^{p-1}(k-j+q)^{\alpha-1}\right],$$

By [4], (cf. [23], theorem 3.7), the term is bounded from above by $C_{\alpha,p}(k+1)^{p+\alpha-1}$, where $C_{\alpha,p}$ is a constant depending on α and p, but not k. □

We have the following theorem for the product trapezoidal formula that we have used for the corrector. The proof is very similar to theorem 4.3, so we omit the details. Th following theorem can be found in [4],

Theorem 4.4. *(a) If $z \in C^2[0,T]$ then there is a constant C_α^{Tr} depending only on α such that*

$$\left| \int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} z(t)dt - \sum_{j=0}^{k+1} a_{j,k+1} z(t_j) \right| \leq C_\alpha^{Tr} ||z''||_\infty t_{k+1}^\alpha h^2.$$

(b) Let $z \in C^1[0,T]$ and assume that z' fulfils a Lipschitz condition of order μ for some $\mu \in (0,1)$. Then, there exist positive constants $B_{\alpha,\mu}^{Tr}$ (depending only on α and μ) and $M(z,\mu)$ (depending only on z and μ) such that

$$\left| \int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} z(t)dt - \sum_{j=0}^{k+1} a_{j,k+1} z(t_j) \right| \leq B_{\alpha,\mu}^{Tr} M(z,\mu) t_{k+1}^\alpha h^{1+\mu}.$$

(c) Let $z(t) = t^p$ for some $p \in (0,2)$ and $\varrho := min(2, p+1)$. Then,

$$\left| \int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} z(t)dt - \sum_{j=0}^{k+1} a_{j,k+1} z(t_j) \right| \leq C_{\alpha,p}^{Tr} t_{k+1}^{\alpha+p-\varrho} h^\varrho,$$

where $C_{\alpha,p}^{Tr}$ is a constant that depends only on α and p.

4.2 Error Analysis

In this section we will cover the main theorem concerning the error of the method and the general convergence result. We will first start with a lemma concerning the convergence of the method, found in [4].

Lemma 4.5. Assume that the solution y of the initial value problem is,

$$\left| \int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} D_*^\alpha y(t)dt - \sum_{j=0}^{k} b_{j,k+1} D_*^\alpha y(t_j) \right| \leq C_1 t_{k+1}^{\gamma_1} h^{\delta_1},$$

and,

$$\left|\int_0^{t_{k+1}}(t_{k+1}-t)^{\alpha-1}D_*^\alpha y(t)dt-\sum_{j=0}^{k+1}a_{j,k+1}D_*^\alpha y(t_j)\right|\le C_2t_{k+1}^{\gamma_2}h^{\delta_2},$$

with some $\gamma_1,\gamma_2\ge 0$ and $\delta_1,\delta_2>0$. Then, for some suitably chosen $T>0$, we have,

$$\max_{0\le j\le N}|y(t_j)-y_j|=O(h^q).$$

where $q=min\{\delta_1+\alpha,\delta_2\}$ and $N=\lfloor T/h\rfloor$.

Proof. We will show that , for sufficiently small h

$$|y(t_j)-y_j|\le Ch^q, \tag{3.24}$$

for all $j\in\{0,1,...,N\}$, where C is a suitable constant. We will use proof my mathematical induction. When $j=0$, $|y(t_0)-y_0|=0$ because of the initial conditions from (3.1). Now assume that (3.24) is true for $j=0,1,...,k$ for some $k\le N-1$. So we need to prove that the inequality holds true fro $j=k+1$. We can show this by first looking at the error of the predictor, y_{k+1}^P. From (3.2) and (3.19), we will get,

$$\begin{aligned}
\left|y(t_{k+1})-y_{k+1}^P\right| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_{k+1}}(t_{k+1}-t)^{\alpha-1}f(t,y(t))dt-\sum_{j=0}^{k}b_{j,k+1}f(t_j,y_j)\right|,\\
&= \frac{1}{\Gamma(\alpha)}\Bigg|\int_0^{t_{k+1}}(t_{k+1}-t)^{\alpha-1}D_*^\alpha y(t)dt+\Bigg(\sum_{j=0}^{k}b_{j,k+1}f(t_j,y(t_j))\\
&\quad-\sum_{j=0}^{k}b_{j,k+1}D_*^\alpha y(t_j)\Bigg)-\sum_{j=0}^{k}b_{j,k+1}f(t_j,y_j)\Bigg|,\\
&\le \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_{k+1}}(t_{k+1}-t)^{\alpha-1}D_*^\alpha y(t)dt-\sum_{j=0}^{k}b_{j,k+1}D_*^\alpha y(t_j)\right|\\
&\quad+\frac{1}{\Gamma(\alpha)}\left|\sum_{j=0}^{k}b_{j,k+1}f(t_j,y(t_j))-\sum_{j=0}^{k}b_{j,k+1}f(t_j,y_j)\right|,\\
&= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_{k+1}}(t_{k+1}-t)^{\alpha-1}D_*^\alpha y(t)dt-\sum_{j=0}^{k}b_{j,k+1}D_*^\alpha y(t_j)\right|\\
&\quad+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}b_{j,k+1}\left|f(t_j,y(t_j))-f(t_j,y_j)\right|.
\end{aligned}$$

By using the assumption above, the fact that f satisfies the Lipschitz condition, Equation (3.24) and $b_{j,k+1}>0$, we will get,

$$\le\frac{C_1t_{k+1}^{\gamma_1}}{\Gamma(\alpha)}h^{\delta_1}+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}b_{j,k+1}LCh^q.$$

Note: $\sum_{j=0}^{k} b_{j,k+1} = \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} dt = \frac{1}{\alpha} t_{k+1}^{\alpha} \leq \frac{1}{\alpha} T^{\alpha}$, because $t_{k+1} \leq T$ as $t \in [0, T]$.

$$\leq \frac{C_1 T^{\gamma_1}}{\Gamma(\alpha)} h^{\delta_1} + \frac{CLT^{\alpha}}{\Gamma(\alpha+1)} h^q. \tag{3.25}$$

Therefore we have a bound for the predictor error, (3.25). We can now calculate the corrector error from (3.2) and (3.19),

$$\begin{aligned}
|y(t_{k+1}) - y_{k+1}| &= \frac{1}{\Gamma(\alpha)} \bigg| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} f(t, y(t)) dt - \sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) \\
&\quad - a_{k+1,k+1} f(t_{k+1}, y_{k+1}^P) \bigg|, \\
&= \frac{1}{\Gamma(\alpha)} \bigg| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} D_*^{\alpha} y(t) dt + \bigg(\sum_{j=0}^{k} a_{j,k+1} f(t_j, y(t_j)) \\
&\quad + a_{k+1,k+1} f(t_{k+1}, y(t_{k+1})) - \sum_{j=0}^{k+1} a_{j,k+1} f(t_j, y(t_j)) \bigg) - \sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) \\
&\quad - a_{k+1,k+1} f(t_{k+1}, y_{k+1}^P) \bigg|, \\
&\leq \frac{1}{\Gamma(\alpha)} \bigg| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} D_*^{\alpha} y(t) dt - \sum_{j=0}^{k+1} a_{j,k+1} f(t_j, y(t_j)) \bigg| \\
&\quad + \frac{1}{\Gamma(\alpha)} \bigg| \sum_{j=0}^{k} a_{j,k+1} f(t_j, y(t_j)) - \sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) \bigg| \\
&\quad + \frac{1}{\Gamma(\alpha)} \bigg| a_{k+1,k+1} f(t_{k+1}, y(t_{k+1})) - a_{k+1,k+1} f(t_{k+1}, y_{k+1}^P) \bigg|, \\
&= \frac{1}{\Gamma(\alpha)} \bigg| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} D_*^{\alpha} y(t) dt - \sum_{j=0}^{k+1} a_{j,k+1} f(t_j, y(t_j)) \bigg| \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} a_{j,k+1} \bigg| f(t_j, y(t_j)) - f(t_j, y_j) \bigg| \\
&\quad + \frac{1}{\Gamma(\alpha)} a_{k+1,k+1} \bigg| f(t_{k+1}, y(t_{k+1})) - f(t_{k+1}, y_{k+1}^P) \bigg|.
\end{aligned}$$

We know $a_{k+1,k+1}$ from (3.15), when $j = k+1$ and f satisfies the Lipschitz condition, we get,

$$\leq \frac{C_2 t_{k+1}^{\gamma_2} h^{\delta_2}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} a_{j,k+1} L \bigg| y(t_j) - y_j \bigg| + \frac{1}{\Gamma(\alpha)} \frac{h^{\alpha}}{\alpha(\alpha+1)} L \bigg| y(t_{k+1}) - y_{k+1}^P \bigg|.$$

Then by (3.24) and (3.25),

$$\leq \frac{C_2 T^{\gamma_2} h^{\delta_2}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} a_{j,k+1} LCh^q + \frac{Lh^{\alpha}}{\Gamma(\alpha+2)} \bigg(\frac{C_1 T^{\gamma_1}}{\Gamma(\alpha)} h^{\delta_1} + \frac{CLT^{\alpha}}{\Gamma(\alpha+1)} h^q \bigg),$$

$$= \frac{C_2 T^{\gamma_2}}{\Gamma(\alpha)} h^{\delta_2} + \frac{LCT^{\alpha}}{\Gamma(\alpha+1)} h^q + \frac{C_1 T^{\gamma_1} L}{\Gamma(\alpha)\Gamma(\alpha+2)} h^{\alpha} h^{\delta_1} + \frac{CL^2 T^{\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha+2)} h^{\alpha} h^q,$$

We know that $\delta_1, \delta_2 > 0$ and because $q = min\{\delta_1 + \alpha, \delta_2\}$, $q \geq \delta_2$ and $q \geq \delta_1 + \alpha$. Hence,

$$\leq \left(\frac{C_2 T^{\gamma_2}}{\Gamma(\alpha)} + \frac{LCT^{\alpha}}{\Gamma(\alpha+1)} + \frac{C_1 T^{\gamma_1} L}{\Gamma(\alpha)\Gamma(\alpha+2)} + \frac{CL^2 T^{\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha+2)} h^{\alpha} \right) h^q$$

By choosing T sufficiently small, we can make sure that the second summand in the parenthesis is bounded by $C/2$. Having fixed this value for T, we can then make the sum of the remaining expressions in the parentheses smaller than $C/2$ too by choosing C sufficiently large, [4]. Hence,

$$|y(t_{k+1}) - y_{k+1}| \leq Ch^q.$$

□

Theorem 4.6. *Let $\alpha > 0$ and assume $D_*^{\alpha} y \in C^2[0,T]$ for some suitable T. Then,*

$$\max_{0 \leq j \leq N} |y(t_j) - y_j| = \begin{cases} O(h^2), & \text{if } \alpha \geq 1, \\ O(h^{\alpha+1}), & \text{if } \alpha < 1. \end{cases}$$

Proof. Theorems 4.3 and 4.4 states,

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} z(t) dt - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \leq \frac{1}{\alpha} ||z'||_{\infty} t_{k+1}^{\alpha} h,$$

and,

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} z(t) dt - \sum_{j=0}^{k+1} a_{j,k+1} z(t_j) \right| \leq C_{\alpha}^{Tr} ||z''||_{\infty} t_{k+1}^{\alpha} h^2,$$

respectively. From observation we can see that the inequalites above correspond to the inequalities in lemma 4.5, where $C_1 = \frac{1}{\alpha}||z'||_{\infty}$, $\gamma_1 = \alpha$, $\delta_1 = 1$, $C_2 = C_{\alpha}^{Tr}||z'||_{\infty}$, $\gamma_2 = \alpha$, $\delta_2 = 2$. Lemma 4.5 states,

$$\max_{0 \leq j \leq N} |y(t_j) - y_j| = O(h^q).$$

where $q = min\{\delta_1 + \alpha, \delta_2\}$. So by replacing the values we know, we will find,

$$q = min\{\delta_1 + \alpha, \delta_2\} = min\{1 + \alpha, 2\} = \begin{cases} 2, & \text{if } \alpha \geq 1, \\ 1 + \alpha, & \text{if } \alpha < 1. \end{cases}$$

□

Chapter 4

Detailed Error Analysis for a Fractional Adams Method with Graded Meshes

1 Introduction

By using a similar idea to [4] but instead of using uniform meshes like above, we can introduce the following graded meshes, [25]. Let $0 = t_0^r < t_1^r < ... < t_N^r = T$ be the graded meshes on [0,T], where

$$t_n^r = T\left(\frac{n}{N}\right)^r, \tag{4.1}$$

and $r \geq 1$, $n = 0, 1, 2, ..., N$. By inserting the graded meshes into (3.2), you will get,

$$y(t_{n+1}^r) = \sum_{k=0}^{\lceil\alpha\rceil-1} (y_0^r)^{(k)} \frac{(t_{n+1}^r)^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1} g(s) ds. \tag{4.2}$$

2 Fractional Adams-Bashforth-Moulton Method with Graded Meshes

We can solve this equation by using linear interpolation as before and in particular the trapezoidal rule (cf. [4]). Hence,

$$\begin{aligned}\int_0^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1} g(s) ds &\approx \sum_{j=0}^{n} \Bigg[\frac{g(t_j^r)}{t_j^r - t_{j+1}^r} \int_{t_j^r}^{t_{j+1}^r} (t_{n+1}^r - s)^{\alpha-1} (s - t_{j+1}^r) ds \\ &\quad + \frac{g(t_{j+1}^r)}{t_{j+1}^r - t_j^r} \int_{t_j^r}^{t_{j+1}^r} (t_{n+1}^r - s)^{\alpha-1} (s - t_j^r) ds \Bigg], \\ &= \sum_{j=0}^{n+1} a_{j,n+1}^r g(t_j^r),\end{aligned} \tag{4.3}$$

where the weights $a_{j,n+1}^r$, can be calculated as follows,

$$= \sum_{j=0}^{n} \Bigg[\underbrace{\frac{g(t_j^r)}{t_j^r - t_{j+1}^r} \int_{t_j^r}^{t_{j+1}^r} (t_{n+1}^r - s)^{\alpha-1}(s - t_{j+1}^r)ds}_{③} + \underbrace{\frac{g(t_{j+1}^r)}{t_{j+1}^r - t_j^r} \int_{t_j^r}^{t_{j+1}^r} (t_{n+1}^r - s)^{\alpha-1}(s - t_j^r)ds}_{④} \Bigg]$$

where,

$$\begin{aligned}
③ &= \frac{g(t_j^r)}{t_j^r - t_{j+1}^r} \int_{t_j^r}^{t_{j+1}^r} (t_{n+1}^r - s)^{\alpha-1}(s - t_{j+1}^r)ds, \\
&= \frac{g(t_j^r)}{t_j^r - t_{j+1}^r} \int_{t_j^r}^{t_{j+1}^r} (t_{n+1}^r - s)^{\alpha-1}[(s - t_{n+1}^r) + (t_{n+1}^r - t_{j+1}^r)]ds, \\
&= \frac{g(t_j^r)}{t_j^r - t_{j+1}^r} \Bigg[- \int_{t_j^r}^{t_{j+1}^r} (t_{n+1}^r - s)^{\alpha}ds + (t_{n+1}^r - t_{j+1}^r) \int_{t_j^r}^{t_{j+1}^r} (t_{n+1}^r - s)^{\alpha-1}ds \Bigg], \\
&= \frac{g(t_j^r)}{t_j^r - t_{j+1}^r} \Bigg[\frac{(t_{n+1}^r - s)^{\alpha+1}}{\alpha+1}\Bigg|_{t_j^r}^{t_{j+1}^r} - (t_{n+1}^r - t_{j+1}^r)\frac{(t_{n+1}^r - s)^{\alpha}}{\alpha}\Bigg|_{t_j^r}^{t_{j+1}^r} \Bigg], \\
&= \frac{g(t_j^r)}{t_j^r - t_{j+1}^r} \Bigg[\frac{(t_{n+1}^r - s)^{\alpha+1}}{\alpha+1}\Bigg|_{t_j^r}^{t_{j+1}^r} - (t_{n+1}^r - t_{j+1}^r)\frac{(t_{n+1}^r - s)^{\alpha}}{\alpha}\Bigg|_{t_j^r}^{t_{j+1}^r} \Bigg], \\
&= \frac{g(t_j^r)}{t_j^r - t_{j+1}^r} \Bigg[\frac{(t_{n+1}^r - t_{j+1}^r)^{\alpha+1}}{\alpha+1} - \frac{(t_{n+1}^r - t_j^r)^{\alpha+1}}{\alpha+1} - (t_{n+1}^r - t_{j+1}^r)\frac{(t_{n+1}^r - t_{j+1}^r)^{\alpha}}{\alpha} \\
&\quad + (t_{n+1}^r - t_{j+1}^r)\frac{(t_{n+1}^r - t_j^r)^{\alpha}}{\alpha} \Bigg], \\
&= \frac{g(t_j^r)}{\alpha(\alpha+1)(t_j^r - t_{j+1}^r)} \Bigg[\alpha(t_{n+1}^r - t_{j+1}^r)^{\alpha+1} - \alpha(t_{n+1}^r - t_j^r)^{\alpha+1} \\
&\quad - (\alpha+1)(t_{n+1}^r - t_{j+1}^r)^{\alpha+1} + (\alpha+1)(t_{n+1}^r - t_{j+1}^r)(t_{n+1}^r - t_j^r)^{\alpha} \Bigg], \\
&= \frac{g(t_j^r)}{\alpha(\alpha+1)(t_j^r - t_{j+1}^r)} \Bigg[- \alpha(t_{n+1}^r - t_j^r)^{\alpha+1} - (t_{n+1}^r - t_{j+1}^r)^{\alpha+1} \\
&\quad + (\alpha+1)(t_{n+1}^r - t_{j+1}^r)(t_{n+1}^r - t_j^r)^{\alpha} \Bigg].
\end{aligned}$$

We know that from (4.1), $t_n^r = T\left(\frac{n}{N}\right)^r$, hence,

$$\begin{aligned}
&= \frac{g(t_j^r)}{TN^{-r}\alpha(\alpha+1)(j^r - (j+1)^r)} \Bigg[- \alpha T^{\alpha+1} N^{-r(\alpha+1)} \Big((n+1)^r - j^r \Big)^{\alpha+1} \\
&\quad - T^{\alpha+1} N^{-r(\alpha+1)} \Big((n+1)^r - (j+1)^r \Big)^{\alpha+1} \\
&\quad + T^{\alpha+1} N^{-r(\alpha+1)} (\alpha+1) \Big((n+1)^r - (j+1)^r \Big) \Big((n+1)^r - j^r \Big)^{\alpha} \Bigg], \\
&= \frac{g(t_j^r) T^{\alpha} N^{-r\alpha}}{\alpha(\alpha+1)(j^r - (j+1)^r)} \Bigg[- \alpha \Big((n+1)^r - j^r \Big)^{\alpha+1} - \Big((n+1)^r - (j+1)^r \Big)^{\alpha+1} \\
&\quad + (\alpha+1) \Big((n+1)^r - (j+1)^r \Big) \Big((n+1)^r - j^r \Big)^{\alpha} \Bigg],
\end{aligned}$$

$$
\begin{aligned}
&= \frac{g(t_j^r)T^\alpha N^{-r\alpha}}{\alpha(\alpha+1)(j^r-(j+1)^r)}\Big[-\Big((n+1)^r-(j+1)^r\Big)^{\alpha+1} \\
&\quad + \Big((n+1)^r-j^r\Big)^\alpha\Big((\alpha+1)\Big((n+1)^r-(j+1)^r\Big)-\alpha\Big((n+1)^r-j^r\Big)\Big)\Big], \\
&= \frac{g(t_j^r)T^\alpha N^{-r\alpha}}{\alpha(\alpha+1)(j^r-(j+1)^r)}\Big[-\Big((n+1)^r-(j+1)^r\Big)^{\alpha+1} \\
&\quad + \Big((n+1)^r-j^r\Big)^\alpha\Big(-\alpha(j+1)^r+(n+1)^r-(j+1)^r+\alpha j^r+j^r-j^r\Big)\Big], \\
&= \frac{g(t_j^r)T^\alpha N^{-r\alpha}}{\alpha(\alpha+1)(j^r-(j+1)^r)}\Big[-\Big((n+1)^r-(j+1)^r\Big)^{\alpha+1} \\
&= \frac{g(t_j^r)T^\alpha N^{-r\alpha}}{\alpha(\alpha+1)(j^r-(j+1)^r)}\Big[\Big((n+1)^r-j^r\Big)^{\alpha+1}-\Big((n+1)^r-(j+1)^r\Big)^{\alpha+1} \\
&\quad + (\alpha+1)\Big((n+1)^r-j^r\Big)^\alpha\Big].
\end{aligned}
$$

By using the same idea, we know that,

$$
\begin{aligned}
④ &= \frac{g(t_{j+1}^r)}{t_{j+1}^r-t_j^r}\int_{t_j^r}^{t_{j+1}^r}(t_{n+1}^r-s)^{\alpha-1}(s-t_j^r)ds, \\
&= \frac{g(t_{j+1}^r)}{t_{j+1}^r-t_j^r}\int_{t_j^r}^{t_{j+1}^r}(t_{n+1}^r-s)^{\alpha-1}[(s-t_{n+1}^r)+(t_{n+1}^r-t_j^r)]ds, \\
&= \frac{g(t_{j+1}^r)}{t_{j+1}^r-t_j^r}\Big[-\int_{t_j^r}^{t_{j+1}^r}(t_{n+1}^r-s)^\alpha ds+(t_{n+1}^r-t_j^r)\int_{t_j^r}^{t_{j+1}^r}(t_{n+1}^r-s)^{\alpha-1}ds\Big], \\
&= \frac{g(t_{j+1}^r)}{t_{j+1}^r-t_j^r}\Big[\frac{(t_{n+1}^r-s)^{\alpha+1}}{\alpha+1}\Big|_{t_j^r}^{t_{j+1}^r}-(t_{n+1}^r-t_j^r)\frac{(t_{n+1}^r-s)^\alpha}{\alpha}\Big|_{t_j^r}^{t_{j+1}^r}\Big], \\
&= \frac{g(t_{j+1}^r)}{t_{j+1}^r-t_j^r}\Big[\frac{(t_{n+1}^r-s)^{\alpha+1}}{\alpha+1}\Big|_{t_j^r}^{t_{j+1}^r}-(t_{n+1}^r-t_j^r)\frac{(t_{n+1}^r-s)^\alpha}{\alpha}\Big|_{t_j^r}^{t_{j+1}^r}\Big], \\
&= \frac{g(t_{j+1}^r)}{t_{j+1}^r-t_j^r}\Big[\frac{(t_{n+1}^r-t_{j+1}^r)^{\alpha+1}}{\alpha+1}-\frac{(t_{n+1}^r-t_j^r)^{\alpha+1}}{\alpha+1}-(t_{n+1}^r-t_j^r)\frac{(t_{n+1}^r-t_{j+1}^r)^\alpha}{\alpha} \\
&\quad + (t_{n+1}^r-t_j^r)\frac{(t_{n+1}^r-t_j^r)^\alpha}{\alpha}\Big], \\
&= \frac{g(t_{j+1}^r)}{\alpha(\alpha+1)(t_{j+1}^r-t_j^r)}\Big[\alpha(t_{n+1}^r-t_{j+1}^r)^{\alpha+1}-\alpha(t_{n+1}^r-t_j^r)^{\alpha+1} \\
&\quad -(\alpha+1)(t_{n+1}^r-t_j^r)(t_{n+1}^r-t_{j+1}^r)^\alpha+(\alpha+1)(t_{n+1}^r-t_j^r)^{\alpha+1}\Big], \\
&= \frac{g(t_{j+1}^r)}{\alpha(\alpha+1)(t_{j+1}^r-t_j^r)}\Big[\alpha(t_{n+1}^r-t_{j+1}^r)^{\alpha+1}+(t_{n+1}^r-t_j^r)^{\alpha+1} \\
&\quad -(\alpha+1)(t_{n+1}^r-t_j^r)(t_{n+1}^r-t_{j+1}^r)^\alpha\Big].
\end{aligned}
$$

