The Application of Lyapunov Method for the Investigation of Global Stability of Some Population and Epidemiology Models

by

Sara Bate

Submitted to the Department of Mathematics
Faculty of Science and Engineering
in partial fulfilment for the degree of
Master of Science in Mathematics
at the
University of Chester

October 2015

Abstract

The primary purpose of this thesis is to determine the global behaviour of some population and epidemiology models through the application of Lyapunov functions. Using Lyapunov functions and applying these to mathematical models of ODE systems representing different predator-prey models, we were able to determine global assymptotic stability for their equilibrium points. Similarly, for the investigation into the stability of epidemiological models, we were able to analyse various $SIRS$, SIR, SIS and $SEIR$ models to also conclude global assymptotic stability by implementing the Lyapunov direct method. We then continue our investigation by the application of Lyapunov functions to PDE systems representing reaction-diffusion systems of various predator-prey and epidemiological models. We have also been able to conclude global assymptotic stability for their correspondng equilibriums in these cases. We then proceeded to create our own reaction-diffusion system from a previously constructed ODE system and have been able to prove that for both cases they have a globally assymptotically stable endemic equilibrium.

Acknowledgements

I would like to take this opportunity to thank everyone who has helped to make the completion of this thesis possible. I would firstly like to offer my special thanks to my supervisor Dr. Nikos Kavallaris whose patience and guidance helped me to achieve my best. I would also like to express my graditute to the lecturers in the Mathematics department who have helped me through the last four years to reach this point. I would next like to express my very great appreciation to my family: my mum and dad for their love, constant support and motivating me when I needed it the most; my brother for always inspiring me; and to my grandparents for the extra encouragement I never knew I needed. I would finally like to acknowledge the help provided by the students on my course, particularly Gerard Coleman for keeping me balanced throughout the stress of this past year.

Contents

Chapter 1

Introduction

1.1 Mathematical Modelling

Mathematical modelling, as described by Andrews and McLone [1], is the representation of real world problems in mathematical terms so that we may obtain a more precise understanding of its significant properties. Mathematical models can be created for a vast amount of situations and, as explained by Kalman [7], can be so accurate that they can actually take the place of building and testing physical models. This has proven to be essential in some research where it may be time consuming, unethical or expensive for practical experimentation.

Kapur [8] provides a twelve-point procedure for solving problems through mathematical modelling:

1. Be clear about the real world situation to be investigated. Find all its essential characteristics relevant to the situation and find those aspects which are irrelevant or whose relevance is minimal. It is important to decide what aspects must be considered and what aspects can be ignored.

2. Think about all the physical, chemical, biological, social, economic laws that may be relevant to the situation. If necessary collect some data and analyse it to get some initial insight into this situation.

3. Formulate the problem in problem language.

4. Think about all the variables $x_1, x_2, \ldots, x_n$ and parameters $a_1, a_2, \ldots, a_m$ involved. Classify these into known and unknown ones.

5. Think of the most appropriate mathematical model and translate the problem suitably into mathematical language in the form
$$f_j\left(x_i, a_h, \frac{\partial}{\partial x_i}, \int \ldots \mathrm{d}x_i, d\right) \leq 0$$
i.e. in terms of algebraic, transcendental, differential, difference, integral, integro-differential, differential-difference equations or inequalities.

6. Think of all possible ways of solving the equations of the model. The methods may be analytical, numerical or simulation. Try to get as far as possible analytically, supplement this with numerical and computer methods when necessary and use simulation when warranted.

7. If a reasonable change in the assumptions makes analytical solution possible, investigate the possibility. If new methods are required to solve the equations of the model, try to develop these methods.

8. Make an error analysis of the method used. If the error is not within acceptable limits, change the method of solution.

9. Translate into problem language.

10. Compare the predictions with available observations or data. If agreement is good, accept the model. If the agreement is not good, examine the assumptions and approximations and change them in light of the discrepancies observed and proceed as before.

11. Continue the process until a satisfactory model is obtained which explains all earlier data and observations.

12. Deduce conclusions from your model and test these conclusions against earlier data and additional data that may be collected and see if the agreement still continues to be good.

This twelve-point procedure will aid in producing an accurate mathematical model for any real world problem being investigated.

Throughout this thesis we will not be creating models but concentrating on models already produced by other researchers.

1.2 Stability

This work will be investigating the stability of equilibrium points of the systems given.

Definition 1.2.1 (Equilibrium Points) *Given a set of nonlinear first-order differential equations*

$$\dot{x}_i = f_i\left(x_1, \ldots, x_n\right) \qquad \text{for } i = 1, \ldots, n, \tag{1.1}$$

where $x_i = x_i(t)$, for time $t \geq 0$, and $\dot{x}_i = \dot{x}_i(t)$ stands for the time derivative of x_i at time t whereas f_i are analytic functions. The equilibrium point, also known as an extreme point, is a solution that does not change with time and thus has a derivative of zero. An equilibrium point of system (1.1) *is therefore a point $x^* = ({x_1}^*, \ldots, {x_n}^*)$ for which $f_i\left({x_1}^*, \ldots, {x_n}^*\right) = 0$.*

Given a set of nonlinear first-order differential equations

$$\dot{x}_i = f_i\left(x_1, \ldots, x_n\right) \qquad \text{for } i = 1, \ldots, n, \tag{1.2}$$

where $x_i = x_i(t)$ for time t and $\dot{x}_i$ stands for the time derivative of x_i for $i = 1, \ldots, n$. Whereas f_i are analytic functions such that $f_i(0 \ldots, 0) = 0$ for $i = 1, \ldots, n$ so that the origin $x = 0$ is an equilibrium point. Roughly speaking, an equilibrium point is classed as stable if all solutions starting at close points stay close. A more mathematical definition is the following:

Definition 1.2.2 (Stability) *The equilibrum point $x = 0$ of the system* (1.2) *is stable if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x(0)\| < \delta$ implies that $\|x(t)\| < \varepsilon$, for all $t \geq 0$. The equilibrium point is unstable if it is not stable. The equilibrium point is asymptotically stable if it is stable and there exists $c > 0$ such that $\lim\limits_{t \to +\infty} x(t) = 0$ for all solutions with $\|x(0)\| \leq c$.*

1.3 Lyapunov Stability

In 1892, the Russian mathematician Aleksandr Mikhailovich Lyapunov developed a method to examine the stability of equilibria of ordinary differential equations, known as the direct Lyapunov method. O'Regan, Kelly, Korobeinikov, O'Callaghan and Pokrovskii [16] describe the direct method as one of the most powerful techniques for qualitative analysis of a dynamical system. This method is the most successful approach to establish global properties of nonlinear systems, however, an auxiliary function with specific properties, a Lyapunov function, is required and finding such a function can be a difficult and time consuming process.

Definition 1.3.1 (Lyapunov Test Function) *For a function $V(x)$, where $x = (x_1, x_2, \ldots, x_n)$, if the following conditions are satisfied:*

1. *$V(x)$ and $\frac{\partial V}{\partial x_i}$ are continuous, for all $x \in \mathbb{R}^n$ and $i = 1, \ldots, n$, not necessarily at the origin,*

2. $V(0) = 0$,

then we say that $V(x)$ is a possible Lyapunov test function for system (1.2).

Theorem 1.3.1 (Lyapunov's Direct Method) *Using an appropriate Lyapunov test function it may be possible to investigate the stability of an equilibrium point of the system of nonlinear differential equations (1.2), as explained by Parks [17], by examining the rate of change with respect to time of $V(x)$ calculated as the Lyapunov derivative:*

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x).$$

We can then interpret stability from $\dot{V}$ as follows:

Definition 1.3.2 (Stable) *If $V(x) > 0$ for $x \neq 0$ and $\dot{V}(x) \leq 0$, then we say that $V(x)$ is positive definite and the origin $(0,0)$ of the system of ordinary differential equations (1.2) is stable.*

Definition 1.3.3 (Asymptotically Stable) *If $V(x) > 0$ for $x \neq 0$ and $\dot{V}(x) < 0$, then we say that $V(x)$ is positive definite and the origin $(0,0)$ of the system of ordinary differential equations (1.2) is asymptotically stable.*

Definition 1.3.4 (Unstable) *If $\dot{V}(x) > 0$, then we say that $\dot{V}(x)$ is positive definite and the origin $(0,0)$ of the system of ordinary differential equations (1.2) is unstable.*

1.4 LaSalle Invariance Principle

The LaSalle invariance principle provides a generalization of Lyapunov criteria for asymptotic stability. This principle allows us to determine if an equilibrium point is asymptotically stable when we have a negative semi-definite Lyapunov derivative.

Definition 1.4.1 (Invariant Set) *A set $M \subset \mathbb{R}^n$, $n \geq 1$ is invariant with respect to system (1.2) if $x(0) \in M \Rightarrow x(t) \in M$, for any $t \in \mathbb{R}^n$.*

Definition 1.4.2 (Trajectory) *A trajectory of system (1.2) is the curve $\{(\phi_1(t), \phi_2(t), \ldots, \phi_n(t)) : \alpha < t < \beta\}$, where $\phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))$ is a solution of system (1.2).*

Theorem 1.4.1 (LaSalle's Invariance Principle) *Assume that $V(x)$ is a Lyapunov function of (1.2) on a subset $G \subset \mathbb{R}^n, n \geq 1$. Define $S = \left\{x \in \bar{G} : \dot{V}(x) = 0\right\}$, where $\bar{G}$ is the closure of G. Let M be the maximal invariant subset of S. Then, for $t \geq 0$, every bounded trajectory of (1.2) that remains in G approaches the set M as $t \to +\infty$.*

Henceforth we will concentrate on the cases $n = 2, 3$.

As stated by O'Regan et al. [16], while the intention of Lyapunov was to study the stability of motion, the direct Lyapunov method and the idea of an auxiliary function have found a large number of applications. The main purpose of the current thesis is to concentrate on the construction of Lyapunov functionals for different systems applied to various fields from population dynamics to epidemiology.

Chapter 2

Predator-Prey Models

When species interact the population dynamics of each species is affected, in general there is a whole web of interacting species. For this thesis we consider populations in a predator-prey situation i.e., the growth of one population is decreased whilst the other is increased. Throughout this investigation we will concentrate on systems involving either two or three species.

2.1 A Predator-Prey Model Given by Hsu [5]

The first predator-prey work we will concentrate on is given by Hsu [5].

Let $x(t), y(t)$ be the population densities of prey and predator at time t, respectively. Consider the following Gause-type predator-prey system given by Hsu [5]:

$$\begin{aligned} x' &= xg(x) - cp(x)y, \\ y' &= (p(x) - d)y, \\ x(0) &> 0, \quad y(0) > 0, \end{aligned} \tag{2.1}$$

where $g(x)$ is the rate of change of population size of the prey species for which the population grows under ideal conditions, $p(x)$ is the specific growth rate of predator species, $c > 0$ is the conversion rate and $d > 0$ is the death rate of predator species. We assume $g(x)$ and $p(x)$ satisfy the following:

1. $g(0) > 0$ and there exists $K > 0$ such that $g(K) = 0, \quad g(x) > 0$ for $0 \leq x < K$,

2. $p(0) = 0, p'(x) > 0$ for $0 \leq x \leq K$,

where K is the maximum population size of the prey species that a given area can sustain indefinitely with the resources available, known as the carrying capacity of the prey species.

From system (2.1), we can calculate the equilibrium as follows:

$$\begin{aligned} x' = xg(x) - cp(x)y = 0 \\ \Rightarrow cp(x)y = xg(x) \\ y = \frac{xg(x)}{cp(x)} \end{aligned}$$

and

$$\Rightarrow y' = (p(x) - d)y = 0$$

$y = \dfrac{xg(x)}{cp(x)}$ will therefore give:

$$(p(x) - d)\frac{xg(x)}{cp(x)} = 0$$

which is true if and only if $p(x) - d = 0 \Rightarrow p(x) = d$.

We can therefore see that $x' = 0$ if and only if $y = \frac{xg(x)}{cp(x)}$ and $y' = 0$ if and only if $x = x^*$ where x^* satisfies $p(x^*)=d$. We note that the curve $y = \frac{xg(x)}{cp(x)}$ the prey isocline and the curve $x = x^*$ predator isocline.

To show that (x^*,y^*) is globally stable in the first quadrant of xy-plane we introduce, see [5], the following Lyapunov test function:

$$V(x,y) = \int_{x^*}^{x} \frac{p(\varepsilon) - d}{p(\varepsilon)} d\varepsilon + c \int_{y^*}^{y} \frac{\eta - y^*}{\eta} d\eta. \tag{2.2}$$

Then it follows that

$$\begin{aligned}\dot{V}(x,y) &= \frac{p(x) - d}{p(x)}(xg(x) - cp(x)y) + c\frac{y - y^*}{y}(p(x) - d)y \\ &= \frac{p(x) - d}{p(x)}(xg(x) - cp(x)y) + c(y - y^*)(p(x) - d)\end{aligned}$$

Which can be rearranged to give:

$$\begin{aligned}\dot{V}(x,y) &= \frac{p(x) - d}{p(x)}(xg(x) - cp(x)y^* - cp(x)(y - y^*)) + c(y - y^*)(p(x) - d) \\ &= (p(x) - p(x^*))\left(\frac{xg(x)}{p(x)} - cy^* - c(y - y^*) + c(y - y^*)\right) \\ &= c(p(x) - p(x^*))\left(\frac{xg(x)}{cp(x)} - y^*\right) \leq 0,\end{aligned} \tag{2.3}$$

if the horizontal line $y = y^*$ and the vertical line $x = x^*$ seperate the prey isocline $y = \frac{xg(x)}{cp(x)}$ into two disjoint parts (see Fig.(2.1)).

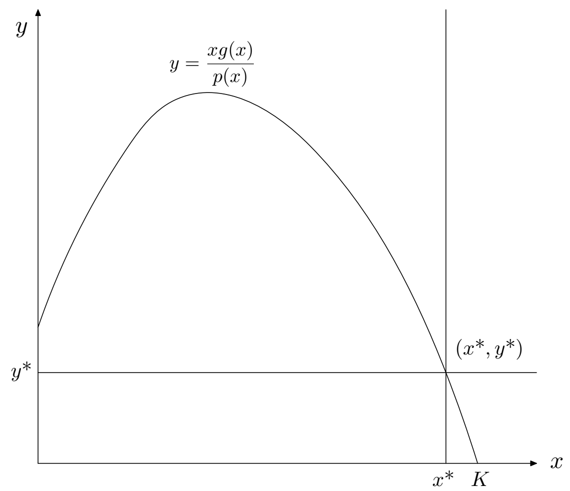

Figure 2.1:

We could also consider, see [5], the following mixed type Lyapunov function:

$$V(x,y) = y^\theta \int_{x^*}^{x} \frac{p(\varepsilon) - d}{p(\varepsilon)} d\varepsilon + \int_{y^*}^{y} \eta^{\theta-1}(\eta - y^*) d\eta. \tag{2.4}$$

Using system (2.1) it follows that

$$\begin{aligned}
\dot{V}(x,y) &= \left[y^{\theta}\left(\frac{p(x)-d}{p(x)}\right)\right](xg(x)-p(x)y) + \left[\theta y^{\theta-1}\int_{x^*}^{x}\frac{p(\varepsilon)-d}{p(\varepsilon)}d\varepsilon + y^{\theta-1}(y-y^*)\right](p(x)-d)y \\
&= y^{\theta}(p(x)-d)\left[\frac{xg(x)-p(x)y}{p(x)} + \theta\int_{x^*}^{x}\frac{p(\varepsilon)-d}{p(\varepsilon)}d\varepsilon + (y-y^*)\right] \\
&= y^{\theta}(p(x)-d)\left[\frac{xg(x)}{p(x)} - y + \theta\int_{x^*}^{x}\frac{p(\varepsilon)-d}{p(\varepsilon)}d\varepsilon + y - y^*\right]. \\
&= y^{\theta}(p(x)-d)\left[\frac{xg(x)}{p(x)} - y^* + \theta\int_{x^*}^{x}\frac{p(\varepsilon)-d}{p(\varepsilon)}d\varepsilon\right].
\end{aligned}$$

Thus we have:

$$\dot{V}(x,y) = y^{\theta}(p(x)-d)\left[\frac{xg(x)}{p(x)} - y^* + \theta\int_{x^*}^{x}\frac{p(\varepsilon)-d}{p(\varepsilon)}d\varepsilon\right].$$

We note that if we choose $\theta = 0$ in the Lyapunov function (2.4) then $V(x,y)$ becomes the Lyapunov function (2.2).

2.2 A Competing Predator-Prey Model Given by Chiu [3]

The final predator-prey work we will concentrate on is given by Chiu [3]. This paper studies the global asymptotic behaviour of the following three-dimensional predator-prey system which consists of two predator species x_1, x_2 competing for a single prey species S. Thus Chiu [3] provides the system:

$$\begin{cases}
\dfrac{dS}{dt} = Sg(S) - p_1(S)x_1 - p_2(S)x_2, \\[2ex]
\dfrac{dx_1}{dt} = q_1(S)x_1, \\[2ex]
\dfrac{dx_2}{dt} = q_2(S)x_2, \\[2ex]
S(0) > 0, \quad x_1(0) > 0, \quad x_2(0) > 0.
\end{cases} \tag{2.5}$$

Chiu [3] provides the basic assumptions for system (4.12):

(H1) $p_i(\cdot) \in C([0,\infty),\mathbb{R}),\ p_i(0) = 0$, and $p_i'(S) > 0$ for all $S > 0,\ i = 1,2$.

(H2) $g(\cdot) \in C([0,\infty),\mathbb{R})$ and there exists $K > 0$ such that $g(K) = 0$ and $(S-K)g(S) < 0$ for all $S > 0$ with $S \neq K$.

(H3) $q_i(\cdot) \in C([0,\infty),\mathbb{R}),\ q_i'(S) > 0$ for all $S > 0$ and there exists $0 < \lambda_i < K$ such that $q_i(\lambda_i) = 0$ and $(S-\lambda_i)q_i(S) > 0$ for all $S > 0$ with $S \neq \lambda_i,\ i = 1,2$.

In [3] it is proven that the solution $(S(t), x_1(t), x_2(t))$ of system (4.12) is bounded and positive for all $t \geq 0$. Moreover, $S(t) \leq K$ for t sufficiently large.

Under the assumptions (H1)-(H3) listed previously, there exists an equilibrium $E_* = (\lambda_1, \hat{x}_1, 0)$ of system (4.12) with $\hat{x}_1 = h(\lambda_1)$. Let us define, see [3],

$$F(S) = \frac{\hat{x}_1 - h(S)}{\int_{\lambda_1}^{S}\left(\frac{q_1(\varepsilon)}{p_1(\varepsilon)}\right)\mathrm{d}\varepsilon} \ : \ (0,\lambda_1)\cup(\lambda_1,K) \to \mathbb{R}, \tag{2.6}$$

then the following result holds, see also theorem 3.4.1 in [3].

Theorem 2.2.1 *Let the conditions (H1)-(H3) hold and let* $(S(t), x_1(t), x_2(t))$ *be the solution of system* (4.12). *If* $q_2(\lambda_1) \leq 0$ *and there exists:*

1. $\theta \in \mathbb{R}$ *such that*

$$\theta \geq F(S), \text{ for all } S \in (0, \lambda_1), \text{ and } \theta \leq F(S), \text{ for all } S \in (\lambda_1, K), \tag{2.7}$$

2. $c > 0$ *such that for all* $S > 0$ *with* $S \neq \lambda_1$,

$$\Delta(S) = -\frac{q_1(S)}{p_1(S)} p_2(S) + c\left[\theta q_1(S) + (q_2(S) - q_2(\lambda_1))\right] < 0. \tag{2.8}$$

Then $(S(t), x_1(t), x_2(t)) \to (\lambda_1, \hat{x}_1, 0)$ *as* $t \to \infty$.

Proof. We consider the potential Lyapunov function, see also [3],

$$V(S, x_1, x_2) = \int_{\hat{x}_1}^{x_1} \varepsilon^{\theta-1}(\varepsilon - \hat{x}_1) d\varepsilon + x_1^\theta \int_{\lambda_1}^{S} \frac{q_1(\varepsilon)}{p_1(\varepsilon)} d\varepsilon + c x_1^\theta x_2.$$

It can be seen that $V(S, x_1, x_2) \in C^1(\mathbb{R}_+^3, \mathbb{R})$, $V(\lambda_1, \hat{x}_1, 0) = 0$, and $V(S, x_1, x_2) > 0$ for $(S, x_1, x_2) \in \mathbb{R}_+^3 - \{(\lambda_1, \hat{x}_1, 0)\}$. Then the derivative of $V(S, x_1, x_2)$ along the trajectory (see definition 1.4.2) of system (4.12) is

$$\begin{aligned}
\dot{V}(S, x_1, x_2) = &\left(x_1^\theta \frac{q_1(S)}{p_1(S)}\right)(Sg(S) - p_1(S)x_1 - p_2(S)x_2) \\
&+ \left(x_1^{\theta-1}(x_1 - \hat{x}_1) + \theta x_1^{\theta-1} \int_{\lambda_1}^{S} \frac{q_1(\varepsilon)}{p_1(\varepsilon)} d\varepsilon + \theta c x_1^{\theta-1} x_2\right) q_1(S) x_1 \\
&+ c_1 x_1^\theta q_2(S) x_2
\end{aligned}$$

Which can be rearranged to give:

$$\begin{aligned}
\dot{V}(S, x_1, x_2) = x_1^\theta &\left(\frac{q_1(S)}{p_1(S)}\right)(Sg(S) - p_1(S)x_1 - p_2(S)x_2) \\
&+ \left(x_1^\theta - \hat{x}_1 x_1^{\theta-1} \int_{\lambda_1}^{S} \frac{q_1(\varepsilon)}{p_1(\varepsilon)} d\varepsilon + c\theta x_1^{\theta-1} x_2\right) q_1(S) x_1 \\
&+ c x_1^\theta q_2(S) x_2
\end{aligned}$$

After further calculations we obtain:

$$\begin{aligned}
\dot{V}(S, x_1, x_2) &= x_1^\theta \left(\frac{q_1(S)}{p_1(S)}\right)(Sg(S) - p_1(S)x_1) \\
&\quad + \left(x_1^\theta - \hat{x}_1 x_1^{\theta-1} \int_{\lambda_1}^{S} \frac{q_1(\varepsilon)}{p_1(\varepsilon)} d\varepsilon\right) q_1(S) x_1 \\
&\quad - x_1^\theta \frac{q_1(S)}{p_1(S)} p_2(S) x_2 + c\theta x_1^\theta q_1(S) x_2 + c x_1^\theta q_2(S) x_2 \\
&= x_1^\theta \frac{q_1(S)}{p_1(S)} Sg(S) - x_1^\theta \frac{q_1(S)}{p_1(S)} p_1(S) x_1 + x_1^\theta q_1(S) x_1 \\
&\quad - \hat{x}_1 x_1^{\theta-1} q_1(S) x_1 + \theta x_1^{\theta-1} \int_{\lambda_1}^{S} \frac{q_1(\varepsilon)}{p_1(\varepsilon)} d\varepsilon q_1(S) x_1 - x_1^\theta \frac{q_1(S)}{p_1(S)} p_2(S) x_2 \\
&\quad + c\theta x_1^\theta q_1(S) x_2 + c x_1^\theta q_2(S) x_2 \\
&= x_1^\theta \frac{q_1(S)}{p_1(S)} Sg(S) - x_1^{\theta+1} q_1(S) + x_1^{\theta+1} q_1(S) - x_1^\theta \hat{x}_1 q_1(S) + \theta x_1^\theta q_1(S) \int_{\lambda_1}^{S} \frac{q_1(\varepsilon)}{p_1(\varepsilon)} d\varepsilon
\end{aligned}$$

$$
\begin{aligned}
&- x_1^\theta \frac{q_1(S)}{p_1(S)} p_2(S) x_2 + c\theta x_1^\theta x_2 + c x_1^\theta q_2(S) x_2 \\
&= x_1^\theta q_1(S) \left(\frac{Sg(S)}{p_1(S)} - \hat{x}_1 + \theta \int_{\lambda_1}^{S} \frac{q_1(\varepsilon)}{p_1(\varepsilon)} d\varepsilon \right) \\
&\quad + x_1^\theta x_2 \left(-\frac{q_1(S)}{p_1(S)} p_2(S) + c\left(\theta q_1(S) + q_2(S)\right) \right) \\
&= x_1^\theta q_1(S) \left(\frac{Sg(S)}{p_1(S)} - \hat{x}_1 + \theta \int_{\lambda_1}^{S} \frac{q_1(\varepsilon)}{p_1(\varepsilon)} d\varepsilon \right) \\
&\quad + x_1^\theta x_2 \left[-\frac{q_1(S)}{p_1(S)} p_2(S) + c\left(\theta q_1(S) + q_2(S)\right) \right] \\
&\quad + c x_1^\theta q_2(\lambda_1) x_2 - c x_1^\theta q_2(\lambda_1) x_2 \\
&= x_1^\theta q_1(S) \left(\frac{Sg(S)}{p_1(S)} - \hat{x}_1 + \theta \int_{\lambda_1}^{S} \frac{q_1(\varepsilon)}{p_1(\varepsilon)} d\varepsilon \right) + c x_1^\theta q_2(\lambda_1) x_2 \\
&\quad + x_1^\theta x_2 \left[-\frac{q_1(S)}{p_1(S)} p_2(S) + c\big[\theta q_1(S) + \left(q_2(S) - q_2(\lambda_1)\right)\big] \right].
\end{aligned}
$$

From the assumption $q_2(\lambda_1) \leq 0$ and using (4.15), (4.16), it follows that

$$
\dot{V}(S, x_1, x_2) \leq 0, \quad for\ 0 < S < K,\ x_1 > 0,\ x_2 > 0.
$$

Hence we complete the proof of theorem (3.3.3) by LaSalle's invariance principle (see definition **??**).

