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AN ANALYTIC APPROACH TO THE NORMALIZED RICCI FLOW-LIKE EQUATION: REVISITED

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Abstract. In this paper we revisit Hamilton's normalized Ricci flow, which was thoroughly studied via a PDE approach in [10]. Here we provide an improved convergence result compared to the one presented [10] for the critical case $\lambda = 8\pi$. We actually prove that the convergence towards the stationary solution is realized through any time sequence.

1. Introduction-Preliminaries

In the current work we revisit the problem

$$\frac{\partial e^w}{\partial t} = \Delta w + \lambda \left(\frac{e^w}{\int_\Omega e^w \, dx} - \frac{1}{|\Omega|} \right), \qquad x \in \Omega, \quad t > 0, \tag{1.1}$$

$$w(x,0) = w_0(x), \qquad x \in \Omega, \tag{1.2}$$

which we first studied, using a PDE approach, in [10]. Here $\lambda > 0$ is a constant and Ω is a compact Riemannian surface without boundary. When $\lambda = 8\pi$ and Ω equals to the unit sphere $S^2 = \{x \in \mathbb{R}^2 : ||x|| < 1\}$ then (1.1) - (1.2) describes the evolution of the metric associated with the normalized Ricci, [9].

First note that integrating equation (1.1) over Ω, taking also into account that Ω is a compact manifold without boundary, we obtain the total mass conservation

$$\int_\Omega e^{w(x,t)} \, dx = \int_\Omega e^{w_0(x)} \, dx = \lambda, \quad \text{for} \quad t > 0. \tag{1.3}$$

Also if we consider the functional

$$\begin{aligned} J_\lambda(w) &= \frac{1}{2}||\nabla w||_2^2 - \lambda \left\{ \log \int_\Omega e^w \, dx - \frac{1}{|\Omega|} \int_\Omega w \, dx \right\} \\ &= \frac{1}{2} \left\| \nabla (w - \overline{w}) \right\|_2^2 - \lambda \log \int_\Omega e^{w - \overline{w}} dx, \end{aligned} \tag{1.4}$$

where $\overline{w} = \frac{1}{|\Omega|} \int_\Omega w dx$, then there holds that

$$\frac{d}{dt} J_\lambda(w(\cdot,t)) = \int_\Omega \nabla w \cdot \nabla w_t dx - \lambda \int_\Omega \left(\frac{e^w}{\int_\Omega e^w dx} - \frac{1}{|\Omega|} \right) w_t dx = -\int_\Omega e^w w_t^2 dx, \tag{1.5}$$

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and thus $J_\lambda(w)$ is a Lyapunov functional for problem (1.1) - (1.3).

The corresponding steady-state problem is, therefore,

$$-\Delta\phi = \lambda\left(\frac{e^{\phi}}{\int_\Omega e^{\phi}\,dx} - \frac{1}{|\Omega|}\right), \qquad \int_\Omega e^{\phi}\,dx = \int_\Omega e^{w_0}\,dx\ (=\lambda). \tag{1.6}$$

In [10] we proved that for the so called subcritical case, $0<\lambda<8\pi$, problem (1.1) - (1.3) has a global-in-time solution which actually converges to a unique steady state. More precisely the following result was obtained.

Theorem 1.1. *([10]) If $0<\lambda<8\pi$, the solution $w=w(x,t)$ to (1.1) - (1.3) satisfies the uniform estimates*

$$\sup_{t\geq 0}\left\{||e^{w(\cdot,t)}||_\infty + ||e^{-w(\cdot,t)}||_\infty\right\} < \infty,$$

and hence exists globally in time. Moreover

$$\omega(w_0) := \{\psi\in C^2(\Omega)\mid \text{there exists } t_k\uparrow\infty \text{ such that } ||w(\cdot,t_k;u_0)-\psi||_{C^2(\Omega)}\to 0\}$$

is a non-empty connected compact set in $C^2(\Omega)$ contained in the steady-state solution set

$$E_\lambda = \left\{\phi\in C^2(\Omega)\mid \phi \text{ satisfies (1.6) when parameter is equal to } \lambda\right\}.$$

The above result actually yields that for any $0<\lambda<8\pi$ the solution orbit stays in the center manifold, hence there exists $w_\infty\in E_\lambda$ such that

$$w(\cdot,t)\to w_\infty \quad \text{in} \quad C^2(\Omega) \quad \text{as} \quad t\to\infty. \tag{1.7}$$

On the other hand, in the "critical case" $\lambda=8\pi$ although we established the global-in-time existence of solutions, due the lack of some estimates from below for $\overline{w}$, we were unable to derive convergence towards steady-state through any time sequence. Indeed, in [10] was derived.