Then by substituting (4.1), $t_n^r = T\left(\frac{n}{N}\right)^r$, we get,

$$
\begin{aligned}
&= \frac{g(t_{j+1}^r)}{TN^{-r}\alpha(\alpha+1)((j+1)^r - j^r)}\Bigg[\alpha T^{\alpha+1}N^{-r(\alpha+1)}\Big((n+1)^r-(j+1)^r\Big)^{\alpha+1} \\
&\quad + T^{\alpha+1}N^{-r(\alpha+1)}\Big((n+1)^r - j^r\Big)^{\alpha+1} \\
&\quad - T^{\alpha+1}N^{-r(\alpha+1)}(\alpha+1)\Big((n+1)^r - j^r\Big)\Big((n+1)^r-(j+1)^r\Big)^{\alpha}\Bigg], \\
&= \frac{g(t_{j+1}^r)T^{\alpha}N^{-r\alpha}}{\alpha(\alpha+1)((j+1)^r - j^r)}\Bigg[\alpha\Big((n+1)^r-(j+1)^r\Big)^{\alpha+1} + \Big((n+1)^r - j^r\Big)^{\alpha+1} \\
&\quad - (\alpha+1)\Big((n+1)^r - j^r\Big)\Big((n+1)^r-(j+1)^r\Big)^{\alpha}\Bigg], \\
&= \frac{g(t_{j+1}^r)T^{\alpha}N^{-r\alpha}}{\alpha(\alpha+1)((j+1)^r - j^r)}\Bigg[\Big((n+1)^r - j^r\Big)^{\alpha+1} \\
&\quad + \Big((n+1)^r-(j+1)^r\Big)^{\alpha}\Big(\alpha\Big((n+1)^r-(j+1)^r\Big) - (\alpha+1)\Big((n+1)^r - j^r\Big)\Big)\Bigg], \\
&= \frac{g(t_{j+1}^r)T^{\alpha}N^{-r\alpha}}{\alpha(\alpha+1)((j+1)^r - j^r)}\Bigg[\Big((n+1)^r - j^r\Big)^{\alpha+1} \\
&\quad + \Big((n+1)^r-(j+1)^r\Big)^{\alpha}\Big(\alpha\Big((n+1)^r-(j+1)^r\Big) - (\alpha+1)\Big((n+1)^r - j^r\Big)\Big)\Bigg], \\
&= \frac{g(t_{j+1}^r)T^{\alpha}N^{-r\alpha}}{\alpha(\alpha+1)((j+1)^r - j^r)}\Bigg[\Big((n+1)^r - j^r\Big)^{\alpha+1} + \Big((n+1)^r-(j+1)^r\Big)^{\alpha} \\
&\quad \times\Big(-(n+1)^r - \alpha(j+1)^r + \alpha j^r + j^r - (j+1)^r + (j+1)^r\Big)\Bigg], \\
&= \frac{g(t_{j+1}^r)T^{\alpha}N^{-r\alpha}}{\alpha(\alpha+1)((j+1)^r - j^r)}\Bigg[\Big((n+1)^r - j^r\Big)^{\alpha+1} - \Big((n+1)^r-(j+1)^r\Big)^{\alpha+1} \\
&\quad - (\alpha+1)\Big((n+1)^r-(j+1)^r\Big)^{\alpha}\Bigg].
\end{aligned}
$$

Therefore we will get,

$$
\begin{aligned}
\sum_{j=0}^{n}\Big[③ + ④\Big] = \frac{T^{\alpha}N^{-r\alpha}}{\alpha(\alpha+1)}\sum_{j=0}^{n}\Bigg[&g(t_j^r)\Bigg(\frac{\Big((n+1)^r - j^r\Big)^{\alpha+1} - \Big((n+1)^r-(j+1)^r\Big)^{\alpha+1}}{j^r-(j+1)^r} \\
&+ (\alpha+1)\Big((n+1)^r - j^r\Big)^{\alpha}\Bigg) \\
&+ g(t_{j+1}^r)\Bigg(\frac{\Big((n+1)^r - j^r\Big)^{\alpha+1} - \Big((n+1)^r-(j+1)^r\Big)^{\alpha+1}}{(j+1)^r-j^r} \\
&- (\alpha+1)\Big((n+1)^r-(j+1)^r\Big)^{\alpha}\Bigg)\Bigg].
\end{aligned} \tag{4.4}
$$

By separating each term of the summation, we will get,

$$= \frac{T^\alpha N^{-r\alpha}}{\alpha(\alpha+1)}\Bigg[g(t_0^r)\Big[\Big((n+1)^r-1\Big)^{\alpha+1} - (n+1)^{r(\alpha+1)} + (\alpha+1)(n+1)^{r\alpha}\Big]$$

$$+ g(t_1^r)\Bigg[\frac{\Big((n+1)^r-0^r\Big)^{\alpha+1} - \Big((n+1)^r-1^r\Big)^{\alpha+1}}{1^r-0^r} - (\alpha+1)\Big((n+1)^r-1^r\Big)^\alpha$$

$$+ \frac{\Big((n+1)^r-1^r\Big)^{\alpha+1} - \Big((n+1)^r-2^r\Big)^{\alpha+1}}{1^r-2^r} + (\alpha+1)\Big((n+1)^r-1^r\Big)^\alpha\Bigg]$$

$$+ g(t_2^r)\Bigg[\frac{\Big((n+1)^r-1^r\Big)^{\alpha+1} - \Big((n+1)^r-2^r\Big)^{\alpha+1}}{2^r-1^r} - (\alpha+1)\Big((n+1)^r-2^r\Big)^\alpha$$

$$+ \frac{\Big((n+1)^r-2^r\Big)^{\alpha+1} - \Big((n+1)^r-3^r\Big)^{\alpha+1}}{2^r-3^r} + (\alpha+1)\Big((n+1)^r-2^r\Big)^\alpha\Bigg]$$

$$+ \ldots +$$

$$+ g(t_n^r)\Bigg[\frac{\Big((n+1)^r-(n-1)^r\Big)^{\alpha+1} - \Big((n+1)^r-n^r\Big)^{\alpha+1}}{n^r-(n-1)^r}$$

$$- (\alpha+1)\Big((n+1)^r-n^r\Big)^\alpha + \frac{\Big((n+1)^r-n^r\Big)^{\alpha+1} - \Big((n+1)^r-(n+1)^r\Big)^{\alpha+1}}{n^r-(n+1)^r}$$

$$+ (\alpha+1)\Big((n+1)^r-n^r\Big)^\alpha\Bigg] + g(t_{n+1}^r)\Big((n+1)^r-n^r\Big)^\alpha.$$

Giving us the desired weights, $a_{j,n+1}^r$ (cf. [25], where $T=1$),

$$a_{j,n+1}^r = \frac{T^\alpha N^{-r\alpha}}{\alpha(\alpha+1)} \times \begin{cases} \Big([(n+1)^r-1]^{\alpha+1} - (n+1)^{r(\alpha+1)} \\ \qquad +(\alpha+1)(n+1)^{r\alpha}\Big), & \text{if } j=0, \\ \Big(\frac{[(n+1)^r-(j-1)^r]^{\alpha+1}-[(n+1)^r-j^r]^{\alpha+1}}{j^r-(j-1)^r} \\ \qquad +\frac{[(n+1)^r-(j+1)^r]^{\alpha+1}-[(n+1)^r-j^r]^{\alpha+1}}{(j+1)^r-j^r}\Big), & \text{if } 1\le j\le n, \\ \Big((n+1)^r-n^r\Big)^\alpha, & \text{if } j=n+1. \end{cases} \tag{4.5}$$

Note: If $r=1$ we will get (3.15).

By replacing (4.3) in (4.2) and the approximation $y_j^r \approx y(t_j^r)$ when $j=0,1,...,n+1$, we will obtain,

$$y_{n+1}^r = \sum_{k=0}^{\lceil\alpha\rceil-1} (y_0^r)^{(k)} \frac{(t_{n+1}^r)^k}{k!} + \frac{1}{\Gamma(\alpha)}\Bigg(\sum_{j=0}^{n} a_{j,n+1}^r f(t_j^r, y_j^r) + a_{n+1,n+1}^r f(t_{n+1}^r, y_{n+1}^r)\Bigg). \tag{4.6}$$

This is an implicit method which is difficult to solve, hence we use the predictor method which involves substituting y^r_{n+1} with $y^{r,P}_{n+1}$. To calculate the predictor, $y^{r,P}_{n+1}$, we can use linear interpolation but with the rectangle rule instead.

$$\begin{aligned}\int_0^{t^r_{n+1}} (t^r_{n+1}-s)^{\alpha-1}g(s)ds &\approx \sum_{j=0}^{n+1} g(t^r_j)\Big[\frac{1}{t^r_j - t^r_{j+1}}\int_{t^r_j}^{t^r_{j+1}} (t^r_{n+1}-s)^{\alpha-1}(s-t^r_{j+1})ds \\ &\quad + \frac{1}{t^r_{j+1}-t^r_j}\int_{t^r_j}^{t^r_{j+1}} (t^r_{n+1}-s)^{\alpha-1}(s-t^r_j)ds\Big], \\ &= \sum_{j=0}^{n} b^r_{j,n+1}g(t^r_j),\end{aligned} \tag{4.7}$$

where the weights $b^r_{j,n+1}$, can be calculated by inserting $g(t^r_{j+1}) = g(t^r_j)$ in (4.4),

$$b^r_{j,n+1} = \frac{T^\alpha N^{-r\alpha}}{\alpha}\left[\Big((n+1)^r - j^r\Big)^\alpha - \Big((n+1)^r - (j+1)^r\Big)^\alpha\right]. \tag{4.8}$$

The predictor is calculated the same as the Fractional Adams-Bashforth-Moulton method in section 3 except we are using graded meshes. So,

$$y^{r,P}_{n+1} = \sum_{k=0}^{\lceil\alpha\rceil-1}(y^r_0)^{(k)}\frac{(t^r_{n+1})^k}{k!} + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n} b^r_{j,n+1}f(t^r_j, y^r_j). \tag{4.9}$$

Giving us the following Fractional Adams-Bashforth-Moulton method with graded meshes,

$$\begin{cases} y^{r,P}_{n+1} = \sum_{k=0}^{\lceil\alpha\rceil-1}(y^r_0)^{(k)}\frac{(t^r_{n+1})^k}{k!} + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n} b^r_{j,n+1}f(t^r_j, y^r_j). \\ y^r_{n+1} = \sum_{k=0}^{\lceil\alpha\rceil-1}(y^r_0)^{(k)}\frac{(t^r_{n+1})^k}{k!} + \frac{1}{\Gamma(\alpha)}\big(\sum_{j=0}^{n} a^r_{j,n+1}f(t^r_j, y^r_j) + a^r_{n+1,n+1}f(t^r_{n+1}, y^r_{n+1})\big) \\ y^r_0,\ (y^r_0)^{(1)},\ \dots,\ (y^r_0)^{(\lceil\alpha\rceil-1)} \text{ is given.}\end{cases} \tag{4.10}$$

where the weights $a^r_{j,n+1}$ and $b^r_{j,n+1}$ are (4.5) and (4.8), respectively.

Example 2.1. If we choose $0 < \alpha \leq 1$, the predictor-corrector method (4.10), gives us,

$$\begin{cases} y^{r,P}_{n+1} = y^r_0 + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n} b^r_{j,n+1}f(t^r_j, y^r_j). \\ y^r_{n+1} = y^r_0 + \frac{1}{\Gamma(\alpha)}\big(\sum_{j=0}^{n} a^r_{j,n+1}f(t^r_j, y^r_j) + a^r_{n+1,n+1}f(t^r_{n+1}, y^r_{n+1})\big) \\ y^r_0 \text{ is given.}\end{cases} \tag{4.11}$$

Example 2.2. If we choose $1 \leq \alpha < 2$, the predictor-corrector method (4.10), gives us,

$$\begin{cases} y^{r,P}_{n+1} = y^r_0 + (y^r_0)^{(1)}\frac{t^r_{n+1}}{1!} + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n} b^r_{j,n+1}f(t^r_j, y^r_j). \\ y^r_{n+1} = y^r_0 + (y^r_0)^{(1)}\frac{t^r_{n+1}}{1!} + \frac{1}{\Gamma(\alpha)}\big(\sum_{j=0}^{n} a^r_{j,n+1}f(t^r_j, y^r_j) + a^r_{n+1,n+1}f(t^r_{n+1}, y^r_{n+1})\big) \\ y^r_0, (y^r_0)^{(1)} \text{ is given.}\end{cases} \tag{4.12}$$

3 Error Analysis for the Adams-Bashforth-Moulton Method with Graded Meshes

From theorems 4.1 and 4.2 we can see that the smoothness of one of these functions will imply non-smoothness of the other unless some special conditions are fulfilled, [25]. From [25], we will get the following results,

Assumption 3.1. Let $0 < \sigma < 1$ and let $g(t) = D_*^\alpha y(t)$ with $0 < \alpha < 2$. Then there exists a constant $c > 0$ such that,

$$|g(t)| \leq ct^\sigma, \qquad |g'(t)| \leq ct^{\sigma-1}, \qquad |g''(t)| \leq ct^{\sigma-2}.$$

Theorem 3.2. *Let $0 < \alpha < 2$ and assume that $g = D_*^\alpha y \in C^2[0,T]$ for some suitable T. Assume that y_j^r and $y(t_j^r)$ are the solutions of (3.2) and (4.10) for $0 < \alpha \leq 1$. Let $r = 1$ so we have uniform meshes. Then,*

$$\max_{0\leq j\leq N} |y(t_j) - y_j| \leq \begin{cases} CN^{-2}, & \text{if } 1 \leq \alpha < 2, \\ CN^{-(1+\alpha)}, & \text{if } 0 < \alpha \leq 1, \end{cases} \tag{4.13}$$

Lemma 3.3. Let $0 < \alpha < 2$. Assume that g satisfies assumption 3.1.
(1) If $0 < \alpha \leq 1$, then,

$$\left|\int_0^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1}(g(s) - P_1^r(s))ds\right| \leq \begin{cases} CN^{-r(\alpha+\sigma)}, & \text{if } r(\alpha+\sigma) < 2, \\ CN^{-2}ln(N), & \text{if } r(\alpha+\sigma) = 2, \\ CN^{-2}, & \text{if } r(\alpha+\sigma) > 2. \end{cases}$$

(2) If $1 \leq \alpha < 2$, then,

$$\left|\int_0^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1}(g(s) - P_1^r(s))ds\right| \leq \begin{cases} CN^{-r(1+\sigma)}, & \text{if } r(\alpha+\sigma) < 2, \\ CN^{-2}ln(N), & \text{if } r(\alpha+\sigma) = 2, \\ CN^{-2}, & \text{if } r(\alpha+\sigma) > 2. \end{cases}$$

where P_1^r is the linear interpolation,

$$P_1^r = \frac{s - t_{j+1}^r}{t_j^r - t_{j+1}^r} g(t_j^r) + \frac{s - t_j^r}{t_{j+1}^r - t_j^r} g(t_{j+1}^r). \tag{4.14}$$

Proof. Let $n = 0, 1, 2, ..., N-1$, then by separating the integral we will get,

$$\int_0^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1}(g(s) - P_1^r(s))ds = \underbrace{\int_0^{t_1^r} (t_{n+1}^r - s)^{\alpha-1}(g(s) - P_1^r(s))ds}_{I_1}$$

$$+ \underbrace{\sum_{j=1}^{n-1} \int_{t_j^r}^{t_{j+1}^r} (t_{n+1}^r - s)^{\alpha-1}(g(s) - P_1^r(s))ds}_{I_2} + \underbrace{\int_{t_n^r}^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1}(g(s) - P_1^r(s))ds}_{I_3},$$

$$= I_1 + I_2 + I_3.$$

From assumption 3.1, we know that,

$$|I_1| \leq \int_0^{t_1^r} (t_{n+1}^r - s)^{\alpha-1} |g(s)| \, ds + \int_0^{t_1^r} (t_{n+1}^r - s)^{\alpha-1} |P_1^r(s)| \, ds,$$

We can find an upper bound for P_1^r, because we know that $s \in [t_j^r, t_{j+1}^r]$ so we know that $s \leq t_{j+1}^r$, so $P_1^r \leq g(t_{j+1}^r)$. Hence for this particular interval, $[0, t_1^r]$, $P_1^r \leq g(t_1^r)$. Then with assumption 3.1, we will get,

$$\begin{aligned}
|I_1| &\leq C \int_0^{t_1^r} (t_{n+1}^r - s)^{\alpha-1} s^{\sigma} ds + C \int_0^{t_1^r} (t_{n+1}^r - s)^{\alpha-1} (t_1^r)^{\sigma} ds, \\
&\leq C(t_{n+1}^r - t_1^r)^{\alpha-1} (t_1^r)^{\sigma} \int_0^{t_1^r} \cdot \, ds + C(t_{n+1}^r - t_1^r)^{\alpha-1} (t_1^r)^{\sigma} \int_0^{t_1^r} \cdot \, ds, \\
&= C(t_{n+1}^r - t_1^r)^{\alpha-1} (t_1^r)^{\sigma+1} + C(t_{n+1}^r - t_1^r)^{\alpha-1} (t_1^r)^{\sigma+1}, \\
&\leq C(t_{n+1}^r - t_1^r)^{\alpha-1} (t_1^r)^{\sigma+1}, \\
&\leq C(t_{n+1}^r)^{\alpha-1} (t_1^r)^{\sigma+1}, \\
&\leq C(t_n^r)^{\alpha-1} (t_1^r)^{\sigma+1}, \\
&= C n^{r(\alpha-1)} N^{-r(\alpha-1)} N^{-r(\sigma+1)}, \\
&= C n^{r(\alpha-1)} N^{-r(\sigma+\alpha)}.
\end{aligned}$$

As $0 < \alpha \leq 1$ then $n^{r(\alpha-1)}$ is a decimal because the indices is negative. Hence, for $0 < \alpha \leq 1$,

$$|I_1| \leq C N^{-r(\sigma+\alpha)}. \tag{4.15}$$

For $1 \leq \alpha < 2$, we know $n \leq N$ so,

$$\begin{aligned}
|I_1| &\leq C N^{r(\alpha-1)} N^{-r(\sigma+\alpha)}, \\
&= C N^{-r(\sigma+1)}.
\end{aligned} \tag{4.16}$$

For I_2, let $\zeta_j^r \in (t_j^r, t_{j+1}^r)$, $j = 1, 2, ..., n-1$ and $n = 2, 3, ..., N-1$. Hence,

$$|I_2| = \left| \sum_{j=1}^{n-1} \int_{t_j^r}^{t_{j+1}^r} (t_{n+1}^r - s)^{\alpha-1} g''(\zeta_j^r)(s - t_j^r)(s - t_{j+1}^r) ds \right|.$$

From 3.1 and [22], section 5.2, we will get,

$$\begin{aligned}
|I_2| &\leq C \left| \sum_{j=1}^{n-1} (t_{j+1}^r - t_j^r)^2 (t_j^r)^{\sigma-2} \int_{t_j^r}^{t_{j+1}^r} (t_{n+1}^r - s)^{\alpha-1} ds \right|, \\
&= C \left| \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} (t_{j+1}^r - t_j^r)^2 (t_j^r)^{\sigma-2} \int_{t_j^r}^{t_{j+1}^r} (t_{n+1}^r - s)^{\alpha-1} ds \right|, \\
&\quad + C \left| \sum_{j=\lceil \frac{n-1}{2} \rceil}^{n-1} (t_{j+1}^r - t_j^r)^2 (t_j^r)^{\sigma-2} \int_{t_j^r}^{t_{j+1}^r} (t_{n+1}^r - s)^{\alpha-1} ds \right|, \\
&= I_{21} + I_{22},
\end{aligned}$$

where $\lceil \frac{n-1}{2} \rceil$ is the smallest integer bigger than or equal to $\frac{n-1}{2}$. We will first consider I_{21} when $0 < \alpha \leq 1$ and when $n \geq 4$. Knowing that $s \leq t_{j+1}^r$ from above, we get,

$$\begin{aligned}
I_{21} &\leq C \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} (t_{j+1}^r - t_j^r)^2 (t_j^r)^{\sigma-2} (t_{n+1}^r - t_{j+1}^r)^{\alpha-1} \int_{t_j^r}^{t_{j+1}^r} \cdot ds, \\
&= C \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} (t_{j+1}^r - t_j^r)^2 (t_j^r)^{\sigma-2} (t_{n+1}^r - t_{j+1}^r)^{\alpha-1} (t_{j+1}^r - t_j^r), \\
&= C \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} (t_{j+1}^r - t_j^r)^3 (t_j^r)^{\sigma-2} (t_{n+1}^r - t_{j+1}^r)^{\alpha-1}.
\end{aligned}$$

Note 1: Let $\zeta_j \in [j, j+1]$, $j = 1, 2, ..., \lceil \frac{n-1}{2} \rceil - 1$, we have,

$$t_{j+1}^r - t_j^r = \frac{(j+1)^r - j^r}{N^r},$$

then from the Mean Value Theorem and the fact that $\zeta_j \leq j+1$,

$$\begin{aligned}
t_{j+1}^r - t_j^r &= \frac{r\zeta_j^{r-1}}{N^r}, \\
&\leq \frac{r(j+1)^{r-1}}{N^r}, \\
&\leq \frac{Cj^{r-1}}{N^r}.
\end{aligned} \tag{4.17}$$