Chapter 3

Epidemiological Models

3.1 Modelling Epidemics

Definition 3.1.1 (Classical Assumptions) *The host population is subdivided into distinct classes according to the health of its members. The basic variables identifying the state of the population are the three epidemiological classes:*

- $S(t)$ *the number of susceptibles at time* t*; the individuals who are healthy and can be infected,*
- $I(t)$ *the number of infectives at time* t*; the individuals who are infected and are able to transmit the disease,*
- $R(t)$ *the number of immune at time* t*; the individuals who are immune because they have been infected and have now either recovered or died.*

These classical assumptions will be used throughout all of the models investigated throughout this thesis. After the basic state variables introduced above, further simplification may be introduced concerning the disease progression and effect. A basic distinction may be done between the diseases that impart lifelong immunity, temporary immunity and no immunity which lead to the SIR models, $SIRS$ models and SIS models respectively.

Kermack and McKendrick [9] summarise the problem as follows: one (or more) infected person is introduced into a community of individuals susceptible to the disease in question. The disease spreads from the affected to the unaffected by contact infection. Each infected person runs through the course of his sickness and is removed from the number of those who are sick, by recovery or by death.

The method of Lyapunov functions has been used extensively in mathematical biology and one of the main areas it is applied to is epidemiology. We can use Lyapunov functions to investigate the stability of infection-free and endemic equilibriums of the different systems. Throughout this thesis we will concentrate on a small number of papers which look at a variety of different epidemiological systems.

3.2 $SIRS$ Models

The first type of epidemiological model we will concentrate on is the $SIRS$ model. Here we divide the population, of size N, into the subpopulations defined in (3.1). We assume that an individual begins in the susceptibles compartment, after infection moves to the infective compartment and then continues into the removed compartment as a result of recovery. For our $SIRS$ models we assume that recovery implies temporary immunity and thus the individuals in the removed compartment would return to the susceptible compartment after the recovery period. We will also assume that the population size N is constant which implies that deaths are balanced by births.

3.2.1 An $SIRS$ Model Given by Korobeinikov and Wake [13]

The $SIRS$ model we will investigate is given by Korobeinikov and Wake [13] and is described via the transfer diagram in Figure 3.1. In this model the births are proportional to the population size N with a birth rate γ, all disease-associated deaths are from the R compartment and the susceptibles and the infectives whom die from causes not connected with the disease are modelled with $\sigma \geq 0$. Furthermore, we assume that the vaccination of the susceptibles is proportional to the susceptibles population which therefore gives the rate σ as the sum of the death rate of susceptibles and of the vaccination rate. In this case, σ does not necessarily equal γ.

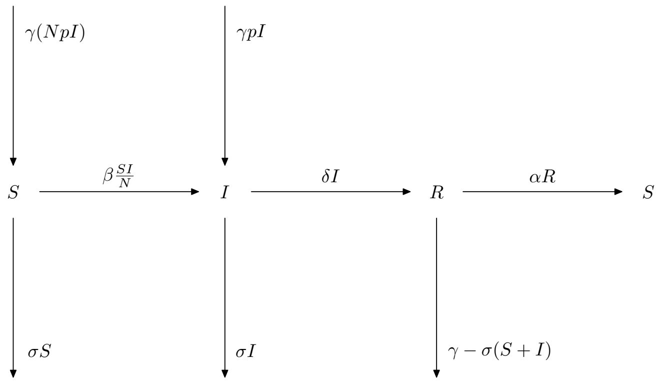

Figure 3.1: Transfer diagram of the $SIRS$ model given by Korobeinikov and Wake [13].

Korobeinikov and Wake [13] explain that there are two different types of transmission to be modelled; horizontal transmission and vertical transmission. An infection can be transmitted through contacts between the infectives and the susceptibles which is an example given of horizontal transmission for which, in this model, we assume to occur according to the mass action incidence $\frac{\beta SI}{N}$. An example for vertical transmission can be, for some diseases, the transmission of the disease passed on from an infective parent to their offspring. This has been incorporated into this model by assuming that a fraction p of the offspring from the infectives are infected at birth, and hence, a part of birth flux, $p\gamma I$, enters the infective compartment while the remaining births, $\gamma N - p\gamma I$, come to the susceptibles compartment (see Figure 3.1).

For this model Korobeinikov and Wake [13] have given an average life expectancy of the susceptibles $\frac{1}{\sigma}$, an average infective period $\frac{1}{\delta}$ and an average period of immunity $\frac{1}{\alpha}$ which results in the following differential system

$$\begin{aligned} \dot{S} &= (\gamma + \alpha)N - \beta\frac{SI}{N} - (\alpha + p\gamma)I - (\alpha + \sigma)S, \\ \dot{I} &= \beta\frac{SI}{N} - (\delta + \sigma - p\gamma)I. \end{aligned} \tag{3.1}$$

For this system we do not need an equation for the removed class R as we have $N = S + I + R$ which is a constant.

We will now study the stability of the equilibrium points of system (3.5). This system has two equilibria: an infection-free equilibrium and an endemic equilibrium.

Finding the infection-free equilibrium:

Infection-free $\Rightarrow I_0 = 0$.
We need to find S_0 such that $\dot{S} = 0$:

$$\dot{S} = (\gamma + \alpha)N - \beta\frac{S_0 I_0}{N} - (\alpha + p\gamma)I_0 - (\alpha + \sigma)S_0 = 0$$

$$\begin{aligned} \Rightarrow (\gamma+\alpha)N - \beta\frac{S_0(0)}{N} - (\alpha+p\gamma)(0) - (\alpha+\sigma)S_0 = 0 \\ \Rightarrow (\gamma+\alpha)N - (\alpha+\sigma)S_0 = 0 \\ \Rightarrow (\alpha+\sigma)S_0 = (\gamma+\alpha)N \\ \Rightarrow S_0 = \left(\frac{\gamma+\alpha}{\alpha+\sigma}\right)N. \end{aligned}$$

Thus we have an infection-free equilibrium $E_0 = (S_0, I_0)$, with

$$S_0 = \left(\frac{\gamma+\alpha}{\alpha+\sigma}\right)N, \qquad I_0 = 0.$$

Finding the endemic equilibrium:

First we need to find S^* such that $\dot{I} = 0$:

$$\begin{aligned} \dot{I} = \beta\frac{S^*I^*}{N} - (\delta+\sigma-p\gamma)I^* = 0 \\ \Rightarrow \beta\frac{S^*I^*}{N} = (\delta+\sigma-p\gamma)I^* \\ \Rightarrow S^* = \frac{(\delta+\sigma-p\gamma)N}{\beta}. \end{aligned}$$

Given $R_0 = \dfrac{\beta(\alpha+\gamma)}{(\alpha+\sigma)(\delta+\sigma-p\gamma)}$, we can rearrange S^* as follows:

$$\begin{aligned} S^* &= \left(\frac{\delta+\sigma-p\gamma}{\beta}\right)R_0\frac{N}{R_0} \\ &= \left(\frac{\delta+\sigma-p\gamma}{\beta}\right)\left(\frac{\beta(\alpha+\gamma)}{(\alpha+\sigma)(\delta+\sigma-p\gamma)}\right)\frac{N}{R_0} \\ &= \left(\frac{\alpha+\gamma}{\alpha+\sigma}\right)\frac{N}{R_0}. \end{aligned}$$

Next we need to find I^* such that $\dot{S} = 0$:

$$\dot{S} = (\gamma+\alpha)N - \beta\frac{S^*I^*}{N} - (\alpha+p\gamma)I^* - (\alpha+\sigma)S^* = 0$$

$$\begin{aligned} \Rightarrow \beta\frac{S^*I^*}{N} + (\alpha+p\gamma)I^* = (\gamma+\alpha)N - (\alpha+\sigma)S^* \\ \Rightarrow I^*\left(\beta\frac{S^*}{N} + \alpha + p\gamma\right) = (\gamma+\alpha)N - (\alpha+\sigma)S^* \\ \Rightarrow I^* = \frac{(\gamma+\alpha)N - (\alpha+\sigma)S^*}{\beta\frac{S^*}{N} + \alpha + p\gamma} \end{aligned}$$

Substituting in $S^* = \left(\frac{\alpha+\gamma}{\alpha+\sigma}\right)\frac{N}{R_0} = \frac{(\delta+\sigma-p\gamma)N}{\beta}$, we derive:

$$\begin{aligned} I^* &= \frac{(\gamma+\alpha)N - (\alpha+\sigma)\left(\frac{(\delta+\sigma-p\gamma)N}{\beta}\right)}{\frac{\beta}{N}\left(\frac{(\delta+\sigma-p\gamma)N}{\beta}\right) + \alpha + p\gamma} \\ &= \left(\frac{\beta(\gamma+\alpha) - (\alpha+\sigma)(\delta+\sigma-p\gamma)}{\beta(\delta+\sigma-p\gamma+\alpha+p\gamma)}\right)N \\ &= \left(\frac{\beta(\gamma+\alpha) - (\alpha+\sigma)(\delta+\sigma-p\gamma)}{\beta(\alpha+\delta+\sigma)}\right)N. \end{aligned}$$

Rearranging gives:

$$\begin{aligned} I^* &= \frac{\alpha+\gamma}{\alpha+\delta+\sigma}\left(\frac{\beta(\alpha+\gamma)-(\alpha+\sigma)(\delta+\sigma-p\gamma)}{\beta(\alpha+\gamma)}\right)N \\ &= \frac{\alpha+\gamma}{\alpha+\delta+\sigma}\left(1-\frac{(\alpha+\sigma)(\delta+\sigma-p\gamma)}{\beta(\alpha+\gamma)}\right)N. \end{aligned}$$

Given $R_0 = \dfrac{\beta(\alpha+\gamma)}{(\alpha+\sigma)(\delta+\sigma-p\gamma)}$, we can rearrange I^* as follows:

$$I^* = \frac{\alpha+\gamma}{\alpha+\delta+\sigma}\left(1-\frac{1}{R_0}\right)N.$$

Thus we have an endemic equilibrium $E^* = (S^*, I^*)$, with

$$S^* = \left(\frac{\alpha+\gamma}{\alpha+\sigma}\right)\frac{N}{R_0}, \quad I^* = \frac{\alpha+\gamma}{\alpha+\delta+\sigma}\left(1-\frac{1}{R_0}\right)N.$$

For this model, the basic reproduction number, that is an average number of secondary cases produced by a single infective introduced into an entirely susceptible population, is given as the parameter

$$R_0 = \frac{\beta(\alpha+\gamma)}{(\alpha+\sigma)(\delta+\sigma-p\gamma)}.$$

Korobeinikov and Wake [13] give the condition $R_0 > 1$ as this will ensure the existence of the positive endemic equilibrium state E^* and thus we will henceforth assume that this condition holds.

For $\alpha, p \neq 0$ the positive quadrant $\mathbb{R}^2_+$ of the SI plane is not an invariant set of system (3.1) as at $S = 0$ we have $\dot{S} < 0$ for all $I > \left(\dfrac{\alpha+\gamma}{\alpha+p\gamma}\right)N$. Consequently the boundary $S = 0$ is penetrable from $\mathbb{R}^2_+$. To avoided this, Korobeinikov and Wake [13] suggest the use of the substitution $(S, I) \to (P, I)$ where $P = S + \left(\dfrac{\alpha+p\gamma}{\beta}\right)N$.

We will now create a new system of differential equations using these substitutions.
First calculating $\dot{P}$:

$$\dot{P} = \dot{S} = (\gamma+\alpha)N - \beta\frac{SI}{N} - (\alpha+p\gamma)I - (\alpha+\sigma)S.$$

Note that we have

$$P = S + \left(\frac{\alpha+p\gamma}{\beta}\right)N \Rightarrow S = P - \left(\frac{\alpha+p\gamma}{\beta}\right)N$$

and thus substituting in S we derive:

$$\begin{aligned} \dot{P} &= (\gamma+\alpha)N - \beta\frac{I}{N}\left[P - \left(\frac{\alpha+p\gamma}{\beta}\right)N\right] - (\alpha+p\gamma)I - (\alpha+\sigma)\left[P - \left(\frac{\alpha+p\gamma}{\beta}\right)N\right] \\ &= (\gamma+\alpha)N - \beta\frac{PI}{N} + (\alpha+p\gamma)I - (\alpha+p\gamma)I - (\alpha+\sigma)P + \left(\frac{(\alpha+\sigma)(\alpha+p\gamma)}{\beta}\right)N \\ &= (\gamma+\alpha)N + \left(\frac{(\alpha+\sigma)(\alpha+p\gamma)}{\beta}\right)N - \beta\frac{PI}{N} - (\alpha+\sigma)P \\ &= \left[\gamma+\alpha+\frac{(\alpha+\sigma)(\alpha+p\gamma)}{\beta}\right]N - \beta\frac{PI}{N} - (\alpha+\sigma)P \\ &= \hat{\gamma}N - \beta\frac{PI}{N} - \hat{\sigma}P \end{aligned}$$

where $\hat{\gamma} = \gamma + \alpha + \dfrac{(\alpha+\sigma)(\alpha+p\gamma)}{\beta}$ and $\hat{\sigma} = \alpha + \sigma$.
Now calculating $\dot{I}$ we have:

$$\dot{I} = \beta \frac{SI}{N} - (\delta + \sigma - p\gamma)I$$

and substituting in $S = P - \left(\dfrac{\alpha + p\gamma}{\beta}\right)N$ we derive:

$$\begin{aligned}
\dot{I} &= \beta \frac{I}{N}\left[P - \left(\frac{\alpha+p\gamma}{\beta}\right)N\right] - (\delta + \sigma - p\gamma)I \\
&= \beta \frac{PI}{N} - (\alpha + p\gamma)I - (\delta + \sigma - p\gamma)I \\
&= \beta \frac{PI}{N} - (\alpha + p\gamma + \delta + \sigma - p\gamma)I \\
&= \beta \frac{PI}{N} - (\alpha + \delta + \sigma)I \\
&= \beta \frac{PI}{N} - \hat{\delta} I
\end{aligned}$$

where $\hat{\delta} = \alpha + \delta + \sigma$.

Thus in the new variables, we have the following differential system

$$\begin{aligned}
\dot{P} &= \hat{\gamma} N - \beta \frac{PI}{N} - \hat{\sigma} P, \\
\dot{I} &= \beta \frac{PI}{N} - \hat{\delta} I,
\end{aligned} \tag{3.2}$$

where $\hat{\gamma} = \gamma + \alpha + \dfrac{(\alpha+\sigma)(\alpha+p\gamma)}{\beta}$, $\quad \hat{\delta} = \alpha + \delta + \sigma \quad$ and $\quad \hat{\sigma} = \alpha + \sigma$.
System (3.2) obtained by the shift of system (3.1) along the S axis inherits the global properties of system (3.1) and vice versa. When $\alpha, p = 0$ i.e., for considering the SIR model, system (3.2) coincides with system (3.1). In the new variables, we can derive the endemic equilibrium as follows:

First we need to find P^* such that $\dot{I} = 0$:

$$\begin{aligned}
\dot{I} = \beta \frac{P^* I^*}{N} - \hat{\delta} I^* &= 0 \\
\Rightarrow \beta \frac{P^* I^*}{N} &= \hat{\delta} I^* \\
\Rightarrow P^* &= \frac{\hat{\delta} N}{\beta}
\end{aligned}$$

Given $R_0 = \dfrac{\beta\hat{\gamma}}{\hat{\sigma}\hat{\delta}}$, we can rearrange P^* as follows:

$$\begin{aligned}
P^* &= \frac{\hat{\delta}}{\beta} R_0 \frac{N}{R_0} \\
&= \frac{\hat{\delta}\beta\hat{\gamma}}{\beta\hat{\sigma}\hat{\delta}} \frac{N}{R_0} \\
&= \frac{\hat{\gamma}}{\hat{\sigma}} \frac{N}{R_0}
\end{aligned}$$

Next we need to find I^* such that $\dot{P} = 0$:

$$\begin{aligned}\dot{P} = \hat{\gamma}N - \beta\frac{P^*I^*}{N} - \hat{\sigma}P^* = 0 \\ \Rightarrow \beta\frac{P^*I^*}{N} = \hat{\gamma}N - \hat{\sigma}P^* \\ \Rightarrow I^* = \left(\frac{\hat{\gamma}N - \hat{\sigma}P^*}{\beta P^*}\right)N \\ = \left(\frac{\hat{\gamma}N}{\beta P^*} - \frac{\hat{\sigma}}{\beta}\right)N\end{aligned}$$

Substituting in $P^* = \frac{\hat{\gamma}}{\hat{\sigma}}\frac{N}{R_0} = \frac{\hat{\delta}N}{\beta}$ we derive:

$$\begin{aligned}I^* &= \left(\frac{\hat{\gamma}N}{\beta}\left[\frac{\beta}{\hat{\delta}N}\right] - \frac{\hat{\sigma}}{\beta}\right)N \\ &= \left(\frac{\hat{\gamma}}{\hat{\delta}} - \frac{\hat{\sigma}}{\beta}\right)N \\ &= \frac{\hat{\gamma}}{\hat{\delta}}\left(1 - \frac{\hat{\sigma}\hat{\delta}}{\beta\hat{\gamma}}\right)N \\ &= \frac{\hat{\gamma}}{\hat{\delta}}\left(1 - \frac{1}{R_0}\right)N.\end{aligned}$$

By noting that $R_0 = \frac{\beta\hat{\gamma}}{\hat{\sigma}\hat{\delta}}$, in the new variables the endemic equilibrium state $E^* = (P^*, I^*)$ has coordinates

$$P^* = \frac{\hat{\gamma}}{\hat{\sigma}}\frac{N}{R_0}, \qquad I^* = \frac{\hat{\gamma}}{\hat{\delta}}\left(1 - \frac{1}{R_0}\right)N \tag{3.3}$$

and is therefore the only equilibrium point for $P, I > 0$.

It follows from (3.2) that

$$\beta\frac{P^*I^*}{N} = \frac{\beta}{N}\left(\frac{\hat{\delta}N}{\beta}\right)I^* = \hat{\delta}I^*$$

and

$$\begin{aligned}\hat{\delta}I^* &= \hat{\gamma}\left(1 - \frac{1}{R_0}\right)N = \hat{\gamma}\left(1 - \frac{\hat{\sigma}\hat{\delta}}{\beta\hat{\gamma}}\right)N \\ &= \gamma\left(\frac{\beta\hat{\gamma} - \hat{\sigma}\hat{\delta}}{\beta\hat{\gamma}}\right)N = \left(\frac{\beta\hat{\gamma} - \hat{\sigma}\hat{\delta}}{\beta}\right)N \\ &= \hat{\gamma}N - \hat{\sigma}\frac{\hat{\delta}N}{\beta} = \hat{\gamma}N - \hat{\sigma}P^*.\end{aligned}$$

Thus we have

$$\beta\frac{P^*I^*}{N} = \hat{\gamma}N - \hat{\sigma}P^* = \hat{\delta}I^*. \tag{3.4}$$

Global properties of system (3.2), and therefore system (3.1), are given by Korobeinikov and Wake [13] in the following theorem:

Theorem 3.2.1 *The endemic equilibrium state E^* of system* (3.2) *(and hence, that of system* (3.1)*) is globally stable.*

Proof. We consider the Lyapunov test function

$$V(P, I) = P^* \left(\frac{P}{P^*} - \ln \frac{P}{P^*} \right) + I^* \left(\frac{I}{I^*} - \ln \frac{I}{I^*} \right) \tag{3.5}$$

which is defined and continuous for all $P, I > 0$ and satisfies

$$\frac{\partial V}{\partial P} = 1 - \frac{P^*}{P} \quad \text{and} \quad \frac{\partial V}{\partial I} = 1 - \frac{I^*}{I}.$$

Note that here we can determine the type of equilibrium point.

Definition 3.2.1 (Extrema: Maximum or Minimum) *Extreme points are points of a function whose derivative is zero and can be classified into maximum or minimum points, either locally or globally. Given $x = (x_1, \ldots, x_n)$ and a function $f(x)$, defined on the domain $\Omega \subset \mathbb{R}^n$, with an extreme point $x^* = ({x_1}^*, \ldots, {x_n}^*)$ we can define:*

1. *$f(x)$ has a **local minimum** at x^* if $f(x) \geq f(x^*)$ for every x in an open interval containing x^*.*
2. *$f(x)$ has a **global minimum** at x^* if $f(x) \geq f(x^*)$ for every x in Ω.*
3. *$f(x)$ has a **local maximum** at x^* if $f(x) \leq f(x^*)$ for every x in an open interval containing x^*.*
4. *$f(x)$ has a **global maximum** at x^* if $f(x) \leq f(x^*)$ for every x in Ω.*

For the Lyapunov function (3.5) we can see that $\frac{\partial V}{\partial P} = \frac{\partial V}{\partial I} = 0$ at E^* as defined in (3.3). To determine if the Lyapunov function has a maximum or minimum extreme point at E^*, we use the Hessian matrix to complete the second derivative test explained in the following definitions.

Definition 3.2.2 (Hessian Matrix) *A Hessian matrix is a square matrix containing second-order partial derivatives of a given function $f(x_i)$ and we denote as:*

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}. \tag{3.6}$$

The Hessian matrix describes the local curvature of a function of many variables and can be used to determine if an equilibrium point is a maximum or minimum extremum using the second derivative test.

Definition 3.2.3 (The Second Derivative Test) *Given $x = (x_1, \ldots, x_n)$ and a function $f(x)$, defined on the domain $\Omega \subset \mathbb{R}^n$, with an extreme point $x^* = ({x_1}^*, \ldots, {x_n}^*)$. Using the Hessian matrix defined in section 3.2.2, we can determine the type of extreme point.*

Firstly we denote D_i is the determinant of the Hessian matrix with i rows and columns, where $1 \leq i \leq n$ i.e.,

$$D_1 = \frac{\partial^2 f}{\partial x_1^2}, \quad D_2 = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix}, \ldots, D_n = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{vmatrix}.$$

1. *If all $D_i > 0$, for $1 \le i \le n$, at the extreme point x^*, then $f(x)$ has a **minimum** extreme point at x^* and the function $f(x)$ is concave upward at the extreme point x^*.*

2. *If all $(-1)^i D_i > 0$, for $1 \le i \le n$, at the extreme point x^*, then $f(x)$ has a **maximum** extreme point at x^* and the function $f(x)$ is concave downward (or convex) at the extreme point x^*.*

For the Lyapunov function (3.5) we can derive the second derivatives as follows:

$$\frac{\partial^2 V}{\partial P^2} = \frac{P^*}{P^2} > 0, \qquad \frac{\partial^2 V}{\partial I^2} = \frac{I^*}{I^2} > 0 \text{ and } \frac{\partial^2 V}{\partial P \partial I} = \frac{\partial^2 V}{\partial I \partial P} = 0.$$

Thus, by the second derivative test we have

$$D_1 = \frac{\partial^2 V}{\partial P^2} > 0 \text{ and } D_2 = \begin{vmatrix} \frac{\partial^2 V}{\partial P^2} & \frac{\partial^2 V}{\partial P \partial I} \\ \frac{\partial^2 V}{\partial I \partial P} & \frac{\partial^2 V}{\partial I^2} \end{vmatrix} > 0. \tag{3.7}$$

We can therefore conclude that the endemic equilibrium state $E^*= (P^*, I^*)$ is the only local stationary point of the function and is a minimum extreme point. As $V(P,I) \to \infty$ at the boundary, we have $V(P,I) \ge V(P^*, I^*)$ and thus the function is bounded from below. Consequently E^*, as defined in (3.3), is the only extremum and the global minimum of the function in $\mathbb{R}^2_+$ and hence the function (3.5) is indeed a Lyapunov function.