Theorem 1.2. *([10]) If $\lambda=8\,\pi$, then problem (1.1) - (1.3) still has a global-time solution. Furthermore, there exists $t_k\uparrow\infty$ and $w_\infty\in E_{8\pi}$ such that $w(\cdot,t_k)\to w_\infty$ in $C^2(\Omega)$ as $k\to\infty$.*

The purpose of this paper is to improve the convergence result provided by Theorem 1.2. In particular in the next section we are proving that the convergence in (1.7) is still true even for the case $\lambda=8\pi$, which then cannot be characterized as critical.

2. Main Result

The main result of the current work, improving Theorem 1.2, is.

Theorem 2.1. *If $\lambda=8\,\pi$, then there holds*

$$\sup_{t\geq 0}\left\{||e^{w(\cdot,t)}||_\infty + ||e^{-w(\cdot,t)}||_\infty\right\} < \infty, \tag{2.1}$$

and thus $\omega(w_0)\subset E_{8\pi}$.

Before proceeding to the proof of Theorem 2.1 we need to provide some auxiliary results. First of all we note that by Fontana-Moser-Trudinger's inequality, [8, 14], there holds

$$\begin{aligned} J_{8\pi}(w) &= \frac{1}{2}\|\nabla w\|_2^2 - 8\pi \log\left(\int_\Omega e^{w-\overline{w}}dx\right) \\ &= \frac{1}{2}\|\nabla w\|_2^2 + 8\pi\overline{w} - 8\pi\log(8\pi) \geq -C. \end{aligned} \tag{2.2}$$

Moreover, Jensen's inequality implies

$$exp\left(\frac{1}{|\Omega|}\int_\Omega w dx\right) \leq \frac{1}{|\Omega|}\int_\Omega e^w dx$$

and thus

$$\overline{w}(t) \leq \log\left(\frac{8\pi}{|\Omega|}\right), \tag{2.3}$$

using also (1.3).

The following lemma provides a fundamental lower estimate of $\overline{w}$ was not obtained in [10].

Lemma 2.2. *If $\lambda = 8\pi$, then there holds*

$$\liminf_{t\uparrow\infty} \overline{w}(t) > -\infty. \tag{2.4}$$

Proof. Assume that $\liminf_{t\uparrow\infty} \overline{w}(t) = -\infty$, then there exist $t_k \uparrow \infty,\ \delta > 0$ with $t_{k+1} > t_k + \delta$ such that $\lim_{k\to\infty} \overline{w}(t_k) = -\infty$ and thus by virtue of the Benilan-Crandall's type estimate $w_t \leq \dfrac{e^t}{e^t-1}$, see Lemma 3 in [10], we obtain

$$8\pi\overline{w}(t) \leq -k - C \quad \text{for any} \quad t \in (t_k-\delta, t_k). \tag{2.5}$$

The entropy defined by

$$\mathcal{H}(t) := \int_\Omega e^w w\, dx$$

satisfies the obvious estimate

$$\mathcal{H}(t) \geq -e|\Omega|. \tag{2.6}$$

On the other hand, by following similar calculations as in Lemma 11 in [10], we deduce

$$\begin{aligned} \frac{d\mathcal{H}}{dt} &= \mathcal{H}(t) - \|\nabla w\|_2^2 - 8\pi\overline{w}(t) \\ &= \mathcal{H}(t) + 8\pi\overline{w}(t) + O(1) \end{aligned}$$

taking also into account (2.2).

The latter combined with (2.5) reads

$$\frac{d\mathcal{H}}{dt} \leq \mathcal{H}(t) - k$$

or

$$\frac{d\left(e^{-t}\mathcal{H}\right)}{dt} \leq -ke^{-t} \quad \text{for any} \quad t \in (t_k - \delta, t_k),$$

and integrating over $(t_k - \delta, t_k - \delta/2)$ we deduce by virtue of (2.6)

$$\mathcal{H}(t) \geq e^{t-t_k}\mathcal{H}(t_k) + k\left(1 - e^{t-t_k}\right) \geq -e^{1+\delta}|\Omega| + k\left(1 - e^{-\delta/2}\right),$$

which actually yields

$$\lim_{k\to\infty} \inf_{t\in(t_k-\delta, t_k-\delta/2)} \mathcal{H}(t) = +\infty. \tag{2.7}$$

