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The elegance of differential forms in vector calculus and electromagnetics

by

Christian Parkinson B.Sc(Hons) AMIMA

Submitted to the School of Computer of Science and Mathematics,
Faculty of Science and Engineering
in partial fulfilment of the requirements for the degree of
Master of Science in Mathematics
at the
UNIVERSITY OF CHESTER
September 2014

Certified by Dr. Graham Roberts, Senior Lecturer, Dissertation Supervisor

Declaration of Authorship

I, Christian Parkinson, declare that this thesis titled, 'The elegance of differential forms in vector calculus and electromagnetics' and the work presented in it are my own. I confirm that:

- This work was undertaken completely while in candidature for a Masters of Science degree at this University.
- Where I have consulted the published work of others, this is always clearly attributed.
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Signed: Christian Parkinson B.Sc(Hons) AMIMA

Date: September 29, 2014

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Abstract

In the chapter one of this text we give an introduction to, and discuss the main integral theorems, of vector calculus; Green's theorem, Stokes' theorem and Gauss' Divergence theorem. Note that the main resource used for this chapter is [8]. Chapter two introduces differential forms and exterior calculus; in it we discuss exterior multiplication and exterior differentiation giving proofs for properties of both. We discuss the integration of differential forms in chapter three and provide definitions of the Divergence, Gradient and Curl and main integral theorems of vector calculus including the Generalised Stokes' theorem that encloses them all in terms of such forms. Further we give a proof of the Generalised Stokes', Green's, Stokes' and Gauss' Divergence theorems. Given the elegance of differential forms that enables us to write the integral theorems of vector calculus as one theorem, the Generalised Stokes' theorem, we show a second elegance by deducing and proving Maxwell's equations, whilst reducing them from four equations to just two. Finally we provide some current research involving differential forms.

Please note that where some proofs may be found in other texts, [6, 26, 29, 31], this paper provides extra steps and details.

Dissertation Supervisor: Dr. Graham Roberts
Title: Senior Lecturer in Mathematics

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"For the things of this world cannot be made known without a knowledge of mathematics." — Roger Bacon [5]

Contents

Chapter 1

Calculus in vectors

1.1 Div, grad and curl

1.1.1 The gradient and directional derivative

In multivariable calculus we define the operator $\nabla^{(\mathrm{i})}$, to be the n-dimensional differential operator in vector form [1, 13],

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\hat{\mathbf{e}}_j.$$

Using this we can define the *gradient* of a scalar field $\phi(x_1, x_2, \ldots, x_n)$ with the following notation, (1.1).

$$\nabla\phi = \operatorname{grad}\phi = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \ldots, \frac{\partial \phi}{\partial x_n}\right) = \sum_{j=1}^{n} \frac{\partial \phi}{\partial x_j}\hat{\mathbf{e}}_j \tag{1.1}$$

The above notation suggests that we view the gradient of ϕ as the result of multiplying the vector ∇ by the scalar field $\phi^{(\mathrm{ii})}$. The application of the gradient, $\nabla\phi$ returns a vector whose direction at a given point, points in the direction that the scalar field ϕ is increasing most rapidly. The magnitude of this vector is the rate of change of the scalar field ϕ at that point. In fact, the direction of $\nabla\phi$ is the orientation in which the directional derivative has the largest value and $|\nabla\phi|$ is the value of that directional derivative, see the succeeding paragraph. The following conditions are commonplace with use of the gradient, ∇.

(i) $\forall\, j,\ 1 \leq j \leq n,\ \dfrac{\partial \phi}{\partial x_j} \equiv 0$ if ϕ is independent of x_j,

(ii) $\nabla\phi \equiv 0$ if ϕ is constant.

The *directional derivative* is a vector form of the usual derivative used to calculate the rate of change of a differentiable scalar field $\phi(x_1, x_2, \ldots, x_n)$ of n variables in the direction of

(i) ∇ can be pronounced as *Del* or *Nabla*.