In the case of system (3.2), using (3.4), the function $V(P,I)$ satisfies

$$\begin{aligned}
\dot{V}(P,I) &= \left(1 - \frac{P^*}{P}\right)\left(\hat{\gamma}N - \beta\frac{PI}{N} - \hat{\sigma}P\right) + \left(1 - \frac{I^*}{I}\right)\left(\beta\frac{PI}{N} - \hat{\delta}I\right). \\
&= \hat{\gamma}N - \beta\frac{PI}{N} - \hat{\sigma}P - \hat{\gamma}N\frac{P^*}{P} + \beta\frac{P^*}{N}I + \hat{\sigma}P^* + \beta\frac{PI}{N} - \hat{\delta}I - \beta\frac{PI^*}{N} + \hat{\delta}I^* \\
&= \hat{\gamma}N - \hat{\gamma}N\frac{P^*}{P} + \hat{\sigma}P^* + \hat{\delta}I^* - \hat{\sigma}P - \hat{\delta}I + \beta\frac{P^*}{N}I - \beta\frac{PI^*}{N}
\end{aligned}$$

Note that $\hat{\gamma}N - \hat{\sigma}P^* = \hat{\delta}I^* \Rightarrow \hat{\sigma}P^* + \hat{\delta}I^* = \hat{\gamma}N$, therefore:

$$\dot{V}(P,I) = \hat{\gamma}N - \hat{\gamma}N\frac{P^*}{P} + \hat{\gamma}N - \hat{\sigma}P - \hat{\delta}I + \beta\frac{P^*}{N}I + \beta\frac{PI}{N} - \beta\frac{PI}{N} - \beta\frac{PI^*}{N}$$

Also note that $\hat{\gamma}N - \hat{\sigma}P^* = \hat{\delta}I^* \Rightarrow \hat{\sigma}P^* = \hat{\gamma}N - \hat{\delta}I^* \Rightarrow \hat{\sigma}P = \hat{\sigma}P^*\frac{P}{P^*} = \frac{P}{P^*}(\hat{\gamma}N - \hat{\delta}I^*)$, thus:

$$\begin{aligned}
\dot{V}(P,I) &= \hat{\gamma}N - \hat{\gamma}N\frac{P^*}{P} + \hat{\gamma}N - \left[\frac{P}{P^*}(\hat{\gamma}N - \hat{\delta}I^*)\right] - \hat{\delta}I + \beta\frac{P^*}{N}I - \beta\frac{PI^*}{N} \\
&= \hat{\gamma}N - \hat{\gamma}N\frac{P^*}{P} + \hat{\gamma}N - \hat{\gamma}N\frac{P}{P^*} + \hat{\delta}I^*\frac{P}{P^*} - \hat{\delta}I + \beta\frac{P^*I^*}{N}\frac{I}{I^*} - \beta\frac{P^*I^*}{N}\frac{P}{P^*} \\
&= \hat{\gamma}N\left(1 - \frac{P^*}{P} - \frac{P}{P^*} + 1\right) + \hat{\delta}I^*\frac{P}{P^*} - \hat{\delta}I^*\frac{I}{I^*} + \hat{\delta}I\frac{I}{I^*} - \hat{\delta}I^*\frac{P}{P^*} \\
&= -\hat{\gamma}N\left(\frac{P^*}{P} + \frac{P}{P^*} - 2\right) \\
&= -\hat{\gamma}N\frac{P^*}{P}\left(1 - 2\frac{P}{P^*} + \left(\frac{P}{P^*}\right)^2\right) \\
&= -\hat{\gamma}N\frac{P^*}{P}\left(1 - \frac{P}{P^*}\right)^2.
\end{aligned}$$

Thus we have

$$\dot{V}(P,I) = -\hat{\gamma}N\frac{P^*}{P}\left(1 - \frac{P}{P^*}\right)^2 \le 0, \qquad \text{for all} \quad P, I \ge 0. \tag{3.8}$$

The equality $\dot{V}(P, I) = 0$ holds only on the straight line $P = P^*$ (see [13]). Since the endemic equilibrium state E^* is the only invariant set of system (3.2) on the straight line $P = P^*$, by the asymptotic stability theorem 1.3.3, the equilibrium E^*, as defined in (3.3), is globally asymptotically stable. The theorem is proven.

Note that if we were to consider permanent immunity for the model (3.1) this implies that average period of immunity $\frac{1}{\alpha}$ is infinite and thus $\alpha = 0$ which would therefore reduce the $SIRS$ model to the SIR model. Further examples of SIR models follow.

3.3 SIR Models

The second type of epidemiological model we will concentrate on is the SIR model. Following classical assumptions we divide the population, of size N, into the subpopulations as defined in (3.1). We assume that an individual begins in the susceptibles compartment, after infection moves to the infective compartment and then continues into the removed compartment as a result of recovery, isolation or death by disease. For our SIR models we assume that recovery implies permanent immunity. For simplicity, we will normalize the population size N to 1 for each of these models; i.e. now S, I and R are, respectively, the fractions of the susceptibles, the infectives and the removed in the population, and thus $S + I + R = 1$ holds.

3.3.1 An SIR Model Given by Korobeinikov and Maini [11]

The first SIR model we will concentrate on is a model with nonlinear incidence. Korobeinikov and Maini [11] note that there are a variety of reasons why the standard bilinear incidence rate may require modification. The incidence rate of the form $\beta I^p S^q$, where S and I are respectively the number of susceptible and infective individuals in the population (or the fractions of the susceptible and infective), and β, p and q are positive constants, is the most common nonlinear incidence rate and thus is what we will investigate.

Applying the direct Lyapunov method, we can consider the global properties of SIR models with the incidence rate of the form $\beta I^p S^q$ for the particular case $p \leq 1$. Korobeinikov and Maini [11] construct a Lyapunov function for models with bilinear incidence and we will use their approach to show that the condition $p < 1$ is a sufficient condition for global stability.

For the incidence rate of the form $\beta I^p S^q$, the basic SIR model is given by Korobeinikov and Maini [11] as:

$$\begin{aligned} \dot{S} &= b - \beta I^p S^q - \mu S, \\ \dot{I} &= \beta I^p S^q - \delta I, \end{aligned} \tag{3.9}$$

Here b is the birth rate, μ is the susceptible death rate and δ is the infective removal rate (including mortality rate). We omit the equations for the recovered population R; the constant population size assumption enables us to do so.

If $0 < p < 1$ holds, then system (3.9) has two equilibrium states: an infection-free equilibrium and an endemic equilibrium.

Finding the infection-free equilibrium:

Infection-free $\Rightarrow I_0 = 0$.
First we need to find S_0 such that $\dot{S} = 0$:

$$\begin{aligned} \dot{S} = b - \beta I_0^p S_0^q - \mu S_0 = 0 \\ \Rightarrow b - \beta (0)^p S_0^q - \mu S_0 = 0 \\ \Rightarrow b - \mu S_0 = 0 \end{aligned}$$

$$\Rightarrow \mu S_0 = b$$
$$\Rightarrow S_0 = \frac{b}{\mu}.$$

Thus we have an infection-free equilibrium Q_0, see also [11], with the coordinates $S_0 = \dfrac{b}{\mu}, I_0 = 0$.
Finding the endemic equilibrium:

First we will consider $\dot{I} = 0$:

$$\dot{I} = \beta(I^*)^p(S^*)^q - \delta I^* = 0$$
$$\Rightarrow \delta I^* = \beta(I^*)^p(S^*)^q. \tag{3.10}$$

Now we will consider $\dot{S} = 0$:

$$\dot{S} = b - \beta(I^*)^p(S^*)^q - \mu S^* = 0$$
$$\Rightarrow \mu S^* + \beta(I^*)^p(S^*)^q = b. \tag{3.11}$$

Substituting (3.10) into (3.11):

$$\mu S^* + \delta I^* = b. \tag{3.12}$$

Thus, using (3.10) and (3.12), we have the endemic equilibrium state $Q^*= (S^*, I^*)$, such that

$$\delta I^* = \beta(I^*)^p(S^*)^q, \qquad \mu S^* + \delta I^* = b. \tag{3.13}$$

We will now consider the following theorem, as given by Korobeinikov and Maini [11], and apply it instead to the SIR system (3.9) to confirm their conclusion.

Theorem 3.3.1 *If $p \leq 1$, then the endemic equilibrium state Q^* of the model* (3.9) *is globally asymptotically stable. The stability does not depend on the value of the parameter q.*

Proof. Assume that $p, q \neq 1$. Then for the SIR model, we consider a test Lyapunov function of the form

$$V(S,I) = S\left(1 + \frac{1}{q-1}\left(\frac{S^*}{S}\right)^q\right) + I\left(1 + \frac{1}{p-1}\left(\frac{I^*}{I}\right)^p\right). \tag{3.14}$$

This function is defined and continuous for all $S, I > 0$ and satisfies

$$\frac{\partial V}{\partial S} = 1 - \left(\frac{S^*}{S}\right)^q \quad \text{and} \quad \frac{\partial V}{\partial I} = 1 - \left(\frac{I^*}{I}\right)^p.$$

Note that for this function we have $\frac{\partial V}{\partial S} = \frac{\partial V}{\partial I} = 0$ at Q^*, as defined in (3.13), and thus we can determine the type of equilibrium point. For the possible Lyapunov function (3.14) we can derive the second derivatives as follows:

$$\frac{\partial^2 V}{\partial S^2} = \frac{q}{S}\left(\frac{S^*}{S}\right)^q > 0, \qquad \frac{\partial^2 V}{\partial I^2} = \frac{p}{I}\left(\frac{I^*}{I}\right)^p > 0 \quad \text{and} \quad \frac{\partial^2 V}{\partial S \partial I} = 0.$$

Thus, by the second derivative test, see definition 3.2.3, we can conclude that the endemic equilibrium state $Q^*= (S^*, I^*)$ is the only extremum and the global minimum of the function in the positive octant $\mathbb{R}^3_+$ and hence $V(S,I) \geq V(S^*, I^*)$. Consequently, the function (3.14) is indeed a Lyapunov function.

Using the equalities

$$b = \mu S^* + \delta I^*, \qquad \beta(I^*)^p(S^*)^q = \delta I^* \tag{3.15}$$

for the equilibrium state Q^*, as defined in (3.13), the Lyapunov function (3.14) satisfies:

$$\dot{V}(S,I) = \left(1 - \left(\frac{S^*}{S}\right)^q\right)(b - \beta I^p S^q - \mu S) + \left(1 - \left(\frac{I^*}{I}\right)^p\right)(\beta I^p S^q - \delta I)$$

$$
\begin{aligned}
&= b - \beta I^p S^q - \mu S - b\left(\frac{S^*}{S}\right)^q \\
&\quad + \beta I^p (S^*)^q + \mu S\left(\frac{S^*}{S}\right)^q + \beta I^p S^q - \delta I - \beta (I^*)^p S^q + \delta I\left(\frac{I^*}{I}\right)^p \\
&= b - \mu S - b\left(\frac{S^*}{S}\right)^q + \beta I^p (S^*)^q + \mu S\left(\frac{S^*}{S}\right)^q - \delta I - \beta (I^*)^p S^q + \delta I\left(\frac{I^*}{I}\right)^p
\end{aligned}
$$

Using now (3.15) we obtain:

$$
\begin{aligned}
\dot{V}(S,I) &= [\mu S^* + \delta I^*] - \mu S - [\mu S^* + \delta I^*]\left(\frac{S^*}{S}\right)^q + \delta I^*\left(\frac{I}{I^*}\right)^p + \mu S\left(\frac{S^*}{S}\right)^q \\
&\quad - \delta I - \delta I^*\left(\frac{S}{S^*}\right)^q + \delta I\left(\frac{I^*}{I}\right)^p \\
&= \mu S^* + \delta I^* - \mu S - \mu S^*\left(\frac{S^*}{S}\right)^q - \delta I^*\left(\frac{S^*}{S}\right)^q + \delta I^*\left(\frac{I}{I^*}\right)^p + \mu S\left(\frac{S^*}{S}\right)^q \\
&\quad - \delta I - \delta I^*\left(\frac{S}{S^*}\right)^q + \delta I\left(\frac{I^*}{I}\right)^p \\
&= \mu S^*\left[1 - \frac{S}{S^*} - \left(\frac{S^*}{S}\right)^q + \frac{S}{S^*}\left(\frac{S^*}{S}\right)^q\right] \\
&\quad + \delta I^*\left[1 - \left(\frac{S^*}{S}\right)^q + \left(\frac{I}{I^*}\right)^p - \frac{I}{I^*} - \left(\frac{S}{S^*}\right)^q + \frac{I}{I^*}\left(\frac{I^*}{I}\right)^p\right] \\
&= \mu S^*\left[1 - \frac{S}{S^*} - \left(\frac{S^*}{S}\right)^q + \frac{S}{S^*}\left(\frac{S^*}{S}\right)^q\right] \\
&\quad + \delta I^*\left[2 - \left(\frac{S^*}{S}\right)^q - \left(\frac{S}{S^*}\right)^q\right] \\
&\quad + \delta I^*\left[+\left(\frac{I}{I^*}\right)^p - \frac{I}{I^*} + \frac{I}{I^*}\left(\frac{I^*}{I}\right)^p - 1\right] \\
&= \mu S^*\left(1 - \frac{S}{S^*}\right)\left(1 - \left(\frac{S^*}{S}\right)^q\right) \\
&\quad - \delta I^*\left(\frac{S}{S^*}\right)^q\left[\left(\left(\frac{S^*}{S}\right)^q\right)^2 - 2\left(\frac{S^*}{S}\right)^q + 1\right] \\
&\quad + \delta I^*\left(1 - \left(\frac{I^*}{I}\right)^p\right)\left(\left(\frac{I}{I^*}\right)^p - \frac{I}{I^*}\right) \\
&= \mu S^*\left(1 - \frac{S}{S^*}\right)\left(1 - \left(\frac{S^*}{S}\right)^q\right) \\
&\quad - \delta I^*\left(\frac{S}{S^*}\right)^q\left(\left(\frac{S^*}{S}\right)^q - 1\right)^2 \\
&\quad + \delta I^*\left(1 - \left(\frac{I^*}{I}\right)^p\right)\left(\left(\frac{I}{I^*}\right)^p - \frac{I}{I^*}\right).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\dot{V}(S,I) = \mu S^* &\left(1 - \frac{S}{S^*}\right)\left(1 - \left(\frac{S^*}{S}\right)^q\right) \\
&- \delta I^*\left(\frac{S}{S^*}\right)^q\left(\left(\frac{S^*}{S}\right)^q - 1\right)^2 \\
&+ \delta I^*\left(1 - \left(\frac{I^*}{I}\right)^p\right)\left(\left(\frac{I}{I^*}\right)^p - \frac{I}{I^*}\right) \le 0, \quad \forall p < 1.
\end{aligned}
\tag{3.16}
$$

Therefore, as explained by Korobeinikov and Maini [11], the condition $p < 1$ ensures that $\frac{dV}{dt} \le 0$ for all $S, I > 0$, where the equality holds only at the equilibrium point $Q^*=(S^*,I^*)$ as defined in (3.13). By the

Lyapunov asymptotic stability theorem 1.3.3, the equilibrium point Q^* is globally asymptotically stable. This result is valid for the whole of $\mathbb{R}^2_+$.

3.3.2 An SIR Model Given by Korobeinikov and Maini [12]

The second SIR model we will concentrate on is also a model with nonlinear incidence.
We consider, as in [12], a function $f(S, I, N)$, where

$$f(S, 0, N) = f(0, I, N) = 0 \tag{3.17}$$

and

$$\frac{\partial f(S, I, N)}{\partial I} > 0, \quad \frac{\partial f(S, I, N)}{\partial S} > 0, \tag{3.18}$$

holds for all $S, I > 0$ and we also assume that the function $f(S, I, N)$ is concave with respect to the variable I, i.e.

$$\frac{\partial^2 f(S, I, N)}{\partial I^2} \leq 0, \quad \text{for all} \quad S, I > 0. \tag{3.19}$$

For the incidence rate of the form $h(S)g(I)$ satisfying the conditions (3.17)-(3.18), the direct Lyapunov method enables us to prove global stability for some models. Korobeinikov and Maini [12] claims that the direct Lyapunov method so far is the most effective method of global analysis.

The basic SIR model is given by Korobeinikov and Maini [12] as:

$$\begin{aligned} \dot{S} &= \mu - h(S)g(I) - \mu S, \\ \dot{I} &= h(S)g(I) - (\delta + \mu)I. \end{aligned} \tag{3.20}$$

Here the equation for the recovered population R is omitted, and as usual we assume that the incidence rate satisfies the conditions (3.17)-(3.19). The condition (3.19) ensures this systems has two equilibrium states: an infection-free equilibrium Q_0 and an endemic equilibrium Q^*.

Finding the infection-free equilibrium:

Infection-free$\Rightarrow I_0 = 0$.
We need to find S_0 such that $\dot{S} = 0$:

$$\begin{aligned} \Rightarrow \dot{S} = \mu - h(S_0)g(I_0) - \mu S_0 = 0 \\ \Rightarrow \mu - h(S_0)g(0) - \mu S_0 = 0 \\ \Rightarrow \mu - \mu S_0 = 0 \\ \Rightarrow \mu S_0 = \mu \\ \Rightarrow S_0 = 1. \end{aligned}$$

Thus we have an infection-free equilibrium $Q_0 = (1, 0)$.

Finding the endemic equilibrium:
First we consider $\dot{I} = 0$:

$$\begin{aligned} \dot{I} = h(S^*)g(I^*) - (\delta + \mu)I^* = 0 \\ \Rightarrow (\delta + \mu)I^* = h(S^*)g(I^*). \end{aligned}$$

Now considering $\dot{S} = 0$:

$$\begin{aligned} \dot{S} = \mu - h(S^*)g(I^*) - \mu S^* = 0 \\ \Rightarrow \mu S^* + h(S^*)g(I^*) = \mu. \end{aligned}$$

Substituting in $(\delta+\mu)I^* = h(S^*)g(I^*)$:

$$\mu S^* + (\delta+\mu)I^* = \mu.$$

Thus we have an endemic equilibrium $Q^*= (S^*, I^*)$, such that

$$(\delta+\mu)I^* = h(S^*)g(I^*), \quad \mu S^* + (\delta+\mu)I^* = \mu. \tag{3.21}$$

For this SIR model, Korobeinikov and Maini [12] give the possible Lyapunov function:

$$V(S,I) = S - h(S^*)\int_a^S \frac{\mathrm{d}\tau}{h(\tau)} + \left(I - g(I^*)\int_a^I \frac{\mathrm{d}\tau}{g(\tau)}\right). \tag{3.22}$$

Here the parameter a, such that $0 < a \ll 1$, is an arbitrary positive constant which is not fixed and can be made sufficiently small. The function $V(S,I)$ is defined and continuous for all $S, I \geq a$ and satisfies

$$\frac{\partial V}{\partial S} = 1 - \frac{h(S^*)}{h(S)}, \quad \text{and} \quad \frac{\partial V}{\partial I} = B\left(1 - \frac{g(I^*)}{g(I)}\right).$$

Since the function $h(S)$ and $g(I)$ grow monotonically, the partial derivatives $\frac{\partial V}{\partial S}$ and $\frac{\partial V}{\partial I}$ grow monotonically as well. Note that for this function we have $\frac{\partial V}{\partial S} = \frac{\partial V}{\partial I} = 0$ at Q^*, as defined in (3.21), and thus we can determine the type of equilibrium point. For the possible Lyapunov function (3.22) we can derive the second derivatives as follows:

$$\frac{\partial^2 V}{\partial S^2} = \frac{h(S^*)}{(h(S))^2}\frac{\partial h(S)}{\partial S} > 0, \quad \frac{\partial^2 V}{\partial I^2} = \frac{g(I^*)}{(g(I))^2}\frac{\partial g(I)}{\partial I} > 0, \quad \text{and} \quad \frac{\partial^2 V}{\partial S \partial I} = 0.$$

Thus, by the second derivative test, see definition 3.2.3, we can conclude that the endemic equilibrium state $Q^*= (S^*, I^*)$ is the only local stationary point of the function and is a minimum extreme point. As $V(S,I) \to \infty$ at the boundary, we have $V(S,I) \geq V(S^*, I^*)$ and thus the function is bounded from below. Consequently Q^* is the only extremum and the global minimum of the function in $\mathbb{R}^2_+$ and hence the function (3.22) is indeed a Lyapunov function.

The following theorem, given by Korobeinikov and Maini [12], provides global properties of the system (3.20).

Theorem 3.3.2

1. *If the incidence rate satisfies the conditions* (3.17)-(3.19)*, and if* $R_0 > 1$*, then the endemic equilibrium state* Q^* *is globally asymptotically stable.*
2. *If* $R_0 \leq 1$*, then there is no positive equilibrium state* Q^**, and the infection-free equilibrium state* Q_0 *is globally asymptotically stable.*

Proof of 1. Using the equalities

$$(\delta+\mu)I^* = h(S^*)g(I^*), \quad \mu = \mu S^* + (\delta+\mu)I^*, \tag{3.23}$$

for the equilibrium state Q^*, the Lyapunov function $V(S,I)$ satisfies

$$\begin{aligned}\frac{\mathrm{d}V(S,I)}{\mathrm{d}t} &= \left(1 - \frac{h(S^*)}{h(S)}\right)(\mu - h(S)g(I) - \mu S) + \left(1 - \frac{g(I^*)}{g(I)}\right)(h(S)g(I) - (\delta+\mu)I) \\ &= \mu - h(S)g(I) - \mu S - \mu\frac{h(S^*)}{h(S)} + g(I)h(S^*) + \mu S\frac{h(S^*)}{h(S)} \\ &\quad + h(S)g(I) - (\delta+\mu)I - g(I^*)h(S) + (\delta+\mu)I\frac{g(I^*)}{g(I)}.\end{aligned}$$

Using the relations (3.23) we get:

$$\begin{aligned}
\frac{\mathrm{d}V(S,I)}{\mathrm{d}t} &= [\mu S^* + (\delta+\mu)I^*] - \mu S^*\frac{S}{S^*} - [\mu S^* + (\delta+\mu)I^*]\,\frac{h(S^*}{h(S)} + (\delta+\mu)I^*\frac{g(I)}{g(I^*)} + \mu S\frac{h(S^*)}{h(S)} \\
&\quad - (\delta+\mu)I - (\delta+\mu)I^*\frac{h(S)}{h(S^*)} + (\delta+\mu)I\frac{g(I^*)}{g(I)} \\
&= (\delta+\mu)I^*\left(1 - \frac{h(S^*)}{h(S)} - \frac{h(S)}{h(S^*)}\right) + \mu S^*\left(1 - \frac{h(S^*)}{h(S)} - \frac{S}{S^*} + \frac{S}{S^*}\frac{h(S^*)}{h(S)}\right) \\
&\quad + (\delta+\mu)I^*\left(\frac{I}{I^*}\frac{g(I^*)}{g(I)} + \frac{g(I)}{g(I^*)} - \frac{I}{I^*}\right) \\
&= (\delta+\mu)I^*\left(2 - \frac{h(S^*)}{h(S)} - \frac{h(S)}{h(S^*)}\right) + \mu S^*\left(1 - \frac{h(S^*)}{h(S)} - \frac{S}{S^*} + \frac{S}{S^*}\frac{h(S^*)}{h(S)}\right) \\
&\quad + (\delta+\mu)I^*\left(\frac{I}{I^*}\frac{g(I^*)}{g(I)} + \frac{g(I)}{g(I^*)} - \frac{I}{I^*} - 1\right) \\
&= -(\delta+\mu)I^*\frac{h(S^*)}{h(S)}\left(1 - 2\frac{h(S)}{h(S^*)} + \left(\frac{h(S)}{h(S^*)}\right)^2\right) + \mu S^*\left(1 - \frac{h(S^*)}{h(S)} - \frac{S}{S^*} + \frac{S}{S^*}\frac{h(S^*)}{h(S)}\right) \\
&\quad - (\delta+\mu)I^*\left(1 - \frac{I}{I^*}\frac{g(I^*)}{g(I)} - \frac{g(I)}{g(I^*)} + \frac{I}{I^*} - 1\right),
\end{aligned}$$

which can be rearranged to give:

$$\begin{aligned}
\frac{\mathrm{d}V(S,I)}{\mathrm{d}t} &= -(\delta+\mu)I^*\frac{h(S^*)}{h(S)}\left(1 - 2\frac{h(S)}{h(S^*)}\right)^2 + \mu S^*\left(1 - \frac{S}{S^*)}\right)\left(1 - \frac{h(S^*)}{h(S)}\right) \\
&\quad - (\delta+\mu)I^*\left(\frac{g(I^*)}{g(I)} - 1\right)\left(\frac{g(I)}{g(I^*)} - \frac{I}{I^*}\right).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{\mathrm{d}V(S,I)}{\mathrm{d}t} &= -(\delta+\mu)I^*\frac{h(S^*)}{h(S)}\left(1 - 2\frac{h(S)}{h(S^*)}\right)^2 + \mu S^*\left(1 - \frac{S}{S^*)}\right)\left(1 - \frac{h(S^*)}{h(S)}\right) \\
&\quad - (\delta+\mu)I^*\left(\frac{g(I^*)}{g(I)} - 1\right)\left(\frac{g(I)}{g(I^*)} - \frac{I}{I^*}\right) \le 0, \quad \text{for all } S, I > a.
\end{aligned} \tag{3.24}$$

Hence the endemic equilibrium state is globally asymptotically stable.