Note 2: We know that $j \leq \lceil \frac{n-1}{2} \rceil - 1$, hence,

$$\begin{aligned}
(t_{n+1}^r - t_{j+1}^r)^{\alpha-1} &= \left(\frac{N^r}{(n+1)^r - (j+1)^r} \right)^{1-\alpha}, \\
&\leq \left(\frac{N^r}{(n+1)^r - \lceil \frac{n-1}{2} \rceil^r} \right)^{1-\alpha},
\end{aligned}$$

and we also know that $\lceil \frac{n-1}{2} \rceil \leq \lceil \frac{n+1}{2} \rceil$, so,

$$\begin{aligned}
(t_{n+1}^r - t_{j+1}^r)^{\alpha-1} &\leq \left(\frac{N^r}{(n+1)^r - \lceil \frac{n+1}{2} \rceil^r} \right)^{1-\alpha}, \\
&\leq \left(\frac{N^r}{(n+1)^r} \right)^{1-\alpha}, \\
&\leq C \left(\frac{N^r}{(n+1)^r} \right)^{1-\alpha}, \\
&\leq C \left(\frac{N^r}{n^r} \right)^{1-\alpha}.
\end{aligned} \tag{4.18}$$

From (4.17) and (4.18), we get,

$$
\begin{aligned}
I_{21} &\leq C \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil -1} \left(\frac{j^{r-1}}{N^r}\right)^3 \left(\frac{j}{N}\right)^{r(\sigma-2)} \left(\frac{N}{n}\right)^{r(1-\alpha)}, \\
&= C \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil -1} j^{r-3+r\sigma+(r\alpha-r\alpha)} N^{-r(\sigma+\alpha)} n^{-r(1-\alpha)}, \\
&= C \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil -1} j^{r(\sigma+\alpha)-3} N^{-r(\sigma+\alpha)} \left(\frac{j}{n}\right)^{r(1-\alpha)},
\end{aligned}
$$

For $\left(\frac{j}{n}\right)^{r(1-\alpha)}$, we know $\alpha \leq 1$, so $\left(\frac{j}{n}\right)^{r(1-\alpha)} \leq 1$, so,

$$
I_{21} \leq CN^{-r(\sigma+\alpha)} \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil -1} j^{r(\sigma+\alpha)-3}.
$$

Case 1: If $r(\sigma+\alpha) < 2$, we know $j \leq n-1$, and $n-1 \leq n$, giving us

$$
\begin{aligned}
I_{21} &\leq CN^{-r(\sigma+\alpha)} \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil -1} j^{r(\sigma+\alpha)-3}, \\
&\leq CN^{-r(\sigma+\alpha)} n^{r(\sigma+\alpha)-3}.
\end{aligned}
$$

From $r(\sigma+\alpha) < 2$ and $r(\sigma+\alpha)-3 < -1$, $N^{-r(\sigma+\alpha)}$ and $n^{r(\sigma+\alpha)-3}$ is a fraction. Giving us,

$$
I_{21} \leq CN^{-r(\sigma+\alpha)}.
$$

Case 2: If $r(\sigma+\alpha) = 2$,

$$
\begin{aligned}
I_{21} \leq &CN^{-r(\sigma+\alpha)} \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil -1} j^{r(\sigma+\alpha)-3}, \\
&\leq CN^{-2} \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil -1} j^{-1}, \\
&\leq CN^{-2} \left(1 + \frac{1}{2} + ... + \frac{1}{N}\right).
\end{aligned}
$$

We know that $1 + \frac{1}{2} + ... + \frac{1}{N} \leq \int_1^N \frac{1}{x} dx = ln(N)$. Hence,

$$
I_{21} \leq CN^{-2} ln(N).
$$

Case 3: If $r(\sigma+\alpha) > 2$,

$$
\begin{aligned}
I_{21} &\leq CN^{-r(\sigma+\alpha)} \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil -1} j^{r(\sigma+\alpha)-3}, \\
&= CN^{-r(\sigma+\alpha)} \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil -1} \frac{j^{r(\sigma+\alpha)-3}}{n^{r(\sigma+\alpha)-3}} \times \frac{n^{r(\sigma+\alpha)-2}}{n}.
\end{aligned}
$$

We know that $j \leq n-1$, $n \leq N-1$ and $\frac{b-1}{b} \leq 1$, where $b = n,\ N$. Hence,

$$\begin{aligned}
I_{21} &\leq CN^{-r(\sigma+\alpha)} \frac{(n-1)^{r(\sigma+\alpha)-3}}{n^{r(\sigma+\alpha)-3}} \times \frac{n^{r(\sigma+\alpha)-2}}{n}, \\
&\leq CN^{-r(\sigma+\alpha)} \frac{n^{r(\sigma+\alpha)-2}}{n}, \\
&\leq CN^{-r(\sigma+\alpha)} n^{r(\sigma+\alpha)-2}, \\
&= C\left(\frac{n}{N}\right)^{r(\sigma+\alpha)-2} N^{-2}, \\
&\leq C\left(\frac{N-1}{N}\right)^{r(\sigma+\alpha)-2} N^{-2}, \\
&\leq CN^{-2}.
\end{aligned}$$

Therefore, when $0 < \alpha \leq 1$, we get,

$$I_{21} \leq \begin{cases} CN^{-r(\alpha+\sigma)}, & \text{if } r(\alpha+\sigma) < 2, \\ CN^{-2} ln(N), & \text{if } r(\alpha+\sigma) = 2, \\ CN^{-2}, & \text{if } r(\alpha+\sigma) > 2. \end{cases}$$

We can now find I_{21} when $1 \leq \alpha < 2$. Hence,

$$\begin{aligned}
I_{21} &\leq C \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} (t_{j+1}^r - t_j^r)^3 (t_j^r)^{\sigma-2} (t_{n+1}^r - t_{j+1}^r)^{\alpha-1}, \\
&\leq C \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} (t_{j+1}^r - t_j^r)^3 (t_j^r)^{\sigma-2} (t_{n+1}^r)^{\alpha-1}.
\end{aligned}$$

From (4.17), we get,

$$I_{21} \leq C \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} \left(\frac{j^{r-1}}{N^r}\right)^3 \left(\frac{j}{N}\right)^{r(\sigma-2)} \left(\frac{n+1}{N}\right)^{r(\alpha-1)}.$$

We know that $n \leq N-1$, so,

$$\begin{aligned}
I_{21} &\leq C \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} \left(\frac{j^{r-1}}{N^r}\right)^3 \left(\frac{j}{N}\right)^{r(\sigma-2)}, \\
&= CN^{-r(1+\sigma)} \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} j^{r(1+\sigma)-3}.
\end{aligned}$$

Using the same idea in the previous cases you will find when $1 \leq \alpha < 2$,

$$I_{21} \leq \begin{cases} CN^{-r(1+\sigma)}, & \text{if } r(1+\sigma) < 2, \\ CN^{-2} ln(N), & \text{if } r(1+\sigma) = 2, \\ CN^{-2}, & \text{if } r(1+\sigma) > 2. \end{cases}$$

Note: We know $j \le n-1$ and $n \le N-1$, hence,

$$(t_j^r)^{\sigma-2} = \left(\frac{j}{N}\right)^{r(\sigma-2)} = \left(\frac{N}{j}\right)^{r(2-\sigma)} \le C\left(\frac{N}{n}\right)^{r(2-\sigma)}.$$

From (4.17), we can now calculate I_{22} like so,

$$\begin{aligned}
I_{22} &\le C \sum_{j=\lceil \frac{n-1}{2} \rceil}^{n-1} (t_{j+1}^r - t_j^r)^2 (t_j^r)^{\sigma-2} \int_{t_j^r}^{t_{j+1}^r} (t_{n+1}^r - s)^{\alpha-1} ds, \\
&\le C \sum_{j=\lceil \frac{n-1}{2} \rceil}^{n-1} \left(\frac{n^{r-1}}{N^{-r}}\right)^2 \left(\frac{N}{n}\right)^{r(\sigma-2)} \int_{t_j^r}^{t_{j+1}^r} (t_{n+1}^r - s)^{\alpha-1} ds, \\
&\le C n^{r\sigma-2} N^{-r\sigma} \int_{t_{\lceil \frac{n-1}{2} \rceil}^r}^{t_n^r} (t_{n+1}^r - s)^{\alpha-1} ds, \\
&= -C n^{r\sigma-2} N^{-r\sigma} \frac{(t_{n+1}^r - s)^\alpha}{\alpha} \Bigg|_{t_{\lceil \frac{n-1}{2} \rceil}^r}^{t_n^r}, \qquad (4.19) \\
&= C n^{r\sigma-2} N^{-r\sigma} \frac{(t_{n+1}^r - t_{\lceil \frac{n-1}{2} \rceil}^r)^\alpha - (t_{n+1}^r - t_n^r)^\alpha}{\alpha}, \\
&\le C n^{r\sigma-2} N^{-r\sigma} \frac{(t_{n+1}^r - t_{\lceil \frac{n-1}{2} \rceil}^r)^\alpha}{\alpha}, \\
&\le C n^{r\sigma-2} N^{-r\sigma} \frac{(t_{n+1}^r)^\alpha}{\alpha}, \\
&= C n^{r\sigma-2} N^{-r\sigma} \frac{(n+1)^{r\alpha}}{N^{r\alpha}\alpha},
\end{aligned}$$

knowing that $\frac{n+1}{n} = 1 + \frac{1}{n} \le 2$, we will get $n+1 \le 2n$, hence,

$$\begin{aligned}
I_{22} &\le C n^{r\sigma-2} N^{-r\sigma} \frac{n^{r\alpha}}{N^{r\alpha}}, \\
&= C n^{r(\sigma+\alpha)-2} N^{-r(\sigma+\alpha)},
\end{aligned}$$

when $n \ge 2$ and $0 < \alpha < 2$.

Case 1: If $r(\sigma+\alpha) < 2$, then $r(\sigma+\alpha) - 2 < 0$, hence $n^{r(\sigma+\alpha)-2}$ is a fraction, so,

$$\begin{aligned}
I_{22} &\le C n^{r(\sigma+\alpha)-2} N^{-r(\sigma+\alpha)}, \\
&\le C N^{-r(\sigma+\alpha)}.
\end{aligned}$$

Case 2: If $r(\sigma+\alpha) \ge 2$, we know that $n \le N-1$ and $\frac{N-1}{N} < 1$, so,

$$\begin{aligned}
I_{22} &\le C n^{r(\sigma+\alpha)-2} N^{-r(\sigma+\alpha)} = C \frac{n}{N}^{r(\sigma+\alpha)-2} N^{-2}, \\
&\le C \left(\frac{N-1}{N}\right)^{r(\sigma+\alpha)-2} N^{-2} \\
&\le C N^{-2}.
\end{aligned}$$

For I_3, we have $\zeta_n \in (t_n^r, t_{n+1}^r)$, $n = 1, 2, ..., N-1$,

$$
\begin{aligned}
|I_3| &= \left| \int_{t_n^r}^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1} (g(s) - P_1^r(s)) ds \right|, \\
&= \left| \int_{t_n^r}^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1} g''(\zeta_n^r)(s - t_n^r)(s - t_{n+1}^r) ds \right|.
\end{aligned}
$$

By assumption 3.1 and (4.17), when $0 < \alpha < 2$,

$$
\begin{aligned}
|I_3| &\le \left| \int_{t_n^r}^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1} g''(\zeta_n^r)(s - t_n^r)(s - t_{n+1}^r) ds \right|, \\
&\le C(t_{n+1}^r - t_n^r)^2 (t_j^r)^{\sigma-2} \int_{t_n^r}^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1} ds, \\
&= C(t_{n+1}^r - t_n^r)^2 (t_j^r)^{\sigma-2} \frac{1}{\alpha} (t_{n+1}^r - t_n^r)^{\alpha}, \\
&= C(t_{n+1}^r - t_n^r)^{2+\alpha} (t_j^r)^{\sigma-2}, \\
&\le C(n^{r-1} N^{-r})^{2+\alpha} \left(\frac{n}{N} \right)^{r(\sigma-2)}, \\
&= C n^{r(\sigma+\alpha)-(2+\alpha)} N^{-r(\alpha+\sigma)}.
\end{aligned}
$$

Case 1: If $r(\sigma+\alpha) < 2+\alpha$, then $r(\sigma+\alpha) - (2+\alpha) < 0$, hence $n^{r(\sigma+\alpha)-(2+\alpha)}$ is a fraction, so,

$$
\begin{aligned}
I_{22} &\le C n^{r(\sigma+\alpha)-(2+\alpha)} N^{-r(\sigma+\alpha)}, \\
&\le C N^{-r(\sigma+\alpha)}.
\end{aligned}
$$

Case 2: If $r(\sigma+\alpha) \ge 2+\alpha$, we know that $n \le N-1$ and $\frac{N-1}{N} < 1$, so,

$$
\begin{aligned}
I_{22} &\le C n^{r(\sigma+\alpha)-(2+\alpha)} N^{-r(\sigma+\alpha)}, \\
&= C \frac{n}{N}^{r(\sigma+\alpha)-(2+\alpha)} N^{-(2+\alpha)}, \\
&\le C \left(\frac{N-1}{N} \right)^{r(\sigma+\alpha)-(2+\alpha)} N^{-(2+\alpha)}, \\
&\le C N^{-(2+\alpha)}.
\end{aligned}
$$

From [25], the bound for I_3 is stronger than I_{21}. Giving us the results we were trying to prove for lemma 3.3. □

Lemma 3.4. Let $\alpha > 0$. We have,

(1) $a_{j,n+1}^r > 0$, $j = 0, 1, 2, ..., n+1$, where $a_{j,n+1}^r$ are the weights defined in (4.5).

(2) $b_{j,n+1}^r > 0$, $j = 0, 1, 2, ..., n$, where $b_{j,n+1}^r$ are the weights defined in (4.8).

Proof. We know $\alpha > 0$, $r \ge 0$, $n \ge 0$ and $\frac{N^{-r\alpha}}{\alpha(1+\alpha)} > 0$. Hence, $a_{0,n+1}^r > 0$. We also know that $n+1 > n$ so $(n+1)^r - n^r$ is always positive. Hence, $a_{n+1,n+1}^r > 0$. For $a_{j,n+1}^r$ when $1 \le j \le n$,

$$a_{j,n+1}^r = \frac{N^{-r\alpha}}{\alpha(1+\alpha)}\Bigg(\frac{[(n+1)^r-(j-1)^r]^{\alpha+1}-[(n+1)^r-j^r]^{\alpha+1}}{j^r-(j-1)^r} + \frac{[(n+1)^r-(j+1)^r]^{\alpha+1}-[(n+1)^r-j^r]^{\alpha+1}}{(j+1)^r-j^r}\Bigg).$$

We know that $j^r > (j-1)^r$, so $j^r-(j-1)^r > 0$ and $(j+1)^r > j^r$, so $(j+1)^r - j^r > 0$ when $r \geq 1$. Hence, both the numerators and denominators in the parenthesis are bigger than 0. So, $a_{j,n+1}^r > 0$ when $1 \leq j \leq n$.

$$b_{j,n+1}^r = \frac{T^\alpha N^{-r\alpha}}{\alpha}\Bigg[\Big((n+1)^r - j^r\Big)^\alpha - \Big((n+1)^r-(j+1)^r\Big)^\alpha\Bigg].$$

As before, we know that $(j+1)^r > j^r$ hence $b_{j,n+1}^r$ is also positive. □

Lemma 3.5. Let $\alpha > 0$. We have, with $n = 0, 1, 2, ..., N-1$,

$$a_{n+1,n+1}^r \leq CN^{-r\alpha} n^{(r-1)\alpha},$$

where $a_{n+1,n+1}^r$ is defined in (4.5).

Proof. Let $\zeta_n \in (n, n+1)$ and we know $n+1 \leq 2n$. We have,

$$\begin{aligned} a_{n+1,n+1}^r &= \frac{T^\alpha N^{-r\alpha}}{\alpha(\alpha+1)}\Big((n+1)^r - n^r\Big)^\alpha, \\ &= CN^{-r\alpha}\Big((n+1)^r-n^r\Big)^\alpha, \\ &= CN^{-r\alpha}\Big(r\zeta_n^{r-1}\Big)^\alpha, \\ &\leq CN^{-r\alpha}\Big(r(n+1)^{r-1}\Big)^\alpha, \\ &\leq CN^{-r\alpha}n^{(r-1)\alpha}. \end{aligned}$$

□

Lemma 3.6. Let $0 < \alpha < 2$. Assume that g satisfies assumption 3.1.
(1) If $0 < \alpha \leq 1$, then,

$$\Bigg|a_{n+1,n+1}^r \int_0^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1}(g(s) - P_0^r(s))ds\Bigg| \leq \begin{cases} CN^{-r(\alpha+\sigma)}, & \text{if } r(\alpha+\sigma) < 1+\alpha, \\ CN^{-r(\alpha+\sigma)} ln(N), & \text{if } r(\alpha+\sigma) = 1+\alpha, \\ CN^{-1-\alpha}, & \text{if } r(\alpha+\sigma) > 1+\alpha. \end{cases}$$

(2) If $1 \leq \alpha < 2$, then,

$$\Bigg|a_{n+1,n+1}^r \int_0^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1}(g(s) - P_0^r(s))ds\Bigg| \leq \begin{cases} CN^{-r(\alpha+\sigma)}, & \text{if } r(\alpha+\sigma) < 1+\alpha, \\ CN^{-1-\alpha}, & \text{if } r(\alpha+\sigma) \geq 1+\alpha. \end{cases}$$

where P_0^r is the linear interpolation,

$$P_0^r = g(t_j^r)\left(\frac{s-t_{j+1}^r}{t_j^r - t_{j+1}^r} + \frac{s-t_j^r}{t_{j+1}^r - t_j^r}\right). \tag{4.20}$$

Proof. We can separate the integral,

$$\begin{aligned}
&a_{n+1,n+1}^r \int_0^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1}(g(s) - P_0^r(s))ds \\
&= a_{n+1,n+1}^r \left(\int_0^{t_1^r} + \sum_{j=1}^{n-1}\int_{t_j^r}^{t_{j+1}^r} + \int_{t_n^r}^{t_{n+1}^r}\right)(t_{n+1}^r - s)^{\alpha-1}(g(s) - P_0^r(s))ds, \\
&= I_1' + I_2' + I_3'.
\end{aligned}$$

By assumption 3.1 and lemma 3.5, we have,

$$|I_1'| \le a_{n+1,n+1}^r \left(\int_0^{t_1^r}(t_{n+1}^r - s)^{\alpha-1}|g(s)|ds + \int_0^{t_1^r}(t_{n+1}^r - s)^{\alpha-1}|P_0^r(s)|ds\right).$$

This is similar to lemma 3.3, except $P_0^r \le g(t_j^r)$ and for this particular interval, $[0, t_1^r]$, $P_0^r \le g(t_0^r) = g(0)$. Giving us,

$$\begin{aligned}
|I_1'| &\le \left(CN^{-r\alpha}n^{(r-1)\alpha}\right)\left(\int_0^{t_1^r}(t_{n+1}^r - s)^{\alpha-1}s^\sigma ds + \int_0^{t_1^r}(t_{n+1}^r - s)^{\alpha-1}0^\sigma ds\right), \\
&= (CN^{-r\alpha}n^{(r-1)\alpha})\int_0^{t_1^r}(t_{n+1}^r - s)^{\alpha-1}s^\sigma ds.
\end{aligned}$$

If $0 < \alpha \le 1$, we know that $s \le t_1^r$, $n \le N-1$ and from (4.15), we will get,

$$\begin{aligned}
|I_1'| &\le (CN^{-r\alpha}n^{(r-1)\alpha})(t_{n+1}^r - t_1^r)^{\alpha-1}(t_1^r)^{\sigma+1} \\
&\le (CN^{-r\alpha}n^{(r-1)\alpha})(CN^{-r(\alpha+\sigma)}), \\
&= CN^{-r(\alpha+\sigma)}\left(\frac{n}{N}\right)^{r\alpha} n^{-\alpha}, \\
&\le CN^{-r(\alpha+\sigma)}\left(\frac{N-1}{N}\right)^{r\alpha} n^{-\alpha}, \\
&\le CN^{-r(\alpha+\sigma)}.
\end{aligned}$$

If $1 \le \alpha < 2$, from (**??**), we have,

$$\begin{aligned}
|I_1'| &\le (CN^{-r\alpha}n^{(r-1)\alpha})(t_{n+1}^r - t_1^r)^{\alpha-1}(t_1^r)^{\sigma+1} \\
&\le (CN^{-r\alpha}n^{(r-1)\alpha})(CN^{-r(1+\sigma)}), \\
&= CN^{-r(1+\sigma)}\left(\frac{n}{N}\right)^{\alpha(r-1)} N^{-\alpha}, \\
&\le CN^{-r(1+\sigma)}\left(\frac{N-1}{N}\right)^{\alpha(r-1)} N^{-\alpha}, \\
&\le CN^{-r(1+\sigma)-\alpha}, \\
&\le CN^{-1-\alpha}.
\end{aligned}$$

For I_2', we have $\zeta_j^r \in (t_j^r, t_{j+1}^r)$, $j = 1, 2, ..., n-1$,

$$|I_2'| \leq a_{n+1,n+1}^r \sum_{j=1}^{n-1} \int_{t_j^r}^{t_{j+1}^r} (t_{n+1}^r - s)^{\alpha-1} |f'(\zeta_j^r)| (s - t_j^r) ds.$$

From assumption 3.1, and the fact $s \leq t_{j+1}^r$, we will get,

$$\begin{aligned}
|I_2'| \leq a_{n+1,n+1}^r &\sum_{j=1}^{n-1} \int_{t_j^r}^{t_{j+1}^r} (t_{n+1}^r - s)^{\alpha-1} |f'(\zeta_j^r)| (s - t_j^r) ds, \\
&\leq C a_{n+1,n+1}^r \sum_{j=1}^{n-1} (t_{j+1}^r - t_j^r)(t_j^r)^{\sigma-1}(t_{n+1}^r - t_{j+1}^r)^{\alpha-1}(t_{j+1}^r - t_j^r), \\
&= C a_{n+1,n+1}^r \sum_{j=1}^{n-1} (t_{j+1}^r - t_j^r)^2 (t_j^r)^{\sigma-1}(t_{n+1}^r - t_{j+1}^r)^{\alpha-1}, \\
&= C a_{n+1,n+1}^r \left(\sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} + \sum_{j=\lceil \frac{n-1}{2} \rceil}^{n-1} \right) (t_{j+1}^r - t_j^r)^2 (t_j^r)^{\sigma-1}(t_{n+1}^r - t_{j+1}^r)^{\alpha-1}, \\
&= I_{21}' + I_{22}'.
\end{aligned}$$

If $0 < \alpha \leq 1$, and by using lemma 3.5, (4.17) and (4.17), we get for I_{21}', when $n \geq 4$,

$$\begin{aligned}
I_{21}' &\leq C(N^{-r\alpha} n^{(r-1)\alpha}) \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} (t_{j+1}^r - t_j^r)^2 (t_j^r)^{\sigma-1}(t_{n+1}^r - t_{j+1}^r)^{\alpha-1}, \\
&\leq C(N^{-r\alpha} n^{(r-1)\alpha}) \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} (j^{r-1} N^{-r})^2 \left(\frac{j}{N}\right)^{r(\sigma-1)} \left(\frac{N}{n}\right)^{r(1-\alpha)}, \\
&= C \left(\frac{n}{N}\right)^{r\alpha} \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} j^{r(\alpha+\sigma)-(2+\alpha)} \left(\frac{j}{n}\right)^{\alpha} \left(\frac{j}{N}\right)^{r(1-\alpha)} N^{-r(\alpha+\sigma)}.
\end{aligned}$$