(ii) Note that the gradient is not commutative, $\nabla\phi \neq \phi\nabla$.

$\mathbf{u} = \big(u_1, u_2, \dots, u_n\big) \in \mathbb{R}^n$ with definition (1.2) at a given point [22].

$$\nabla_{\mathbf{u}}\phi = \nabla\phi \cdot \hat{\mathbf{u}} = \frac{1}{\sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}} \sum_{j=1}^{n} u_j \frac{\partial \phi}{\partial x_j} \tag{1.2}$$

The directional derivative satisfies the following properties where $\lambda \in \mathbb{R}$, $\boldsymbol{\xi} \in \mathbb{R}^n$ and ϕ and ψ are n-variable scalar fields.

$$\nabla_{\mathbf{u}}\big(\lambda\phi\big) = \lambda\nabla_{\mathbf{u}}\phi \tag{1.3a}$$

$$\nabla_{\mathbf{u}}\big(\phi + \psi\big) = \nabla_{\mathbf{u}}\phi + \nabla_{\mathbf{u}}\psi \tag{1.3b}$$

$$\nabla_{\mathbf{u}}\big(\phi\psi\big) = \phi\nabla_{\mathbf{u}}\psi + \psi\nabla_{\mathbf{u}}\phi \tag{1.3c}$$

$$\nabla_{\mathbf{u}}\big(\phi \circ \psi\big)(\boldsymbol{\xi}) = \phi'\big(\psi(\boldsymbol{\xi})\big)\nabla_{\mathbf{u}}\psi(\boldsymbol{\xi}) \tag{1.3d}$$

1.1.2 Divergence and the Laplacian

Where the gradient is seen as the multiplication of the vector ∇ by the scalar field ϕ, the *divergence* is seen as the dot-product between the vector ∇ and the vector field $\vec{\Phi}$. If $\phi_1, \phi_2, \dots, \phi_n$ are the component (scalar) fields of $\vec{\Phi}$ the divergence of ϕ is defined to be,

$$\nabla \cdot \vec{\Phi} = \operatorname{div}\vec{\Phi} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right) \cdot \big(\phi_1, \phi_2, \dots, \phi_n\big) = \sum_{j=1}^{n} \frac{\partial \phi_j}{\partial x_j}, \tag{1.4}$$

which is in turn, itself a scalar field. The divergence defines the outward flow of the vector field at every point in space and with this, we have the following two definitions.

(i) If the divergence of the vector field we are calculating is greater than zero then it is called a source. I.e. If $\nabla \cdot \vec{\Phi} > 0$ we have a source,

(ii) If the divergence of the vector field we are calculating is less than zero then it is called a sink. I.e. If $\nabla \cdot \vec{\Phi} > 0$ we have a sink.

If ϕ is a twice differentiable scalar field, then its *Laplacian* is defined to be,

$$\nabla^2\phi = \nabla \cdot (\nabla\phi) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right) \cdot \left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \dots, \frac{\partial\phi}{\partial x_n}\right) = \sum_{j=1}^{n} \frac{\partial^2 \phi}{\partial x_j^2} = (\nabla \cdot \nabla)\phi \tag{1.5}$$

and the scalar field is called *harmonic* if the following condition, (1.6) is satisfied.

$$\nabla^2\phi = 0 \tag{1.6}$$

If we apply the dot-product of the gradient $m \in \mathbb{N}$ times then we yield the following, (1.7).

$$\nabla^m = \underbrace{\nabla \cdot \nabla \cdot \cdots \cdot \nabla}_{m \text{ times}} = \sum_{j=1}^{n} \frac{\partial^m}{\partial x_j^m} \tag{1.7}$$

1.1.3 Curl

We introduce the *curl* of a vector field $\vec{\Phi} = P\big(x_1, x_2, x_3\big)\hat{\mathbf{i}} + Q\big(x_1, x_2, x_3\big)\hat{\mathbf{j}} + R\big(x_1, x_2, x_3\big)\hat{\mathbf{k}} \in \mathbb{R}^3$ by simply giving its definition, (1.8).