Due to (1.5) and (2.2) we also have

$$\sum_{k=1}^{\infty} \int_{t_k-\delta}^{t_k-\delta/2} dt \int_{\Omega} e^w w_t^2 \, dx \leq \int_0^{\infty} dt \int_{\Omega} e^w w_t^2 \, dx < \infty,$$

thus

$$\lim_{k\to\infty} \int_{t_k-\delta}^{t_k-\delta/2} dt \int_{\Omega} e^w w_t^2 \, dx = 0. \tag{2.8}$$

By virtue of (2.7) and (2.8) we can extract a sequence $t_k' \in (t_k - \delta, t_k - \delta/2)$ such that

$$\lim_{k\to\infty} \mathcal{H}(t_k') = +\infty \tag{2.9}$$

and

$$\lim_{k\to\infty} \int_{\Omega} e^{w(\cdot,t_k')} w_t^2(\cdot, t_k') \, dx = 0,$$

whereby the latter relation we deduce

$$\left\| \frac{\partial e^w}{\partial t}(\cdot, t_k') \right\|_1 \leq \left\| e^{w(\cdot,t_k')} \right\|_1 \left\| e^{w(\cdot,t_k')} w_t^2(\cdot, t_k') \right\|_1 \to 0 \quad \text{as} \quad k \to \infty. \tag{2.10}$$

Applying now the concentration result of Lemma 10 in [10] we derive

$$w(\cdot, t_k') \to G(\cdot, x_\infty) \qquad \text{in } W^{1,q}(\Omega), \quad 1 \leq q < 2, \tag{2.11}$$

for some $x_\infty \in \Omega$, using L^1-estimate in [3] as well as (2.9) and Brezis-Merle's estimate [2]. Here $G = G(x, x')$ denotes the Green's function of $(-\Delta_{JL})^{-1}$ where the operator $(-\Delta_{JL})^{-1}$ is defined by

$$(-\Delta_{JL})^{-1}u = v \tag{2.12}$$

if and only if

$$-\Delta v = u - \frac{1}{|\Omega|}\int_{\Omega} u \, dx, \quad x \in \Omega, \qquad \int_{\Omega} v \, dx = 0.$$

But (2.11) via Fatou's Lemma implies

$$\int_{\Omega} e^{w(x,t_k')} dx \to \infty \quad \text{as} \quad k \to \infty$$

contradicting to the fact that $\int_{\Omega} e^w \, dx = 8\pi$. □

Proof of Theorem 2.1 The uniform estimate (2.1) is obtained by employing Moser's iteration scheme as in Theorem in [10] and taking also into account estimate (2.4). Then the inclusion $\omega(w_0) \subset E_{8\pi}$ is obtained due to the compactness of the orbit, guaranteed by (2.1) and parabolic regularity, as well as due to classical dynamical systems theory taking also into account dissipation relation (1.5). $\square$

Remark 2.3. *Theorem 2.1 verifies the convergence of the normalized Ricci flow towards its steady state, see* [1, 9]*, since when* $\Omega = S^2$ *then* $E_{8\pi} = \{w_\infty = \log(8\pi/|S^2|)\}$ *by the results in* [6, 7, 12]*. Furthermore, in the flat torus case, i.e when* $\Omega = \mathbb{T}^2 = \mathbb{R}^2/a\mathbb{Z} \times b\mathbb{Z}$ *with* $\frac{b}{a} > \frac{\pi}{4}$ *then again* $E_{8\pi} = \{w_\infty = \log(8\pi/|\mathbb{T}^2|)\}$, [13]*, and thus* $w(\cdot, t) \to \log(8\pi/|\mathbb{T}^2|)$ *as* $t \to \infty$ *in* $C^2(\mathbb{T}^2)$.

Remark 2.4. *Similar results to Theorems 1.1 and 2.1 hold if* Ω *is a bounded domain in* $\mathbf{R}^2$ *with smooth boundary, and the Neumann boundary condition is provided with (1.1)-(1.2). That is, the global-in-time solution arises with compact orbit, as far as* $0 < \lambda \leq 4\pi$*. Actually, we have only to use Chang-Yang's inequality,* [4, 5]*, instead of Fontana's inequality for the proof.*

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