We know that $j \leq n-1$ and $n \leq N-1$, so,

$$\begin{aligned}
I_{21}' &\leq C N^{-r(\alpha+\sigma)} \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} j^{r(\alpha+\sigma)-(2+\alpha)}, \\
&\leq \begin{cases} C N^{-r(\alpha+\sigma)}, & \text{if } r(\alpha+\sigma) < 1+\alpha, \\ C N^{-r(\alpha+\sigma)} ln(N), & \text{if } r(\alpha+\sigma) = 1+\alpha, \\ C N^{-1-\alpha}, & \text{if } r(\alpha+\sigma) > 1+\alpha. \end{cases}
\end{aligned}$$

For $1 \leq \alpha < 2$,

$$
\begin{aligned}
I'_{21} &\leq C(N^{-r\alpha} n^{(r-1)\alpha}) \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} (t^r_{j+1} - t^r_j)^2 (t^r_j)^{\sigma-1} (t^r_{n+1})^{\alpha-1}, \\
&\leq C(N^{-r\alpha} n^{(r-1)\alpha}) \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} (j^{r-1} N^{-r})^2 \left(\frac{j}{N}\right)^{r(\sigma-1)} \left(\frac{N}{n}\right)^{r(1-\alpha)}, \\
&= C\left(\frac{n}{N}\right)^{(r-1)\alpha} N^{-\alpha} N^{-r(\sigma+1)} \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} j^{r(1+\sigma)-2}, \\
&\leq C N^{-r(\sigma+1)-\alpha} \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} j^{r(1+\sigma)-2}.
\end{aligned}
$$

Note: We know $r \geq 1$ and $0 < \sigma < 1$. So, $r(1+\sigma) - 2 > -1$. Giving us,

$$
\begin{aligned}
I'_{21} &\leq C N^{-r(\sigma+1)-\alpha} N^{1-1} \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} j^{r(1+\sigma)-2} j^{1-1}, \\
&= C N^{-r(\sigma+1)+1} N^{-\alpha-1} \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} j^{r(1+\sigma)-2+1} j^{-1}, \\
&= C N^{-\alpha-1} \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil - 1} \left(\frac{j}{N}\right)^{r(\sigma+1)-2+1} j^{-1}, \\
&\leq C N^{-\alpha-1}.
\end{aligned}
$$

For I'_{22}, we have,

$$
I'_{22} \leq (C N^{-r\alpha} n^{(r-1)\alpha}) \sum_{j=\lceil \frac{n-1}{2} \rceil}^{n-1} (t^r_{j+1} - t^r_j)(t^r_j)^{\sigma-1} \int_{t^r_j}^{t^r_{j+1}} (t^r_{n+1} - s)^{\alpha-1} ds,
$$

Note: We know $j \leq n-1$ and $n \leq N-1$, hence,

$$
(t^r_j)^{\sigma-1} = \left(\frac{j}{N}\right)^{r(\sigma-1)} = \left(\frac{N}{j}\right)^{r(1-\sigma)} \leq C\left(\frac{N}{n}\right)^{r(1-\sigma)}.
$$

By (4.17) and (4.19), when $0 < \alpha < 2$,

$$
\begin{aligned}
I'_{22} &\leq (C N^{-r\alpha} n^{(r-1)\alpha}) \sum_{j=\lceil \frac{n-1}{2} \rceil}^{n-1} (C n^{r-1} N^{-r})(N/n)^{r(1-\sigma)} \int_{t^r_j}^{t^r_{j+1}} (t^r_{n+1} - s)^{\alpha-1} ds, \\
&\leq (C N^{-r\alpha} n^{(r-1)\alpha})(C n^{r-1} N^{-r})(N/n)^{r(1-\sigma)} C(n/N)^{r\alpha}, \\
&= C N^{-r\alpha} n^{(r-1)\alpha} N^{-r\sigma} n^{-1+r\sigma} C(n/N)^{r\alpha}, \\
&= C n^{r(\alpha+\sigma)-\alpha-1} N^{-r(\alpha+\sigma)}, \\
&\leq \begin{cases} C N^{-r(\alpha+\sigma)}, & \text{if } r(\alpha+\sigma) < 1+\alpha, \\ C N^{-1-\alpha}, & \text{if } r(\alpha+\sigma) \geq 1+\alpha. \end{cases}
\end{aligned}
$$

For I'_3, when $0 < \alpha < 2$ and using (4.17),

$$\begin{aligned}
|I'_3| &\le C(N^{-r\alpha} n^{(r-1)\alpha})(t^r_{n+1} - t^r_n)(t^r_n)^{\sigma-1}(t^r_{n+1} - t^r_n)^{\alpha}, \\
&= C(N^{-r\alpha} n^{(r-1)\alpha})(t^r_n)^{\sigma-1}(t^r_{n+1} - t^r_n)^{\alpha+1}, \\
&\le C(N^{-r\alpha} n^{(r-1)\alpha})(n/N)^{r(\sigma-1)}(n^{r-1}N^{-r})^{\alpha+1}, \\
&= C(n/N)^{r\alpha} n^{-\alpha} n^{r(\alpha+\sigma)-\alpha-1} N^{-r(\alpha+\sigma)}, \\
&\le C n^{r(\alpha+\sigma)-\alpha-1} N^{-r(\alpha+\sigma)}, \\
&\le \begin{cases} CN^{-r(\alpha+\sigma)}, & \text{if } r(\alpha+\sigma) < 1+\alpha, \\ CN^{-1-\alpha}, & \text{if } r(\alpha+\sigma) \ge 1+\alpha. \end{cases}
\end{aligned}$$

Giving us the required estimates. □

Lemma 3.7. Let $\alpha > 0$. There exists a positive constant C such that,

$$\sum_{j=0}^{n} a^r_{j,n+1} \le CT^{\alpha} \tag{4.21}$$

$$\sum_{j=0}^{n} b^r_{j,n+1} \le CT^{\alpha} \tag{4.22}$$

where $a^r_{j,n+1}$ and $b^r_{j,n+1}$, $j = 0, 1, 2, ..., n$, are defined by (4.5) and (4.8), respectively.

Proof. We know,

$$\int_0^{t^r_{n+1}} (t^r_{n+1} - s)^{\alpha-1} g(s) ds \approx \sum_{j=0}^{n+1} a^r_{j,n+1} g(t_j),$$

this can also be viewed as, from [25],

$$\int_0^{t^r_{n+1}} (t^r_{n+1} - s)^{\alpha-1} g(s) ds = \sum_{j=0}^{n+1} a^r_{j,n+1} g(t_j) + R_1,$$

where R_1 is the remainder term. We know,

$$\sum_{j=0}^{n+1} a^r_{j,n+1} = \int_0^{t^r_{n+1}} (t^r_{n+1} - s)^{\alpha-1} \cdot 1 ds = \frac{1}{\alpha}(t^r_{n+1})^{\alpha} = \frac{1}{\alpha} T^{\alpha} \left(\frac{n}{N}\right)^{r\alpha} \le CT^{\alpha},$$

as $n \le N$. We also know from lemma 3.4 that $a^r_{n+1,n+1} > 0$. Hence,

$$\sum_{j=0}^{n} a^r_{j,n+1} \le CT^{\alpha} - a^r_{n+1,n+1} \le CT^{\alpha}.$$

We can use a similar technique to find (4.22). □

By using all the lemma's above we can now prove the main theorem from [25].

Theorem 3.8. *Let $0 < \alpha < 2$ and assume that $g := D_*^\alpha y$ satisfies assumption 3.1.*
(1) If $0 < \alpha \leq 1$, assume that y_j^r and $y(t_j^r)$ are the solutions of (3.2) and (4.10), respectively, then we have,

$$\max_{0 \leq j \leq N} |y(t_j^r) - y_j^r| \leq \begin{cases} CN^{-r(\alpha+\sigma)}, & \text{if } r(\alpha+\sigma) < 1+\alpha, \\ CN^{-r(\alpha+\sigma)} ln(N), & \text{if } r(\alpha+\sigma) = 1+\alpha, \\ CN^{-1-\alpha}, & \text{if } r(\alpha+\sigma) > 1+\alpha. \end{cases}$$

(2) If $1 \leq \alpha < 2$, assume that y_j^r and $y(t_j^r)$ are the solutions of (3.2) and (4.10), respectively, then we have,

$$\max_{0 \leq j \leq N} |y(t_j^r) - y_j^r| \leq \begin{cases} CN^{-r(1+\sigma)}, & \text{if } r(\alpha+\sigma) < 2, \\ CN^{-2} ln(N), & \text{if } r(\alpha+\sigma) = 2, \\ CN^{-2}, & \text{if } r(\alpha+\sigma) > 2. \end{cases}$$

Proof. By using (4.2), we have,

$$y(t_{n+1}^r) = y_0^r + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1} g(s) ds, \qquad \text{for } 0 < \alpha \leq 1, \tag{4.23}$$

and

$$y(t_{n+1}^r) = y_0^r + (y_0^r)^{(1)} \frac{t_{n+1}^r}{1!} + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1} g(s) ds, \quad \text{for } 1 \leq \alpha < 2. \tag{4.24}$$

By subtracting (4.23) from (4.11) for $0 < \alpha \leq 1$ or subtracting (4.24) from (4.12) for $1 \leq \alpha < 2$, we have,

$$\begin{aligned} y(t_{n+1}^r) - y_{n+1}^r = & \frac{1}{\Gamma(\alpha)} \Bigg[\int_0^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1} \big(f(s, y(s)) - P_1^r(s) \big) ds \\ & + \sum_{j=0}^{n} a_{j,n+1}^r \big(f(t_j^r, y(t_j^r)) - f(t_j^r, y_j^r) \big) \\ & + a_{n+1,n+1}^r \big(f(t_{n+1}^r, y(t_{n+1}^r)) - f(t_{n+1}^r, y_{n+1}^{r,P}) \big) \Bigg], \\ = & \frac{1}{\Gamma(\alpha)} (I + II + III). \end{aligned}$$

We already know I from lemma 3.3. We can find II by lemma 3.4 and the Lipschitz condition of f,

$$\begin{aligned} |II| = & \left| \sum_{j=0}^{n} a_{j,n+1}^r \big(f(t_j^r, y(t_j^r)) - f(t_j^r, y_j^r) \big) \right| \leq \sum_{j=0}^{n} a_{j,n+1}^r \left| f(t_j^r, y(t_j^r)) - f(t_j^r, y_j^r) \right|, \\ \leq & L \sum_{j=0}^{n} a_{j,n+1}^r \left| y(t_j^r) - y_j^r \right|. \end{aligned}$$

We can also find III by lemma 3.4 and the Lipschitz condition of f,

$$|III| = \left|a^r_{n+1,n+1}\big(f(t^r_j, y(t^r_{n+1})) - f(t^r_j, y^{r,P}_{n+1})\big)\right| \le a^r_{n+1,n+1}\left|f(t^r_j, y(t^r_{n+1})) - f(t^r_j, y^{r,P}_{n+1})\right|,$$
$$\le La^r_{n+1,n+1}\left|y(t^r_{n+1}) - y^{r,P}_{n+1}\right|.$$

We have by (4.23) and (4.9),

$$y(t^r_{n+1}) - y^{r,P}_{n+1} = \frac{1}{\Gamma(\alpha)}\bigg[\int_0^{t^r_{n+1}}(t^r_{n+1}-s)^{\alpha-1}\big(f(s,y(s)) - P^r_0(s)\big)ds$$
$$+ \sum_{j=0}^{n} b^r_{j,n+1}\big(f(t^r_j, y(t^r_j)) - f(t^r_j, y^r_j)\big)\bigg],$$

Thus,

$$|III| \le CLa^r_{n+1,n+1}\bigg|\int_0^{t^r_{n+1}}(t^r_{n+1}-s)^{\alpha-1}\big(f(s,y(s)) - P^r_0(s)\big)ds$$
$$+ \sum_{j=0}^{n} b^r_{j,n+1}\big(f(t^r_j, y(t^r_j)) - f(t^r_j, y^r_j)\big)\bigg|,$$
$$\le CLa^r_{n+1,n+1}\int_0^{t^r_{n+1}}(t^r_{n+1}-s)^{\alpha-1}\big|f(s,y(s)) - P^r_0(s)\big|ds$$
$$+ CLa^r_{n+1,n+1}\sum_{j=0}^{n} b^r_{j,n+1}\big|f(t^r_j, y(t^r_j)) - f(t^r_j, y^r_j)\big|,$$
$$= III_1 + III_2.$$

We already know III_1 by lemma 3.6. For III_2, we can use lemmas 3.4 and 3.5,

$$III_2 \le Ca^r_{n+1,n+1}\sum_{j=0}^{n} b^r_{j,n+1}\big|f(t^r_j, y(t^r_j)) - f(t^r_j, y^r_j)\big|,$$
$$\le CL(CN^{-r\alpha}n^{(r-1)\alpha})\sum_{j=0}^{n} b^r_{j,n+1}\big|y(t^r_j) - y^r_j\big|,$$
$$= C((n/N)^{(r-1)\alpha})N^{-\alpha}\sum_{j=0}^{n} b^r_{j,n+1}\big|y(t^r_j) - y^r_j\big|,$$
$$= CN^{-\alpha}\sum_{j=0}^{n} b^r_{j,n+1}\big|y(t^r_j) - y^r_j\big|,$$

Hence,

$$|y(t^r_{n+1}) - y^r_{n+1}| \le C\left(|I| + |II| + |III|\right),$$
$$\le C|I| + C\sum_{j=0}^{n} a^r_{j,n+1}\left|y(t^r_j) - y^r_j\right| + C|III_1 + III_2|,$$
$$\le C|I| + C\sum_{j=0}^{n} a^r_{j,n+1}\left|y(t^r_j) - y^r_j\right| + C|III_1|$$
$$+ CN^{-\alpha}\sum_{j=0}^{n} b^r_{j,n+1}\big|y(t^r_j) - y^r_j\big|,$$

We will now use mathematical induction to complete the proof.

Proof by Mathematical Induction.
We will first consider $0 < \alpha \le 1$. We will get the following four cases,

Case 1: Let $r(\sigma+\alpha) > max\{2, 1+\alpha\} = 2$. Assume that there exists a constant $C_0 > 0$ such that, with $j = 0, 1, 2, ..., n$ and $n = 0, 1, 2, ..., N-1$,

$$|y(t_j^r) - y_j^r| \le C_0 N^{-1-\alpha},$$

we shall show that

$$|y(t_{n+1}^r) - y_{n+1}^r| \le C_0 N^{-1-\alpha}.$$

By lemmas 3.3, 3.6 and 3.7, we have,

$$\begin{aligned} |y(t_{n+1}^r) - y_{n+1}^r| &\le CN^{-2} + C\sum_{j=0}^{n} a_{j,n+1}^r \left|y(t_j^r) - y_j^r\right| + CN^{-1-\alpha} \\ &\quad + CN^{-\alpha}\sum_{j=0}^{n} b_{j,n+1}^r \left|y(t_j^r) - y_j^r\right|, \\ &\le CN^{-2} + CT^{\alpha}C_0N^{-1-\alpha} + CN^{-1-\alpha} + CN^{-\alpha}T^{\alpha}C_0N^{-1-\alpha}. \end{aligned} \tag{4.25}$$

By using the same idea as 4.5, (cf. [7], p.41) and [25], we can choose T to be sufficiently small such that the second term on the right hand side is less than $\frac{C_0}{2}N^{-1-\alpha}$. We can then choose N and C_0 to be sufficiently large such that the summation of the other terms on the right hand side is also less than $\frac{C_0}{2}N^{-1-\alpha}$. Hence,

$$|y(t_{n+1}^r) - y_{n+1}^r| \le C_0 N^{-1-\alpha}.$$

Case 2: Let $r(\sigma+\alpha) > min\{2, 1+\alpha\} = 1+\alpha$. Assume that there exists a constant $C_0 > 0$ such that, with $j = 0, 1, 2, ..., n$ and $n = 0, 1, 2, ..., N-1$,

$$|y(t_j^r) - y_j^r| \le C_0 N^{-r(\alpha+\sigma)}.$$

By using the same idea as before we can show that,

$$|y(t_{n+1}^r) - y_{n+1}^r| \le C_0 N^{-r(\alpha+\sigma)}.$$

Case 3: Let $1+\alpha < r(\sigma+\alpha) \le 2$. We can show that,

$$|y(t_{n+1}^r) - y_{n+1}^r| \le C_0 N^{-1-\alpha}.$$

Case 4: Let $r(\sigma+\alpha) = 1+\alpha$. We can show that,

$$|y(t_{n+1}^r) - y_{n+1}^r| \le C_0 N^{-1-\alpha} ln(N).$$

We can now consider the four cases for $1 \le \alpha < 2$.

Case 1: Let $r > max\{\frac{1+\alpha}{\sigma+\alpha}, \frac{2}{1+\sigma}\} = \frac{2}{1+\sigma}$. Assume that there exists a constant $C_0 > 0$ such that, with $j = 0, 1, 2, ..., n$ and $n = 0, 1, 2, ..., N-1$,

$$|y(t_j^r) - y_j^r| \leq C_0 N^{-2},$$

we shall show that

$$|y(t_{n+1}^r) - y_{n+1}^r| \leq C_0 N^{-2}.$$

We have,

$$\begin{aligned} |y(t_{n+1}^r) - y_{n+1}^r| &\leq CN^{-2} + C\sum_{j=0}^{n} a_{j,n+1}^r \left|y(t_j^r) - y_j^r\right| + CN^{-1-\alpha} \\ &\quad + CN^{-\alpha}\sum_{j=0}^{n} b_{j,n+1}^r \left|y(t_j^r) - y_j^r\right|, \\ &\leq CN^{-2} + CT^\alpha C_0 N^{-2} + CN^{-1-\alpha} + CN^{-\alpha}T^\alpha C_0 N^{-2}. \end{aligned} \tag{4.26}$$

If we choose a sufficiently small T, the second term on the right hand side is less than $\frac{C_0}{2}N^{-2}$. Then if we choose N and C_0 to be sufficiently large, the other terms of the right hand side will also be less than $\frac{C_0}{2}N^{-2}$. Hence,

$$|y(t_{n+1}^r) - y_{n+1}^r| \leq C_0 N^{-2}.$$

Case 2: Let $r < min\{\frac{1+\alpha}{\sigma+\alpha}, \frac{2}{1+\sigma}\} = \frac{1+\alpha}{\sigma+\alpha}$. Assume that there exists a constant $C_0 > 0$ such that, with $j = 0, 1, 2, ..., n$ and $n = 0, 1, 2, ..., N-1$,

$$|y(t_j^r) - y_j^r| \leq C_0 N^{-r(1+\sigma)}.$$

By using the same idea as before we can show that,

$$|y(t_{n+1}^r) - y_{n+1}^r| \leq C_0 N^{-r(1+\sigma)}.$$

Case 3: Let $\frac{1+\alpha}{\sigma+\alpha} \leq r < \frac{2}{1+\sigma}$. We can show that,

$$|y(t_{n+1}^r) - y_{n+1}^r| \leq C_0 N^{-r(1+\sigma)}.$$

Case 4: Let $r = \frac{2}{1+\sigma}$. We can show that,

$$|y(t_{n+1}^r) - y_{n+1}^r| \leq C_0 N^{-2} ln(N).$$

Giving us the estimates to complete the proof. □

Chapter 5

Higher Order Fractional Adams-Bashforth-Moulton Method with Graded Meshes

1 Introduction

We can now add on to the results from earlier to find a higher order solution to solve (3.1). We will first look at a uniform mesh and then extend the results to find a graded mesh. At $t = t_{n+1}$ and let $n \geq 1$, we have,

$$y(t_{n+1}) = \sum_{k=0}^{\lceil \alpha \rceil - 1} (y_0)^{(k)} \frac{(t_{n+1})^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha - 1} g(s) ds.$$

We will only consider the case $0 < \alpha < 1$, giving us,

$$y(t_{n+1}) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha - 1} g(s) ds. \tag{5.1}$$

2 Higher Order Fractional Adams-Bashforth-Moulton Method with Uniform Meshes

We will first start with a uniform mesh, so we can assume that we are working on a uniform grid, such that $\{t_j = jh : j = 0, 1, ..., N\}$ and $h = T/N$, when $N \in \mathbb{Z}$ and let $0 = t_0 < t_1 < t_2 < ... < t_n = T$ be a partition of $[0, T]$. This is very similar to section 3 except instead of using the trapezoidal rule to interpolate the points, we will use a quadratic interpolation. We already know that by separating the integrals into sections we will get,

$$y(t_{n+1}) = y_0 + \frac{1}{\Gamma(\alpha)} \left[\int_{t_0}^{t_1} (t_{n+1} - s)^{\alpha - 1} g(s) ds + \sum_{j=1}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha - 1} g(s) ds \right]. \tag{5.2}$$

Now by using quadratic interpolation to approximate $g(s)$, ($g(s) \approx P_2(s)$), we will get,

$$
\begin{aligned}
P_2(s) = &\frac{(s-t_1)(s-t_2)}{(t_0-t_1)(t_0-t_2)}g(t_0) + \frac{(s-t_0)(s-t_2)}{(t_1-t_0)(t_1-t_2)}g(t_1) \\
&+ \frac{(s-t_0)(s-t_1)}{(t_2-t_0)(t_2-t_1)}g(t_2),
\end{aligned} \tag{5.3}
$$

and,

$$
\begin{aligned}
P_2(s) = &\frac{(s-t_j)(s-t_{j+1})}{(t_{j-1}-t_j)(t_{j-1}-t_{j+1})}g(t_{j-1}) + \frac{(s-t_{j-1})(s-t_{j+1})}{(t_j-t_{j-1})(t_j-t_{j+1})}g(t_j) \\
&+ \frac{(s-t_{j-1})(s-t_j)}{(t_{j+1}-t_{j-1})(t_{j+1}-t_j)}g(t_{j+1}),
\end{aligned} \tag{5.4}
$$

for $[t_0, t_1]$ and $[t_j, t_{j+1}]$, $j = 1, 2, ..., n$, respectively.