$$\nabla \times \vec{\Phi} = \operatorname{curl} \vec{\Phi} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial x_2} - \frac{\partial Q}{\partial x_3}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial x_3} - \frac{\partial R}{\partial x_1}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2}\right)\hat{\mathbf{k}} \quad (1.8)$$

The curl of a vector field is the circulation per unit area, or rate of rotation; that is, the amount of twisting at a single point. The best way to imagine a curl is with a whirlpool in water. If the whirlpool shrinks to a small size while keeping the force F constant, there will be a lot of force in a small area and hence the curl will be large. However a larger whirlpool, with same force F will have a curl much smaller. Consequently, no whirlpool will result in zero circulation and a curl equal to zero. We observe the following three facts about the curl.

(i) If ϕ is a twice differentiable scalar field then the following condition is satisfied.

$$\operatorname{curl}(\nabla\phi) = \nabla \times (\nabla\Phi) = \vec{0}$$

(ii) A vector field $\vec{\Phi}$ is called conservative if the following condition is satisfied.

$$\operatorname{curl} \vec{\Phi} = \nabla \times \vec{\Phi} = \vec{0}$$

(iii) If $\vec{\Phi} \in \mathbb{R}^3$, it's components have continuous first order partial derivatives and $\nabla \times \vec{\Phi} = \vec{0}$ then $\vec{\Phi}$ is a conservative vector field.

(iv) The divergence of the curl of a vector field $\vec{\Phi}$ is always zero.

$$\nabla \cdot \big(\nabla \times \vec{\Phi}\big) = 0$$

1.2 Line integrals

1.2.1 Line integrals of scalar fields

In calculus, we become accustom to integrating an n-variable scalar field ϕ over a subset of n-dimensional space. Suppose then that instead of integrating over this subset of space, we choose to integrate over some curve γ. In doing so we define a *line integral*. Defining a function on a curve γ allows us to break the curve up into infinitesimally small line segments, multiply the length of these segments by the function value on the segment and sum the products. As the length of these line segments approaches zero, so the summation of the products approaches the value of the line integral. More generally we define the line integral of an n-dimensional scalar field over the curve γ as (1.9), where $\phi(\boldsymbol{\xi}_i)$ represents the value of the scalar field at an n-dimensional point $\boldsymbol{\xi}_i$ and Δs_i is the length of the i^{th} line segment of γ.

$$\int_\gamma \phi.\mathrm{d}s = \lim_{n\to\infty}\sum_{i=1}^{n}\phi(\boldsymbol{\xi}_i)\Delta s_i = \lim_{\Delta s_i\to 0}\sum_i \phi(\boldsymbol{\xi}_i)\Delta s_i \tag{1.9}$$

Where the above definition is not the most efficient for finding exact line integrals, parameterisation becomes somewhat providential. Denoting the parameterisation of a curve γ as a differentiable n-dimensional vector-valued function, $\vec{\mathbf{r}}(t) = \sum_{i=1}^{n} x_i(t)\hat{\mathbf{e}}_i \;\forall\, \alpha \leq t \leq \beta$ we have the following property (1.10) and definition (1.11).

$$\mathrm{d}s = \left\|\frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t}\right\|.\mathrm{d}t = \sqrt{\sum_{i=1}^{n}\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)^2}.\mathrm{d}t \tag{1.10}$$

$$\int_\gamma \phi(x_1, x_2, \dots, x_n).\mathrm{d}s = \int_\alpha^\beta \phi(x_1(t), x_2(t), \dots, x_n(t))\sqrt{\sum_{i=1}^{n}\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)^2}.\mathrm{d}t \tag{1.11}$$