Proof of 2. To prove global stability of the infection-free equilibrium state $Q_0 = (1,0)$ we consider, as in [12], the Lyapunov test function:

$$U(S,I) = S - h(S_0)\int_a^S \frac{\mathrm{d}\tau}{h(\tau)} + I.$$

For system (3.20), the Lyapunov function satisfies

$$\begin{aligned}
\frac{\mathrm{d}U(S,I)}{\mathrm{d}t} &= \left(1 - \frac{h(S_0)}{h(S)}\right)(\mu - h(S)g(I) - \mu S) + (1)\,(h(S)g(I) - (\delta+\mu)I) \\
&= \mu - h(S)g(I) - \mu S - \mu\frac{h(S_0)}{h(S)} + h(S_0)g(I) + \mu S\frac{h(S_0)}{h(S)} + h(S)g(I) - (\delta+\mu)I \\
&= \mu\left(1 - \frac{h(S_0)}{h(S)} - S + S\frac{h(S_0}{h(S)}\right) + h(S_0)g(I) - (\delta+\mu)I \\
&= \mu(1-S)\left(1 - \frac{h(S_0)}{h(S)}\right) + (\delta+\mu)I\left(\frac{h(S_0)}{\delta+\mu}\frac{g(I)}{I} - 1\right).
\end{aligned}$$

Here

$$(1-S)\left(1-\frac{h(S_0)}{h(S)}\right) \leq 0, \quad \text{for all } S > 0,$$

and the conditions (3.17) and (3.19) ensure that $\frac{g(I)}{I} \leq \frac{\partial g(0)}{\partial I}$, for all $I > 0$. Hence, as given by Korobeinikov and Maini [12], using

$$R_0 = \frac{\theta}{(\theta+\mu)(\delta+\mu)} \frac{\partial f(S_0, I_0, N)}{\partial I}, \tag{3.25}$$

we have

$$\frac{h(S_0)}{\delta+\mu}\frac{g(I)}{I} \leq \frac{h(S_0)}{\delta+\mu}\frac{\partial g(0)}{\partial I} = R_0.$$

Therefore, $R_0 \leq 1$ ensures that $\frac{\mathrm{d}V}{\mathrm{d}t} \leq 0$, for all $S, I > a$, and hence by the asymptotic stability theorem (see theorem 1.3.3), the equilibrium state Q_0 is globally asymptotically stable in this case. The theorem is proven.

3.3.3 An SIR Model Given by Korobeinikov [10]

The final SIR model we will concentrate on is also a model with nonlinear incidence. This paper attempts to extend the Lyapunov functions constructed by Korobeinikov and Maini [12] to a more general incidence rate given by an arbitrary function $f(S, I)$.

The basic SIR model is given by Korobeinikov [10] as:

$$\begin{aligned} \dot{S} &= \mu - f(S,I) - \mu S, \\ \dot{I} &= f(S,I) - \delta I. \end{aligned} \tag{3.26}$$

Here μ is the death/birth rate and δ is the sum of the death rate of infected individuals (which is here assumed to be equal to the death rate of susceptibles) and the recovery rate. The equation for the recovery population, which in this case is

$$\dot{R} = (\delta - \mu)\, I - \mu R,$$

is usually omitted-the constant population size assumption enables us to do that.

Korobeinikov [10] explains that $f(S, I)$ is a positive and monotonically growing function for all $S, I > 0$, and $f(0, I) = f(S, 0) = 0$. These conditions solely arise from biological considerations; further we will show that they are not necessary for our analysis.

It is easy to see that the non-negative quadrant of the SI plane is an invariant set of the system. The system has two equilibrium states: an infection-free equilibrium and an endemic equilibrium.

Finding the infection-free equilibrium:

Infection-free $\Rightarrow I_0 = 0$.
We need to find S_0 such that $\dot{S} = 0$:

$$\begin{aligned} \dot{S} = \mu - f(S_0, I_0) - \mu S_0 &= 0 \\ \Rightarrow \mu - f(S_0, 0) - \mu S_0 &= 0 \\ \Rightarrow \mu - (0) - \mu S_0 &= 0 \\ \Rightarrow \mu - \mu S_0 &= 0 \\ \Rightarrow \mu S_0 &= \mu \\ \Rightarrow S_0 &= 1. \end{aligned}$$

Thus we have an infection-free equilibriumQ_0 with the coordinates $S_0 = 1, I_0 = 0$.

Finding the endemic equilibrium:
First we will consider $\dot{S} = 0$:

$$\dot{S} = \mu - f(S^*, I^*) - \mu S^* = 0$$
$$\Rightarrow \mu = f(S^*, I^*) + \mu S^*.$$

Next we will consider $\dot{I} = 0$:

$$\dot{I} = f(S^*, I^*) - \delta I^*$$
$$\Rightarrow \delta I^* = f(S^*, I^*).$$

Thus we have an endemic equilibrium Q^*=(S^*,I^*), such that

$$\mu = f(S^*, I^*) + \mu S^* \quad \text{and} \quad \delta I^* = f(S^*, I^*). \tag{3.27}$$

By the following theorem given by Korobeinikov [10], we will show that, under certain biologically reasonable conditions, if the function $f(S, I)$ is concave with respect to the variable I (that is if $\frac{\partial^2 f}{\partial I^2} \leq 0$ holds for all $I > 0$), then the uniqueness of the equilibrium state Q^* is ensured.

For this model, the basic reproduction number is

$$R_0 = \frac{1}{\delta} \frac{\partial f(S_0, I_0)}{\partial I}.$$

Theorem 3.3.3

1. *Assume that the function $f(S, I)$ monotonically grows with respect to S and I and is concave with respect to the variable I (that is if $\frac{\partial^2 f}{\partial I^2} \leq 0$). Assume also $R_0 > 1$, then the system (3.26) has an unique positive endemic equilibrium state Q^*= (S^*, I^*), as defined in* (3.27)*, which is globally asymptotically stable.*

2. *If $R_0 \leq 1$, then there is no positive endemic equilibrium state, and the infection-free equilibrium state Q_0 is globally asymptotically stable.*

Proof: Existence of a positive equilibrium state. Korobeinikov [10] explains that at a fixed point of the system the relations $\delta I + \mu S = \mu$ and $\delta I = f(S, I)$ hold. These equalities define a negatively sloped straight line q_1 and a curve q_2 on the IS plane (Figure 3.2). The equality $\delta I = f(S, I)$ defines also a function $S = h(I)$. If $\frac{\partial f(S,I)}{\partial S}$ is strictly positive, then, by the implicit function theorem, the function $h(I)$ is defined and continuous for all $I > 0$. It is obvious (see Figure 3.2) that if $S_* = h(0) \leq S_0 = 1$ then there is at least one point of intersection of the lines q_1 and q_2. The function $f(S, I)$ grows monotonically with respect to both its variables, and hence $\dfrac{S_0}{S_*} > 1$ if

$$\lim_{I \to 0} \frac{f(S_0, I)}{f(S_*, I)} = \lim_{I \to 0} \frac{f(S_0, I)}{\delta I} = \frac{1}{\delta} \frac{\partial f(S_0, 0)}{\partial I} = R_0 > 1.$$

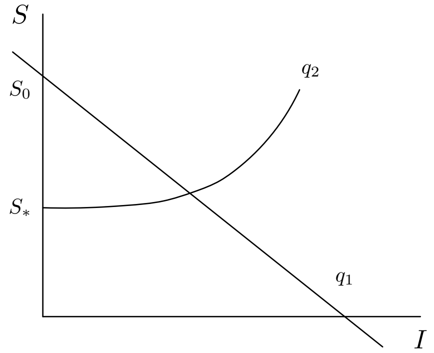

Figure 3.2: The straight line q_1 and the curve q_2.

Proof: Stability of the endemic equilibrium state. We assume that $R_0 > 1$, and hence there exists a positive endemic equilibrium state $Q^*= (S^*, I^*)$, as defined in (3.27). The function, given be Korobeinikov [10]:

$$V(S,I) = S - \int_{\varepsilon}^{S} \frac{f(S^*,I^*)}{f(\tau,I^*)} d\tau + I - \int_{\varepsilon}^{I} \frac{f(S^*,I^*)}{f(S^*,\tau)} d\tau, \tag{3.28}$$

is defined and continuous for all $S, I > \varepsilon$ and satisfies

$$\frac{\partial V}{\partial S} = 1 - \frac{f(S^*,I^*)}{f(S,I^*)} \quad \text{and} \quad \frac{\partial V}{\partial I} = 1 - \frac{f(S^*,I^*)}{f(S^*,I)}. \tag{3.29}$$

Since the function $f(S, I)$ is monotonic with respect to both variables, the partial derivatives $\frac{\partial V}{\partial S}$ and $\frac{\partial V}{\partial I}$ are also monotonic. Note that for this function we have $\frac{\partial V}{\partial S} = \frac{\partial V}{\partial I} = 0$ at $Q^*= (S^*, I^*)$, as defined in (3.27), and thus we can determine the type of equilibrium point. For the possible Lyapunov function (3.28) we can derive the second derivatives as follows:

$$\frac{\partial^2 V}{\partial S^2} = \frac{f(S^*,I^*)}{(f(S,I^*))^2} \frac{\partial f(S,I^*)}{\partial S} > 0, \qquad \frac{\partial^2 V}{\partial I^2} = \frac{f(S^*,I^*)}{(f(S^*,I))^2} \frac{\partial f(S^*,I)}{\partial I} > 0 \quad \text{and} \quad \frac{\partial^2 V}{\partial S \partial I} = 0.$$

Thus, by the second derivative test, see definition 3.2.3, we can conclude that if $f(S, I)$ is monotonic with respect to both variables then the endemic equilibrium state $Q^*= (S^*, I^*)$ is the only local stationary point of the function and is a minimum extreme point. As $V(S, I) \to \infty$ at the boundary, we have $V(S, I) \geq V(S^*, I^*)$ and thus the function is bounded from below. Consequently Q^* is the only extremum and the global minimum of the function in $\mathbb{R}_+^2$ and hence the function (3.28) is indeed a Lyapunov function.

In the case of system (3.26), using

$$\mu = f(S^*,I^*) + \mu S^* \quad \text{and} \quad \delta I^* = f(S^*,I^*)$$

the Lyapunov function (3.28) satisfies

$$\frac{\mathrm{d}V(S,I)}{\mathrm{d}t} = \left(1 - \frac{f(S^*,I^*)}{f(S,I^*)}\right)(\mu - f(S,I) - \mu S) + \left(1 - \frac{f(S^*,I^*)}{f(S^*,I)}\right)(f(S,I) - \delta I),$$

which can be rearranged to give:

$$\begin{aligned}\frac{\mathrm{d}V(S,I)}{\mathrm{d}t} &= \mu - f(S,I) - \mu S - \mu \frac{f(S^*,I^*)}{f(S,I^*)} + \frac{f(S^*,I^*)}{f(S,I^*)} f(S,I) \\ &\quad + \mu S \frac{f(S^*,I^*)}{f(S,I^*)} + f(S,I) - \delta I - \frac{f(S^*,I^*)}{f(S^*,I)} f(S,I) + \delta I \frac{f(S^*,I^*)}{f(S^*,I)}.\end{aligned}$$

Also note that we have $\mu = f(S^*, I^*) + \mu S^*$ and $\delta I^* = f(S^*, I^*) \Rightarrow \delta = \frac{1}{I^*} f(S^*, I^*)$.

Substituting in:

$$\begin{aligned}
\frac{\mathrm{d}V(S,I)}{\mathrm{d}t} &= (f(S^*,I^*)+\mu S^*) - \mu S - (f(S^*,I^*)+\mu S^*)\frac{f(S^*,I^*)}{f(S,I^*)} + \frac{f(S,I)}{f(S^*,I)}f(S^*,I^*) \\
&\quad + \mu S\frac{f(S^*,I^*)}{f(S^*,I)} - \left(\frac{1}{I^*}f(S^*,I^*)\right)I - \frac{f(S,I)}{f(S^*,I)}f(S^*,I^*) + \left(\frac{1}{I^*}f(S^*,I^*)\right)I\frac{f(S^*,I^*)}{f(S^*,I)} \\
&= f(S^*,I^*) + \mu S^* - \mu S - \frac{f(S^*,I^*)}{f(S^*,I)}f(S^*,I^*) - \mu S^*\frac{f(S^*,I^*)}{f(S,I^*)} + \frac{f(S,I)}{f(S,I^*)}f(S^*,I^*) \\
&\quad + \mu S\frac{f(S^*,I^*)}{f(S,I^*)} - \frac{I}{I^*}f(S^*,I^*) - \frac{f(S,I)}{f(S^*,I)}f(S^*,I^*) + \frac{I}{I^*}\frac{f(S^*,I^*)}{f(S^*,I)}f(S^*,I^*) \\
&= \mu S^*\left(1 - \frac{S}{S^*} - \frac{f(S^*,I^*)}{f(S,I^*)} + \frac{S}{S^*}\frac{f(S^*,I^*)}{f(S,I^*)}\right) \\
&\quad + f(S^*,I^*)\left(1 - \frac{f(S^*,I^*)}{f(S,I^*)} + \frac{f(S,I)}{f(S,I^*)} - \frac{I}{I^*} - \frac{f(S,I)}{f(S^*,I)} + \frac{I}{I^*}\frac{f(S^*,I^*)}{f(S^*,I)}\right),
\end{aligned}$$

which via rearrangement gives:

$$\begin{aligned}
\frac{\mathrm{d}V(S,I)}{\mathrm{d}t} &= \mu S^*\left(1 - \frac{S}{S^*} - \frac{f(S^*,I^*)}{f(S,I^*)} + \frac{S}{S^*}\frac{f(S^*,I^*)}{f(S,I^*)}\right) \\
&\quad + f(S^*,I^*)\left(1 - \frac{f(S^*,I^*)}{f(S,I^*)} - \frac{f(S,I)}{f(S^*,I)}\right) \\
&\quad + f(S^*,I^*)\left(-\frac{I}{I^*} + \frac{f(S,I)}{f(S,I^*)} + \frac{I}{I^*}\frac{f(S^*,I^*)}{f(S^*,I)}\right) \\
&= \mu S^*\left(1 - \frac{S}{S^*}\right)\left(1 - \frac{f(S^*,I^*)}{f(S,I^*)}\right) \\
&\quad + f(S^*,I^*)\left(1 - \frac{f(S^*,I^*)}{f(S,I^*)}\right)\left(1 - \frac{f(S,I)}{f(S^*,I)}\right) \\
&\quad + f(S^*,I^*)\left(\frac{I}{I^*} - \frac{f(S,I)}{f(S,I^*)}\right)\left(\frac{f(S^*,I^*)}{f(S^*,I)} - 1\right).
\end{aligned}$$

If $Q^*>0$ and the function $f(S,I)$ is concave with respect to I, then $\frac{\mathrm{d}V}{\mathrm{d}t} \leq 0$ for all $S, I > 0$. Indeed, the monotonicity of $f(S,I)$ with respect to S ensures that

$$\left(1 - \frac{S}{S^*}\right)\left(1 - \frac{f(S^*,I^*)}{f(S,I^*)}\right) \leq 0$$

and

$$\left(1 - \frac{f(S^*,I^*)}{f(S,I^*)}\right)\left(1 - \frac{f(S,I)}{f(S^*,I)}\right) \leq 0$$

hold for all $S, I > 0$ (see also [10]). Furthermore

$$\left(\frac{I}{I^*} - \frac{f(S,I)}{f(S,I^*)}\right)\left(\frac{f(S^*,I^*)}{f(S^*,I)} - 1\right) \leq 0$$

if

$$\begin{aligned}
&\frac{I}{I^*} \leq \frac{f(S,I)}{f(S,I^*)} \quad \text{when} \quad f(S^*,I^*) \geq f(S^*,I), \quad \text{and} \\
&\frac{I}{I^*} \geq \frac{f(S,I)}{f(S,I^*)} \quad \text{when} \quad f(S^*,I^*) \leq f(S^*,I)
\end{aligned} \tag{3.30}$$

holds for all $S, I > 0$. For a monotonic function, $f(S^*,I) \geq f(S^*,I^*)$ implies $I \geq I^*$, and $f(S^*,I) \leq f(S^*,I^*)$ implies $I \leq I^*$, and hence the condition (3.30) is equivalent to the condition

$$\frac{f(S,I)}{f(S,I^*} \geq \frac{I}{I^*} \quad \text{for all} \quad I \leq I^*, \quad \text{and}$$

$$\frac{f(S,I)}{f(S,I^*)} \leq \frac{I}{I^*} \quad \text{for all} \quad I \geq I^*.$$

This condition (and hence condition (3.30)) holds for all functions concave with respect to the variable I (see Figure 3.3), and hence the concavity is sufficient to ensure that $\frac{\mathrm{d}V(S,I)}{\mathrm{d}t} \leq 0$ in $\mathbb{R}^2_+$ (see also [10]).

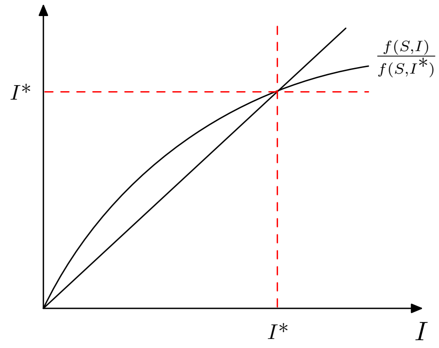

Figure 3.3: A function $f(S,I)$ concave with respect to I.

Proof: Uniqueness of the positive endemic equilibrium state. Following the work of Korobeinikov [10], we now assume that apart from the equilibrium Q^*, the system has another positive equilibrium state $Q_1 = (S_1, I_1)$. Then, substituting S_1 and I_1 into the equalities (3.27), we derive $f(S_1, I_1) + \mu S_1 = \mu$ and $\delta I_1 = f(S_1, I_1)$.

The derivative of a Lyapunov function is equal to zero at any equilibrium state, and therefore $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$ at Q_1. Therefore, S_1 and I_1 must satisfy the equalities

$$\left(1 - \frac{S_1}{S^*}\right)\left(1 - \frac{f(S^*,I^*)}{f(S_1,I^*)}\right) = 0, \tag{3.31}$$

$$\left(1 - \frac{f(S^*,I^*)}{f(S_1,I^*)}\right)\left(1 - \frac{f(S_1,I)}{f(S^*,I)}\right) = 0, \tag{3.32}$$

$$\left(\frac{I_1}{I^*} - \frac{f(S_1,I_1)}{f(S_1,I^*)}\right)\left(\frac{f(S^*,I^*)}{f(S^*,I_1)} - 1\right) = 0. \tag{3.33}$$

It is obvious that for a monotonic function (or a function satisfying condition (3.29)), the equality (3.31) holds only when $S_1 = S^*$. Then $I_1 = I^*$ is necessary to satisfy $f(S^*, I_1) + \mu S^* = \mu$. That is $S_1 = S^*$ and $I_1 = I^*$, and hence Q^* is the only positive fixed point of the system. For a monotonic function (or satisfying condition (3.29)), the point Q^* is the only invariant set (see definition 1.4.1) of the system (3.26) in the set $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$ (see also [10]) and therefore, by LaSalle's invariance principle (see theorem 1.4.1), the point Q^* is asymptotically stable for all $S \geq \varepsilon$.

Note that the parameter ε may be made as small as required, and therefore the endemic equilibrium Q^* is asymptotically stable in the non-negative quadrant $\mathbb{R}^2_+$.

Proof: Stability of the infection free equilibrium state. To prove the global stability of the infection-free equilibrium state $Q_0 = (1, 0)$ we consider the potential Lyapunov function, given by Korobeinikov [10]:

$$U(S,I) = S - \int_{\varepsilon}^{S} \lim_{I \to I_0} \frac{f(S_0,I)}{f(\tau,I)} \mathrm{d}\tau + I.$$

Note that here we cannot consider the function $U(S,I) = S - \int_{\varepsilon}^{S} \frac{f(S_0,I)}{f(\tau,I)} \mathrm{d}\tau + I$, because $f(S,0) = 0$. This Lyapunov function satisfies

$$\frac{\mathrm{d}U(S,I)}{\mathrm{d}t} = \left(1 - \lim_{I \to I_0} \frac{f(S_0,I)}{f(S,I)}\right)(\mu - f(S,I) - \mu S) + (1)\left(f(S,I) - \delta I\right),$$

which can be rearranged to give:

$$\frac{\mathrm{d}U(S,I)}{\mathrm{d}t} = \mu - f(S,I) - \mu S - \mu \lim_{I\to I_0} \frac{f(S_0,I_0)}{f(S,I_0)} + f(S,I) \lim_{I\to I_0} \frac{f(S_0,I_0)}{f(S,I_0)} + \mu S \lim_{I\to I_0} \frac{f(S_0,I_0)}{f(S,I_0)} + f(S,I) - \delta I.$$

As $S_0 = 1$ we can substitute in $S = \frac{S}{S_0}$ to make simplifying the equations later easier. This then gives the equation, see also [10]:

$$\begin{aligned}
\frac{\mathrm{d}U(S,I)}{\mathrm{d}t} &= \mu - \mu\frac{S}{S_0} - \mu \lim_{I\to I_0} \frac{f(S_0,I_0)}{f(S,I_0)} + f(S,I) \lim_{I\to I_0} \frac{f(S_0,I_0)}{f(S,I_0)} + \mu\frac{S}{S_0} \lim_{I\to I_0} \frac{f(S_0,I_0)}{f(S,I_0)} - \delta I \\
&= \mu\left(1 - \frac{S}{S_0} - \lim_{I\to I_0} \frac{f(S_0,I_0)}{f(S,I_0)} + \frac{S}{S_0} \lim_{I\to I_0} \frac{f(S_0,I_0)}{f(S,I_0)}\right) + f(S,I) \lim_{I\to I_0} \frac{f(S_0,I_0)}{f(S,I_0)} - \delta I \\
&= \mu\left(1 - \frac{S}{S_0}\right)\left(1 - \lim_{I\to I_0} \frac{f(S_0,I_0)}{f(S,I_0)}\right) + \delta I\left(\frac{f(S,I)}{\delta I} \lim_{I\to I_0} \frac{f(S_0,I_0)}{f(S,I_0)} - 1\right).
\end{aligned}$$

For a monotonic function

$$\left(1 - \frac{S}{S_0}\right)\left(1 - \lim_{I\to I_0} \frac{f(S_0,I_0)}{f(S,I_0)}\right) \le 0 \quad \text{for all} \quad S > 0,$$

and concavity of the function $f(S,I)$ ensures that $f(S,I) \le I\frac{\partial f(S,0)}{\partial I}$ for all $I > 0$, and hence

$$\frac{f(S,I)}{\delta I} \lim_{I\to I_0} \frac{f(S_0),I)}{f(S,I)} = \frac{f(S,I)}{\delta I} \frac{\frac{\partial f(S_0,I_0)}{\partial I}}{\frac{\partial f(S,I_0)}{\partial I}} \le \frac{1}{\delta}\frac{\partial f(S_0,I_0)}{\partial I} = R_0.$$

Therefore $R_0 \le 1$ ensures that $\frac{\mathrm{d}U(S,I)}{\mathrm{d}t} \le 0$ for all $S, I > \varepsilon$ (see also [10]), and hence by the asymptotic stability theorem (see theorem ??) the equilibrium state Q_0 is globally asymptotically stable in this case. Therefore, as also concluded by Korobeinikov [10], for all $R_0 > 1$ there exists an unique and globally stable positive equilibrium state Q^*, and that for $R_0 < 1$ the infection-free equilibrium Q_0 is globally stable. At $R_0 = 1$ for any function $f(S,I)$ monotonic with respect to S the point at which the two equilibria, Q_0 and Q^*, meet is a transcritical bifurcation point i.e., at this point the equilibrium points exchange their stability. Indeed for a monotonic function $f(S,I), S_*$ (and hence S^*) tends to S_0 as R_0 tends to 1 (see also [10]), and that $R_0 = 1$ implies $S_* = S_0$. At $R_0 = 1$ the point (S_0, I_0) is, therefore, the point of intersection of the lines q_1 and q_2 (see Figure 3.3). For $R_0 > 1$ the equilibrium Q^* moves in the quadrant $S > 0$, $I < 0$. The theorem is proven.