So for the first integral from t_0 to t_1, we are borrowing a point from t_2. Whereas for t_j to t_{j+1}, we are borrowing a point from the point before, t_{j-1}. From (5.2), we will get,

$$
\begin{aligned}
y(t_{n+1}) &= y_0 + \frac{1}{\Gamma(\alpha)}\Bigg[\underbrace{\int_{t_0}^{t_1}(t_{n+1}-s)^{\alpha-1}g(s)ds}_{IIII_1} + \sum_{j=1}^{n}\underbrace{\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}g(s)ds}_{IIII_2}\Bigg] \\
&= y_0 + \frac{1}{\Gamma(\alpha)}\Bigg[IIII_1 + \sum_{j=1}^{n} IIII_2\Bigg].
\end{aligned} \tag{5.5}
$$

We can now calculate the integral of each parts separately. For $IIII_1$,

$$
\begin{aligned}
IIII_1 = &\frac{1}{(t_0-t_1)(t_0-t_2)}\int_{t_0}^{t_1}(t_{n+1}-s)^{\alpha-1}(s-t_1)(s-t_2)g(t_0)ds \\
&+ \frac{1}{(t_1-t_0)(t_1-t_2)}\int_{t_0}^{t_1}(t_{n+1}-s)^{\alpha-1}(s-t_0)(s-t_2)g(t_1)ds \\
&+ \frac{1}{(t_2-t_0)(t_2-t_1)}\int_{t_0}^{t_1}(t_{n+1}-s)^{\alpha-1}(s-t_0)(s-t_1)g(t_2)ds, \\
= &\frac{1}{(t_0-t_1)(t_0-t_2)}\int_{t_0}^{t_1}(t_{n+1}-s)^{\alpha-1}[(s-t_{n+1})+(t_{n+1}-t_1)] \\
&\times[(s-t_{n+1})+(t_{n+1}-t_2)]g(t_0)ds \\
&+ \frac{1}{(t_1-t_0)(t_1-t_2)}\int_{t_0}^{t_1}(t_{n+1}-s)^{\alpha-1}[(s-t_{n+1})+(t_{n+1}-t_0)] \\
&\times[(s-t_{n+1})+(t_{n+1}-t_2)]g(t_1)ds \\
&+ \frac{1}{(t_2-t_0)(t_2-t_1)}\int_{t_0}^{t_1}(t_{n+1}-s)^{\alpha-1}[(s-t_{n+1})+(t_{n+1}-t_0)] \\
&\times[(s-t_{n+1})+(t_{n+1}-t_1)]g(t_2)ds,
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{g(t_0)}{(t_0-t_1)(t_0-t_2)} \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha-1}[(s-t_{n+1})^2 + (2t_{n+1}-t_1-t_2)(s-t_{n+1}) \\
&\quad + (t_{n+1}-t_2)(t_{n+1}-t_1)]ds \\
&\quad + \frac{g(t_1)}{(t_1-t_0)(t_1-t_2)} \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha-1}[(s-t_{n+1})^2 + (2t_{n+1}-t_0-t_2)(s-t_{n+1}) \\
&\quad + (t_{n+1}-t_0)(t_{n+1}-t_2)]ds \\
&\quad + \frac{g(t_2)}{(t_2-t_0)(t_2-t_1)} \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha-1}[(s-t_{n+1})^2 + (2t_{n+1}-t_0-t_1)(s-t_{n+1}) \\
&\quad + (t_{n+1}-t_0)(t_{n+1}-t_1)]ds, \\
&= \frac{g(t_0)}{(t_0-t_1)(t_0-t_2)} \Bigg[\int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha+1} ds - (2t_{n+1}-t_1-t_2) \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha} ds \\
&\quad + (t_{n+1}-t_2)(t_{n+1}-t_1) \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha-1} ds \Bigg] \\
&\quad + \frac{g(t_1)}{(t_1-t_0)(t_1-t_2)} \Bigg[\int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha+1} ds - (2t_{n+1}-t_0-t_2) \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha} ds \\
&\quad + (t_{n+1}-t_0)(t_{n+1}-t_2) \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha-1} ds \Bigg] \\
&\quad + \frac{g(t_2)}{(t_2-t_0)(t_2-t_1)} \Bigg[\int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha+1} ds - (2t_{n+1}-t_0-t_1) \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha} ds \\
&\quad + (t_{n+1}-t_0)(t_{n+1}-t_1) \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha-1} ds \Bigg].
\end{aligned}
$$

After some basic integration we will find,

$$
\begin{aligned}
IIII_1 &= \frac{g(t_0)}{(t_0-t_1)(t_0-t_2)} \Bigg[\frac{1}{\alpha+2}[(t_{n+1}-t_0)^{\alpha+2} - (t_{n+1}-t_1)^{\alpha+2}] \\
&\quad + \frac{2t_{n+1}-t_1-t_2}{\alpha+1}[(t_{n+1}-t_1)^{\alpha+1} - (t_{n+1}-t_0)^{\alpha+1}] \\
&\quad + \frac{(t_{n+1}-t_2)(t_{n+1}-t_1)}{\alpha}[(t_{n+1}-t_0)^{\alpha} - (t_{n+1}-t_1)^{\alpha}] \Bigg] \\
&\quad + \frac{g(t_1)}{(t_1-t_0)(t_1-t_2)} \Bigg[\frac{1}{\alpha+2}[(t_{n+1}-t_0)^{\alpha+2} - (t_{n+1}-t_1)^{\alpha+2}] \\
&\quad + \frac{2t_{n+1}-t_0-t_2}{\alpha+1}[(t_{n+1}-t_1)^{\alpha+1} - (t_{n+1}-t_0)^{\alpha+1}] \\
&\quad + \frac{(t_{n+1}-t_0)(t_{n+1}-t_2)}{\alpha}[(t_{n+1}-t_0)^{\alpha} - (t_{n+1}-t_1)^{\alpha}] \Bigg] \\
&\quad + \frac{g(t_2)}{(t_2-t_0)(t_2-t_1)} \Bigg[\frac{1}{\alpha+2}[(t_{n+1}-t_0)^{\alpha+2} - (t_{n+1}-t_1)^{\alpha+2}] \\
&\quad + \frac{2t_{n+1}-t_0-t_1}{\alpha+1}[(t_{n+1}-t_1)^{\alpha+1} - (t_{n+1}-t_0)^{\alpha+1}] \\
&\quad + \frac{(t_{n+1}-t_0)(t_{n+1}-t_1)}{\alpha}[(t_{n+1}-t_0)^{\alpha} - (t_{n+1}-t_1)^{\alpha}] \Bigg].
\end{aligned} \tag{5.6}
$$

We know that $t_n = nh$ and $t_0 = 0$, hence,

$$\begin{aligned}
IIII_1 = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)} \Bigg[& \frac{g(t_0)}{2}\big[\alpha(\alpha+1)((n+1)^{\alpha+2} - n^{\alpha+2}) \\
& + \alpha(\alpha+2)(2n-1)(n^{\alpha+1} - (n+1)^{\alpha+1}) \\
& + (\alpha+1)(\alpha+2)(n)(n-1)((n+1)^\alpha - n^\alpha)\big] \\
- & g(t_1)\big[\alpha(\alpha+1)((n+1)^{\alpha+2} - n^{\alpha+2}) \\
& + \alpha(\alpha+2)(2n)(n^{\alpha+1} - (n+1)^{\alpha+1}) \\
& + (\alpha+1)(\alpha+2)(n+1)(n-1)((n+1)^\alpha - n^\alpha)\big] \\
+ & \frac{g(t_2)}{2}\big[\alpha(\alpha+1)((n+1)^{\alpha+2} - n^{\alpha+2}) \\
& + \alpha(\alpha+2)(2n+1)(n^{\alpha+1} - (n+1)^{\alpha+1}) \\
& + (\alpha+1)(\alpha+2)(n+1)(n)((n+1)^\alpha - n^\alpha)\big]\Bigg].
\end{aligned}$$

By using the same technique for $IIII_1$. For $IIII_2$, we get,

$$\begin{aligned}
IIII_2 = & \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} \Bigg(\frac{(s-t_j)(s-t_{j+1})}{(t_{j-1}-t_j)(t_{j-1}-t_{j+1})} g(t_{j-1}) \\
& + \frac{(s-t_{j-1})(s-t_{j+1})}{(t_j - t_{j-1})(t_j - t_{j+1})} g(t_j) + \frac{(s-t_{j-1})(s-t_j)}{(t_{j+1}-t_{j-1})(t_{j+1}-t_j)} g(t_{j+1}) \Bigg) ds, \\
= & \frac{1}{(t_{j-1}-t_j)(t_{j-1}-t_{j+1})} \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha-1} [(s - t_{n+1}) + (t_{n+1} - t_j)] \\
& \times [(s - t_{n+1}) + (t_{n+1} - t_{j+1})] g(t_{j-1}) ds \\
& + \frac{1}{(t_j - t_{j-1})(t_j - t_{j+1})} \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha-1} [(s - t_{n+1}) + (t_{n+1} - t_{j-1})] \\
& \times [(s - t_{n+1}) + (t_{n+1} - t_{j+1})] g(t_j) ds \\
& + \frac{1}{(t_{j+1}-t_{j-1})(t_{j+1}-t_j)} \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha-1} [(s - t_{n+1}) + (t_{n+1} - t_{j-1})] \\
& \times [(s - t_{n+1}) + (t_{n+1} - t_j)] g(t_{j+1}) ds, \\
= & \frac{g(t_{j-1})}{(t_{j-1}-t_j)(t_{j-1}-t_{j+1})} \Bigg[\int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha+1} ds - (2t_{n+1} - t_j - t_{j+1}) \\
& \times \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha} ds + (t_{n+1}-t_j)(t_{n+1}-t_{j+1}) \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha-1} ds \Bigg] \\
+ & \frac{g(t_j)}{(t_j - t_{j-1})(t_j - t_{j+1})} \Bigg[\int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha+1} ds - (2t_{n+1} - t_{j-1} - t_{j+1}) \\
& \times \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha} ds + (t_{n+1}-t_{j-1})(t_{n+1}-t_{j+1}) \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha-1} ds \Bigg] \\
+ & \frac{g(t_{j+1})}{(t_{j+1}-t_{j-1})(t_{j+1}-t_j)} \Bigg[\int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha+1} ds - (2t_{n+1} - t_{j-1} - t_j) \\
& \times \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha} ds + (t_{n+1}-t_{j-1})(t_{n+1}-t_j) \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha-1} ds \Bigg].
\end{aligned}$$

After some more basic integration, we will get,

$$
\begin{aligned}
IIII_2 = {} & \frac{g(t_{j-1})}{(t_{j-1}-t_j)(t_{j-1}-t_{j+1})}\Bigg[\frac{1}{\alpha+2}[(t_{n+1}-t_j)^{\alpha+2}-(t_{n+1}-t_{j+1})^{\alpha+2}] \\
& + \frac{2t_{n+1}-t_j-t_{j+1}}{\alpha+1}[(t_{n+1}-t_{j+1})^{\alpha+1}-(t_{n+1}-t_j)^{\alpha+1}] \\
& + \frac{(t_{n+1}-t_j)(t_{n+1}-t_{j+1})}{\alpha}[(t_{n+1}-t_j)^{\alpha}-(t_{n+1}-t_{j+1})^{\alpha}]\Bigg] \\
& + \frac{g(t_j)}{(t_j-t_{j-1})(t_j-t_{j+1})}\Bigg[\frac{1}{\alpha+2}[(t_{n+1}-t_j)^{\alpha+2}-(t_{n+1}-t_{j+1})^{\alpha+2}] \\
& + \frac{2t_{n+1}-t_{j-1}-t_{j+1}}{\alpha+1}[(t_{n+1}-t_{j+1})^{\alpha+1}-(t_{n+1}-t_j)^{\alpha+1}] \\
& + \frac{(t_{n+1}-t_{j-1})(t_{n+1}-t_{j+1})}{\alpha}[(t_{n+1}-t_j)^{\alpha}-(t_{n+1}-t_{j+1})^{\alpha}]\Bigg] \\
& + \frac{g(t_{j+1})}{(t_{j+1}-t_{j-1})(t_{j+1}-t_j)}\Bigg[\frac{1}{\alpha+2}[(t_{n+1}-t_j)^{\alpha+2}-(t_{n+1}-t_{j+1})^{\alpha+2}] \\
& + \frac{2t_{n+1}-t_{j-1}-t_j}{\alpha+1}[(t_{n+1}-t_{j+1})^{\alpha+1}-(t_{n+1}-t_j)^{\alpha+1}] \\
& + \frac{(t_{n+1}-t_{j-1})(t_{n+1}-t_j)}{\alpha}[(t_{n+1}-t_j)^{\alpha}-(t_{n+1}-t_{j+1})^{\alpha}]\Bigg].
\end{aligned} \tag{5.7}
$$

We know $t_k = kh$, $k = 0,1,2,...,n$,

$$
\begin{aligned}
IIII_2 = {} & \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)}\Bigg[\frac{g(t_{j-1})}{2}\big[\alpha(\alpha+1)((n-j+1)^{\alpha+2}-(n-j)^{\alpha+2}) \\
& + \alpha(\alpha+2)(2n-2j+1)((n-j)^{\alpha+1}-(n-j+1)^{\alpha+1}) \\
& + (\alpha+1)(\alpha+2)(n-j+1)(n-j)((n-j+1)^{\alpha}-(n-j)^{\alpha})\big] \\
& - g(t_j)\big[\alpha(\alpha+1)((n-j+1)^{\alpha+2}-(n-j)^{\alpha+2}) \\
& + \alpha(\alpha+2)(2n-2j+2)((n-j)^{\alpha+1}-(n-j+1)^{\alpha+1}) \\
& + (\alpha+1)(\alpha+2)(n-j+2)(n-j)((n-j+1)^{\alpha}-(n-j)^{\alpha})\big] \\
& + \frac{g(t_{j+1})}{2}\big[\alpha(\alpha+1)((n-j+1)^{\alpha+2}-(n-j)^{\alpha+2}) \\
& + \alpha(\alpha+2)(2n-2j+3)((n-j)^{\alpha+1}-(n-j+1)^{\alpha+1}) \\
& + (\alpha+1)(\alpha+2)(n-j+2)(n-j+1)((n-j+1)^{\alpha}-(n-j)^{\alpha})\big].
\end{aligned}
$$

By putting $IIII_1$ and $IIII_2$ back into (5.5), we will get,

$$
\begin{aligned}
y(t_{n+1}) &= y_0 + \frac{1}{\Gamma(\alpha)}\Bigg[IIII_1 + \sum_{j=1}^{n} IIII_2\Bigg], \\
&= y_0 + \frac{h^\alpha}{\Gamma(\alpha+3)}\Bigg[\Big[\frac{g(t_0)}{2}\big[\alpha(\alpha+1)((n+1)^{\alpha+2}-n^{\alpha+2}) \\
& + \alpha(\alpha+2)(2n-1)(n^{\alpha+1}-(n+1)^{\alpha+1}) \\
& + (\alpha+1)(\alpha+2)(n)(n-1)((n+1)^{\alpha}-n^{\alpha})\big]
\end{aligned}
$$

$$
\begin{aligned}
&- g(t_1)\big[\alpha(\alpha+1)((n+1)^{\alpha+2} - n^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n)(n^{\alpha+1} - (n+1)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n+1)(n-1)((n+1)^{\alpha} - n^{\alpha})\big] \\
&+ \frac{g(t_2)}{2}\big[\alpha(\alpha+1)((n+1)^{\alpha+2} - n^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n+1)(n^{\alpha+1} - (n+1)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n+1)(n)((n+1)^{\alpha} - n^{\alpha})\big]\Big] \\
&+ \sum_{j=1}^{n} \Big[\frac{g(t_{j-1})}{2}\big[\alpha(\alpha+1)((n-j+1)^{\alpha+2} - (n-j)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n-2j+1)((n-j)^{\alpha+1} - (n-j+1)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n-j+1)(n-j)((n-j+1)^{\alpha} - (n-j)^{\alpha})\big] \\
&- g(t_j)\big[\alpha(\alpha+1)((n-j+1)^{\alpha+2} - (n-j)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n-2j+2)((n-j)^{\alpha+1} - (n-j+1)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n-j+2)(n-j)((n-j+1)^{\alpha} - (n-j)^{\alpha})\big] \\
&+ \frac{g(t_{j+1})}{2}\big[\alpha(\alpha+1)((n-j+1)^{\alpha+2} - (n-j)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n-2j+3)((n-j)^{\alpha+1} - (n-j+1)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n-j+2)(n-j+1)((n-j+1)^{\alpha} - (n-j)^{\alpha})\big]\Big]\Bigg],
\end{aligned}
$$

We will ignore the first term in the parenthesis for now and expand the summation so we can group terms together,

$$
\begin{aligned}
\sum_{j=1}^{n} IIII_2 = &\Big[\frac{g(t_0)}{2}\big[\alpha(\alpha+1)(n^{\alpha+2} - (n-1)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n-1)((n-1)^{\alpha+1} - n^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n)(n-1)(n^{\alpha} - (n-1)^{\alpha})\big] \\
&- g(t_1)\big[\alpha(\alpha+1)(n^{\alpha+2} - (n-1)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n)((n-1)^{\alpha+1} - n^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n+1)(n-1)(n^{\alpha} - (n-1)^{\alpha})\big] \\
&+ \frac{g(t_2)}{2}\big[\alpha(\alpha+1)(n^{\alpha+2} - (n-1)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n+1)((n-1)^{\alpha+1} - n^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n+1)(n)(n^{\alpha} - (n-1)^{\alpha})\big]\Big] \\
&+ \Big[\frac{g(t_1)}{2}\big[\alpha(\alpha+1)((n-1)^{\alpha+2} - (n-2)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n-3)((n-2)^{\alpha+1} - (n-1)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n-1)(n-2)((n-1)^{\alpha} - (n-2)^{\alpha})\big] \\
&- g(t_2)\big[\alpha(\alpha+1)((n-1)^{\alpha+2} - (n-2)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n-2)((n-2)^{\alpha+1} - (n-1)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n)(n-2)((n-1)^{\alpha} - (n-2)^{\alpha})\big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{g(t_3)}{2}\big[\alpha(\alpha+1)((n-1)^{\alpha+2} - (n-2)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n-1)((n-2)^{\alpha+1} - (n-1)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n)(n-1)((n-1)^{\alpha} - (n-2)^{\alpha})\big]\Big] \\
&+ \Big[\frac{g(t_2)}{2}\big[\alpha(\alpha+1)((n-2)^{\alpha+2} - (n-3)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n-5)((n-3)^{\alpha+1} - (n-2)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n-2)(n-3)((n-2)^{\alpha} - (n-3)^{\alpha})\big] \\
&- g(t_3)\big[\alpha(\alpha+1)((n-2)^{\alpha+2} - (n-3)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n-4)((n-3)^{\alpha+1} - (n-2)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n-1)(n-3)((n-2)^{\alpha} - (n-3)^{\alpha})\big] \\
&+ \frac{g(t_4)}{2}\big[\alpha(\alpha+1)((n-2)^{\alpha+2} - (n-3)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n-3)((n-3)^{\alpha+1} - (n-2)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n-1)(n-2)((n-2)^{\alpha} - (n-3)^{\alpha})\big]\Big] \\
&+ \Big[\frac{g(t_3)}{2}\big[\alpha(\alpha+1)((n-3)^{\alpha+2} - (n-4)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n-7)((n-4)^{\alpha+1} - (n-3)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n-3)(n-4)((n-3)^{\alpha} - (n-4)^{\alpha})\big] \\
&- g(t_4)\big[\alpha(\alpha+1)((n-3)^{\alpha+2} - (n-4)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n-6)((n-4)^{\alpha+1} - (n-3)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n-2)(n-4)((n-3)^{\alpha} - (n-4)^{\alpha})\big] \\
&+ \frac{g(t_5)}{2}\big[\alpha(\alpha+1)((n-3)^{\alpha+2} - (n-4)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n-5)((n-4)^{\alpha+1} - (n-3)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n-2)(n-3)((n-3)^{\alpha} - (n-4)^{\alpha})\big]\Big] \\
&+ \ldots \\
&+ \Big[\frac{g(t_{n-2})}{2}\big[\alpha(\alpha+1)((2)^{\alpha+2} - (1)^{\alpha+2}) + \alpha(\alpha+2)(3)((1)^{\alpha+1} - (2)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(2)(1)((2)^{\alpha} - (1)^{\alpha})\big] \\
&- g(t_{n-1})\big[\alpha(\alpha+1)((2)^{\alpha+2} - (1)^{\alpha+2}) + \alpha(\alpha+2)(4)((1)^{\alpha+1} - (2)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(3)(1)((2)^{\alpha} - (1)^{\alpha})\big] \\
&+ \frac{g(t_n)}{2}\big[\alpha(\alpha+1)((2)^{\alpha+2} - (1)^{\alpha+2}) + \alpha(\alpha+2)(5)((1)^{\alpha+1} - (2)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(3)(2)((2)^{\alpha} - (1)^{\alpha})\big]\Big] \\
&+ \Big[\frac{g(t_{n-1})}{2}\big[\alpha(\alpha+1)((1)^{\alpha+2} - (0)^{\alpha+2}) + \alpha(\alpha+2)(1)((0)^{\alpha+1} - (1)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(1)(0)((1)^{\alpha} - (0)^{\alpha})\big]
\end{aligned}
$$

$$
\begin{aligned}
&- g(t_n)\big[\alpha(\alpha+1)((1)^{\alpha+2} - (0)^{\alpha+2}) + \alpha(\alpha+2)(2)((0)^{\alpha+1} - (1)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(2)(0)((1)^{\alpha} - (0)^{\alpha})\big] \\
&+ \frac{g(t_{n+1})}{2}\big[\alpha(\alpha+1)((1)^{\alpha+2} - (0)^{\alpha+2}) + \alpha(\alpha+2)(3)((0)^{\alpha+1} - (1)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(2)(1)((1)^{\alpha} - (0)^{\alpha})\big]\Big].
\end{aligned}
$$

By combining all of the same coefficients, we will eventually get,

$$
\int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} g(s)ds = \sum_{j=0}^{n+1} c_{j,n+1} g(t_j),
$$

where,

$$
c_{j,n+1} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)} \times \begin{cases} \frac{Y0}{2} + \frac{X0(1)}{2}, & \text{if } j = 0, \\ -Y1 - X1(1) + \frac{X0(2)}{2}, & \text{if } j = 1, \\ \frac{Y2}{2} + \frac{X2(1)}{2} - X1(2) + \frac{X0(3)}{2}, & \text{if } j = 2, \\ \frac{X2(j-1)}{2} - X1(j) + \frac{X0(j+1)}{2}, & \text{if } 3 \le j \le n-1, \\ \frac{X2(n-1)}{2} - X1(n), & \text{if } j = n, \\ \frac{X2(n)}{2}, & \text{if } j = n+1, \end{cases} \tag{5.8}
$$

and,

$$
\begin{aligned}
X0(j) &= \alpha(\alpha+1)((n-j+1)^{\alpha+2} - (n-j)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n-2j+1)((n-j)^{\alpha+1} - (n-j+1)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n-j+1)(n-j)((n-j+1)^{\alpha} - (n-j)^{\alpha}), \\
X1(j) &= \alpha(\alpha+1)((n-j+1)^{\alpha+2} - (n-j)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n-2j+2)((n-j)^{\alpha+1} - (n-j+1)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n-j+2)(n-j)((n-j+1)^{\alpha} - (n-j)^{\alpha}), \\
X2(j) &= \alpha(\alpha+1)((n-j+1)^{\alpha+2} - (n-j)^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n-2j+3)((n-j)^{\alpha+1} - (n-j+1)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n-j+2)(n-j+1)((n-j+1)^{\alpha} - (n-j)^{\alpha}), \\
Y0 &= \alpha(\alpha+1)((n+1)^{\alpha+2} - n^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n-1)(n^{\alpha+1} - (n+1)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n)(n-1)((n+1)^{\alpha} - n^{\alpha}), \\
Y1 &= \alpha(\alpha+1)((n+1)^{\alpha+2} - n^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n)(n^{\alpha+1} - (n+1)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n+1)(n-1)((n+1)^{\alpha} - n^{\alpha}), \\
Y2 &= \alpha(\alpha+1)((n+1)^{\alpha+2} - n^{\alpha+2}) \\
&\quad + \alpha(\alpha+2)(2n+1)(n^{\alpha+1} - (n+1)^{\alpha+1}) \\
&\quad + (\alpha+1)(\alpha+2)(n+1)(n)((n+1)^{\alpha} - n^{\alpha}).
\end{aligned} \tag{5.9}
$$

We need at least $n = 4$, otherwise the weights would be calculated differently for each $n = 1, 2, 3$.