We use ds to represent the change along the curve, γ in (1.9), (1.10) and (1.11) opposed to the change in one of the n-variables of the scalar field ϕ. Where the notation s in ds is seen in a line integral, we say that the line integral of ϕ is computed with respect to *arc length*. If we define $-\gamma$ to be the curve with the same points as γ but in the opposite direction, such that the starting and end points of γ become the end and starting points of $-\gamma$ respectively, then we have the following property, (1.12).

$$\int_\gamma \phi.\mathrm{d}s = \int_{-\gamma} \phi.\mathrm{d}s \tag{1.12}$$

Piecewise smooth curves

A *piecewise smooth curve* γ is any curve written as the union of a finite number of m smooth curves, $\gamma_1, \gamma_2, \dots, \gamma_m$, where the endpoint of the curve γ_i is the starting point of γ_{i+1}.

$$\gamma = \bigcup_{i=1}^{m}\gamma_i$$

The line integral over a piecewise smooth curve γ can be expressed as the summation of the line integral over each piece of the curve separately, (1.13).

$$\int_\gamma \phi.\mathrm{d}s = \sum_{i=1}^{m}\int_{\gamma_i}\phi.\mathrm{d}s \tag{1.13}$$

Suppose then that we wish to find the line integral of $\phi\big(x_1, x_2, \ldots, x_n\big)$ with respect to one of its dependent variables x_i, then (1.10) and (1.11) become (1.14) and (1.15) respectively

$$\mathrm{d}x_i = \frac{\mathrm{d}x_i}{\mathrm{d}t}.\mathrm{d}t \tag{1.14}$$

$$\int_\gamma \phi\big(x_1, x_2, \ldots, x_n\big).\mathrm{d}x_i = \int_\alpha^\beta \phi\big(x_1(t), x_2(t), \ldots, x_n(t)\big)\frac{\mathrm{d}x_i}{\mathrm{d}t}.\mathrm{d}t \tag{1.15}$$

with the above known as a line integral of ϕ with respect to a dependent variable t.

The following short hand notation is commonly used for the summation of the line integrals of n, n–variable functions $\phi_1, \phi_2, \ldots, \phi_n$ with respect to n dependent variables, $x_i\ \forall\, x = 1, 2, \ldots, n$.

$$\int_\gamma \sum_{i=1}^{n} \phi_i.\mathrm{d}x_i = \sum_{i=1}^{n} \int_\gamma \phi_i.\mathrm{d}x_i \tag{1.16}$$

By (1.12) we have that changing the direction of the curve for a line integral with respect to arc length does not change the resulting value but for line integral with respect to a dependent variable the following properties are satisfied for a scalar field ϕ.

$$\int_{-\gamma} \phi.\mathrm{d}x_i = -\int_\gamma \phi.\mathrm{d}x_i \qquad \forall\, x_i = 1, 2, \ldots, n \tag{1.17}$$

$$\int_{-\gamma} \sum_{i=1}^{n} \phi_i.\mathrm{d}x_i = -\sum_{i=1}^{n} \int_\gamma \phi_i.\mathrm{d}x_i \tag{1.18}$$

1.2.2 Line integrals of vector fields

In the previous section we introduced the line integral of a scalar field over a curve γ with respect to arc length and dependent variables. We now advance further into evaluating a line integral of a vector field over a given smooth curve. If we introduce the n–dimensional vector field,

$$\vec{\Phi}\big(x_1, x_2, \ldots, x_n\big) = \sum_{i=1}^{n} \phi_i\big(x_1, x_2, \ldots, x_n\big)\hat{\mathbf{e}}_i$$

and the curve γ parameterised by the the vector-valued function,

$$\gamma : \vec{\mathbf{r}}(t) = \sum_{i=1}^{n} x_i(t)\hat{\mathbf{e}}_i \qquad \forall\, \alpha \le t \le \beta$$

then the line integral of the vector field $\vec{\Phi}$ along the curve γ is given by the following definition, (1.19).