3.4 SIS Models

Some infections (e.g., gonorrhea) do not give rise to acquired immunity in the host and so the third type of epidemiological model we will concentrate on is the SIS model. Here we divide the population, of size N, into the subpopulations defined in (3.1) however for this model we do not require the removed $R(t)$ compartment . For the SIS model, we assume that an individual begins in the susceptibles compartment, after infection moves to the infective compartment and once the individual has recovered, we assume no immunity, and thus the individual will return to the susceptibles compartment. We will again also assume that the population size $N = S + I$ is constant, therefore implying that deaths are balanced by births.

3.4.1 An SIS Model Given by Korobeinikov and Wake [13]

Note that the SIS model, as also commented by Korobeinikov and Wake [13], can be regarded as the limiting case of the $SIRS$ model with no period of immunity. For example, using system (3.1) we would let $\dfrac{1}{\alpha} \to 0$ which would enable us to produce an SIS model.

Korobeinikov and Wake [13] considers the SIS model with vertical transmission (see Figure 3.4) with the differential equations:

$$\begin{aligned}\dot{S} &= \gamma N - \beta\frac{SI}{N} - p\gamma I + \delta I - \sigma S,\\ \dot{I} &= \beta\frac{SI}{N} - (\delta + \sigma + \varepsilon - p\gamma)I,\end{aligned} \tag{3.34}$$

where δ is the rate of recovery, σ and ε are the rates of natural and disease-associated mortality and other parameters are the same as for the $SIRS$ system (3.1). We again assume that the population size N is constant.

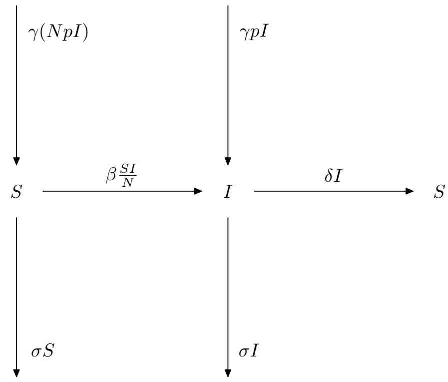

Figure 3.4: Transfer diagram of the SIS model (3.34) given by Korobeinikov and Wake [13].

The system has two equilibria: an infection-free equilibrium state and an endemic equilibrium state.

Finding the infection-free equilibrium:

Infection-free $\Rightarrow I_0 = 0$.
We need to find S_0 such that $\dot{S} = 0$:

$$\begin{aligned}\dot{S} = \gamma N - \beta\frac{S_0 I_0}{N} - p\gamma I_0 + \delta I_0 - \sigma S_0 &= 0\\ \Rightarrow \gamma N - \beta\frac{S_0(0)}{N} - p\gamma(0) + \delta(0) - \sigma S_0 &= 0\\ \Rightarrow \gamma N - \sigma S_0 &= 0\\ \Rightarrow \sigma S_0 &= \gamma N\\ \Rightarrow S_0 &= \frac{\gamma N}{\sigma}.\end{aligned}$$

Thus we have an infection-free equilibrium state $E_0 = (\frac{\gamma N}{\sigma}, 0)$.

Next we will find the endemic equilibrium.
First we need to find S^* such that $\dot{I} = 0$:

$$\begin{aligned}\dot{I} = \beta\frac{S^* I^*}{N} - (\delta + \sigma + \varepsilon - p\gamma)I^* = 0&\\ \Rightarrow \beta\frac{S^* I^*}{N} &= (\delta + \sigma + \varepsilon - p\gamma)I^*\\ \Rightarrow S^* &= \frac{(\delta + \sigma + \varepsilon - p\gamma)N}{\beta}\\ &= \frac{\gamma}{\sigma}\left(\frac{\sigma(\delta + \sigma + \varepsilon - p\gamma)}{\beta\gamma}\right)N\end{aligned}$$

$$= \frac{\gamma}{\sigma}\frac{N}{R_0}$$

where $R_0 = \dfrac{\beta\gamma}{\sigma(\delta+\sigma+\varepsilon-p\gamma)}$.

Next we need to find I^* such that $\dot{S}=0$:

$$\begin{aligned}
\dot{S} = \gamma N - \beta\frac{S^*I^*}{N} - p\gamma I^* + \delta I^* - \sigma S^* = 0 \\
\Rightarrow \beta\frac{S^*I^*}{N} + p\gamma I^* - \delta I^* = \gamma N - \sigma S^* \\
\Rightarrow I^*\left(\frac{\beta S^*}{N} + p\gamma - \delta\right) = \gamma N - \sigma S^* \\
\Rightarrow I^* = \frac{\gamma N - \sigma S^*}{\frac{\beta S^*}{N} + p\gamma - \delta}.
\end{aligned}$$

Substituting in $S^* = \dfrac{\gamma}{\sigma}\dfrac{N}{R_0} = \dfrac{(\delta+\sigma+\varepsilon-p\gamma)N}{\beta}$:

$$\begin{aligned}
I^* &= \frac{\gamma N - \sigma\frac{(\delta+\sigma+\varepsilon-p\gamma)N}{\beta}}{\frac{\beta}{N}\left(\frac{(\delta+\sigma+\varepsilon-p\gamma)N}{\beta}\right) + p\gamma - \delta} \\
&= \left(\frac{\beta\gamma - \sigma(\delta+\sigma+\varepsilon-p\gamma)}{\beta(\delta+\sigma+\varepsilon-p\gamma+p\gamma-\delta)}\right)N \\
&= \left(\frac{\beta\gamma - \sigma(\delta+\sigma+\varepsilon-p\gamma)}{\beta(\sigma+\varepsilon)}\right)N \\
&= \frac{\gamma}{\sigma+\varepsilon}\left(\frac{\beta\gamma - \sigma(\delta+\sigma+\varepsilon-p\gamma)}{\beta\gamma}\right)N \\
&= \frac{\gamma}{\sigma+\varepsilon}\left(1 - \frac{\sigma(\delta+\sigma+\varepsilon-p\gamma)}{\beta\gamma}\right)N \\
&= \frac{\gamma}{\sigma+\varepsilon}\left(1 - \frac{1}{R_0}\right)N
\end{aligned}$$

where $R_0 = \dfrac{\beta\gamma}{\sigma(\delta+\sigma+\varepsilon-p\gamma)}$.

Thus we have an endemic equilibrium $E^* = (S^*, I^*)$, with

$$S^* = \frac{\gamma}{\sigma}\frac{N}{R_0} \qquad \text{and} \qquad I^* = \frac{\gamma}{\sigma+\varepsilon}\left(1-\frac{1}{R_0}\right)N, \tag{3.35}$$

where $R_0 = \dfrac{\beta\gamma}{\sigma(\delta+\sigma+\varepsilon-p\gamma)}$. The positive endemic equilibrium exists if $R_0 > 1$.

Note that, using

$$S^* = \frac{(\delta+\sigma+\varepsilon-p\gamma)N}{\beta} \qquad \text{and} \qquad I^* = \left(\frac{\beta\gamma - \sigma(\delta+\sigma+\varepsilon-p\gamma)}{\beta(\sigma+\varepsilon)}\right)N,$$

we have

$$\beta\frac{S^*I^*}{N} = \frac{\beta}{N}\left(\frac{(\delta+\sigma+\varepsilon-p\gamma)N}{\beta}\right)I^* = (\delta+\sigma+\varepsilon-p\gamma)I^*$$

and

$$\gamma N + (\delta - p\gamma)I^* - \sigma S^* = \gamma N + (\delta - p\gamma)\left(\frac{\beta\gamma - \sigma(\delta+\sigma+\varepsilon-p\gamma)}{\beta(\sigma+\varepsilon)}\right)N - \sigma\left(\frac{(\delta+\sigma+\varepsilon-p\gamma)N}{\beta}\right)$$

$$
\begin{aligned}
&= \frac{\beta\gamma N(\sigma+\varepsilon) + (\delta - p\gamma)(\beta\gamma - \sigma(\delta+\sigma+\varepsilon-p\gamma))N - \sigma(\delta+\sigma+\varepsilon-p\gamma)N(\sigma+\varepsilon)}{\beta(\sigma+\varepsilon)} \\
&= \frac{N(\sigma+\varepsilon)(\beta\gamma - \sigma(\delta+\sigma+\varepsilon-p\gamma)) + (\delta-p\gamma)(\beta\gamma-\sigma(\delta+\sigma+\varepsilon-p\gamma))N}{\beta(\sigma+\varepsilon)} \\
&= \left(\frac{(\delta+\sigma+\varepsilon-p\gamma)(\beta\gamma-\sigma(\delta+\sigma+\varepsilon-p\gamma))}{\beta(\sigma+\varepsilon)}\right)N \\
&= (\delta+\sigma+\varepsilon-p\gamma)\left[\left(\frac{\beta\gamma-\sigma(\delta+\sigma+\varepsilon-p\gamma)}{\beta(\sigma+\varepsilon)}\right)N\right] \\
&= (\delta+\sigma+\varepsilon-p\gamma)I^*.
\end{aligned}
$$

Thus we have

$$
\beta\frac{S^*I^*}{N} = \gamma N + (\delta - p\gamma)I^* - \sigma S^* = (\delta+\sigma+\varepsilon-p\gamma)I^*. \tag{3.36}
$$

After a small alteration, the Lyapunov function (3.5) can be applied to system (3.34) and we can therefore derive the following, as given by Korobeinikov and Wake [13].

Theorem 3.4.1 *The endemic equilibrium state $E^*=(S^*, I^*)$ of system* (3.34) *is globally stable.*

Proof. We consider the potential Lyapunov function

$$
U(S,I) = S^*\left(\frac{S}{S^*} - \ln\frac{S}{S^*}\right) + \frac{\sigma+\varepsilon}{\delta+\sigma+\varepsilon-p\gamma}I^*\left(\frac{I}{I^*} - \ln\frac{I}{I^*}\right), \tag{3.37}
$$

which is a modification of function (3.5). This function is defined and continuous for all $S, I > 0$ and satisfies

$$
\frac{\partial U}{\partial S} = 1 - \frac{S^*}{S} \quad \text{and} \quad \frac{\partial U}{\partial I} = \frac{\sigma+\varepsilon}{\delta+\sigma+\varepsilon-p\gamma}\left(1 - \frac{I^*}{I}\right).
$$

Note that for this function we have $\frac{\partial U}{\partial S} = \frac{\partial U}{\partial I} = 0$ at E^*, as defined in (3.35), and thus we can determine the type of equilibrium point. For the possible Lyapunov function (3.37) we can derive the second derivatives as follows:

$$
\frac{\partial^2 U}{\partial S^2} = \frac{S^*}{S^2} > 0, \qquad \frac{\partial^2 U}{\partial I^2} = \left(\frac{\sigma+\varepsilon}{\delta+\sigma+\varepsilon-p\gamma}\right)\frac{I^*}{I^2} > 0 \quad \text{and} \quad \frac{\partial^2 U}{\partial S \partial I} = 0.
$$

Thus, by the second derivative test, see definition 3.2.3, we can conclude that the endemic equilibrium state $E^*= (S^*, I^*)$ is the only local stationary point of the function and is a minimum extreme point. As $U(S,I) \to \infty$ at the boundary, we have $U(S,I) \geq U(S^*, I^*)$ and thus the function is bounded from below. Consequently E^* is the only extremum and the global minimum of the function in $\mathbb{R}^2_+$ and hence the function (3.37) is indeed a Lyapunov function.

In the case of system (3.34), using (3.36), the derivative of the function satisfies:

$$
\begin{aligned}
\dot{U}(S,I) &= \left(1 - \frac{S^*}{S}\right)\left(\gamma N - \beta\frac{SI}{N} - p\gamma I + \delta I - \sigma S\right) + \frac{\sigma+\varepsilon}{\delta+\sigma+\varepsilon-p\gamma}\left(1 - \frac{I^*}{I}\right)\left(\beta\frac{SI}{N} - (\delta+\sigma+\varepsilon-p\gamma)I\right) \\
&= \gamma N - \beta\frac{SI}{N} - p\gamma I + \delta I - \sigma S - \gamma N\frac{S^*}{S} + \beta\frac{S^*}{N}I + p\gamma\frac{S^*}{S}I - \delta\frac{S^*}{S}I + \sigma S^* \\
&\quad + \frac{\sigma+\varepsilon}{\delta+\sigma+\varepsilon-p\gamma}\left(\beta\frac{SI}{N} - (\delta+\sigma+\varepsilon-p\gamma)I - \beta\frac{SI^*}{N} + (\delta+\sigma+\varepsilon-p\gamma)I^*\right) \\
&= \gamma N - \beta\frac{SI}{N} + (\delta - p\gamma)I - \sigma S - \gamma N\frac{S^*}{S} + \beta\frac{S^*}{N}I + (p\gamma - \delta)\frac{S^*}{S}I + \sigma S^* \\
&\quad + \frac{\sigma+\varepsilon}{\delta+\sigma+\varepsilon-p\gamma}\left(\beta\frac{SI}{N} - \beta\frac{SI^*}{N}\right) + \frac{\sigma+\varepsilon}{\delta+\sigma+\varepsilon-p\gamma}\left[(I^* - I)(\delta+\sigma+\varepsilon-p\gamma)\right].
\end{aligned}
$$

Which can be rearranged to give:

$$\begin{aligned}\dot{U}(S,I) &= \gamma N - \beta\frac{SI}{N} + (\delta - p\gamma)I - \sigma S - \gamma N\frac{S^*}{S} + \beta\frac{S^*}{N}I - (\delta - p\gamma)\frac{S^*}{S}I + \sigma S^* \\ &\quad + \frac{\sigma+\varepsilon}{\delta+\sigma+\varepsilon-p\gamma}\left(\beta\frac{SI}{N} - \beta\frac{I^*}{N}S\right) + (\sigma+\varepsilon)(I - I^*).\end{aligned}$$

Note that we have:

$$\begin{aligned}\gamma N + (\delta - p\gamma)I^* - \sigma S^* &= (\delta+\sigma+\varepsilon-p\gamma)I^* \\ \Rightarrow \sigma S^* = \gamma N + (\delta - p\gamma)I^* - (\delta+\sigma+\varepsilon-p\gamma)I^* &= \gamma N - (\sigma+\varepsilon)I^*.\end{aligned}$$

Then we have:

$$\begin{aligned}\dot{U}(S,I) &= \gamma N - \beta\frac{SI}{N} + (\delta - p\gamma)I - \sigma S - \gamma N\frac{S^*}{S} + \beta\frac{S^*}{N}I - (\delta - p\gamma)\frac{S^*}{S}I + \gamma N - (\sigma+\varepsilon)I^* \\ &\quad + \frac{\sigma+\varepsilon}{\delta+\sigma+\varepsilon-p\gamma}\left(\beta\frac{SI}{N} - \beta\frac{I^*}{N}S\right) - (\sigma+\varepsilon)(I - I^*) \\ &= 2\gamma N - \beta\frac{SI}{N} + (\delta - p\gamma)I - \sigma S - \gamma N\frac{S^*}{S} + \beta\frac{S^*}{N}I - (\delta - p\gamma)\frac{S^*}{S}I \\ &\quad + \frac{\sigma+\varepsilon}{\delta+\sigma+\varepsilon-p\gamma}\left(\beta\frac{SI}{N} - \beta\frac{I^*}{N}S\right) - (\sigma+\varepsilon)I.\end{aligned}$$

Note that:

$$\beta\frac{S^*I^*}{N} = (\delta+\sigma+\varepsilon-p\gamma)I^* \Rightarrow \beta\frac{S^*}{N} = (\delta+\sigma+\varepsilon-p\gamma) \Rightarrow \sigma+\varepsilon = \beta\frac{S^*}{N} - (\delta - p\gamma).$$

Then we have

$$\begin{aligned}\dot{U}(S,I) &= 2\gamma N - \beta\frac{SI}{N} + (\delta - p\gamma)I - \sigma S - \gamma N\frac{S^*}{S} + \beta\frac{S^*}{N}I - (\delta - p\gamma)\frac{S^*}{S}I \\ &\quad + \left[\left(\beta\frac{S^*}{N} - (\delta - p\gamma)\right)\left(\frac{N}{\beta S^*}\right)\right]\left(\beta\frac{SI}{N} - \beta\frac{I^*}{N}S\right) - \left(\beta\frac{S^*}{N} - (\delta - p\gamma)\right)I \\ &= 2\gamma N - \beta\frac{SI}{N} + (\delta - p\gamma)I - \sigma S - \gamma N\frac{S^*}{S} + \beta\frac{S^*}{N}I - (\delta - p\gamma)\frac{S^*}{S}I \\ &\quad + \left[\left(\beta\frac{S^*}{N} - (\delta - p\gamma)\right)\left(\frac{S}{S^*}I - \frac{S}{S^*}I^*\right)\right] - \left(\beta\frac{S^*}{N} - (\delta - p\gamma)\right)I \\ &= 2\gamma N - \beta\frac{SI}{N} + (\delta - p\gamma)I - \sigma S - \gamma N\frac{S^*}{S} + \beta\frac{S^*}{N}I - (\delta - p\gamma)\frac{S^*}{S}I \\ &\quad + \beta\frac{SI}{N} - \beta\frac{S}{N}I^* - (\delta - p\gamma)\frac{S}{S^*}I + (\delta - p\gamma)\frac{S}{S^*}I^* - \beta\frac{S^*}{N}I + (\delta - p\gamma)I \\ &= 2\gamma N + 2(\delta - p\gamma)I - \sigma S - \gamma N\frac{S^*}{S} - (\delta - p\gamma)\frac{S^*}{S}I - \beta\frac{S}{N}I^* - (\delta - p\gamma)\frac{S}{S^*}I + (\delta - p\gamma)\frac{S}{S^*}I^*.\end{aligned}$$

Since

$$\begin{aligned}\beta\frac{S^*I^*}{N} &= \gamma N + (\delta - p\gamma)I^* - \sigma S^* \\ \Rightarrow \sigma S^* = \gamma N + (\delta - p\gamma)I^* - \beta\frac{S^*I^*}{N} &\Rightarrow \sigma S = \gamma N\frac{S}{S^*} + (\delta - p\gamma)\frac{S}{S^*}I^* - \beta\frac{S}{N}I^*.\end{aligned}$$

Then we have:

$$\begin{aligned}\dot{U}(S,I) &= 2\gamma N + 2(\delta - p\gamma)I - \gamma N\frac{S}{S^*} - (\delta - p\gamma)\frac{S}{I}{}^*S^* + \beta\frac{S}{N}I^* \\ &\quad - \gamma N\frac{S^*}{S} - (\delta - p\gamma)\frac{S^*}{S}I - \beta\frac{S}{N}I^* - (\delta - p\gamma)\frac{S}{S^*}I + (\delta - p\gamma)\frac{S}{S^*}I^*\end{aligned}$$

$$= 2\gamma N - \gamma N\frac{S}{S^*} - \gamma N\frac{S^*}{S} + 2(\delta - p\gamma)I - (\delta - p\gamma)\frac{S}{S^*}I - (\delta - p\gamma)\frac{S^*}{S}I$$

Which can be rearranged, as also shown by Korobeinikov and Wake [13], to give:

$$\begin{aligned}\dot{U}(S,I) &= \gamma N\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + (\delta - p\gamma)I\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right)\\ &= (\gamma N + (\delta - p\gamma)I)\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right)\\ &= -(\gamma N + (\delta - p\gamma)I)\frac{S}{S^*}\left(\left(\frac{S^*}{S}\right)^2 - 2\frac{S^*}{S} + 1\right)\\ &= -\left(\gamma N + (\delta - p\gamma)I\right)\frac{S}{S^*}\left(1 - \frac{S^*}{S}\right)^2.\end{aligned}$$

Thus we have

$$\dot{U}(S,I) = -\left(\gamma N + (\delta - p\gamma)I\right)\frac{S}{S^*}\left(1 - \frac{S^*}{S}\right)^2 \leq 0. \tag{3.38}$$

That is, $\dot{U}(S,I) \leq 0$ for all $S, I \geq 0$ ensured by $\delta - p\gamma \geq 0$. Since $\dot{U}(S,I) = 0$ holds only for $S = S^*$ and the endemic equilibrium state E^* is the only invariant set of the system on the line $S = S^*$ (see also [13]), by the asymptotic stability theorem 1.3.3 the equilibrium state E^* is globally asymptotically stable.
As explained in [13], although the case $\delta - p\gamma < 0$ is hardly biologically feasible, the theorem holds in this case as well. In this case, an approach also used for the $SIRS$ model in section 3.2, a shift of the system to the right can be applied.

We will use the substitution $(S,I) \to (P,I)$, where $P = S - \frac{\delta - p\gamma}{\beta}N$,
Hence we have

$$\dot{P} = \dot{S} = \gamma N - \beta\frac{SI}{N} - p\gamma I + \delta I - \sigma S.$$

Since

$$P = S - \frac{\delta - p\gamma}{\beta}N \Rightarrow S = P + \frac{\delta - p\gamma}{\beta}N,$$

we can substitute in to obtain:

$$\begin{aligned}\dot{P} &= \gamma N - \beta\frac{I}{N}\left(P + \frac{\delta - p\gamma}{\beta}N\right) - p\gamma I + \delta I - \sigma\left(P + \frac{\delta - p\gamma}{\beta}N\right)\\ &= \gamma N - \beta\frac{PI}{N} - (\delta - p\gamma)I + (\delta - p\gamma)I - \sigma P - \sigma\frac{\delta - p\gamma}{\beta}N\\ &= \left(\gamma + \sigma\frac{p\gamma - \delta}{\beta}\right)N - \beta\frac{PI}{N} - \sigma P\\ &= \hat{\gamma}N - \beta\frac{PI}{N} - \sigma P,\end{aligned}$$

where $\hat{\gamma} = \gamma + \sigma\frac{p\gamma - \delta}{\beta}$.
We will now calculate $\dot{I}$. We have:

$$\dot{I} = \beta\frac{SI}{N} - (\delta + \sigma + \varepsilon - p\gamma)I$$

and substituting in $S = P + \frac{\delta - p\gamma}{\beta}N$ we obtain:

$$\dot{P} = \beta\frac{I}{N}\left(P + \frac{\delta - p\gamma}{\beta}N\right) - (\delta + \sigma + \varepsilon - p\gamma)I$$

$$= \beta\frac{PI}{N} + (\delta - p\gamma)I - (\delta + \sigma + \varepsilon - p\gamma)I$$
$$= \beta\frac{PI}{N} - (\sigma + \varepsilon)I.$$

Thus in the new variables we have:

$$\dot{P} = \hat{\gamma}N - \beta\frac{PI}{N} - \sigma P, \qquad \dot{I} = \beta\frac{PI}{N} - (\sigma + \varepsilon)I, \tag{3.39}$$

where $\hat{\gamma} = \gamma + \sigma\frac{p\gamma - \delta}{\beta} > 0$.