Let $y_j \approx y(t_j)$ be the approximation of $y(t_j)$. Then from (5.1), and by replacing the function g with $f(t_j, y(t_j))$, we will get,

$$
\begin{aligned}
y_{n+1} &= y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n+1} c_{j,n+1} f(t_j, y_j), \\
&= y_0 + \frac{1}{\Gamma(\alpha)} \Bigg(\sum_{j=0}^{n} c_{j,n+1} f(t_j, y_j) + c_{n+1,n+1} f(t_{n+1}, y_{n+1}^P) \Bigg).
\end{aligned}
$$

By approximating $y_{n+1}^P \approx y_{n+1}$, we have changed the type of equation to an explicit type. We can now find y_{n+1}^P by applying trapezoidal linear interpolation like before. We know that from section 3 and (4.5) when $r = 1$,

$$
\int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha - 1} g(s) ds = \sum_{j=0}^{n+1} a_{j,n+1} g(t_j),
$$

where,

$$
a_{j,n+1} = \frac{h^\alpha}{\alpha(\alpha+1)} \times \begin{cases} (n^{\alpha+1} - (n-\alpha)(n+1)^\alpha), & \text{if } j = 0, \\ ((n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} \\ \quad -2(n-j+1)^{\alpha+1}), & \text{if } 1 \leq j \leq n, \\ 1, & \text{if } j = n+1. \end{cases} \tag{5.10}
$$

Giving us the following equation to find y_{n+1}^P,

$$
\begin{aligned}
y_{n+1}^P &= y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n+1} a_{j,n+1} f(t_j, y_j), \\
&= y_0 + \frac{1}{\Gamma(\alpha)} \Bigg(\sum_{j=0}^{n} a_{j,n+1} f(t_j, y_j) + a_{n+1,n+1} f(t_{n+1}, y_{n+1}^{PP}) \Bigg).
\end{aligned}
$$

So we can now do the same as before but to find y_{n+1}^{PP} by using the rectangle rule to interpolate $g(s)$ in (5.1). The calculations for this can be found in section 3 and (4.8) when $r = 1$,

$$
\int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha - 1} g(s) ds = \sum_{j=0}^{n} b_{j,n+1} g(t_j),
$$

where,

$$
b_{j,n+1} = \frac{h^\alpha}{\alpha} \Bigg((n-j+1)^\alpha - (n-j)^\alpha \Bigg). \tag{5.11}
$$

Giving us the following equation to find y_{n+1}^{PP},

$$
\begin{aligned}
y_{n+1}^{PP} &= y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_j), \\
&= y_0 + \frac{1}{\Gamma(\alpha)} \Bigg(\sum_{j=0}^{n} b_{j,n+1} f(t_j, y_j) \Bigg).
\end{aligned}
$$

Giving us a higher order fractional Adams-Bashforth-Moulton method for $0 < \alpha \leq 1$,

$$\begin{cases} y_{n+1}^{PP} = y_0 + \frac{1}{\Gamma(\alpha)}\left(\sum_{j=0}^{n} b_{j,n+1} f(t_j, y_j)\right), \\ y_{n+1}^{P} = y_0 + \frac{1}{\Gamma(\alpha)}\left(\sum_{j=0}^{n} a_{j,n+1} f(t_j, y_j) + a_{n+1,n+1} f(t_{n+1}, y_{n+1}^{PP})\right), \\ y_{n+1} = y_0 + \frac{1}{\Gamma(\alpha)}\left(\sum_{j=0}^{n} c_{j,n+1} f(t_j, y_j) + c_{n+1,n+1} f(t_{n+1}, y_{n+1}^{P})\right), \\ y_0, y_1 \text{ is given.} \end{cases} \tag{5.12}$$

where the weights $b_{j,n+1}$, $a_{j,n+1}$ and $c_{j,n+1}$ are given by (5.11), (5.10) and (5.8), respectively.

Calculate Predictor (y_{n+1}^{PP}) → Evaluate $f(t_{n+1}, y_{n+1}^{PP})$ → Calculate Predictor (y_{n+1}^{P}) → Evaluate $f(t_{n+1}, y_{n+1}^{P})$ → Calculate Corrector (y_{n+1}) → Evaluate $f(t_{n+1}, y_{n+1})$

The diagram above shows how the method would be calculated to find the next step y_{n+1} using (5.12). We can apply this to any α by using the same idea but instead using (3.2), this will give us,

$$\begin{cases} y_{n+1}^{PP} = \sum_{k=0}^{\lceil\alpha\rceil-1} (y_0)^{(k)} \frac{(t_{n+1})^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_j). \\ y_{n+1}^{P} = \sum_{k=0}^{\lceil\alpha\rceil-1} (y_0)^{(k)} \frac{(t_{n+1})^k}{k!} + \frac{1}{\Gamma(\alpha)} \big(\sum_{j=0}^{n} a_{j,n+1} f(t_j, y_j) + a_{n+1,n+1} f(t_{n+1}, y_{n+1}^{PP})\big), \\ y_{n+1} = \sum_{k=0}^{\lceil\alpha\rceil-1} (y_0)^{(k)} \frac{(t_{n+1})^k}{k!} + \frac{1}{\Gamma(\alpha)} \big(\sum_{j=0}^{n} c_{j,n+1} f(t_j, y_j) + c_{n+1,n+1} f(t_{n+1}, y_{n+1}^{P})\big), \\ y_0,\ (y_0)^{(1)},\ \dots,\ (y_0)^{(\lceil\alpha\rceil-1)} \text{ and,} \\ y_1,\ (y_1)^{(1)},\ \dots,\ (y_1)^{(\lceil\alpha\rceil-1)} \text{ is given.} \end{cases} \tag{5.13}$$

3 Higher Order Fractional Adams-Bashforth-Moulton Method with Graded Meshes

Let $0 = t_0^r < t_1^r < ... < t_N^r = T$, be the graded meshes on [0,T], where

$$t_n^r = T\left(\frac{n}{N}\right)^r, \tag{5.14}$$

and $r \geq 1$, $n = 0, 1, 2, ..., N$. We will let $T = 1$ for this section for convenience. By inserting the graded meshes into (3.2), you will get,

$$y(t_{n+1}^r) = \sum_{k=0}^{\lceil\alpha\rceil-1} (y_0^r)^{(k)} \frac{(t_{n+1}^r)^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1} g(s) ds. \tag{5.15}$$

By using a similar technique to section 2, but by using a graded mesh defined by (**??**), we can find Now by using quadratic interpolation to approximate $g(s)$, ($g(s) \approx P_2(s)$), we will get,

$$P_2(s) = \frac{(s - t_1^r)(s - t_2^r)}{(t_0^r - t_1^r)(t_0^r - t_2^r)} g(t_0^r) + \frac{(s - t_0^r)(s - t_2^r)}{(t_1^r - t_0^r)(t_1^r - t_2^r)} g(t_1^r) + \frac{(s - t_0^r)(s - t_1^r)}{(t_2^r - t_0^r)(t_2^r - t_1^r)} g(t_2^r), \tag{5.16}$$

and,

$$P_2(s) = \frac{(s-t_j^r)(s-t_{j+1}^r)}{(t_{j-1}^r-t_j^r)(t_{j-1}^r-t_{j+1}^r)}g(t_{j-1}^r) + \frac{(s-t_{j-1}^r)(s-t_{j+1}^r)}{(t_j^r-t_{j-1}^r)(t_j^r-t_{j+1}^r)}g(t_j^r) + \frac{(s-t_{j-1}^r)(s-t_j^r)}{(t_{j+1}^r-t_{j-1}^r)(t_{j+1}^r-t_j^r)}g(t_{j+1}^r), \tag{5.17}$$

for $[t_0^r, t_1^r]$ and $[t_j^r, t_{j+1}^r]$, $j = 1, 2, ..., n$, respectively. By using t_k^r instead of t_k, when $k = 0, 1, 2, ..., n+1$, (5.6) and (5.7) will become,

$$\begin{aligned} IIII_{1r} = {} & \frac{g(t_0^r)}{(t_0^r-t_1^r)(t_0^r-t_2^r)}\Bigg[\frac{1}{\alpha+2}[(t_{n+1}^r-t_0^r)^{\alpha+2}-(t_{n+1}^r-t_1^r)^{\alpha+2}] \\ & + \frac{2t_{n+1}^r-t_1^r-t_2^r}{\alpha+1}[(t_{n+1}^r-t_1^r)^{\alpha+1}-(t_{n+1}^r-t_0^r)^{\alpha+1}] \\ & + \frac{(t_{n+1}^r-t_2^r)(t_{n+1}^r-t_1^r)}{\alpha}[(t_{n+1}^r-t_0^r)^{\alpha}-(t_{n+1}^r-t_1^r)^{\alpha}]\Bigg] \\ & + \frac{g(t_1^r)}{(t_1^r-t_0^r)(t_1^r-t_2^r)}\Bigg[\frac{1}{\alpha+2}[(t_{n+1}^r-t_0^r)^{\alpha+2}-(t_{n+1}^r-t_1^r)^{\alpha+2}] \\ & + \frac{2t_{n+1}^r-t_0^r-t_2^r}{\alpha+1}[(t_{n+1}^r-t_1^r)^{\alpha+1}-(t_{n+1}^r-t_0^r)^{\alpha+1}] \\ & + \frac{(t_{n+1}^r-t_0^r)(t_{n+1}^r-t_2^r)}{\alpha}[(t_{n+1}^r-t_0^r)^{\alpha}-(t_{n+1}^r-t_1^r)^{\alpha}]\Bigg] \\ & + \frac{g(t_2^r)}{(t_2^r-t_0^r)(t_2^r-t_1^r)}\Bigg[\frac{1}{\alpha+2}[(t_{n+1}^r-t_0^r)^{\alpha+2}-(t_{n+1}^r-t_1^r)^{\alpha+2}] \\ & + \frac{2t_{n+1}^r-t_0^r-t_1^r}{\alpha+1}[(t_{n+1}^r-t_1^r)^{\alpha+1}-(t_{n+1}^r-t_0^r)^{\alpha+1}] \\ & + \frac{(t_{n+1}^r-t_0^r)(t_{n+1}^r-t_1^r)}{\alpha}[(t_{n+1}^r-t_0^r)^{\alpha}-(t_{n+1}^r-t_1^r)^{\alpha}]\Bigg]. \end{aligned} \tag{5.18}$$

$$\begin{aligned} IIII_{2r} = {} & \frac{g(t_{j-1}^r)}{(t_{j-1}^r-t_j^r)(t_{j-1}^r-t_{j+1}^r)}\Bigg[\frac{1}{\alpha+2}[(t_{n+1}^r-t_j^r)^{\alpha+2}-(t_{n+1}^r-t_{j+1}^r)^{\alpha+2}] \\ & + \frac{2t_{n+1}^r-t_j^r-t_{j+1}^r}{\alpha+1}[(t_{n+1}^r-t_{j+1}^r)^{\alpha+1}-(t_{n+1}^r-t_j^r)^{\alpha+1}] \\ & + \frac{(t_{n+1}^r-t_j^r)(t_{n+1}^r-t_{j+1}^r)}{\alpha}[(t_{n+1}^r-t_j^r)^{\alpha}-(t_{n+1}^r-t_{j+1}^r)^{\alpha}]\Bigg] \\ & + \frac{g(t_j^r)}{(t_j^r-t_{j-1}^r)(t_j^r-t_{j+1}^r)}\Bigg[\frac{1}{\alpha+2}[(t_{n+1}^r-t_j^r)^{\alpha+2}-(t_{n+1}^r-t_{j+1}^r)^{\alpha+2}] \\ & + \frac{2t_{n+1}^r-t_{j-1}^r-t_{j+1}^r}{\alpha+1}[(t_{n+1}^r-t_{j+1}^r)^{\alpha+1}-(t_{n+1}^r-t_j^r)^{\alpha+1}] \\ & + \frac{(t_{n+1}^r-t_{j-1}^r)(t_{n+1}^r-t_{j+1}^r)}{\alpha}[(t_{n+1}^r-t_j^r)^{\alpha}-(t_{n+1}^r-t_{j+1}^r)^{\alpha}]\Bigg] \\ & + \frac{g(t_{j+1}^r)}{(t_{j+1}^r-t_{j-1}^r)(t_{j+1}^r-t_j^r)}\Bigg[\frac{1}{\alpha+2}[(t_{n+1}^r-t_j^r)^{\alpha+2}-(t_{n+1}^r-t_{j+1}^r)^{\alpha+2}] \\ & + \frac{2t_{n+1}^r-t_{j-1}^r-t_j^r}{\alpha+1}[(t_{n+1}^r-t_{j+1}^r)^{\alpha+1}-(t_{n+1}^r-t_j^r)^{\alpha+1}] \\ & + \frac{(t_{n+1}^r-t_{j-1}^r)(t_{n+1}^r-t_j^r)}{\alpha}[(t_{n+1}^r-t_j^r)^{\alpha}-(t_{n+1}^r-t_{j+1}^r)^{\alpha}]\Bigg]. \end{aligned} \tag{5.19}$$

After we substitute (5.14) into both equations we will obtain,

$$
\begin{aligned}
IIII_{1r} = \frac{N^{-r\alpha}}{\alpha(\alpha+1)(\alpha+2)} \Bigg[\frac{g(t_0^r)}{2^r} \Big[& \alpha(\alpha+1)[(n+1)^{r(\alpha+2)} - ((n+1)^r - 1)^{\alpha+2}] \\
& + \alpha(\alpha+2)(2(n+1)^r - 1 - 2^r)[((n+1)^r - 1)^{\alpha+1} - (n+1)^{r(\alpha+1)}] \\
& + (\alpha+1)(\alpha+2)((n+1)^r - 2^r)((n+1)^r - 1)[(n+1)^{r\alpha} - ((n+1)^r - 1)^{\alpha}] \Big] \\
+ \frac{g(t_1^r)}{(1-2^r)} \Big[& \alpha(\alpha+1)[(n+1)^{r(\alpha+2)} - ((n+1)^r - 1)^{\alpha+2}] \\
& + \alpha(\alpha+2)(2(n+1)^r - 2^r)[((n+1)^r - 1)^{\alpha+1} - (n+1)^{r(\alpha+1)}] \\
& + (\alpha+1)(\alpha+2)((n+1)^r)((n+1)^r - 2^r)[((n+1)^{r\alpha} - ((n+1)^r - 1)^{\alpha}] \Big] \\
+ \frac{g(t_2^r)}{(2^r)(2^r-1)} \Big[& \alpha(\alpha+1)[((n+1)^{r(\alpha+2)} - ((n+1)^r - 1)^{\alpha+2}] \\
& + \alpha(\alpha+2)(2(n+1)^r - 1)[((n+1)^r - 1)^{\alpha+1} - (n+1)^{r(\alpha+1)}] \\
& + (\alpha+1)(\alpha+2)((n+1)^r)((n+1)^r - 1)[((n+1)^{r\alpha} - ((n+1)^r - 1)^{\alpha}] \Big] \Bigg].
\end{aligned}
$$

and,

$$
\begin{aligned}
IIII_{2r} = \frac{N^{-r\alpha}}{\alpha(\alpha+1)(\alpha+2)} \Bigg[& \frac{g(t_{j-1}^r)}{((j-1)^r - j^r)((j-1)^r - (j+1)^r)} \\
& \times \Big[\alpha(\alpha+1)[((n+1)^r - j^r)^{\alpha+2} - ((n+1)^r - (j+1)^r)^{\alpha+2}] \\
& + \alpha(\alpha+2)(2(n+1)^r - j^r - (j+1)^r) \\
& \times [((n+1)^r - (j+1)^r)^{\alpha+1} - ((n+1)^r - j^r)^{\alpha+1}] \\
& + (\alpha+1)(\alpha+2)((n+1)^r - j^r)((n+1)^r - (j+1)^r) \\
& \times [((n+1)^r - j^r)^{\alpha} - ((n+1)^r - (j+1)^r)^{\alpha}] \Big] \\
+ & \frac{g(t_j^r)}{(j^r - (j-1)^r)(j^r - (j+1)^r)} \\
& \times \Big[\alpha(\alpha+1)[((n+1)^r - j^r)^{\alpha+2} - ((n+1)^r - (j+1)^r)^{\alpha+2}] \\
& + \alpha(\alpha+2)(2(n+1)^r - (j-1)^r - (j+1)^r) \\
& \times [((n+1)^r - (j+1)^r)^{\alpha+1} - ((n+1)^r - j^r)^{\alpha+1}] \\
& + (\alpha+1)(\alpha+2)((n+1)^r - (j-1)^r)((n+1)^r - (j+1)^r) \\
& \times [((n+1)^r - j^r)^{\alpha} - ((n+1)^r - (j+1)^r)^{\alpha}] \Big]
\end{aligned}
\tag{5.20}
$$

$$
\begin{aligned}
&+ \frac{g(t_{j+1}^r)}{((j+1)^r-(j-1)^r)((j+1)^r-j^r)} \\
&\times \Big[\alpha(\alpha+1)[((n+1)^r-j^r)^{\alpha+2}-((n+1)^r-(j+1)^r)^{\alpha+2}] \\
&+\alpha(\alpha+2)(2(n+1)^r-(j-1)^r-j^r) \\
&\times [((n+1)^r-(j+1)^r)^{\alpha+1}-((n+1)^r-j^r)^{\alpha+1}] \\
&+(\alpha+1)(\alpha+2)((n+1)^r-(j-1)^r)((n+1)^r-j^r) \\
&\times [((n+1)^r-j^r)^{\alpha}-((n+1)^r-(j+1)^r)^{\alpha}]\Big]\Big].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
y(t_{n+1}^r) &= y_0^r + \frac{1}{\Gamma(\alpha)}\Big[\int_{t_0^r}^{t_1^r}(t_{n+1}^r-s)^{\alpha-1}g(s)ds + \sum_{j=1}^{n}\int_{t_j^r}^{t_{j+1}^r}(t_{n+1}^r-s)^{\alpha-1}g(s)ds\Big], \\
&= y_0^r + \frac{1}{\Gamma(\alpha)}\Big[IIII_{1r} + \sum_{j=1}^{n} IIII_{2r}\Big].
\end{aligned}
\tag{5.21}
$$

Now we have found $IIII_{1r}$ and $IIII_{2r}$, we can put these back into (5.21). First we will calculate the summation part of the parenthesis like so,