$$\int_\gamma \vec{\Phi} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_\alpha^\beta \vec{\Phi}\big(\vec{\mathbf{r}}(t)\big) \cdot \frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t}.\mathrm{d}t \tag{1.19}$$

It is possible to write the line integral of a vector field as a line integral with respect to arc length,

$$\int_\gamma \vec{\Phi}\cdot \mathrm{d}\vec{\mathbf{r}} = \int_\gamma \vec{\Phi}\cdot \vec{T}.\mathrm{d}s$$

where $\vec{T}$ the unit *tangent vector* given by,

$$\vec{T} = \left\|\frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t}\right\|^{-1}\frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t}.$$

Finally, if we evaluate the right-hand-side of (1.19) we are able to rewrite the line integral of a vector field with respect to its dependent variables as

$$\int_\alpha^\beta \sum_{i=1}^n \phi_i\big(x_1(t),x_2(t),\ldots,x_n(t)\big)\frac{\mathrm{d}x_i}{\mathrm{d}t}.\mathrm{d}t = \sum_{i=1}^n \int_\gamma \phi_i\big(x_1(t),x_2(t),\ldots,x_n(t)\big).\mathrm{d}x_i \quad (1.20)$$

which by (1.18) shows that the line integral of a vector field over a curve γ satisfies the following property, (1.21).

$$\int_{-\gamma}\vec{\Phi}\cdot \mathrm{d}\vec{\mathbf{r}} = -\int_\gamma \vec{\Phi}\cdot\mathrm{d}\vec{\mathbf{r}} \quad (1.21)$$

Fundamental theorem of line integrals

Given a smooth curve γ parameterised by $\vec{\mathbf{r}}(t), \alpha \le t \le \beta$ and scalar field ϕ whose gradient vector $\nabla\phi$ is continuous on γ then the following condition is satisfied.

$$\int_\gamma \nabla\phi\cdot\mathrm{d}\vec{\mathbf{r}} = \phi\big(\vec{\mathbf{r}}(\beta)\big) - \phi\big(\vec{\mathbf{r}}(\alpha)\big)$$

Properties of line integrals

(i) By the fundamental theorem of line integrals $\int_\gamma \nabla\phi\cdot\mathrm{d}\vec{\mathbf{r}}$ is independent of any curve.

(ii) If $\vec{\Phi}$ is a conservative vector field (see section 2.4.3) then $\int_\gamma \vec{\Phi}\cdot\mathrm{d}\vec{\mathbf{r}}$ is independent of any curve.

(iii) If $\int_\gamma \vec{\Phi}\cdot\mathrm{d}\vec{\mathbf{r}}$ is independent of any curve then $\int_\gamma \vec{\Phi}\cdot\mathrm{d}\vec{\mathbf{r}} = 0$ for every closed[(iii)] curve γ.

1.2.3 Green's theorem

Green's theorem relates the line integral of a vector field around a closed curve $\gamma \in R^2$ to a double integral over the region D enclosed by γ [8]. Let γ be a positively oriented, piecewise smooth, simple, closed curve parameterised by $\vec{\mathbf{r}}(t)$ and let D be the region enclosed by γ. Then with the differentiable vector field $\vec{\Phi}\big(x_1,x_2\big) = M\big(x_1,x_2\big)\hat{\mathbf{i}} + N\big(x_1,x_2\big)\hat{\mathbf{j}}$ this elegant connection between the one and two-dimensional integrals can be defined as, (1.22)[(iv)] and

(iii) A curve is called *closed* iff its starting and end points are the same point.