In the new variables, the endemic equilibrium can be found as follows:
First we need to find P^* such that $\dot{I} = 0$:

$$\dot{I} = \beta\frac{P^*I^*}{N} - (\sigma + \varepsilon)I^* = 0$$
$$\Rightarrow \beta\frac{P^*I^*}{N} = (\sigma + \varepsilon)I^*$$
$$\Rightarrow P^* = \frac{\sigma + \varepsilon}{\beta}N.$$

Next we need to find I^* such that $\dot{P} = 0$:

$$\dot{P} = \hat{\gamma}N - \beta\frac{P^*I^*}{N} - \sigma P^* = 0$$
$$\Rightarrow \beta\frac{P^*I^*}{N} = \hat{\gamma}N - \sigma P^*$$
$$\Rightarrow I^* = \left(\frac{\hat{\gamma}N - \sigma P^*}{\beta P^*}\right)N.$$

Substituting in $P^* = \frac{\sigma + \varepsilon}{\beta}N$ we derive:

$$I^* = \left(\frac{\hat{\gamma}N - \sigma\left(\frac{\sigma+\varepsilon}{\beta}N\right)}{\beta\left(\frac{\sigma+\varepsilon}{\beta}N\right)}\right)N$$
$$= \left(\frac{\hat{\gamma} - \sigma\left(\frac{\sigma+\varepsilon}{\beta}\right)}{\sigma + \varepsilon}\right)N$$
$$= \left(\frac{\beta\hat{\gamma} - \sigma\left(\sigma + \varepsilon\right)}{\beta(\sigma + \varepsilon)}\right)N.$$

Thus the endemic equilibrium state of system (3.39), see also [13], is given by

$$P^* = \frac{\sigma + \varepsilon}{\beta}N, \qquad I^* = \left(\frac{\beta\hat{\gamma} - \sigma\left(\sigma + \varepsilon\right)}{\beta(\sigma + \varepsilon)}\right)N.$$

The Lyapunov function (3.5) can therefore be straightforwardly applied to system (3.39). The derivative of the function satisfies

$$\dot{V}(P, I) = \left(1 - \frac{P^*}{P}\right)\left(\hat{\gamma}N - \beta\frac{PI}{N} - \sigma P\right) + \left(1 - \frac{I^*}{I}\right)\left(\beta\frac{PI}{N} - (\sigma + \varepsilon)I\right),$$

which can be rearranged to give:

$$\dot{V}(P, I) = \hat{\gamma}N - \beta\frac{PI}{N} - \sigma P - \hat{\gamma}N\frac{P^*}{P} + \beta\frac{P^*}{N}I + \sigma P^* + \beta\frac{PI}{N} - (\sigma + \varepsilon)I - \beta\frac{I^*}{N}P + (\sigma + \varepsilon)I^*$$

$$= \hat{\gamma}N - \sigma P - \hat{\gamma}N\frac{P^*}{P} + \beta\frac{P^*}{N}I + \sigma P^* - (\sigma+\varepsilon)I - \beta\frac{I^*}{N}P + (\sigma+\varepsilon)I^*.$$

Substituting in $I^* = \left(\frac{\beta\hat{\gamma} - \sigma(\sigma+\varepsilon)}{\beta(\sigma+\varepsilon)}\right)N$ we derive:

$$\begin{aligned}
\dot{V}(P,I) &= \hat{\gamma}N - \sigma P - \hat{\gamma}N\frac{P^*}{P} + \beta\frac{P^*}{N}I + \sigma P^* - (\sigma+\varepsilon)I \\
&\quad - \beta\frac{P}{N}\left[\left(\frac{\beta\hat{\gamma} - \sigma(\sigma+\varepsilon)}{\beta(\sigma+\varepsilon)}\right)N\right] + (\sigma+\varepsilon)\left[\left(\frac{\beta\hat{\gamma} - \sigma(\sigma+\varepsilon)}{\beta(\sigma+\varepsilon)}\right)N\right] \\
&= \hat{\gamma}N - \sigma P - \hat{\gamma}N\frac{P^*}{P} + \beta\frac{P^*}{N}I + \sigma P^* - (\sigma+\varepsilon)I - \frac{\beta\hat{\gamma}}{\sigma+\varepsilon}P + \sigma P + \hat{\gamma}N - \sigma\frac{(\sigma+\varepsilon)}{\beta}N \\
&= 2\hat{\gamma}N - \hat{\gamma}N\frac{P^*}{P} + \beta\frac{P^*}{N}I + \sigma P^* - (\sigma+\varepsilon)I - \frac{\beta\hat{\gamma}}{\sigma+\varepsilon}P - \sigma\frac{(\sigma+\varepsilon)}{\beta}N.
\end{aligned}$$

Since $P^* = \frac{\sigma+\varepsilon}{\beta}N \Rightarrow \sigma+\varepsilon = \beta\frac{P^*}{N}$ and $\frac{1}{P^*} = \frac{\beta}{(\sigma+\varepsilon)N} \Rightarrow \frac{\beta\hat{\gamma}}{\sigma+\varepsilon}P = \hat{\gamma}N\frac{P}{P^*}$ then:

$$\begin{aligned}
\dot{V}(P,I) &= 2\hat{\gamma}N - \hat{\gamma}N\frac{P^*}{P} + \beta\frac{P^*}{N}I + \sigma P^* - \beta\frac{P^*}{N}I - \hat{\gamma}N\frac{P}{P^*} - \sigma P^* \\
&= 2\hat{\gamma}N - \hat{\gamma}N\frac{P^*}{P} - \hat{\gamma}N\frac{P}{P^*}.
\end{aligned}$$

Be rearranging we obtain:

$$\begin{aligned}
\dot{V}(P,I) &= \hat{\gamma}N\left(2 - \frac{P}{P^*} - \frac{P^*}{P}\right) \\
&= -\hat{\gamma}N\frac{P}{P^*}\left(\left(\frac{P^*}{P}\right)^2 - 2\frac{P^*}{P} + 1\right) \\
&= -\hat{\gamma}N\frac{P}{P^*}\left(1 - \frac{P^*}{P}\right)^2.
\end{aligned}$$

Thus we have

$$\dot{V}(P,I) = -\hat{\gamma}N\frac{P}{P^*}\left(1 - \frac{P^*}{P}\right)^2 \leq 0, \qquad \text{for all } P, I \geq 0 \tag{3.40}$$

Hence, by the asymptotic stability theorem 1.3.3, for the case $\delta - p\gamma < 0$, the endemic equilibrium state of system (3.39) and consequently that of system (3.34) is also globally asymptotically stable. The theorem is proven.

3.5 *SEIR* Models

The final type of epidemiological model we will concentrate on is the *SEIR* model. Here we divide the population, of size N, into the subpopulations defined in (3.1) however for this model we include an additional compartment $E(t)$ which represents exposed individuals whom have have been infected but are not yet infectious themselves. For the this model, we assume that an individual begins in the susceptibles compartment, once exposed to the infection move into the exposed compartment, after becoming infectious the individual then moves to the infective compartment and then continues into the removed compartment as a result of recovery, isolation or death by disease. For our *SEIR* models we assume that recovery implies permanent immunity. For simplicity, we will again normalize the population size N to 1; i.e. now $S + E + I + R = 1$ holds. Our investigation in the current section is based on [11] and [12] and focuses on the effect of the nonlinear incidence on the global stability of SEIR models.

3.5.1 An $SEIR$ Model Given by Korobeinikov and Maini [11]

Continuing the same method as used in section 3.2 for the work in [11], we will apply the direct Lyapunov method to consider the global properties of $SEIR$ models with the incidence rate of the form $\beta I^p S^q$ for the particular case $p \leq 1$. Korobeinikov and Maini [11] construct a Lyapunov function for models with bilinear incidence and we will use their approach to show that the condition $p \leq 1$ is a sufficient condition for global stability, and that the global properties of the systems do not depend on the value of q.

For the incidence rate of the form $\beta I^p S^q$, the basic $SEIR$ model is given by Korobeinikov and Maini [11] as:

$$\begin{aligned} \dot{S} &= b - \beta I^p S^q - \mu S, \\ \dot{E} &= \beta I^p S^q - \sigma E \\ \dot{I} &= \theta E - \delta I. \end{aligned} \tag{3.41}$$

Here b is the birth rate, μ is the susceptible death rate, δ is the infective removal rate (including mortality rate) and θ is the rate with which the exposed population moves into the infective class; $\sigma \geq \theta$ includes also mortality of the exposed individuals. We omit the equations for the recovered population R; the constant population size assumption enables us to do so.

If $0 < p < 1$ holds, then each of these systems has two equilibrium states: an infection-free equilibrium and an endemic equilibrium.

Finding the infection-free equilibrium:

Infection-free $\Rightarrow I_0 = 0$.
First we need to find S_0 such that $\dot{S} = 0$:

$$\begin{aligned} \dot{S} = b - \beta I_0^p S_0^q - \mu S_0 &= 0 \\ \Rightarrow b - \beta (0)^p S_0^q - \mu S_0 &= 0 \\ \Rightarrow b - \mu S_0 &= 0 \\ \Rightarrow \mu S_0 &= b \\ \Rightarrow S_0 &= \frac{b}{\mu}. \end{aligned}$$

Next we need to find E_0 such that $\dot{E} = 0$:

$$\begin{aligned} \dot{E} = \beta I_0^p S_0^q - \sigma E_0 &= 0 \\ \Rightarrow \beta (0)^p S_0^q - \sigma E_0 &= 0 \\ \Rightarrow \sigma E_0 &= 0 \\ E_0 &= 0. \end{aligned}$$

Thus we have an infection-free equilibrium Q_0 with the coordinates $S_0 = \dfrac{b}{\mu}, E_0 = I_0 = 0$.
Finding the endemic equilibrium:
First we will consider $\dot{I} = 0$:

$$\begin{aligned} \dot{I} = \theta E^* - \delta I^* &= 0 \\ \Rightarrow \delta I^* &= \theta E^*. \end{aligned} \tag{3.42}$$

Next we will consider $\dot{E} = 0$:

$$\dot{E} = \beta (I^*)^p (S^*)^q - \sigma E^* = 0$$

$$\Rightarrow \beta(I^*)^p(S^*)^q = \sigma E^*. \tag{3.43}$$

Substituting (3.42) into (3.43):

$$\sigma E^* = \frac{\sigma}{\theta}\delta I^* = \beta(I^*)^p(S^*)^q. \tag{3.44}$$

Finally we will consider $\dot{S} = 0$:

$$\begin{aligned} \dot{S} = b - \beta(I^*)^p(S^*)^q - \mu S^* = 0 \\ \Rightarrow \mu S^* + \beta(I^*)^p(S^*)^q = b. \end{aligned} \tag{3.45}$$

Substituting (3.44) into (3.45):

$$\mu S^* + \frac{\sigma}{\theta}\delta I^* = b. \tag{3.46}$$

Thus, using (3.42), (3.44) and (3.46), we have the endemic equilibrium state $Q^*= (S^*, E^*, I^*)$, such that

$$\frac{\sigma}{\theta}\delta I^* = \beta(I^*)^p(S^*)^q, \quad \mu S^* + \frac{\sigma}{\theta}\delta I^* = b, \quad \delta I^* = \theta E^*. \tag{3.47}$$

We will now consider the following theorem as given by Korobeinikov and Maini [11].

Theorem 3.5.1 *If $p \leq 1$, then the endemic equilibrium states Q^* of the model* (3.41) *is globally asymptotically stable. The stability does not depend on the value of the parameter q.*

Proof. Assume that $p, q \neq 1$. Then for the $SEIR$ model, we consider a test Lyapunov function of the form

$$V(S,E,I) = S\left(1 + \frac{1}{q-1}\left(\frac{S^*}{S}\right)^q\right) + BI\left(1 + \frac{1}{p-1}\left(\frac{I^*}{I}\right)^p\right) + (E - E^*\ln E), \tag{3.48}$$

where $B = \dfrac{\sigma}{\theta}$. This function is defined and continuous for all $S, E, I > 0$ and satisfies

$$\frac{\partial V}{\partial S} = \left(1 - \left(\frac{S^*}{S}\right)^q\right), \quad \frac{\partial V}{\partial E} = \left(1 - \frac{E^*}{E}\right) \quad \text{and} \quad \frac{\partial V}{\partial I} = B\left(1 - \left(\frac{I^*}{I}\right)^p\right).$$

Note that for this function we have $\frac{\partial V}{\partial S} = \frac{\partial V}{\partial E} = \frac{\partial V}{\partial I} = 0$ at Q^*, as defined in (3.47), and thus we can determine the type of equilibrium point. For the possible Lyapunov function (3.48) we can derive the second derivatives as follows:

$$\frac{\partial^2 V}{\partial S^2} = \frac{q}{S}\left(\frac{S^*}{S}\right)^q > 0, \quad \frac{\partial^2 V}{\partial E^2} = \frac{E^*}{E^2} > 0, \quad \frac{\partial^2 V}{\partial I^2} = \frac{Bp}{I}\left(\frac{I^*}{I}\right)^p > 0,$$

and

$$\frac{\partial^2 V}{\partial S \partial E} = \frac{\partial^2 V}{\partial S \partial I} = \frac{\partial^2 V}{\partial E \partial I} = 0.$$

Thus, by the second derivative test, see definition 3.2.3, we can conclude that the endemic equilibrium state $Q^*= (S^*, E^*, I^*)$ is the only extremum and the global minimum of the function in the positive octant $\mathbb{R}^3_+$ and hence $V(S,E,I) \geq V(S^*, E^*, I^*)$. Consequently, the function (3.48) is indeed a Lyapunov function.

Using

$$b = \mu S^* + B\delta I^*, \quad \beta(I^*)^p(S^*)^q = B\delta I^*, \quad \delta I^* = \theta E^*, \quad B\theta = \sigma \tag{3.49}$$

for the equilibrium state Q^*, as defined in (3.47), the Lyapunov function (3.48) satisfies

$$\dot{V}(S,E,I) = \left(1 - \left(\frac{S^*}{S}\right)^q\right)(b - \beta I^p S^q - \mu S) + \left(1 - \frac{E^*}{E}\right)(\beta I^p S^q - \sigma E)$$

$$+ B\left(1-\left(\frac{I^*}{I}\right)^p\right)(\theta E-\delta I)$$
$$= b-\beta I^p S^q-\mu S-b\left(\frac{S^*}{S}\right)^q+\beta I^p(S^*)^q+\mu S\left(\frac{S^*}{S}\right)^q+\beta I^p S^q-\sigma E$$
$$-\beta I^p S^q\left(\frac{E^*}{E}\right)+\sigma E^*+B\left[\theta E-\delta I-\theta E\left(\frac{I^*}{I}\right)^p+\delta I\left(\frac{I^*}{I}\right)^p\right],$$

Which can be rearranged to give:

$$\begin{aligned}\dot V(S,E,I) &= b-\beta I^p S^q-\mu S-b\frac{(S^*)^q}{S^q}+\beta I^p(S^*)^q+\mu\frac{(S^*)^q}{S^{q-1}}\\ &+\beta I^p S^q-\sigma E-\beta I^p S^q\frac{E^*}{E}+\sigma E^*+B\left(\theta E-\delta I-\theta\frac{(I^*)^p}{I^p}E+\delta\frac{(I^*)^p}{I^{p-1}}\right).\end{aligned}$$

Using now relations (3.49) we derive:

$$\begin{aligned}\dot V(S,E,I) &= \mu S^*+B\delta I^*-\mu S-(\mu S^*+B\delta I^*)\frac{(S^*)^q}{S^q}+\beta I^p(S^*)^q+\mu\frac{(S^*)^q}{S^{q-1}}-\sigma E\\ &-\beta I^p S^q\frac{E^*}{E}+\sigma E^*+B\theta E-B\delta I-B\theta\frac{(I^*)^p}{I^p}E+\delta\frac{(I^*)^p}{I^{p-1}}.\\ &= \mu S^*+B\delta I^*-\mu S-\mu S^*\frac{(S^*)^q}{S^q}-B\delta I^*\frac{(S^*)^q}{S^q}+\beta I^p(S^*)^q+\mu\frac{(S^*)^q}{S^{q-1}}\\ &-B\delta\frac{E}{E^*}I^*-\beta I^p S^q\frac{E^*}{E}+B\delta I^*+B\delta\frac{E}{E^*}I^*-\beta I(I^*)^{p-1}(S^*)^q-B\delta I^*\frac{(I^*)^p}{I^p}\frac{E}{E^*}+B\delta\frac{(I^*)^p}{I^{p-1}}\\ &= \mu S^*+B\delta I^*-\mu S^*\frac{S}{S^*}-\mu S^*\frac{(S^*)^q}{S^q}-B\delta I^*\frac{(S^*)^q}{S^q}+B\delta I^*\frac{I^p}{(I^*)^p}+\mu S^*\frac{(S^*)^{q-1}}{S^{q-1}}\\ &-B\delta I^*\frac{E^*}{E}\frac{S^q}{(S^*)^q}\frac{I^p}{(I^*)^p}+B\delta I^*-B\delta I^*\frac{I}{I^*}-B\delta I^*\frac{E}{E^*}\frac{(I^*)^p}{I^p}+B\delta I^*\frac{(I^*)^{p-1}}{I^{p-1}}.\end{aligned}$$

Rearranging now the latter relation as in [11] we get:

$$\begin{aligned}\dot V(S,E,I) &= B\delta I^*\left(2-\frac{(S^*)^q}{S^q}+\frac{I^p}{(I^*)^p}-\frac{E^*}{E}\frac{S^q}{(S^*)^q}\frac{I^p}{(I^*)^p}-\frac{I}{I^*}-\frac{E}{E^*}\frac{(I^*)^p}{I^p}+\left(\frac{I}{I^*}\right)^{1-p}\right)\\ &+\mu S^*\left(1-\frac{S}{S^*}-\frac{(S^*)^q}{S^q}+\left(\frac{S}{S^*}\right)^{1-q}\right).\end{aligned}$$

Let $u=\frac{S}{S^*}$, $\quad v=\frac{I}{I^*}$ and $\quad w=\frac{E(I^*)^p}{E^*I^p}$ then:

$$\begin{aligned}\dot V(S,E,I) &= B\delta I^*\left(2-u^{-q}+v^p-\frac{u^q}{w}-v-w+v^{1-p}\right)+\mu S^*\left(1-u-u^{-q}+u^{1-q}\right)\\ &= B\delta I^*\left(v^p-v+v^{1-p}-1\right)+B\delta I^*\left(3-u^{-q}-\frac{u^q}{w}-w\right)+\mu S^*\left(1-u-u^{-q}+u^{1-q}\right),\end{aligned}$$

which under rearrangement gives:

$$\dot V(S,E,I) = -B\delta I^*\left(v^{1-p}-1\right)\left(v^p-1\right)+B\delta I^*\left(3-u^{-q}-\frac{u^q}{w}-w\right)+\mu S^*(1-u)\left(1-\frac{1}{u^q}\right).$$

If $p<1$, then

$$h(v)=\left(v^p-1\right)\left(v^{1-p}-1\right)\ge 0 \quad \text{for all } \ u,w,q>0$$

where the equality holds only if $u = w = 1$. Furthermore,

$$(1-u)\left(1-\frac{1}{u^q}\right) \leq 0 \qquad \text{for all} \quad u, q > 0.$$

Therefore, as explained by Korobeinikov and Maini [11], the condition $p < 1$ ensures that $\frac{dV}{dt} \leq 0$ for all $S, E, I > 0$, where the equality holds only at the equilibrium point $Q^*=(S^*,E^*,I^*)$ as defined in (3.47). By the Lyapunov asymptotic stability theorem 1.3.3, the equilibrium point Q^* is globally asymptotically stable. This result is valid for the whole positive octant $\mathbb{R}^3_+$.

3.5.2 An $SEIR$ Model Given by Korobeinikov and Maini [12]

Continuing the same method as used in section 3.3 for the work in [12], we will apply the direct Lyapunov method to consider the global properties of $SEIR$ models with the incidence rate of the form $h(S)g(I)$ satisfying the conditions (3.17)-(3.18) given in section 3.3. Then the basic $SEIR$ model is given by Korobeinikov and Maini [12] as:

$$\begin{aligned}
\dot{S} &= \mu - h(S)g(I) - \mu S, \\
\dot{E} &= h(S)g(I) - (\theta+\mu)E, \\
\dot{I} &= \theta E - (\delta+\mu)I.
\end{aligned} \tag{3.50}$$

Here the equations for the recovered population R are omitted, and we note that condition (3.19) ensures that each of these systems has two equilibrium states: an infection-free equilibrium Q_0 and an endemic equilibrium Q^*.

Finding the infection-free equilibrium:
Infection-free$\Rightarrow I_0 = 0$.
We need to find S_0 such that $\dot{S} = 0$:

$$\begin{aligned}
\dot{S} = \mu - h(S_0)g(I_0) - \mu S_0 &= 0 \\
\Rightarrow \mu - h(S_0)g(0) - \mu S_0 &= 0 \\
\Rightarrow \mu - \mu S_0 &= 0 \\
\Rightarrow \mu S_0 &= \mu \\
S_0 &= 1.
\end{aligned}$$

Thus we have an infection-free equilibrium $Q_0 = (1, 0)$.

Finding the endemic equilibrium:
First we consider $\dot{I} = 0$:

$$\begin{aligned}
\dot{I} = \theta E^* - (\delta+\mu)I^* &= 0 \\
\Rightarrow \theta E^* &= (\delta+\mu)I^* \\
\Rightarrow E^* &= \frac{(\delta+\mu)}{\theta} I^*.
\end{aligned}$$

Now considering $\dot{E} = 0$:

$$\begin{aligned}
\dot{E} = h(S^*)g(I^*) - (\theta+\mu)E^* &= 0 \\
\Rightarrow (\theta+\mu)E^* &= h(S^*)g(I^*).
\end{aligned}$$

Substituting in $E^* = \frac{(\delta+\mu)}{\theta} I^*$:

$$\Rightarrow (\theta+\mu)E^* = \frac{(\theta+\mu)}{\theta}(\delta+\mu)I^*$$

and thus:

$$\frac{(\theta+\mu)}{\theta}(\delta+\mu)I^* = h(S^*)g(I^*).$$

Finally we will consider $\dot{S}=0$:

$$\begin{aligned}\dot{S} &= \mu - h(S^*)g(I^*) - \mu S^* = 0 \\ &\Rightarrow \mu S^* + h(S^*)g(I^*) = \mu.\end{aligned}$$

Substituting in $h(S^*)g(I^*) = \frac{(\theta+\mu)}{\theta}(\delta+\mu)I^*$:

$$\mu S^* + \frac{(\theta+\mu)}{\theta}(\delta+\mu)I^* = \mu.$$

Thus we have an endemic equilbrium $Q^*= (S^*, E^*, I^*)$, such that

$$\frac{(\theta+\mu)}{\theta}(\delta+\mu)I^* = h(S^*)g(I^*), \quad \mu S^* + \frac{(\theta+\mu)}{\theta}(\delta+\mu)I^* = \mu, \quad \frac{(\theta+\mu)}{\theta}(\delta+\mu)I^* = (\theta+\mu)E^*. \tag{3.51}$$

For this $SEIR$ model, Korobeinikov and Maini [12] give the possible Lyapunov function:

$$V(S,E,I) = S - h(S^*)\int_a^S \frac{\mathrm{d}\tau}{h(\tau)} + B\left(I - g(I^*)\int_a^I \frac{\mathrm{d}\tau}{g(\tau)}\right) + E - E^*\ln(E). \tag{3.52}$$

where $B = \frac{\theta+\mu}{\theta}$ and the parameter a, such that $0 < a \ll 1$, is an arbitrary positive constant which is not fixed and can be made sufficiently small. The function $V(S,E,I)$ is defined and continuous for all $S, E, I \geq a$ and satisfies

$$\frac{\partial V}{\partial S} = 1 - \frac{h(S^*)}{h(S)}, \quad \frac{\partial V}{\partial E} = \left(1 - \frac{E^*}{E}\right), \quad \frac{\partial V}{\partial I} = B\left(1 - \frac{g(I^*)}{g(I)}\right).$$

Since the function $h(S)$ and $g(I)$ grow monotonically, the partial derivatives $\frac{\partial V}{\partial S}$ and $\frac{\partial V}{\partial I}$ grow monotonically as well. Note that for this function we have $\frac{\partial V}{\partial S} = \frac{\partial V}{\partial E} = \frac{\partial V}{\partial I} = 0$ at Q^*, as defined in (3.51), and thus we can determine the type of equilibrium point. For the possible Lyapunov function (3.52) we can derive the second derivatives as follows:

$$\frac{\partial^2 V}{\partial S^2} = \frac{h(S^*)}{(h(S))^2}\frac{\partial h(S)}{\partial S} > 0, \quad \frac{\partial^2 V}{\partial E^2} = \frac{E^*}{E^2} > 0, \quad \frac{\partial^2 V}{\partial I^2} = B\frac{g(I^*)}{(g(I))^2}\frac{\partial g(I)}{\partial I} > 0,$$

and

$$\frac{\partial^2 V}{\partial S \partial E} = \frac{\partial^2 V}{\partial S \partial I} = \frac{\partial^2 V}{\partial E \partial I} = 0.$$

Thus, by the second derivative test, see definition 3.2.3, we can conclude that the endemic equilibrium state $Q^*= (S^*, E^*, I^*)$ is the only local stationary point of the function and is a minimum extreme point. As $V(S,E,I) \to \infty$ at the boundary, we have $V(S,E,I) \geq V(S^*,E^*,I^*)$ and thus the function is bounded from below. Consequently Q^* is the only extremum and the global minimum of the function in the positive octant $\mathbb{R}^3_+$ and hence the function (3.52) is indeed a Lyapunov function.