$$
\begin{aligned}
\sum_{j=1}^{n} IIII_{2r} &= \frac{N^{-r\alpha}}{\alpha(\alpha+1)(\alpha+2)}\Bigg[\frac{g(t_0^r)}{2^r}\Big[\alpha(\alpha+1)[((n+1)^r-1)^{\alpha+2}-((n+1)^r-2^r)^{\alpha+2}] \\
&+\alpha(\alpha+2)(2(n+1)^r-1-2^r)[((n+1)^r-2^r)^{\alpha+1}-((n+1)^r-1)^{\alpha+1}] \\
&+(\alpha+1)(\alpha+2)((n+1)^r-1)((n+1)^r-2^r)[((n+1)^r-1)^{\alpha}-((n+1)^r-2^r)^{\alpha}]\Big] \\
&+\frac{g(t_1^r)}{1-2^r}\Big[\alpha(\alpha+1)[((n+1)^r-1)^{\alpha+2}-((n+1)^r-2^r)^{\alpha+2}] \\
&+\alpha(\alpha+2)(2(n+1)^r-2^r)[((n+1)^r-2^r)^{\alpha+1}-((n+1)^r-1)^{\alpha+1}] \\
&+(\alpha+1)(\alpha+2)((n+1)^r)((n+1)^r-2^r)[((n+1)^r-1)^{\alpha}-((n+1)^r-2^r)^{\alpha}]\Big] \\
&+\frac{g(t_2^r)}{(2^r)(2^r-1)}\Big[\alpha(\alpha+1)[((n+1)^r-1)^{\alpha+2}-((n+1)^r-2^r)^{\alpha+2}] \\
&+\alpha(\alpha+2)(2(n+1)^r-1)[((n+1)^r-2^r)^{\alpha+1}-((n+1)^r-1)^{\alpha+1}] \\
&+(\alpha+1)(\alpha+2)((n+1)^r)((n+1)^r-1)[((n+1)^r-1)^{\alpha}-((n+1)^r-2^r)^{\alpha}]\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{g(t_1^r)}{(1-2^r)(1-3^r)}\Big[\alpha(\alpha+1)[((n+1)^r-2^r)^{\alpha+2}-((n+1)^r-3^r)^{\alpha+2}] \\
&+\alpha(\alpha+2)(2(n+1)^r-2^r-3^r)[((n+1)^r-3^r)^{\alpha+1}-((n+1)^r-2^r)^{\alpha+1}] \\
&+(\alpha+1)(\alpha+2)((n+1)^r-2^r)((n+1)^r-3^r)[((n+1)^r-2^r)^{\alpha}-((n+1)^r-3^r)^{\alpha}]\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{g(t_2^r)}{(2^r-1)(2^r-3^r)}\Big[\alpha(\alpha+1)[((n+1)^r-2^r)^{\alpha+2}-((n+1)^r-3^r)^{\alpha+2}] \\
&\quad + \alpha(\alpha+2)(2(n+1)^r-1-3^r)[((n+1)^r-3^r)^{\alpha+1}-((n+1)^r-2^r)^{\alpha+1}] \\
&\quad + (\alpha+1)(\alpha+2)((n+1)^r-1)((n+1)^r-3^r)[((n+1)^r-2^r)^{\alpha}-((n+1)^r-3^r)^{\alpha}]\Big] \\
&+ \frac{g(t_3^r)}{(3^r-1)(3^r-2^r)}\Big[\alpha(\alpha+1)[((n+1)^r-2^r)^{\alpha+2}-((n+1)^r-3^r)^{\alpha+2}] \\
&\quad + \alpha(\alpha+2)(2(n+1)^r-1-2^r)[((n+1)^r-3^r)^{\alpha+1}-((n+1)^r-2^r)^{\alpha+1}] \\
&\quad + (\alpha+1)(\alpha+2)((n+1)^r-1)((n+1)^r-2^r)[((n+1)^r-2^r)^{\alpha}-((n+1)^r-3^r)^{\alpha}]\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{g(t_2^r)}{(2^r-3^r)(2^r-4^r)}\Big[\alpha(\alpha+1)[((n+1)^r-3^r)^{\alpha+2}-((n+1)^r-4^r)^{\alpha+2}] \\
&\quad + \alpha(\alpha+2)(2(n+1)^r-3^r-4^r)[((n+1)^r-4^r)^{\alpha+1}-((n+1)^r-3^r)^{\alpha+1}] \\
&\quad + (\alpha+1)(\alpha+2)((n+1)^r-3^r)((n+1)^r-4^r)[((n+1)^r-3^r)^{\alpha}-((n+1)^r-4^r)^{\alpha}]\Big] \\
&+ \frac{g(t_3^r)}{(3^r-2^r)(3^r-4^r)}\Big[\alpha(\alpha+1)[((n+1)^r-3^r)^{\alpha+2}-((n+1)^r-4^r)^{\alpha+2}] \\
&\quad + \alpha(\alpha+2)(2(n+1)^r-2^r-4^r)[((n+1)^r-4^r)^{\alpha+1}-((n+1)^r-3^r)^{\alpha+1}] \\
&\quad + (\alpha+1)(\alpha+2)((n+1)^r-2^r)((n+1)^r-4^r)[((n+1)^r-3^r)^{\alpha}-((n+1)^r-4^r)^{\alpha}]\Big] \\
&+ \frac{g(t_4^r)}{(4^r-2^r)(4^r-3^r)}\Big[\alpha(\alpha+1)[((n+1)^r-3^r)^{\alpha+2}-((n+1)^r-4^r)^{\alpha+2}] \\
&\quad + \alpha(\alpha+2)(2(n+1)^r-2^r-3^r)[((n+1)^r-4^r)^{\alpha+1}-((n+1)^r-3^r)^{\alpha+1}] \\
&\quad + (\alpha+1)(\alpha+2)((n+1)^r-2^r)((n+1)^r-3^r)[((n+1)^r-3^r)^{\alpha}-((n+1)^r-4^r)^{\alpha}]\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{g(t_3^r)}{(3^r-4^r)(3^r-5^r)}\Big[\alpha(\alpha+1)[((n+1)^r-4^r)^{\alpha+2}-((n+1)^r-5^r)^{\alpha+2}] \\
&\quad + \alpha(\alpha+2)(2(n+1)^r-4^r-5^r)[((n+1)^r-5^r)^{\alpha+1}-((n+1)^r-4^r)^{\alpha+1}] \\
&\quad + (\alpha+1)(\alpha+2)((n+1)^r-4^r)((n+1)^r-5^r)[((n+1)^r-4^r)^{\alpha}-((n+1)^r-5^r)^{\alpha}]\Big] \\
&+ \frac{g(t_4^r)}{(4^r-3^r)(4^r-5^r)}\Big[\alpha(\alpha+1)[((n+1)^r-4^r)^{\alpha+2}-((n+1)^r-5^r)^{\alpha+2}] \\
&\quad + \alpha(\alpha+2)(2(n+1)^r-4^r-5^r)[((n+1)^r-5^r)^{\alpha+1}-((n+1)^r-4^r)^{\alpha+1}] \\
&\quad + (\alpha+1)(\alpha+2)((n+1)^r-3^r)((n+1)^r-5^r)[((n+1)^r-4^r)^{\alpha}-((n+1)^r-5^r)^{\alpha}]\Big] \\
&+ \frac{g(t_5^r)}{(5^r-4^r)(5^r-4^r)}\Big[\alpha(\alpha+1)[((n+1)^r-4^r)^{\alpha+2}-((n+1)^r-5^r)^{\alpha+2}] \\
&\quad + \alpha(\alpha+2)(2(n+1)^r-3^r-4^r)[((n+1)^r-5^r)^{\alpha+1}-((n+1)^r-4^r)^{\alpha+1}] \\
&\quad + (\alpha+1)(\alpha+2)((n+1)^r-3^r)((n+1)^r-4^r)[((n+1)^r-4^r)^{\alpha}-((n+1)^r-5^r)^{\alpha}]\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \ldots \\
&+ \frac{g(t^r_{n-2})}{((n-2)^r-(n-1)^r)((n-2)^r-n^r)} \\
&\quad \times \Big[\alpha(\alpha+1)[((n+1)^r-(n-1)^r)^{\alpha+2}-((n+1)^r-n^r)^{\alpha+2}] \\
&\quad + \alpha(\alpha+2)(2(n+1)^r-(n-1)^r-n^r) \\
&\quad \times [((n+1)^r-n^r)^{\alpha+1}-((n+1)^r-(n-1)^r)^{\alpha+1}] \\
&\quad + (\alpha+1)(\alpha+2)((n+1)^r-(n-1)^r)((n+1)^r-n^r) \\
&\quad \times [((n+1)^r-(n-1)^r)^{\alpha}-((n+1)^r-n^r)^{\alpha}]\Big] \\
&+ \frac{g(t^r_{n-1})}{((n-1)^r-(n-2)^r)((n-1)^r-n^r)} \\
&\quad \times \Big[\alpha(\alpha+1)[((n+1)^r-(n-1)^r)^{\alpha+2}-((n+1)^r-n^r)^{\alpha+2}] \\
&\quad + \alpha(\alpha+2)(2(n+1)^r-(n-2)^r-n^r) \\
&\quad \times [((n+1)^r-n^r)^{\alpha+1}-((n+1)^r-(n-1)^r)^{\alpha+1}] \\
&\quad + (\alpha+1)(\alpha+2)((n+1)^r-(n-2)^r)((n+1)^r-n^r) \\
&\quad \times [((n+1)^r-(n-1)^r)^{\alpha}-((n+1)^r-n^r)^{\alpha}]\Big] \\
&+ \frac{g(t^r_{n})}{(n^r-(n-2)^r)(n^r-(n-1)^r)} \\
&\quad \times \Big[\alpha(\alpha+1)[((n+1)^r-(n-1)^r)^{\alpha+2}-((n+1)^r-n^r)^{\alpha+2}] \\
&\quad + \alpha(\alpha+2)(2(n+1)^r-(n-2)^r-(n-1)^r) \\
&\quad \times [((n+1)^r-n^r)^{\alpha+1}-((n+1)^r-(n-1)^r)^{\alpha+1}] \\
&\quad + (\alpha+1)(\alpha+2)((n+1)^r-(n-2)^r)((n+1)^r-(n-1)^r) \\
&\quad \times [((n+1)^r-(n-1)^r)^{\alpha}-((n+1)^r-n^r)^{\alpha}]\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{g(t^r_{n-1})}{((n-1)^r-n^r)((n-1)^r-(n+1)^r)}\Big[\alpha(\alpha+1)[((n+1)^r-n^r)^{\alpha+2}] \\
&\quad - \alpha(\alpha+2)((n+1)^r-n^r)[((n+1)^r-n^r)^{\alpha+1}]\Big] \\
&+ \frac{g(t^r_{n})}{(n^r-(n-1)^r)(n^r-(n+1)^r)}\Big[\alpha(\alpha+1)[((n+1)^r-n^r)^{\alpha+2}] \\
&\quad - \alpha(\alpha+2)((n+1)^r-(n-1)^r)[((n+1)^r-n^r)^{\alpha+1}]\Big] \\
&+ \frac{g(t^r_{n+1})}{((n+1)^r-(n-1)^r)((n+1)^r-n^r)}\Big[\alpha(\alpha+1)[((n+1)^r-n^r)^{\alpha+2}] \\
&\quad - \alpha(\alpha+2)(2(n+1)^r-(n-1)^r-n^r)[((n+1)^r-n^r)^{\alpha+1}] \\
&\quad + (\alpha+1)(\alpha+2)((n+1)^r-(n-1)^r)((n+1)^r-n^r)[((n+1)^r-n^r)^{\alpha}]\Big]\Bigg],
\end{aligned}
$$

Then if we collect all the same terms together from the parenthesis in (5.22), we will get,

$$c_{j,n+1}^r = \frac{N^{-r\alpha}}{\alpha(\alpha+1)(\alpha+2)} \times \begin{cases} Y0^r + X0(1)^r, & \text{if } j=0, \\ Y1^r + X1(1)^r + X0(2)^r, & \text{if } j=1, \\ Y2^r + X2(1)^r + X1(2)^r + X0(3)^r, & \text{if } j=2, \\ X2(j-1)^r + X1(j)^r + X0(j+1)^r, & \text{if } 3 \le j \le n-1, \\ X2(n-1)^r + X1(n)^r, & \text{if } j=n, \\ X2(n)^r, & \text{if } j=n+1, \end{cases} \tag{5.22}$$

and,

$$\begin{aligned} X0(j)^r = {} & \frac{1}{((j-1)^r - j^r)((j-1)^r - (j+1)^r)} \Big(\alpha(\alpha+1)[((n+1)^r - j^r)^{\alpha+2} \\ & - ((n+1)^r - (j+1)^r)^{\alpha+2}] + \alpha(\alpha+2)(2(n+1)^r - j^r - (j+1)^r) \\ & \times [((n+1)^r - (j+1)^r)^{\alpha+1} - ((n+1)^r - j^r)^{\alpha+1}] \\ & + (\alpha+1)(\alpha+2)((n+1)^r - j^r)((n+1)^r - (j+1)^r) \\ & \times [((n+1)^r - j^r)^{\alpha} - ((n+1)^r - (j+1)^r)^{\alpha}] \Big), \\ X1(j)^r = {} & \frac{1}{((j^r - (j-1)^r)(j^r - (j+1)^r)} \Big(\alpha(\alpha+1)[((n+1)^r - j^r)^{\alpha+2} \\ & - ((n+1)^r - (j+1)^r)^{\alpha+2}] \\ & + \alpha(\alpha+2)(2(n+1)^r - (j-1)^r - (j+1)^r) \\ & \times [((n+1)^r - (j+1)^r)^{\alpha+1} - ((n+1)^r - j^r)^{\alpha+1}] \\ & + (\alpha+1)(\alpha+2)((n+1)^r - (j-1)^r)((n+1)^r - (j+1)^r) \\ & \times [((n+1)^r - j^r)^{\alpha} - ((n+1)^r - (j+1)^r)^{\alpha}] \Big), \\ X2(j)^r = {} & \frac{1}{((j+1)^r - (j-1)^r)((j+1)^r - j^r)} \Big(\alpha(\alpha+1)[((n+1)^r - j^r)^{\alpha+2} \\ & - ((n+1)^r - (j+1)^r)^{\alpha+2}] \\ & + \alpha(\alpha+2)(2(n+1)^r - (j-1)^r - j^r) \\ & \times [((n+1)^r - (j+1)^r)^{\alpha+1} - ((n+1)^r - j^r)^{\alpha+1}] \\ & + (\alpha+1)(\alpha+2)((n+1)^r - (j-1)^r)((n+1)^r - j^r) \\ & \times [((n+1)^r - j^r)^{\alpha} - ((n+1)^r - (j+1)^r)^{\alpha}] \Big), \\ Y0^r = {} & \frac{1}{2^r} \Big(\alpha(\alpha+1)[(n+1)^{r(\alpha+2)} - ((n+1)^r - 1)^{\alpha+2}] \\ & + \alpha(\alpha+2)(2(n+1)^r - 1 - 2^r)[((n+1)^r - 1)^{\alpha+1} - (n+1)^{r(\alpha+1)}] \\ & + (\alpha+1)(\alpha+2)((n+1)^r - 2^r)((n+1)^r - 1)[(n+1)^{r\alpha} - ((n+1)^r - 1)^{\alpha}] \Big), \end{aligned}$$

$$
\begin{aligned}
Y1^r = \frac{1}{1-2^r}\Big(&\alpha(\alpha+1)[(n+1)^{r(\alpha+2)} - ((n+1)^r-1)^{\alpha+2}] \\
&+ \alpha(\alpha+2)(2(n+1)^r - 2^r)[((n+1)^r-1)^{\alpha+1} - (n+1)^{r(\alpha+1)}] \\
&+ (\alpha+1)(\alpha+2)((n+1)^r)((n+1)^r - 2^r)[((n+1)^{r\alpha} - ((n+1)^r-1)^{\alpha}], \\
Y2^r = \frac{1}{2^r(2^r-1)}\Big(&\alpha(\alpha+1)[((n+1)^{r(\alpha+2)} - ((n+1)^r-1)^{\alpha+2}] \\
&+ \alpha(\alpha+2)(2(n+1)^r - 1)[((n+1)^r-1)^{\alpha+1} - (n+1)^{r(\alpha+1)}] \\
&+ (\alpha+1)(\alpha+2)((n+1)^r)((n+1)^r - 1)[((n+1)^{r\alpha} - ((n+1)^r-1)^{\alpha}].
\end{aligned} \tag{5.23}
$$

Note: If $r = 1$ we will get (5.8) and $n \geq 4$ for these wights to work.

Because,

$$
\int_0^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1} g(s)ds \approx \sum_{j=0}^{n+1} c_{j,n+1}^r g(t_j^r),
$$

we can replace this in (5.15) and use the function $f(s, y(s))$ instead of $g(s)$, to get

$$
y(t_{n+1}^r) = \sum_{k=0}^{\lceil\alpha\rceil-1} (y_0^r)^{(k)} \frac{(t_{n+1}^r)^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n+1} c_{j,n+1}^r f(t_j^r, y(t_j^r)).
$$

Let $y_j^r \approx y(t_j)^r$ be the approximation of $y(t_j)^r$ and let $y_{n+1}^{r,P} \approx y_{n+1}^r$ be the approximation of y_{n+1}^r, so we don't have an implicit equation. Hence,

$$
\begin{aligned}
y_{n+1}^r &= \sum_{k=0}^{\lceil\alpha\rceil-1} (y_0^r)^{(k)} \frac{(t_{n+1}^r)^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n+1} c_{j,n+1}^r f(t_j^r, y_j^r), \\
&= \sum_{k=0}^{\lceil\alpha\rceil-1} (y_0^r)^{(k)} \frac{(t_{n+1}^r)^k}{k!} + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^{n} c_{j,n+1}^r f(t_j^r, y_j^r) + c_{n+1,n+1}^r f(t_{n+1}^r, y_{n+1}^{r,P}) \right),
\end{aligned}
$$

and to calculate $y_{n+1}^{r,P}$ we will use the trapezoidal rule to interpolate (5.15). Using the same idea as section 3 and (4.5), we know that,

$$
\int_0^{t_{n+1}^r} (t_{n+1}^r - s)^{\alpha-1} g(s)ds = \sum_{j=0}^{n+1} a_{j,n+1}^r g(t_j^r),
$$

where,

$$
a_{j,n+1}^r = \frac{N^{-r\alpha}}{\alpha(\alpha+1)} \times \begin{cases}
\Big([(n+1)^r - 1]^{\alpha+1} - (n+1)^{r(\alpha+1)} \\
\qquad +(\alpha+1)(n+1)^{r\alpha}\Big), & \text{if } j = 0, \\
\Big(\frac{[(n+1)^r-(j-1)^r]^{\alpha+1} - [(n+1)^r - j^r]^{\alpha+1}}{j^r - (j-1)^r} \\
\qquad + \frac{[(n+1)^r-(j+1)^r]^{\alpha+1} - [(n+1)^r - j^r]^{\alpha+1}}{(j+1)^r - j^r}\Big), & \text{if } 1 \leq j \leq n, \\
\Big((n+1)^r - n^r\Big)^{\alpha}, & \text{if } j = n+1.
\end{cases} \tag{5.24}
$$

Then by substituting this into (5.15), like before we will get,

$$\begin{aligned}
y_{n+1}^{r,P} &= \sum_{k=0}^{\lceil\alpha\rceil-1}(y_0^r)^{(k)}\frac{(t_{n+1}^r)^k}{k!} + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n+1} a_{j,n+1}^r f(t_j^r, y_j^r),\\
&= \sum_{k=0}^{\lceil\alpha\rceil-1}(y_0^r)^{(k)}\frac{(t_{n+1}^r)^k}{k!} + \frac{1}{\Gamma(\alpha)}\left(\sum_{j=0}^{n} a_{j,n+1}^r f(t_j^r, y_j^r) + a_{n+1,n+1}^r f(t_{n+1}^r, y_{n+1}^{r,PP})\right),
\end{aligned}$$

where $y_{n+1}^{r,PP}$ is the approximation of $y_{n+1}^{r,P}$ so we don't have to solve an implicit equation again. Hence, we can interpolate this to find $y_{n+1}^{r,PP}$ by using the rectangle rule instead of the trapezoidal rule. Like before, the calculations for this can be found in section 3 and (4.8), meaning,

$$\int_0^{t_{n+1}^r}(t_{n+1}^r - s)^{\alpha-1}g(s)ds = \sum_{j=0}^{n} b_{j,n+1}^r g(t_j^r),$$

where,

$$b_{j,n+1}^r = \frac{N^{-r\alpha}}{\alpha}\left[\Big((n+1)^r - j^r\Big)^\alpha - \Big((n+1)^r - (j+1)^r\Big)^\alpha\right]. \tag{5.25}$$

Giving us the following equation to find y_{n+1}^{PP},

$$\begin{aligned}
y_{n+1}^{PP} &= y_0^r + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n} b_{j,n+1}^r f(t_j^r, y_j^r),\\
&= y_0^r + \frac{1}{\Gamma(\alpha)}\left(\sum_{j=0}^{n} b_{j,n+1}^r f(t_j^r, y_j^r)\right).
\end{aligned}$$

Therefore we can calculate the method by,

$$\begin{cases}
y_{n+1}^{r,PP} = \sum_{k=0}^{\lceil\alpha\rceil-1}(y_0^r)^{(k)}\frac{(t_{n+1}^r)^k}{k!} + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n} b_{j,n+1}^r f(t_j^r, y_j^r),\\
y_{n+1}^{r,P} = \sum_{k=0}^{\lceil\alpha\rceil-1}(y_0^r)^{(k)}\frac{(t_{n+1}^r)^k}{k!} + \frac{1}{\Gamma(\alpha)}\big(\sum_{j=0}^{n} a_{j,n+1}^r f(t_j^r, y_j^r) + a_{n+1,n+1}^r f(t_{n+1}^r, y_{n+1}^{r,PP})\big),\\
y_{n+1}^{r} = \sum_{k=0}^{\lceil\alpha\rceil-1}(y_0^r)^{(k)}\frac{(t_{n+1}^r)^k}{k!} + \frac{1}{\Gamma(\alpha)}\big(\sum_{j=0}^{n} c_{j,n+1}^r f(t_j^r, y_j^r) + c_{n+1,n+1}^r f(t_{n+1}^r, y_{n+1}^{r,P})\big),\\
y_0^r,\ (y_0^r)^{(1)},\ \dots,\ (y_0^r)^{(\lceil\alpha\rceil-1)} \text{ and,}\\
y_1^r,\ (y_1^r)^{(1)},\ \dots,\ (y_1^r)^{(\lceil\alpha\rceil-1)} \text{ is given.}
\end{cases} \tag{5.26}$$

where the weights $b_{j,n+1}^r$, $a_{j,n+1}^r$ and $c_{j,n+1}^r$ are given by (5.25), (5.24) and (5.22), respectively.

Chapter 6

Numerical Examples

In this section we will use some examples to compare the lower order fractional Adams-Bashforth-Moutlon method from [25] with a higher order fractional Adams-Bashforth-Moutlon method using uniform and graded meshes.

1 Example 1

From [25] we consider, with $0 < \alpha < 1$,

$$\begin{aligned} D_*^\alpha y(t) &= f(t, y(t)), \qquad t \in (0, T], \\ y(0) &= y_0, \end{aligned}$$

where $y_0 = 0$, and

$$f(t, y) = \frac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha} + (t^3)^2 - y^2.$$

We already know that the exact solution of this is $y(t) = t^3$, and $D_*^\alpha y(t) = \frac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha}$. We know that $D_*^\alpha y(t) \in C^2[0, T]$, which means that $D_*^\alpha y(t)$ has a second derivative. For our code to work we need at least $n = 4$, so for this example we need the initial values for y_0, y_1, y_2, y_3, y_4. We can calculate these by taking the t values at the exact value. Like in the paper, let N be a positive integer and $0 = t_0^r < t_1^r < ... < t_N^r = T$ be the graded mesh on $[0, T]$, where $t_j^r = T(j/N)^r$, $j = 0, 1, 2, ..., N$ with $r \geq 1$. We will choose $T = 1$. We will first choose $r = 1$, uniform mesh to compare the results in Table 1, [25]. For varying values of N, we will find the maximum nodal errors and calculate the experimental order of convergence (EOC), by $log_2(\frac{||e_N||_\infty}{||e_2N||_\infty})$. If the method isn't a higher order method, from Theorem 4.6 we know,

$$\max_{0 \leq j \leq N} |y(t_j) - y_j| = \begin{cases} O(h^2), & \text{if } \alpha \geq 1, \\ O(h^{\alpha+1}), & \text{if } \alpha < 1. \end{cases}$$

We can see how having a higher order method will effect the EOC and errors. The MATLAB code to calculate these results can be found in Appendix 1.1.

	N=40	N=80	N=160	N=320	N=640
$\alpha = 0.1$	3.0241e-02	1.3725e-02	6.2864e-03	2.8806e-03	1.3146e-03
	1.1397	1.1265	1.1259	1.1318	
$\alpha = 0.3$	9.5749e-03	2.9986e-03	9.3581e-04	2.9169e-04	9.1130e-05
	1.6750	1.6800	1.6818	1.6784	
$\alpha = 0.5$	1.5804e-03	3.6840e-04	8.7038e-05	2.0826e-05	5.0373e-06
	2.1009	2.0816	2.0633	2.0476	
$\alpha = 0.7$	2.6697e-04	4.8507e-05	8.9118e-06	1.6512e-06	3.0786e-07
	2.4604	2.4444	2.4322	2.4232	
$\alpha = 0.9$	4.8549e-05	6.8444e-06	9.6858e-07	1.3743e-07	1.9535e-08
	2.8265	2.8210	2.8172	2.8145	

Table 6.1: Maximum nodal errors and orders of convergence when $r = 1$

The convergence orders are indeed almost $O(h^{1+2\alpha})$. Compared to Table 1 in [25], the error's are smaller for a higher order method, meaning our higher order method is more precise and closer to the exact solution, we can also see that the EOC is a lot higher than [25].