(iv) The symbol $\oint_\gamma$ indicates the line integral over a closed curve as a vector reformulation as (1.23).

$$\oint_{\gamma} M.\mathrm{d}x_1 + N.\mathrm{d}x_2 = \iint_D \left(\frac{\partial N}{\partial x_1} - \frac{\partial M}{\partial x_2}\right).\mathrm{d}x_2\mathrm{d}x_1 \tag{1.22}$$

$$\oint_{\gamma} \vec{\Phi} \cdot \mathrm{d}\vec{\mathbf{r}} = \iint_D \left(\nabla \times \vec{\Phi}\right) \cdot \hat{\mathbf{k}}.\mathrm{d}A \tag{1.23}$$

1.3 Surface integrals

1.3.1 Surface integrals of scalar fields

Where in the previous section we integrated over a line or curve, γ, here we define a *surface integral*, that is the integral of a scalar field ϕ along a surface S in three–dimensional space. Defining a function on the surface, we are able to break up the surface into infinitesimally small area segments, multiply the area of these segments by the function value on the segment and sum the products. As the area of these segments approaches zero, so the summation of the products approaches the value of the surface integral and hence we have the following integral representation (1.24), where $\phi(\boldsymbol{\xi}_i, \boldsymbol{\eta}_j)$ represents the value of the scalar field at the point $(\boldsymbol{\xi}_i, \boldsymbol{\eta}_j)$ and ΔA_{ij} is the ij^{th} area segment of S.

$$\iint_D \phi.\mathrm{d}S = \lim_{\substack{n\to\infty \\ m\to\infty}} \sum_{i=1}^{n}\sum_{j=1}^{m} \phi(\boldsymbol{\xi}_i, \boldsymbol{\eta}_j)\Delta A_{ij} = \lim_{\Delta A_{ij}\to 0} \sum_{i,j} \phi(\boldsymbol{\xi}_i, \boldsymbol{\eta}_j)\Delta A_{ij} \tag{1.24}$$

As with the similar representation of the line integral (1.9), the above is not the most efficient construction for finding exact line integrals and again, parameterisation becomes somewhat providential. Denoting the parameterisation of the surface S as a differentiable multivariable vector-valued function $\vec{\mathbf{r}}(s,t) = \sum_{i=1}^{3} x_i(s,t)\hat{\mathbf{e}}_i$, $\forall\, \alpha \leq t \leq \beta, \gamma \leq s \leq \delta$, we define the surface integral of a scalar field $\phi(x_1, x_2, x_3)$ whose domain includes S as the following (1.25) where D represents the region in which the parameters trace the surface S[v],

$$\iint_S \phi(x_1, x_2, x_3).\mathrm{d}S = \iint_D \phi(\vec{\mathbf{r}}(s,t)) \left\| \frac{\partial \vec{\mathbf{r}}}{\partial s} \times \frac{\partial \vec{\mathbf{r}}}{\partial t} \right\| .\mathrm{d}A \tag{1.25}$$

or equivalently (1.26).

$$\iint_S \phi(x_1, x_2, x_3).\mathrm{d}S = \int_{\gamma}^{\delta}\int_{\alpha}^{\beta} \phi(\vec{\mathbf{r}}(s,t)) \left\| \frac{\partial \vec{\mathbf{r}}}{\partial s} \times \frac{\partial \vec{\mathbf{r}}}{\partial t} \right\| .\mathrm{d}t\mathrm{d}s \tag{1.26}$$

Alternatively, where a surface S is represented by the function $\psi(x_1, x_2)$ the surface integral of ϕ along S can be written as (1.27) where again, D represents the region in which

[v] It is possible to evaluate the surface area of S using the surface integral $\iint_S \mathrm{d}S$.

the dependent variables x_1 and x_2 trace the surface.

$$\iint_S \phi\big(x_1, x_2, x_3\big).\mathrm{d}S = \iint_D \phi\Big(x_1, x_2, \psi\big(x_1, x_2\big)\Big)\sqrt{\left(\frac{\partial\psi}{\partial x_1}\right)^2 + \left(\frac{\partial\psi}{\partial x_2}\right)^2 + 1}.\mathrm{d}A \tag{1.27}$$

Where in (1.27), the surface is represented by $\psi\big(x_1, x_2\big)$ with D in the x_1x_2 plane, we have similar representations for surfaces given by $\psi\big(x_1, x_3\big)$ with D in the x_1x_3 plane and $\psi\big(x_2, x_3\big)$ with D in the x_2x_3 plane given as (1.28) and (1.29) respectively.