The following theorem, given by Korobeinikov and Maini [12], provides global properties of the system (3.50).

Theorem 3.5.2

1. *If the incidence rate satisfies the conditions* (3.17)-(3.19)*, and if* $R_0 > 1$*, then the endemic equilibrium state* Q^* *is globally asymptotically stable.*

2. *If $R_0 \leq 1$, then there is no positive equilibrium state Q^*, and the infection-free equilibrium state Q_0 is globally asymptotically stable.*

Proof of 1. Using the equalities

$$h(S^*)g(I^*) = B(\delta+\mu)I^*, \quad \mu = \mu S^* + B(\delta+\mu)I^*, \quad (\theta+\mu)E^* = \frac{(\theta+\mu)}{\theta}(\delta+\mu)I^*.$$

where $B = \frac{\theta+\mu}{\theta}$. For the equilibrium state Q^*, the Lyapunov function $V(S,E,I)$ satisfies

$$\begin{aligned}\frac{\mathrm{d}V(S,E,I)}{\mathrm{d}t} &= \left(1 - \frac{h(S^*}{h(S)}\right)(\mu - h(S)g(I) - \mu S)\\ &+ \left(1 - \frac{E^*}{E}\right)(h(S)g(I) - (\theta+\mu)E)\\ &+ B\left(1 - \frac{g(I^*)}{g(I)}\right)(\theta E - (\delta+\mu)I).\end{aligned}$$

Which can be rearranged to give:

$$\begin{aligned}\frac{\mathrm{d}V(S,E,I)}{\mathrm{d}t} &= \mu - h(S)g(I) - \mu S - \mu\frac{h(S^*)}{h(S)} + h(S^*)g(I) + \mu S\frac{h(S^*)}{h(S)}\\ &+ h(S)g(I) - (\theta+\mu)E - h(S)g(I)\frac{E^*}{E} + (\theta+\mu)E^*\\ &+ B\left(\theta E - (\delta+\mu)I - \theta E\frac{g(I^*)}{g(I)} + (\delta+\mu)I\frac{g(I^*)}{g(I)}\right).\end{aligned}$$

Substituting the relation (3.23) we derive:

$$\begin{aligned}\frac{\mathrm{d}V(S,E,I)}{\mathrm{d}t} &= [\mu S^* + B(\delta+\mu)I^*] - \mu S^*\frac{S}{S^*} - [\mu S^* + B(\delta+\mu)I^*]\frac{h(S^*)}{h(S)} + B(\delta+\mu)I^*\frac{g(I)}{g(I^*)} + \mu S^*\frac{S}{S^*}\frac{h(S^*)}{h(S)}\\ &- (\theta+\mu)E - B(\delta+\mu)I^*\frac{h(S)g(I)E^*}{h(S^*)g(I^*)E} + B(\delta+\mu)I^*\\ &+ (\theta+\mu)E - B(\delta+\mu)I^*\frac{I}{I^*} - B(\delta+\mu)I^*\frac{g(I^*)}{g(I)}\frac{E}{E^*} + B(\delta+\mu)I^*\frac{g(I^*)}{g(I)}\frac{I}{I^*}\\ &= \mu S^*\left(1 - \frac{S}{S} - \frac{h(S^*)}{h(S)} + \frac{S}{S^*}\frac{h(S^*)}{h(S)}\right)\\ &+ B(\delta+\mu)I^*\left(2 - \frac{h(S^*)}{h(S)} + \frac{g(I)}{g(I^*)} - \frac{h(S)g(I)E^*}{h(S^*)g(I^*)E} - \frac{I}{I^*} - \frac{g(I^*)}{g(I)}\frac{E}{E^*} + \frac{g(I^*)}{g(I)}\frac{I}{I^*}\right)\\ &= \mu S^*\left(1 - \frac{S}{S^*} - \frac{h(S^*)}{h(S)} + \frac{S}{S^*}\frac{h(S^*)}{h(S)}\right)\\ &- B(\delta+\mu)I^*\left(\frac{h(S^*)}{h(S)} + \frac{h(S)g(I)E^*}{h(S^*)g(I^*)E} + \frac{g(I^*)E}{g(I)E^*} - 2\right)\\ &+ B(\delta+\mu)I^*\left(\frac{g(I)}{g(I^*)} - \frac{I}{I^*} + \frac{I}{I^*}\frac{g(I^*)}{g(I)}\right)\\ &= \mu S^*\left(1 - \frac{S}{S^*}\right)\left(1 - \frac{h(S^*)}{h(S)}\right)\\ &- B(\delta+\mu)I^*\left(\frac{h(S^*)}{h(S)} + \frac{h(S)g(I)E^*}{h(S^*)g(I^*)E} + \frac{g(I^*)E}{g(I)E^*} - 3\right)\\ &+ B(\delta+\mu)I^*\left(\frac{g(I)}{g(I^*)} - \frac{I}{I^*} + \frac{I}{I^*}\frac{g(I^*)}{g(I)} - 1\right).\end{aligned}$$

Now after rearrangement we obtain:

$$\frac{\mathrm{d}V(S,E,I)}{\mathrm{d}t} = \mu S^*\left(1 - \frac{S}{S^*}\right)\left(1 - \frac{h(S^*)}{h(S)}\right)$$

$$- B(\delta+\mu)I^* \left(\frac{h(S^*)}{h(S)} + \frac{h(S)g(I)E^*}{h(S^*)g(I^*)E} + \frac{g(I^*)E}{g(I)E^*} - 3\right)$$
$$+ B(\delta+\mu)I^* \left(1 - \frac{g(I^*)}{g(I)}\right)\left(\frac{g(I)}{g(I^*)} - \frac{I}{I^*}\right).$$

Korobeinikov and Maini [12] explain that the concavity of the function $g(I)$ ensures that $\frac{dV}{dt} \leq 0$, for all $S, E, I > a$, where the equality holds only at the point Q^*. Indeed, since the arithmetical mean is greater than or equal to the geometrical mean (see also [12],

$$\frac{h(S^*)}{h(S)} + \frac{h(S)g(I)E^*}{h(S^*)g(I^*)E} + \frac{g(I^*)E}{g(I)E^*} \geq 3, \quad \text{for all} \quad S, E, I > 0.$$

Furthermore,

$$\left(1 - \frac{S}{S^*}\right)\left(1 - \frac{h(S^*)}{h(S)}\right) \leq 0, \quad \text{for all} \quad S > 0,$$

since for a monotonically growing function $h(S)$, $h(S) \geq h(S^*)$ when $S \geq S^*$ and $h(S) \leq h(S^*)$ when $S \leq S^*$. Also

$$\left(1 - \frac{g(I^*)}{g(I)}\right)\left(\frac{g(I)}{g(I^*)} - \frac{I}{I^*}\right)$$

if

$$\begin{cases} \frac{g(I)}{g(I^*)} \geq \frac{I}{I^*}, & \text{for } 0 < I \leq I^*, \\ \frac{g(I)}{g(I^*)} \leq \frac{I}{I^*}, & \text{for } I \geq I^*. \end{cases} \tag{3.53}$$

It is easy to see that this condition holds for all concave functions $g(I)$ (see Figure 3.5). Since the parameter a can be chosen sufficiently small, by the asymptotic stability theorem (see theorem 1.3.3), the $SEIR$ system (3.50) is globally asymptotically stable in $\mathbb{R}^3_+$.

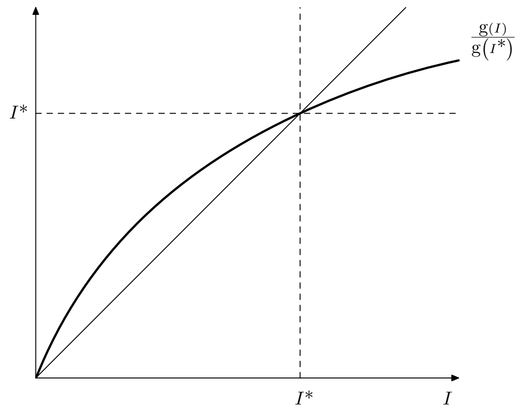

Figure 3.5: A concave function $g(I)$ as shown by Korobeinikov and Maini [12].

Proof of 2. To prove global stability of the infection-free equilibrium state $Q_0 = (1, 0)$ we consider, as in [12], the Lyapunov test function:

$$U(S, I) = S - h(S_0)\int_a^S \frac{d\tau}{h(\tau)} + E + BI,$$

where $B = \frac{\theta+\mu}{\theta}$.

For system (3.50), the Lyapunov function satisfies

$$\frac{dU(S,E,I)}{dt} = \left(1 - \frac{h(S_0)}{h(S)}\right)(\mu - h(S)g(I) - \mu S) + (h(S)g(I) - (\theta_\mu)E) + B(\theta E - (\delta+\mu)I).$$

Note that, for this $SEIR$ system, $B = \frac{\theta+\mu}{\theta}$:

$$\begin{aligned}
\frac{\mathrm{d}U(S,E,I)}{\mathrm{d}t} &= \mu - h(S)g(I) - \mu S - \mu\frac{h(S_0)}{h(S)} + h(S_0)g(I) + \mu S\frac{h(S_0)}{h(S)} \\
&\quad + h(S)g(I) - (\theta+\mu)E + (\theta+\mu)E - B(\delta+\mu)I \\
&= \mu - \mu S - \mu\frac{h(S_0)}{h(S)} + h(S_0)g(I) + \mu S\frac{h(S_0)}{h(S)} - B(\delta+\mu)I \\
&= \mu\left(1 - \frac{h(S_0)}{h(S)} - S + S\frac{h(S_0)}{h(S)}\right) + h(S_0)g(I) - B(\delta+\mu)I,
\end{aligned}$$

which can be rearranged to give:

$$\frac{\mathrm{d}U(S,E,I)}{\mathrm{d}t} = \mu(1-S)\left(1 - \frac{h(S_0)}{h(S)}\right) + B(\delta+\mu)I\left(\frac{h(S_0)}{B(\delta+\mu)}\frac{g(I)}{I} - 1\right).$$

Using the same approach as in section 3.3 for the work in [12], we note that here

$$(1-S)\left(1 - \frac{h(S_0)}{h(S)}\right) \leq 0, \quad \text{for all } S > 0,$$

and the conditions (3.17) and (3.19) ensure that $\frac{g(I)}{I} \leq \frac{\partial g(0)}{\partial I}$, for all $I > 0$. Hence, using R_0, as defined in (3.25), we have

$$\frac{h(S_0)}{B(\delta+\mu)}\frac{g(I)}{I} \leq \frac{h(S_0)}{B(\delta+\mu)}\frac{\partial g(0)}{\partial I} = R_0.$$

Therefore, $R_0 \leq 1$ ensures that $\frac{\mathrm{d}V}{\mathrm{d}t} \leq 0$, for all $S, E, I > a$, and hence by the asymptotic stability theorem (see theorem 1.3.3), the equilibrium state Q_0 is globally asymptotically stable in this case. The theorem is proven.

Chapter 4

Reaction and Diffusion Models

Throughout this section we will investigate reaction-diffusion systems. Following the work of Hsu [5], we will consider the following reaction-diffusion system with Neumann boundary condition

$$\begin{aligned} \frac{\partial u}{\partial t} &= D\Delta u + f(u) \quad \text{in} \quad \Omega \subseteq \mathbb{R}^n, \\ \frac{\partial u}{\partial \nu} &= 0 \qquad\qquad\quad\;\; \text{in} \quad \partial\Omega, \end{aligned} \tag{4.1}$$

where $u = u(x,t) \in \mathbb{R}^n$, $D = \text{diag}(d_1, \ldots, d_n)$, $d_i > 0$, $i = 1, \ldots, n$, $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ is continuously differentiable. We assume there exists a Lyapunov function $V(u)$ for the corresponding ODE system

$$u' = f(u). \tag{4.2}$$

Thus we have that $V(u)$ satisfies

$$\dot{V}(u) = \text{grad}_u V \cdot f(u) \leq 0 \quad \text{for all} \quad u \in \mathbb{R}^n_+. \tag{4.3}$$

To study the global behaviour of system (4.1), Hsu [5] proves that the following functional is a Lyapunov functional for the reaction-diffusion system (4.1).

$$W(t) = \int_\Omega V(u(x,t))dx, \tag{4.4}$$

for the proof, see [5].

We will use this result throughout this section to investigate the global behaviour of reaction-diffusion systems.

4.0.3 A Diffusive Predator-Prey Model Given by Hsu [5]

The first reaction and diffusion system we will concentrate on is the following diffusive predator-prey system investigated in [5]:

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1\Delta u + u(\lambda - \alpha u - \beta v) && \text{in} \quad \Omega, \\ \frac{\partial v}{\partial t} &= d_2\Delta v + \mu v\left(1 - \frac{v}{u}\right) && \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 && \text{in} \quad \partial\Omega, \\ u(x,0) &= u_0(x), \quad v(x,0) = v_0(x) && \text{in} \quad \Omega. \end{aligned} \tag{4.5}$$

Here $u(x,t)$ and $v(x,t)$ respectively represent the species densities of the prey and predator and $\Omega \subset \mathbb{R}^n$ is a fixed bounded domain. Here we recall that ν is the outward unit normal vector on the smooth boundary

$\partial\Omega$. Also note that the initial conditions $u_0(x)$ and $v_0(x)$ are continuous functions on Ω and the constants $d_i \quad (i = 1, 2)$ are the diffusion coefficients corresponding to u and v respectively. All the parameters appearing in (4.5) are assumed to be positive and we will use these assumptions for all of the reaction-diffusion models we investigate.

We first calculate the equilibrium of the corresponding ODE system

$$\begin{aligned} &\frac{du}{dt} = u(\lambda - \alpha u - \beta v) && \text{for} \quad t > 0, \\ &\frac{dv}{dt} = \mu v \left(1 - \frac{v}{u}\right) && \text{for} \quad t > 0, \\ &u(0) = u_0 > 0, \quad v(0) = v_0 > 0. \end{aligned} \tag{4.6}$$

Using (4.6) we can calculate the equilibrium as follows:

First we will consider $\frac{dv}{dt} = 0$:

$$\begin{aligned} \frac{dv}{dt} = \mu v^* \left(1 - \frac{v^*}{u^*}\right) &= 0 \\ \Rightarrow 1 - \frac{v^*}{u^*} &= 0 \\ \Rightarrow v^* &= u^*. \end{aligned}$$

Next we consider $\frac{du}{dt} = 0$:

$$\frac{du}{dt} = u^*(\lambda - \alpha u^* - \beta v^*) = 0.$$

Using v^*=u^* as calculated previously:

$$\begin{aligned} u^*(\lambda - \alpha u^* - \beta u^*) &= 0 \\ \Rightarrow \lambda - (\alpha + \beta) u^* &= 0 \\ \Rightarrow u^* &= \frac{\lambda}{\alpha + \beta}. \end{aligned}$$

Thus we have the unique constant equilibrium (u^*,v^*) where u^*=v^*=$\frac{\lambda}{\alpha+\beta}$.

Given the following Lyapunov function, constructed by Du and Hsu [4]:

$$V(u, v) = \int_{u^*}^{u} \frac{\varepsilon - u^*}{\varepsilon^2} d\varepsilon + \frac{\beta}{\mu} \int_{v^*}^{v} \frac{\eta - v^*}{\eta} d\eta, \tag{4.7}$$

we can analyse the derivative $\dot{V}(u, v)$ as follows.

Denote

$$f(u, v) = u(\lambda - \alpha u - \beta v), \quad g(u, v) = \mu v \left(1 - \frac{v}{u}\right). \tag{4.8}$$

Using the formula, as stated by Hsu [5], we can calculate $\dot{V}(u, v)$ as follows:

$$\begin{aligned} \dot{V}(u, v) &= \frac{\partial V}{\partial u} f(u, v) + \frac{\partial V}{\partial v} g(u, v) \\ &= \frac{u - u^*}{u^2} \left[u(\lambda - \alpha u - \beta v)\right] + \frac{\beta}{\mu} \frac{v - v^*}{v} \left[\mu v \left(1 - \frac{v}{u}\right)\right] \\ &= \frac{u - u^*}{u} (\lambda - \alpha u - \beta v) + \beta (v - v^*) \left(1 - \frac{v}{u}\right). \end{aligned} \tag{4.9}$$

Note that, using the unique constant equilibrium point calculated previously, we have $u^*= \frac{\lambda}{\alpha+\beta} \Rightarrow \lambda = u^*(\alpha+\beta)$, which can be substituted in to give:

$$\dot{V}(u,v) = \frac{u-u^*}{u}\left(u^*(\alpha+\beta) - \alpha u - \beta v\right) + \beta\frac{v-v^*}{u}(u-v).$$

Using $u^*=v^*$, this can be rewritten as follows:

$$\begin{aligned}
\dot{V}(u,v) &= \frac{u-u^*}{u}\left(\alpha u^* - \alpha u\right) + \frac{u-v^*}{u}\left(\beta v^* - \beta v\right) + \beta\frac{v-v^*}{u}(u-v) \\
&= -\alpha\frac{(u-u^*)^2}{u} + \beta\left(v^* - v\right) - \beta\frac{v^*}{u}\left(v^* - v\right) + \beta\left(v - v^*\right) - \beta v\left(\frac{v-v^*}{u}\right) \\
&= -\alpha\frac{(u-u^*)^2}{u} + \beta v^*\frac{v-v^*}{u} - \beta v\frac{v-v^*}{u} \\
&= -\alpha\frac{(u-u^*)^2}{u} - \beta\frac{(v-v^*)^2}{u}.
\end{aligned}$$

Thus we have

$$\dot{V}(u,v) = -\alpha\frac{(u-u^*)^2}{u} - \beta\frac{(v-v^*)^2}{u} \le 0. \tag{4.10}$$

Consider now the potential Lyapunov functional for the reaction-diffusion system (4.6)

$$W(t) = \int_\Omega V(u(x,t), v(x,t))dx,$$

we can calculate $W'(t)$ as follows:

$$\begin{aligned}
W'(t) &= \int_\Omega \left(V_u u_t + V_v v_t\right)dx + \int_\Omega \dot{V}dx \\
&= \int_\Omega \left(\frac{u-u^*}{u^2}d_1\Delta u + \frac{\beta}{\mu}\frac{v-v^*}{v}d_2\Delta v\right)dx + \int_\Omega \dot{V}(u,v)dx.
\end{aligned}$$

Using integration by parts, we have:

$$\begin{aligned}
W'(t) = \Bigg[&\int_{\partial\Omega}\left(d_1\frac{u-u^*}{u^2}\frac{\partial u}{\partial\nu} + \frac{\beta}{\mu}d_2\frac{v-v^*}{v}\frac{\partial v}{\partial\nu}\right)dx \\
&- \int_\Omega\left(-d_1\left(-\frac{1}{u^2} + \frac{2u^*}{u^3}\right)\nabla u\cdot\nabla u + \frac{\beta}{\mu}d_2\frac{v^*}{v}\nabla v\cdot\nabla v\right)dx\Bigg] + \int_\Omega \dot{V}(u,v)dx
\end{aligned}$$

From the initial conditions $\frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0$, given in system (4.5), $\nabla u\cdot\nabla u = |\nabla u|^2$ and $\nabla v\cdot\nabla v = |\nabla v|^2$, we can eliminate the first integral and rearrange to obtain:

$$W'(t) = -\int_\Omega\left(d_1\frac{2u^*-u}{u^3}|\nabla u|^2 + \frac{\beta}{\mu}d_2\frac{v^*}{v}|\nabla v|^2\right)dx + \int_\Omega \dot{V}(u,v)dx. \tag{4.11}$$

If $\alpha > \beta$ then, from $u^*= \frac{\lambda}{\alpha+\beta}$, it can easily be seen that $2u^*= \frac{2\lambda}{\alpha+\beta} > \frac{2\lambda}{2\alpha} = \frac{\lambda}{\alpha}$. From the first equation in (4.5), we have

$$\frac{\partial u}{\partial t} \le d_1\Delta u + u(\lambda - \alpha u).$$

Let $(u(x,t), v(x,t))$ be a positive solution of system (4.5). Using a comparison argument we have $0 < u(x,t) < U(x,t)$ for all $t > 0$ and $x \in \Omega$, where U is the unique solution of

$$\begin{cases} U_t = d_1 \Delta U + U(\lambda - \alpha U) & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial U}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ U(x, 0) = u_0(x). \end{cases} \tag{4.12}$$

Thus from the comparison principle (see [19]), we can deduce that $U(x,t) \to \frac{\lambda}{\alpha}$ as $t \to \infty$ uniformly in x, as also described by Du and Hsu [4]. Hence, from (4.10) and (4.25), we can easily see that $W'(t) < 0$. We can therefore conclude that when $\alpha > \beta$, the constant equilibrium (u^*, v^*) attracts every positive solution of system (4.5).

Next we will show how the restriction $\alpha > \beta$ can be relaxed by using the following Lyapunov function, as given by Hsu [5]:

$$V^*(u,v) = \int_{u^*}^{u} \frac{\varepsilon^2 - (u^*)^2}{\varepsilon^2} d\varepsilon + c \int_{v^*}^{v} \frac{\eta - v^*}{\eta} d\eta, \tag{4.13}$$

with $c > 0$ to be determined.

Using (4.8), we can now calculate $\dot{V}^*(u,v)$ as follows:

$$\begin{aligned} \dot{V}^*(u,v) &= \frac{\partial V^*}{\partial u} f(u,v) + \frac{\partial V^*}{\partial v} g(u,v) \\ &= \frac{u^2 - (u^*)^2}{u^2} \left[u(\lambda - \alpha u - \beta v)\right] + c\frac{v - v^*}{v} \left[\mu v \left(1 - \frac{v}{u}\right)\right] \\ &= \frac{u^2 - (u^*)^2}{u} (\lambda - \alpha u - \beta v) + (v - v^*) \left(1 - \frac{v}{u}\right) \\ &= \frac{1}{u} \left[\left(u^2 - (u^*)^2\right)(\lambda - \alpha u - \beta v) + c\mu(v - v^*)(u - v)\right]. \end{aligned} \tag{4.14}$$

Note that, as calculated previously, we have the unique constant equilibrium point $u^* = \frac{\lambda}{\alpha + \beta} \Rightarrow \lambda = u^*(\alpha + \beta)$:

$$\dot{V}^*(u,v) = \frac{1}{u} \left[(u - u^*)(u + u^*)(u^*(\alpha + \beta) - \alpha u - \beta v) + c\mu(v - v^*)(u - v)\right].$$

Substituting in $u^* = v^*$ and rearranging:

$$\begin{aligned} \dot{V}^*(u,v) &= \frac{1}{u} \left[-\alpha(u - u^*)^2(u + u^*) - \beta(v - v^*)(u - u^*)(u + u^*) + c\mu u(v - v^*)\right. \\ &\qquad \left. - c\mu u^*(v - v^*) + c\mu v^*(v - v^*) - c\mu v(v - v^*)\right] \\ &= \frac{1}{u} \left[-\alpha(u - u^*)^2(u + u^*) - \beta(v - v^*)(u - u^*)(u + u^*) + c\mu(u + u^*)(v - v^*) - c\mu(v - v^*)^2\right] \\ &= \frac{1}{u} \left[-\alpha(u - u^*)^2(u + u^*) + \left[c\mu - \beta(u + u^*)\right](u - u^*)(v - v^*) - c\mu(v - v^*)^2\right]. \end{aligned}$$

Let $\varepsilon = u - u^*$ and $\eta = v - v^*$, then we obtain:

$$\dot{V}^*(u,v) = \frac{1}{u} \left[-\alpha(u + u^*)\varepsilon^2 + \left[c\mu - \beta(u + u^*)\right]\varepsilon\eta - c\mu\eta^2\right].$$

We can see that if

$$\left[c\mu - \beta(u + u^*)\right]^2 - 4\alpha(u + u^*)c\mu < 0, \tag{4.15}$$

then we have

$$-\alpha(u + u^*)\varepsilon^2 + \left[c\mu - \beta(u + u^*)\right]\varepsilon\eta - c\mu\eta^2 < 0 \tag{4.16}$$

unless $\varepsilon = \eta = 0$.