2 Example 2

We can now improve Example 1 and generalise it for t^β. From [25], we will consider, with $0 < \alpha < 1$, $0 < \beta < 1$ and $\alpha < \beta$,

$$\begin{aligned} D_*^\alpha y(t) &= \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha} + t^{2\beta} - y^2, \qquad t \in (0, T], \\ y(0) &= y_0, \end{aligned}$$

where $y_0 = 0$, and the exact solution is t^β. Giving us, $D_*^\alpha y(t) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}$, which implies that the regularity of $D_*^\alpha y(t)$ behaves as $t^{\beta-\alpha}$, Thus we see that $D_*^\alpha y(t)$ satisfies Assumption 3.1. We will choose $\beta = 0.9$ but consider different values of α and r.

	N=40	N=80	N=160	N=320	N=640
$\alpha = 0.1$	7.7660e-03	3.5306e-03	1.6163e-03	7.4237e-04	3.4084e-04
	1.1373	1.1272	1.1225	1.1231	
$\alpha = 0.3$	2.0438e-03	6.5840e-04	2.0914e-04	6.6066e-05	2.0871e-05
	1.6342	1.6545	1.6625	1.6624	
$\alpha = 0.5$	9.4590e-04	5.2498e-04	2.8617e-04	1.5443e-04	8.2982e-05
	0.8494	0.8754	0.8899	0.8961	
$\alpha = 0.7$	3.3974e-03	1.8484e-03	9.9565e-04	5.3448e-04	2.8658e-04
	0.8782	0.8926	0.8975	0.8992	

Table 6.2: Maximum nodal errors and orders of convergence when $r = 1$ and $\beta = 0.9$

	N=40	N=80	N=160	N=320	N=640
$\alpha = 0.1$	9.2326e-03	4.2503e-03	1.9708e-03	9.1333e-04	4.2021e-04
	1.1192	1.1088	1.1096	1.1200	
$\alpha = 0.3$	3.2404e-03	1.0537e-03	3.3587e-04	1.0599e-04	3.3361e-05
	1.6207	1.6495	1.6640	1.6677	
$\alpha = 0.5$	6.1547e-04	1.4528e-04	3.4597e-05	8.5386e-06	3.0187e-06
	2.0828	2.0701	2.0186	1.5001	
$\alpha = 0.7$	3.6217e-04	1.1138e-04	3.4279e-05	1.0550e-05	3.2473e-06
	1.7011	1.7001	1.7000	1.7000	

Table 6.3: Maximum nodal errors and orders of convergence when $r = \frac{1+\alpha}{\beta}$ and $\beta = 0.9$

	N=40	N=80	N=160	N=320	N=640
$\alpha = 0.1$	1.5366e-02	7.1614e-03	3.3400e-03	1.5523e-03	7.1713e-04
	1.1014	1.1004	1.1055	1.1141	
$\alpha = 0.3$	5.1827e-03	1.7040e-03	5.4670e-04	1.7288e-04	5.4365e-05
	1.6048	1.6401	1.6609	1.6691	
$\alpha = 0.5$	8.6202e-04	2.0322e-04	4.8188e-05	1.1535e-05	2.7976e-06
	2.0847	2.0763	2.0626	2.0438	
$\alpha = 0.7$	3.4238e-04	9.8279e-05	2.8222e-05	8.1045e-06	2.3274e-06
	1.8007	1.8001	1.8000	1.8000	

Table 6.4: Maximum nodal errors and orders of convergence when $r = 2$ and $\beta = 0.9$

For Table 6.2 the order of convergence with the uniform mesh ($r = 1$), is $O(h^1)$. We can see from the Tables 6.3 and 6.4 for $\alpha = 0.1, 0.3, 0.5$ the convergence orders are almost $O(h^{1+2\alpha})$.

3 Example 3

From [25], consider, with $0 < \alpha < 1$,

$$\begin{aligned} &D_*^\alpha y(t) + y(t) = 0, \qquad t \in (0, T], \\ &y(0) = y_0, \end{aligned}$$

where $y_0 = 1$. The exact solution is $y(t) = E_{\alpha,1}(-t^\alpha)$ and $D_*^\alpha y(t) = -E_{\alpha,1}(-t^\alpha)$, where $E_{\alpha,\gamma}(z)$ is the Mittag-Leffler function defined by, [25],

$$E_{\alpha,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \gamma)}, \qquad \alpha, \gamma > 0.$$

Hence, we have, [25],

$$D_*^{\alpha}y(t) = -1 - \frac{(-t^{\alpha})}{\Gamma(\alpha+1)} - \frac{(-t^{\alpha})^2}{\Gamma(2\alpha+1)} - ...,$$

which implies that the regularity of $D_*^{\alpha}y(t)$ behaves as t^{α}, $0 < \alpha < 1$. So by Theorem 3.8 with $\sigma = \alpha$, we have when the method isn't higher order,

$$\max_{0 \le j \le N} |y(t_j^r) - y_j^r| \le \begin{cases} CN^{-r(2\alpha)}, & \text{if } r < \frac{1+\alpha}{2\alpha}, \\ CN^{-r(2\alpha)} ln(N), & \text{if } r = \frac{1+\alpha}{2\alpha}, \\ CN^{-1-\alpha}, & \text{if } r > \frac{1+\alpha}{2\alpha}. \end{cases}$$

This is for a uniform mesh so we can see how having a higher order method will change the rate of convergence and error. The MATLAB code to calculate these results can be found in Appendix **??**.

	N=40	N=80	N=160	N=320	N=640
$\alpha = 0.4$	1.0182e-03	5.4761e-04	3.0521e-04	1.7406e-04	1.0054e-04
	0.8948	0.8433	0.8102	0.7919	
$\alpha = 0.6$	2.1768e-04	9.2567e-05	4.0436e-05	1.7813e-05	7.8511e-06
	1.2336	1.1949	1.1827	1.1820	
$\alpha = 0.8$	2.9574e-05	9.7814e-06	3.2720e-06	1.0927e-06	3.6355e-07
	1.5962	1.5799	1.5822	1.5877	

Table 6.5: Maximum nodal errors and orders of convergence when $r = 1$

	N=40	N=80	N=160	N=320	N=640
$\alpha = 0.4$	1.9849e-04	8.7321e-05	3.4842e-05	1.3413e-05	5.0984e-06
	1.1847	1.3255	1.3772	1.3955	
$\alpha = 0.6$	3.1507e-05	1.1064e-05	3.5988e-06	1.1551e-06	3.7232e-07
	1.5098	1.6203	1.6395	1.6334	
$\alpha = 0.8$	4.7023e-06	1.4151e-06	4.3411e-07	1.3052e-07	3.8518e-08
	1.7324	1.7048	1.7338	1.7606	

Table 6.6: Maximum nodal errors and orders of convergence when $r = \frac{1+\alpha}{2\alpha}$

	N=40	N=80	N=160	N=320	N=640
$\alpha = 0.4$	1.9151e-04	6.9713e-05	2.3695e-05	7.8754e-06	2.6000e-06
	1.4580	1.5568	1.5892	1.5988	
$\alpha = 0.6$	4.0601e-05	9.0302e-06	1.9863e-06	4.3479e-07	9.6034e-08
	2.1687	2.1847	2.1917	2.1787	
$\alpha = 0.8$	9.7945e-06	1.7320e-06	3.0100e-07	5.1695e-08	8.8559e-09
	2.4995	2.5246	2.5417	2.5453	

Table 6.7: Maximum nodal errors and orders of convergence when $r = 2$

There are mixed results, some of the maximum nodal errors and EOC's are lower or higher depending on N. When $r = 1$ this corresponds to the uniform meshes. When $r = 2$, the convergence orders indeed are almost $O(h^{1+2\alpha})$.

Chapter 7

Conclusion and Future Work

In this dissertation, we have considered numerical methods for nonlinear fractional differential equations with graded meshes and introduced a higher order method. In future work we could prove the error estimates of the proposed higher order method.

We could also use an even higher Lagrange interpolation polynomials than the quadratic interpolation polynomials to construct a higher order numerical method for solving nonlinear fractional differential equations.

Chapter 8

Appendix

1 Matlab Code

We only include the MATLAB code for Example 1, the MATLAB codes for Examples 2 and 3 are similar.

1.1 Example 1

```
function [  ] = predictor_corrector_graded_meshes( )

%  al=0.25;
%  D^{al} y(t)  =  f(t, y(t)),   0 <= t < =1
%  u(0) =0;
%
%   f(t,y ) = gamma(1+3)/gamma(1+3-al) t^(beta-al) + (t^3)^2 y^2

%  The exact solution is
%  u(t) =  t^3

%
%  To consider the convergence rates error = O (h^(1+2*alpha))
%  We choose
%   1/(5*2), 1/(5*2^2), 1/(5*2^3), .... , 1/(5*2^7)
%  As in the Diethelm and Ford's paper
%

%   The algorithm is the following:
%   y_{n}^{P} = y_{0} + y_{0}^{1} t_{n}/factorial(1)
%    + 1/gamma(al) *sum_{j=0}^{n-1} b_{j, n} f(t_{j}, y_{j});
%   y_{n} = y_{0} + y_{0}^{1} t_{n}/fract(1)
%            + 1/gamma(al) *( sum_{j=0}^{n-1} a_{j, n} f(t_{j}, y_{j})
%                        + a_{n, n) f(t_{n}, y_{n}^{P}));

%   y_{0} =0;
%
%   where
%   a_{j, n} = h^al /(al(al+1)) * ( (n-1)^(al+1) - (n-1-al)(n^{al}),j=0
```

```
%          = h^al /(al(al+1)) * ( (n-1-j+2)^(al+1) + (n-1-j)^{al+1}
%  -2(n-1-j+1)^{al+1}),    1 <=j <=n-1
%          = h^al /(al(al+1))*1,    j=n.
%   where
%   b_{j, n} = h^(al)/al * ((n-j)^(al) - (n-1-j)^(al)).

clear
al  =default('al=the fractional order < 1, (default is h=0.25)', 0.9);
r   =default('r=the graded meshes, (default is r=1)', 1);


t0=0;  t1=1;    % time interval

NN=[40,80,160,320,640];    %different meshes

error=[];
for l=1:length(NN)
   N=NN(l);
 t=t0:(1/N):t1; t=t.^r;  t=t';     % t grids
 error1=predictor_corrector(N,al,t)   % error
 error=[error;error1];
end



format short E
error
format short

ratio=[];
for j=1:length(error)-1
    ratio=[ratio;error(j)/error(j+1)];
end

log2(ratio)





function  [error1] = predictor_corrector(N,al,t)
%exact=t.^8 - 3*t.^(4+al/2)+(9/4)*t.^(al);
exact=t.^3;

y0=exact(1); y1=exact(2); y2=exact(3); y3=exact(4);y4=exact(5);
y=[y0;y1;y2;y3;y4]; %initial value
```

```
for n=5:N
    sum_c=0; sum_a=0; sum_b=0;
        for j=0:n-1
            A=gamma(4)/gamma(4-al)*t(j+1)^(3-al);
            f_j=A+(t(j+1)^3)^2-y(j+1)^2;
            sum_c=sum_c +  c_coeff(j,n,al,t)*f_j;
            sum_a=sum_a +  a_coeff(j,n,al,t)*f_j;
            sum_b=sum_b +  b_coeff(j,n,al,t)*f_j;
        end
        % Find y_PP
         y_PP=y0 +  1/gamma(al)*sum_b;

         %Find y_P
         A=gamma(4)/gamma(4-al)*t(n+1)^(3-al);
            f_PP=A+(t(n+1)^3)^2-y_PP^2;
        f_PP=a_coeff(n,n,al,t)*f_PP;
        y_P=y0 +  1/gamma(al)*(sum_a+f_PP);

        %Find y1
         A=gamma(4)/gamma(4-al)*t(n+1)^(3-al);
            f_P=A+(t(n+1)^3)^2-y_P^2;
        f_P=c_coeff(n,n,al,t)*f_P;
        y5=y0+ 1/gamma(al)*(sum_c+f_P);

        y=[y;y5];
end
 error = y-exact;

 error1=norm(error, inf);

 %error1=abs(error(end))



 %  This the matlab  function to find coefficient c_{j, n}
function [ y ] =  c_coeff(j,n,al,t)



if j==0;
    j=j+2;
    A_0=1/((t(j-1)-t(j))*(t(j-1)-t(j+1))) ...
        *(1/(al+2)*((t(n+1)-t(j))^(al+2) -(t(n+1)-t(j+1))^(al+2))...
        -(2*t(n+1)-t(j)-t(j+1))/(al+1)*((t(n+1)-t(j))^(al+1)-(t(n+1)...
        -t(j+1))^(al+1)) ...
        +(t(n+1)-t(j))*(t(n+1)-t(j+1))/al*((t(n+1)-t(j))^al-(t(n+1)...
        -t(j+1))^al));
    a0=1/((t(1)-t(2))*(t(1)-t(3))) ...
```

```
        *(1/(al+2)*((t(n+1)-t(1))^(al+2) -(t(n+1)-t(2))^(al+2))...
        -(2*t(n+1)-t(2)-t(3))/(al+1)*((t(n+1)-t(1))^(al+1)-(t(n+1)...
        -t(2))^(al+1)) ...
        +(t(n+1)-t(3))*(t(n+1)-t(2))/al*((t(n+1)-t(1))^al-(t(n+1)...
        -t(2))^al));

    y= A_0+a0;
else if j==1;
    j=j+2;
    A_1=1/((t(j-1)-t(j))*(t(j-1)-t(j+1))) ...
        *(1/(al+2)*((t(n+1)-t(j))^(al+2) -(t(n+1)-t(j+1))^(al+2))...
        -(2*t(n+1)-t(j)-t(j+1))/(al+1)*((t(n+1)-t(j))^(al+1)-(t(n+1)...
        -t(j+1))^(al+1)) ...
        +(t(n+1)-t(j))*(t(n+1)-t(j+1))/al*((t(n+1)-t(j))^al...
        -(t(n+1)-t(j+1))^al));
    j=j-1;
    B_1=1/((t(j)-t(j-1))*(t(j)-t(j+1))) ...
        *(1/(al+2)*((t(n+1)-t(j))^(al+2) -(t(n+1)-t(j+1))^(al+2))...
        -(2*t(n+1)-t(j-1)-t(j+1))/(al+1)*((t(n+1)-t(j))^(al+1)-(t(n+1)...
        -t(j+1))^(al+1)) ...
        +(t(n+1)-t(j-1))*(t(n+1)-t(j+1))/al*((t(n+1)-t(j))^al-(t(n+1)...
        -t(j+1))^al));
    a1=1/((t(2)-t(1))*(t(2)-t(3))) ...
        *(1/(al+2)*((t(n+1)-t(1))^(al+2) -(t(n+1)-t(2))^(al+2))...
        -(2*t(n+1)-t(1)-t(3))/(al+1)*((t(n+1)-t(1))^(al+1)-(t(n+1)...
        -t(2))^(al+1)) ...
        +(t(n+1)-t(1))*(t(n+1)-t(3))/al*((t(n+1)-t(1))^al-(t(n+1)...
        -t(2))^al));

    y= A_1+B_1+a1;
    else if j==2;
    j=j+2;
    A_2=1/((t(j-1)-t(j))*(t(j-1)-t(j+1))) ...
        *(1/(al+2)*((t(n+1)-t(j))^(al+2) -(t(n+1)-t(j+1))^(al+2))...
        -(2*t(n+1)-t(j)-t(j+1))/(al+1)*((t(n+1)-t(j))^(al+1)-(t(n+1)...
        -t(j+1))^(al+1)) ...
        +(t(n+1)-t(j))*(t(n+1)-t(j+1))/al*((t(n+1)-t(j))^al-(t(n+1)...
        -t(j+1))^al));
    j=j-1;
    B_2=1/((t(j)-t(j-1))*(t(j)-t(j+1))) ...
        *(1/(al+2)*((t(n+1)-t(j))^(al+2) -(t(n+1)-t(j+1))^(al+2))...
        -(2*t(n+1)-t(j-1)-t(j+1))/(al+1)*((t(n+1)-t(j))^(al+1)-(t(n+1)...
        -t(j+1))^(al+1)) ...
        +(t(n+1)-t(j-1))*(t(n+1)-t(j+1))/al*((t(n+1)-t(j))^al-(t(n+1)...
        -t(j+1))^al));

    j=j-1;
    C_2=1/((t(j+1)-t(j-1))*(t(j+1)-t(j))) ...
```

```
    *(1/(al+2)*((t(n+1)-t(j))^(al+2) -(t(n+1)-t(j+1))^(al+2))...
    -(2*t(n+1)-t(j-1)-t(j))/(al+1)*((t(n+1)-t(j))^(al+1)-(t(n+1)...
    -t(j+1))^(al+1)) ...
    +(t(n+1)-t(j-1))*(t(n+1)-t(j))/al*((t(n+1)-t(j))^al-(t(n+1)...
    -t(j+1))^al));
a3=1/((t(3)-t(1))*(t(3)-t(2))) ...
    *(1/(al+2)*((t(n+1)-t(1))^(al+2) -(t(n+1)-t(2))^(al+2))...
    -(2*t(n+1)-t(1)-t(2))/(al+1)*((t(n+1)-t(1))^(al+1)-(t(n+1)...
    -t(2))^(al+1)) ...
    +(t(n+1)-t(1))*(t(n+1)-t(2))/al*((t(n+1)-t(1))^al-(t(n+1)...
    -t(2))^al));
y= A_2+B_2+C_2+a3;
    else if j==n-1;
j=j+1;
B_n_1=1/((t(j)-t(j-1))*(t(j)-t(j+1))) ...
    *(1/(al+2)*((t(n+1)-t(j))^(al+2) -(t(n+1)-t(j+1))^(al+2))...
    -(2*t(n+1)-t(j-1)-t(j+1))/(al+1)*((t(n+1)-t(j))^(al+1)-(t(n+1)...
    -t(j+1))^(al+1)) ...
    +(t(n+1)-t(j-1))*(t(n+1)-t(j+1))/al*((t(n+1)-t(j))^al-(t(n+1)...
    -t(j+1))^al));
j=j-1;
C_n_1=1/((t(j+1)-t(j-1))*(t(j+1)-t(j))) ...
    *(1/(al+2)*((t(n+1)-t(j))^(al+2) -(t(n+1)-t(j+1))^(al+2))...
    -(2*t(n+1)-t(j-1)-t(j))/(al+1)*((t(n+1)-t(j))^(al+1)-(t(n+1)...
    -t(j+1))^(al+1)) ...
    +(t(n+1)-t(j-1))*(t(n+1)-t(j))/al*((t(n+1)-t(j))^al-(t(n+1)...
    -t(j+1))^al));
y=  B_n_1+C_n_1;
        else if j==n;
C_n=1/((t(j+1)-t(j-1))*(t(j+1)-t(j))) ...
    *(1/(al+2)*((t(n+1)-t(j))^(al+2) -(t(n+1)-t(j+1))^(al+2))...
    -(2*t(n+1)-t(j-1)-t(j))/(al+1)*((t(n+1)-t(j))^(al+1)-(t(n+1)...
    -t(j+1))^(al+1)) ...
    +(t(n+1)-t(j-1))*(t(n+1)-t(j))/al*((t(n+1)-t(j))^al-(t(n+1)...
    -t(j+1))^al));
y=  C_n;
            else
j=j+2;
A_j=1/((t(j-1)-t(j))*(t(j-1)-t(j+1))) ...
    *(1/(al+2)*((t(n+1)-t(j))^(al+2) -(t(n+1)-t(j+1))^(al+2))...
    -(2*t(n+1)-t(j)-t(j+1))/(al+1)*((t(n+1)-t(j))^(al+1)-(t(n+1)...
    -t(j+1))^(al+1)) ...
    +(t(n+1)-t(j))*(t(n+1)-t(j+1))/al*((t(n+1)-t(j))^al-(t(n+1)...
    -t(j+1))^al));
j=j-1;

B_j=1/((t(j)-t(j-1))*(t(j)-t(j+1))) ...
    *(1/(al+2)*((t(n+1)-t(j))^(al+2) -(t(n+1)-t(j+1))^(al+2))...
```

```
        -(2*t(n+1)-t(j-1)-t(j+1))/(al+1)*((t(n+1)-t(j))^(al+1)-(t(n+1)...
        -t(j+1))^(al+1)) ...
        +(t(n+1)-t(j-1))*(t(n+1)-t(j+1))/al*((t(n+1)-t(j))^al-(t(n+1)...
        -t(j+1))^al));
    j=j-1;
    C_j=1/((t(j+1)-t(j-1))*(t(j+1)-t(j))) ...
        *(1/(al+2)*((t(n+1)-t(j))^(al+2) -(t(n+1)-t(j+1))^(al+2))...
        -(2*t(n+1)-t(j-1)-t(j))/(al+1)*((t(n+1)-t(j))^(al+1)-(t(n+1)...
        -t(j+1))^(al+1)) ...
        +(t(n+1)-t(j-1))*(t(n+1)-t(j))/al*((t(n+1)-t(j))^al-(t(n+1)...
        -t(j+1))^al));
    y= A_j+B_j+C_j;
                end
            end
        end
    end
end
```

```
%  This the matlab  function to find coefficient a_{j, n}
function [ y ] =  a_coeff(j,n,al,t)


if j==0;
    y= 1/(al*(al+1))*(1/(t(j+1)-t(j+2)))*(-(t(n+1)-t(2))^(al+1) ...
        +(t(n+1)-t(1))^(al)*(t(n+1)-(1+al)*t(2)));
else if j>=1&& j<=n-1;
        y= 1/(al*(al+1))*(1/(t(j+1)-t(j)))*(t(n+1)-t(j))^(al+1) ...
          - 1/(al*(al+1))*(1/(t(j+1)-t(j+2)))*(t(n+1)-t(j+2))^(al+1)...
       + 1/(al*(al+1))*((t(j+2)-t(j))/((t(j+1)-t(j+2))*(t(j+1)-t(j))))...
       *(t(n+1)-t(j+1))^(al+1);
    else
                y= 1/(al*(al+1))*(t(n+1)-t(n))^al;
    end
end
```

```
%  This the matlab  function to find coefficient b_{j, n}

function [ y ] =  b_coeff( j,n,al,t)
 %where
%  b_{j, n} = h^(al)/al * ((n-j)^(al) - (n-1-j)^(al)).
```

```
y=  1/al*((t(n+1)-t(j+1))^al -(t(n+1)-t(j+2))^al);

function reply = default(query,value)
%default   gets response to IFISS prompt
%   reply = default(query,value);
%   input
%          query   character string: asks a question
%          value   integer: the default response
%
%   IFISS function: AR; 31 August 2005.
% Copyright (c) 2005 D.J. Silvester, H.C. Elman, A. Ramage (see readme.m)
global BATCH FID
if exist('BATCH') & BATCH==1,
   replycell=textscan(FID,'%f%*[^\n]',1);
   reply=deal(replycell{:});
   disp(query)
   disp(reply)
else
   reply=input([query,' : ']);
   if isempty(reply), reply=value; end
end
return
```

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