$$\iint_S \phi\big(x_1, x_2, x_3\big).\mathrm{d}S = \iint_D \phi\Big(x_1, \psi\big(x_1, x_3\big), x_3\Big)\sqrt{\left(\frac{\partial\psi}{\partial x_1}\right)^2 + 1 + \left(\frac{\partial\psi}{\partial x_3}\right)^2}.\mathrm{d}A \tag{1.28}$$

$$\iint_S \phi\big(x_1, x_2, x_3\big).\mathrm{d}S = \iint_D \phi\Big(\psi\big(x_2, x_3\big), x_2, x_3\Big)\sqrt{1 + \left(\frac{\partial\psi}{\partial x_2}\right)^2 + \left(\frac{\partial\psi}{\partial x_3}\right)^2}.\mathrm{d}A \tag{1.29}$$

Piecewise smooth surfaces

As with a piecewise smooth curve, a *piecewise smooth surface* S is any surface written as the union of a finite number of m smooth surfaces, $S_1, S_2, \ldots, S_m$ where each S_i ($i = 1, 2, \ldots, m$) is a region in $\mathbb{R}^2$ consisting of a connected open set, possibly together with some of its boundary points [9].

$$S = \bigcup_{i=1}^{m} S_i$$

Then $S : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ is continuous and has a one-to-one mapping on D, except possibly along $\partial D^{(\mathrm{vi})}$. These conditions on the region D and the surface S ensure that D is a two-dimensional subset of $\mathbb{R}^2$ with a two-dimensional image [8]. The surface integral along a piecewise smooth surface S can be expressed as the summation of the surface integral along each piece of the curve separately, (1.30).

$$\iint_S \phi.\mathrm{d}S = \sum_{i=1}^{m} \iint_{S_i} \phi.\mathrm{d}S \tag{1.30}$$

1.3.2 Flux integrals

A *flux integral*, is a surface integral of a vector field. The flux integral draws its name as it represents the flow, or flux, through a surface. In terms of a fluid, the flux integral represents the amount of fluid flowing through a given surface per unit time.

Oriented surfaces

As with line integrals, where changing the direction of the curve negated the answer, so it does for changing the *orientation* of a surface in surface integrals. Even though it is

(vi) The notation ∂D means he boundary of D for any dimension.

possible for a surface to have a single side, consider the möbius strip [30], we do not consider this case and begin by discussing a surface with two sides. Such a surface will have two sets of unit normal vectors[(vii)]. Each point on the surface will have two respective unit normal vectors, $\vec{n}_1$ and $\vec{n}_2 = -\vec{n}_1$ both in independent sets with the set of vectors chosen providing the surface with an orientation. A surface S is closed if it is the boundary of some solid region V. For such a surface we describe the orientation to be positive or negative if we choose the set of unit normal vectors that point outward from or inward towards the region V respectively.

Suppose that the surface S is defined by the function $x_3 = \psi(x_1, x_2)$. Defining a new function, $\chi(x_1, x_2, x_3) = x_3 - \psi(x_1, x_2)$, the surface can be then given by $\chi(x_1, x_2.x_3) = 0$ with unit normal vector in the positive orientation defined by (1.31).

$$\vec{n} = \frac{\nabla\chi}{\|\nabla\chi\|} = \frac{-\dfrac{\partial\chi}{\partial x_1}\hat{\mathbf{i}} - \dfrac{\partial\chi}{\partial x_2}\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{\left(\dfrac{\partial\chi}{\partial x_1}\right)^2 + \left(\dfrac{\partial\chi}{\partial x_2}\right)^2 + 1}} \tag{1.31}$$