We can now show that the restriction $\alpha > \beta$ can be relaxed by choosing a $c > 0$ such that (4.16) holds. Note that we can rewrite (4.16) as

$$(\mu c)^2 - 2(u + u^*)(\beta + 2\alpha)(\mu c) + \beta^2 (u + u^*)^2 < 0. \tag{4.17}$$

Here we can determine that (4.17) holds if and only if $\mu c \in (c_1, c_2)$ where

$$\begin{aligned} c_1 = c_1(u) &= (u + u^*)\left(\beta + 2\alpha - \sqrt{(\beta + 2\alpha)^2 - \beta^2}\right), \\ c_2 = c_2(u) &= (u + u^*)\left(\beta + 2\alpha + \sqrt{(\beta + 2\alpha)^2 - \beta^2}\right). \end{aligned}$$

To determine a more accurate c we will investigate to find when $c_1\left(\frac{\lambda}{\alpha}\right) < c_2(0)$ holds:

$$\begin{aligned} \left(\frac{\lambda}{\alpha} + \frac{\lambda}{\alpha+\beta}\right)\left(\beta + 2\alpha - \sqrt{(\beta + 2\alpha)^2 - \beta^2}\right) &< \frac{\lambda}{\alpha+\beta}\left(\beta + 2\alpha + \sqrt{(\beta + 2\alpha)^2 - \beta^2}\right) \\ \frac{\lambda}{\alpha}\left(\beta + 2\alpha - \sqrt{(\beta + 2\alpha)^2 - \beta^2}\right) &< \frac{2\lambda}{\alpha+\beta}\sqrt{(\beta + 2\alpha)^2 - \beta^2} \\ (\alpha+\beta)\left(\beta + 2\alpha - \sqrt{(\beta + 2\alpha)^2 - \beta^2}\right) &< 2\alpha\sqrt{(\beta + 2\alpha)^2 - \beta^2} \\ (\alpha+\beta)(\beta+2\alpha) &< (3\alpha+\beta)\sqrt{(\beta + 2\alpha)^2 - \beta^2} \\ (\alpha+\beta)^2(\beta+2\alpha)^2 &< (3\alpha+\beta)^2\left[(\beta + 2\alpha)^2 - \beta^2\right] \\ (\alpha+\beta)^2(\beta+2\alpha)^2 &< (3\alpha+\beta)^2\left[4\alpha(\alpha+\beta)\right] \\ (\alpha+\beta)(\beta+2\alpha)^2 &< 4\alpha(3\alpha+\beta)^2. \end{aligned}$$

This can be then expanded and rearranged to give:

$$\begin{aligned} 32\alpha^3 + 16\alpha^2\beta - \alpha\beta^2 - \beta^3 &> 0 \\ 32\left(\frac{\alpha}{\beta}\right)^3 + 16\left(\frac{\alpha}{\beta}\right)^2 - \frac{\alpha}{\beta} - 1 &> 0. \end{aligned}$$

Let $S = \frac{\alpha}{\beta}$, then we have:

$$h(S) = 32S^3 + 16S^2 - S - 1 > 0.$$

We can see that $h(0) = h'(0) = -1$, thus the cubic $h(S)$ has a unique positive zero S_0 and $h(S) > 0$ when $S > S_0$ (see also [4]). Since $h\left(\frac{1}{5}\right) < 0 < h\left(\frac{1}{4}\right)$, we can conclude that $S_0 \in \left(\frac{1}{5}, \frac{1}{4}\right)$.

Now suppose that we have $\frac{\alpha}{\beta} > S_0$. We can see that $c_1\left(\frac{\lambda}{\alpha}\right) < c_2(0)$ holds and we can therefore choose an $\varepsilon > 0$ small such that $c_1\left(\frac{\lambda}{\alpha} + \varepsilon\right) < c_2(0)$. Hence we can now choose a $c > 0$ such that

$$c_1\left(\frac{\lambda}{\alpha} + \varepsilon\right) < \mu c < c_2(0).$$

Then we can see that

$$c_1(u) \le c_1\left(\frac{\lambda}{\alpha} + \varepsilon\right) < \mu c < c_2(0) \le c_2(u), \quad \forall u \in \left[0, \frac{\lambda}{\alpha} + \varepsilon\right].$$

Therefore, as also explained by Du and Hsu [4], for this choice of c, (4.17) holds for $u \in \left[0, \frac{\lambda}{\alpha} + \varepsilon\right]$. We can then see that

$$Z(u, v) := V_u{}^* f + V_v{}^* g \le 0, \quad \text{for all } \ u \in \left[0, \frac{\lambda}{\alpha} + \varepsilon\right]$$

where this equality holds if and only if $(u, v) = (u^*, v^*)$.

Considering the potential Lyapunov functional

$$W^*(t) = \int_\Omega V^*(u(x,t), v(x,t))dx$$

we can calculate the derivative as follows:

$$\begin{aligned} \frac{d}{dt}W^*(t) &= \int_\Omega (V_u{}^*u_t + V_v{}^*v_t)\, dx + \int_\Omega Z(u,v)dx \\ &= \int_\Omega \left(\frac{u^2 - (u^*)^2}{u^2} d_1\Delta u + c\frac{v - v^*}{v} d_1 \Delta v \right) dx + \int_\Omega Z(u,v)dx. \end{aligned}$$

Using integration by parts we obtain:

$$\begin{aligned} \frac{d}{dt}W^*(t) &= \Bigg[\int_{\partial\Omega} \left(d_1 \frac{u^2 - (u^*)^2}{u^2}\frac{\partial u}{\partial \nu} + cd_2 \frac{v - v^*}{v}\frac{\partial v}{\partial \nu} \right) dx \\ &\quad - \int_\Omega \left(d_1 \frac{2(u^*)^2}{u^3} \nabla u \cdot \nabla u + cd_2 \frac{v^*}{v^2} \nabla v \cdot \nabla v \right) dx \Bigg] + \int_\Omega Z(u,v)dx \\ &= -\int_\Omega \left(d_1 \frac{2(u^*)^2}{u^3} |\nabla u|^2 + cd_2 \frac{v^*}{v^2} |\nabla v|^2 \right) dx + \int_\Omega Z(u,v)dx. \end{aligned}$$

As we know that $u(x,t) < U(x,t) \to \frac{\lambda}{\alpha}$, shown previously, we can thus find $T > 0$ such that $u(x,t) \leq \frac{\lambda}{\alpha} + \varepsilon$ for $t > T$. Thus we have

$$\frac{d}{dt}W^*(t) \leq 0 \ \text{ for } \ t > T \ \text{ and equality holds if and only if } \ (u,v) = (u^*, v^*).$$

We can therefore conclude that for $\frac{\alpha}{\beta} > S_0$, where $S_0 \in (\frac{1}{5}, \frac{1}{4})$ is the unique positive zero of $h(S) = 32S^3 + 16S^2 - S - 1$, (u^*, v^*) attracts every positive solution of system (4.5).

4.0.4 A Diffusive Holling-Tanner Predator-Prey Model Given by Peng and Wang [18]

The next reaction-diffusion system we will concentrate on is the following diffusive predator-prey system found in [18]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + au - u^2 - \dfrac{uv}{m+u} & \text{in } \ \Omega \times (0, \infty), \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + bv - \dfrac{v^2}{\gamma u} & \text{in } \ \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \ \partial\Omega \times (0, \infty), \\ u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) \geq 0, \not\equiv 0, & \text{on } \ \bar{\Omega}. \end{cases} \tag{4.18}$$

Here we use the assumptions as defined in the example previously given by Hsu [5].

Firstly we will calculate the equilibrium of the corresponding ODE system as follows:

We will first consider $\frac{dv}{dt} = 0$:

$$\frac{dv}{dt} = bv^* - \frac{(v^*)^2}{\gamma u^*} = 0$$

$$\Rightarrow v^* \left(b - \frac{v^*}{\gamma u} \right) = 0$$

As we are interested in the positive equilibrium point, we have:

$$v^* = b\gamma u^*. \tag{4.19}$$

Now we will consider $\frac{du}{dt} = 0$:

$$\begin{aligned} \frac{du}{dt} &= au^* - (u^*)^2 - \frac{u^* v^*}{m + u^*} = 0 \\ \Rightarrow u^* &\left(a - u^* - \frac{v^*}{m + u^*} \right) = 0. \end{aligned}$$

As we are interested in the positive equilibrium point, we have:

$$\begin{aligned} a - u^* - \frac{v^*}{m + u^*} &= 0 \qquad (4.20) \\ \Rightarrow u^* (m + u^*) + v^* - a (m + u^*) &= 0. \end{aligned}$$

Substituting in $v^* = b\gamma u^*$ as shown previously, we can rearrange to obtain:

$$(u^*)^2 + (m + b\gamma - a) u^* - am = 0 \tag{4.21}$$

We can therefore solve to obtain the positive equilibrium point:

$$u^* = \frac{1}{2} \left\{ a - m - b\gamma + \sqrt{(m + b\gamma - a)^2 + 4am} \right\}.$$

Thus we obtain the unique positive equilibrium point $(u, v) = (u^*, v^*)$ where

$$u^* = \frac{1}{2} \left\{ a - m - b\gamma + \sqrt{(m + b\gamma - a)^2 + 4am} \right\} \quad \text{and} \quad v^* = b\gamma u^*.$$

Given the Lyapunov function, constructed by Peng and Wang [18]:

$$V(u, v) = \int_{u^*}^{u} \frac{\varepsilon - u^*}{\varepsilon^2} d\varepsilon + \alpha \int_{v^*}^{v} \frac{\eta - v^*}{\eta} d\eta, \tag{4.22}$$

where α is a positive constant to be determined later.

Denote

$$f(u, v) = au - u^2 - \frac{uv}{m + u}, \quad g(u, v) = bv - \frac{v^2}{\gamma u}. \tag{4.23}$$

Using the formula given previously in (4.9), we can calculate $\dot{V}(u, v)$ as follows:

$$\begin{aligned} \dot{V}(u, v) &= \frac{\partial V}{\partial u} f(u, v) + \frac{\partial V}{\partial v} g(u, v) \\ &= \frac{u - u^*}{u^2} \left[au - u^2 - \frac{uv}{m + u} \right] + \frac{v - v^*}{v} \left[bv - \frac{v^2}{\gamma u} \right]. \end{aligned}$$

Using equation (4.19), we can see that $b = \frac{v^*}{\gamma u^*}$ and using equation (4.20), we can see that $a = u^* + \frac{v^*}{m + u^*}$. Substituting and rearranging gives:

$$\dot{V}(u, v) = \frac{u - u^*}{u} \left(u^* + \frac{v^*}{m + u^*} - u - \frac{v}{m + u} \right) + \alpha (v - v^*) \left(\frac{v^*}{\gamma u^*} - \frac{v}{\gamma u} \right)$$

$$
\begin{aligned}
&= (u-u^*)\left(\frac{u^*-u}{u} + \frac{v^*(m+u)-v(m+u^*)}{u(m+u^*)(m+u)}\right) + \alpha(v-v^*)\left(\frac{v^*}{\gamma u^*} - \frac{v}{\gamma u}\right)\\
&= (u-u^*)\left(\frac{u^*-u}{u} + \frac{m(v^*-v)+uv^*-u^*v}{u(m+u^*)(m+u)}\right) + \alpha(v-v^*)\left(\frac{v^*}{\gamma u^*} - \frac{v}{\gamma u}\right)\\
&= (u-u^*)\left(\frac{u^*-u}{u} + \frac{m(v^*-v)+uv^*-u^*v+u^*v^*-u^*v^*}{u(m+u^*)(m+u)}\right) + \alpha(v-v^*)\left(\frac{v^*}{\gamma u^*} - \frac{v}{\gamma u}\right)\\
&= (u-u^*)\left(\frac{u^*-u}{u} + \frac{v^*(u-u^*)-(m+u^*)(v-v^*)}{u(m+u^*)(m+u)}\right) + \alpha(v-v^*)\left(\frac{v^*}{\gamma u^*} - \frac{v}{\gamma u}\right)\\
&= (u-u^*)^2\left(-\frac{1}{u} + \frac{v^*}{u(m+u)(m+u^*)}\right) - \frac{(u-u^*)(v-v^*)}{u(m+u)} + \alpha(v-v^*)\left(\frac{v^*}{\gamma u^*} - \frac{v}{\gamma u} + \frac{v^*}{\gamma u} - \frac{v^*}{\gamma u}\right).
\end{aligned}
$$

Using equation (4.19), we can see that $b = \frac{v^*}{\gamma u^*}$ and using equation (4.20), we can see that $\frac{v^*}{m+u^*} = a - u^*$. Substituting and rearranging gives:

$$
\begin{aligned}
\dot{V}(u,v) &= (u-u^*)^2\left(-\frac{1}{u} + \frac{a-u^*}{u(m+u)}\right) - \frac{(u-u^*)(v-v^*)}{u(m+u)} + \alpha(v-v^*)\left(b - \frac{bu^*}{u} - \frac{v-v^*}{\gamma u}\right)\\
&= (u-u^*)^2\left(-\frac{1}{u} + \frac{a-u^*}{u(m+u)}\right) - \frac{(u-u^*)(v-v^*)}{u(m+u)} + \alpha(v-v^*)\left(\frac{b(u-u^*)}{u} - \frac{v-v^*}{\gamma u}\right)\\
&= (u-u^*)^2\left(-\frac{1}{u} + \frac{a-u^*}{u(m+u)}\right) + (u-u^*)(v-v^*)\left(\frac{b\alpha}{u} - \frac{1}{u(m+u)}\right) - (v-v^*)^2\frac{\alpha}{\gamma u}
\end{aligned}
$$

Since $m > b\gamma$, as stated by Peng and Wang [18], we note that $2u^* - u > 0$ for $x \in \bar{\Omega}$ and $t \gg 1$. We will also suppose that there exists positive constants α and t_0 such that for all $x \in \bar{\Omega}$ and $t \geq t_0$; the solution $(u(x,t), v(x,t))$ of system (4.18) therefore satisfies:

$$
\dot{V}(u,v) = (u-u^*)^2\left(-\frac{1}{u} + \frac{a-u^*}{u(m+u)}\right) + (u-u^*)(v-v^*)\left(\frac{b\alpha}{u} - \frac{1}{u(m+u)}\right) - (v-v^*)^2\frac{\alpha}{\gamma u} \leq 0. \quad (4.24)
$$

Considering the potential Lyapunov functional for (4.18)

$$
W(t) = \int_\Omega V(u(x,t), v(x,t))dx,
$$

we can calculate $W'(t)$ as follows:

$$
\begin{aligned}
W'(t) &= \int_\Omega (V_u u_t + V_v v_t)\,dx + \int_\Omega \dot{V}(u,v)dx\\
&= \int_\Omega \left(\frac{u-u^*}{u^2} d_1 \Delta u + \alpha\frac{v-v^*}{v} d_2 \Delta v\right)dx + \int_\Omega \dot{V}(u,v)dx\\
&= \left[\int_{\partial\Omega}\left(d_1\frac{u-u^*}{u^2}\frac{\partial u}{\partial \eta} + \alpha d_2 \frac{v-v^*}{v}\frac{\partial v}{\partial \eta}\right)dx\right.\\
&\quad \left. - \int_\Omega \left(-d_1\left(-\frac{1}{u^2} + \frac{2u^*}{u^3}\right)\nabla u\cdot\nabla u + \alpha d_2\frac{v^*}{v}\nabla v\cdot\nabla v\right)dx\right] + \int_\Omega \dot{V}(u,v)dx.
\end{aligned}
$$

From the initial conditions $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$, given in system (4.18), $\nabla u \cdot \nabla u = |\nabla u|^2$ and $\nabla v \cdot \nabla v = |\nabla v|^2$, we can eliminate the first integral and rearrange to obtain:

$$
W'(t) = -\int_\Omega \left(d_1 \frac{2u^*-u}{u^3}|\nabla u|^2 + \alpha d_2 \frac{v^*}{v}|\nabla v|^2\right)dx + \int_\Omega \dot{V}(u,v)dx \leq 0 \quad (4.25)
$$

Therefore we can see that $(u(x,t), v(x,t)) \to (u^*, v^*)$ in $[L^\infty(\Omega)]^2$. Thus this shows that (u^*, v^*) attracts all solutions of the system (4.18) and (u^*, v^*) is globally asymptotically stable.

4.0.5 A New Diffusive $SIRS$ System

The final reaction-diffusion system we will investigatee is associated with the SIRS system considered in [13]. This system has been edited throughout the analysis of this system to use variables P and I which can be seen in system (3.2). In particular we consider the following reaction-diffusion system:

$$\begin{aligned} &\frac{\partial P}{\partial t} = d_1 \Delta P + \hat{\gamma} N - \beta \frac{PI}{N} - \hat{\sigma} P && \text{in } \Omega, \\ &\frac{\partial I}{\partial t} = d_2 \Delta I + \beta \frac{PI}{N} - \hat{\delta} I && \text{in } \Omega, \\ &\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 && \text{in } \partial\Omega, \\ &P(x,0) = P_0(x), \quad I(x,0) = I_0(x) && \text{in } \Omega, \end{aligned} \tag{4.26}$$

where we use the assumptions as defined in the example previous given by Hsu [5].

Note that we have calculated the endemic equilibrium point in (3.3) which can therefore be used in this analysis. We recall that the following is a Lyapunov functional for the corresponding ODE system

$$V(P,I) = P^* \left(\frac{P}{P^*} - \ln \frac{P}{P^*} \right) + I^* \left(\frac{I}{I^*} - \ln \frac{I}{I^*} \right), \tag{4.27}$$

since we have already proved $\dot{V} \leq 0$ in (3.8). Now we consider the potential Lyapunov functional for the reaction-diffusion system (4.26)

$$W(t) = \int_\Omega V(u(x,t), v(x,t)) dx,$$

and continue to calculate $W'(t)$ as follows:

$$\begin{aligned} W'(t) &= \int_\Omega (V_P P_t + V_I I_t)\, dx + \int_\Omega \dot{V}(u,v) dx \\ &= \int_\Omega \left(\left(1 - \frac{P^*}{P}\right) d_1 \Delta P + \left(1 - \frac{I^*}{I}\right) d_2 \Delta I \right) dx + \int_\Omega \dot{V}(u,v) dx. \end{aligned}$$

Using integration by parts, we have:

$$\begin{aligned} W'(t) = &\left[\int_{\partial\Omega} \left(d_1 \left(1 - \frac{P^*}{P}\right) \frac{\partial P}{\partial \nu} + d_2 \left(1 - \frac{I^*}{I}\right) \frac{\partial I}{\partial \nu} \right) dx \right. \\ &\left. - \int_\Omega \left(d_1 \frac{P^*}{P^2} \nabla P \cdot \nabla P + d_2 \frac{I^*}{I^2} \nabla I \cdot \nabla I \right) dx \right] + \int_\Omega \dot{V}(u,v) dx. \end{aligned}$$

From the initial conditions $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$, given in system (4.26), $\nabla u \cdot \nabla u = |\nabla u|^2$ and $\nabla v \cdot \nabla v = |\nabla v|^2$, we can eliminate the first integral and rearrange to obtain:

$$W'(t) = -\int_\Omega \left(d_1 \frac{P^*}{P^2} |\nabla P|^2 + d_2 \frac{I^*}{I^2} |\nabla I|^2 \right) dx + \int_\Omega \dot{V}(u,v) dx.$$

Therefore, using $\dot{V} \leq 0$ as previously stated, it can easily be seen that

$$W'(t) = -\int_\Omega \left(d_1 \frac{P^*}{P^2} |\nabla P|^2 + d_2 \frac{I^*}{I^2} |\nabla I|^2 \right) dx + \int_\Omega \dot{V}(u,v) dx \leq 0. \tag{4.28}$$

Thus we can conclude that the reaction and diffusion system (4.26) has a globally asymptotically stable endemic equilibrium point E^* defined in (3.3).

As far as we aware the above result is the first stability result in the literature associated to system (4.26).

Chapter 5

Conclusion and Further Work

Throughout this thesis we have investigated the use of Lyapunov functions to determine global behaviour of ODE and PDE systems for a variety of models. The models we have concentrated on in this work are ODE systems for population models, specifically predator-prey models; ODE systems for epidemiological models, in particular we have investigated $SIRS$, SIR, SIS and $SEIR$ models; and PDE systems for reaction-diffusion models, for predator-prey models and an SIR model. We will further evaluate each of these different models individually and the future work we can achieve through more detailed investigation.

Predator-Prey Models

We have analysed the global behaviour of population dynamics through examples of ODE systems of predator-prey models including a competing predator-prey model. We have been able to determine that each of these models have globally asymptotically stable equilibrium points by using Lyapunov functions previously constructed. Further investigation of this work would include the analysis of alternative interactions. As explained by Murray [15], there are a further two main types of interactions we could explore. The first type of interaction is competition, where the growth rate of each population is decreased and the second type is mutualism for which each population's growth rate is enhanced. Both scenarios would create a greatly differing models for which Lyapunov functions can be created to analyse their global behaviour.

Epidemiological Models

This thesis has used Lyapunov functions to investigate the global behaviour of numerous epidemiological models through examples of ODE systems of SIR, $SIRS$, SIS and $SEIR$ models. Lyapunov functions have been used to prove each of the examples used have a globally asymptotically stable endemic equilibrium points. Further analysis of this work would include investigating different models with convex functions as opposed to concave. Anther aspect of further investigation would include decreasing assumptions to create a better representation of the real world situation. An example of this would be to remove the assumption that birth rates are equal to death rates. An additional possibility of further research would include using data provided on past epidemics to analyse the suitability of different epidemiological models to determine whether the data either follows or contradicts our results of their global behaviour.

Reaction-Diffusion Models

We have used PDE systems of different types of reaction-diffusion models and applied our work on Lyapunov functionals to determine their global behaviour. We have studied diffusive predator-prey models given and determined they have globally asymptotically stable equilibrium points. We have then continued to create our own reaction-diffusion model by expanding on an $SIRS$ model and the corresponding Lyapunov function given. To our knowledge this is the first time this work has been done for this model and we have been able to conclude that this also has a globally asymptotically stable equilibrium point. Further analysis on reaction-diffusion models would include investigating the dynamics of an ODE system in comparison to the dynamics of the corresponding reaction-diffusion system when diffusion is added and determining if diffusion changes the systems global behaviour.

Bibliography

[1] J.G. Andrews and R.R. McLone, *Mathematical Modelling*, Butterworths, London, 1976.

[2] F. Brauer and J.A. Nohel, *The Qualitative Theory of Ordinary Differential Equations: An Introduction*, Dover Publications Inc., New York, 1969.

[3] C. Chiu, *Lyapunov Functions for the Global Stability of Competing Predators*, 1999.

[4] Y. Du and S.B. Hsu, *A Diffusive Predator-prey Model: In Heterogeneous Environment*, 2004.

[5] S. Hsu, *A Survey of Constructing Lyapunov Functions for Mathematical Models in Population Biology*, 2005.

[6] D.W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers*, Oxford University Press, Oxford, 2007.

[7] D. Kalman, *Elementary Mathematical Models: Order Aplenty and a Glimpse of Chaos*, The Mathematical Association of America, Washington, 1998.

[8] J. N. Kapur, *Mathematical Modelling*, New Age International, India, 1988.

[9] W.O. Kermack and A.G. McKendrick, *A Contribution to the Mathematical Theory of Epidemics*, 1927.

[10] A. Korobeinikov, *Lyapunov Functions and Global Stability for SIR and SIRS Epidemiological Models with Non-linear Transmission*, 2006.

[11] A. Korobeinikov and P.K. Maini, *A Lyapunov Function and Global Properties for SIR and SEIR Epidemiological Models with Nonlinear Incidence*, 2004.

[12] ______, *Non-linear Incidence and Stability of Infectious Disease Models*, 2005.

[13] A. Korobeinikov and G.C. Wake, *Lyapunov Functions and Global Stability for SIR, SIRS, and SIS Epidemiological Models*, 2002.

[14] R.I. Leine, *The Historical Development of Classical Stability Concepts: Lagrange, Poisson and Lyapunov Stability*, 2009.

[15] J.D. Murray, *Mathematical Biology*, 3rd ed., Springer-Verlag, Berlin, 2002.

[16] M. O'Regan, T.C. Kelly, A. Korobeinikov, M.J.A. O'Callaghan, and A.V. Pokrovskii, *Lyapunov Functions for SIR and SIRS Epidemic Models*, 2009.

[17] P.C. Parks, *A.M. Lyapunov's Stability Theory-100 Years On*, 1992.

[18] R. Peng and M. Wang, *Global Stability of the Equilibrium of a Diffusive Holling-Tanner Prey-Predator Model*, 2006.

[19] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, 2nd ed., 1983.

[20] J.C. Willems, *The Analysis of Feedback Systems*, The M.I.T. Press, London, 1971.