If the surface S is given by $x_1 = \psi(x_2, x_3)$ or $x_2 = \psi(x_1, x_3)$ then we define the new function by $\chi(x_1, x_2, x_3) = x_1 - \psi(x_2, x_3)$ or $\chi(x_1, x_2, x_3) = x_2 - \psi(x_1, x_3)$ respectively and yield the respective unit normal vectors (1.32) and (1.33).

$$\vec{n} = \frac{\nabla\chi}{\|\nabla\chi\|} = \frac{\hat{\mathbf{i}} - \dfrac{\partial\chi}{\partial x_2}\hat{\mathbf{j}} - \dfrac{\partial\chi}{\partial x_3}\hat{\mathbf{k}}}{\sqrt{1 + \left(\dfrac{\partial\chi}{\partial x_2}\right)^2 + \left(\dfrac{\partial\chi}{\partial x_3}\right)^2}} \tag{1.32}$$

$$\vec{n} = \frac{\nabla\chi}{\|\nabla\chi\|} = \frac{-\dfrac{\partial\chi}{\partial x_1}\hat{\mathbf{i}} + \hat{\mathbf{j}} - \dfrac{\partial\chi}{\partial x_3}\hat{\mathbf{k}}}{\sqrt{\left(\dfrac{\partial\chi}{\partial x_1}\right)^2 + 1 + \left(\dfrac{\partial\chi}{\partial x_3}\right)^2}} \tag{1.33}$$

Now suppose that the surface is parameterised by the differential multivariable vector-valued function $\vec{\mathbf{r}}(s,t) = \sum_{i=1}^{3} x_i(s,t)\hat{\mathbf{e}}_i$, $\forall\, \alpha \leq t \leq \beta, \gamma \leq s \leq \delta$ with unit normal vector defined by (1.34).

$$\vec{n} = \left\| \frac{\partial\vec{\mathbf{r}}}{\partial s} \times \frac{\partial\vec{\mathbf{r}}}{\partial t} \right\|^{-1} \left(\frac{\partial\vec{\mathbf{r}}}{\partial s} \times \frac{\partial\vec{\mathbf{r}}}{\partial t} \right) \tag{1.34}$$

With review of the definitions above of the oriented surface and the associated unit normal vectors we can define the flux integral. Given a vector field $\vec{\Phi}(x_1, x_2, x_3)$ whose domain includes the surface S, then the flux integral of $\vec{\Phi}$ along S is given by (1.35) where S

(vii) A normal vector is a vector perpendicular to a surface.

represents the region in which the parameters trace the surface S.

$$\iint_S \vec{\Phi} \cdot \mathrm{d}\vec{S} = \iint_D \vec{\Phi} \cdot \vec{n}.\mathrm{d}S \tag{1.35}$$

Where a surface is expressed by the function $x_3 = \chi\big(x_1, x_2\big)$ or is parameterised by $\vec{\mathbf{r}}(s,t) = \sum_{i=1}^{3} x_i(s,t)\hat{\mathbf{e}}_i,\ \forall\, \alpha \leq t \leq \beta, \gamma \leq s \leq \delta$ (1.35) becomes (1.36) and (1.37) respectively.

$$\iint_D \vec{\Phi} \cdot \vec{n}.\mathrm{d}S = \iint_D \vec{\Phi} \cdot \big(\nabla\chi\big).\mathrm{d}A \tag{1.36}$$

$$\iint_D \vec{\Phi} \cdot \vec{n}.\mathrm{d}S = \iint_D \vec{\Phi} \cdot \left(\frac{\partial \vec{\mathbf{r}}}{\partial s} \times \frac{\partial \vec{\mathbf{r}}}{\partial t}\right).\mathrm{d}A \tag{1.37}$$

1.3.3 Stokes' theorem

Stokes' theorem, a higher dimensional version of Green's theorem (see subsection 1.2.3, p.6) relates the flux integral of the curl of a differentiable vector field along a piecewise smooth, orientable surface with the line integral of the vector field along the boundary or edge of the surface, the *boundary curve* [8]. As with line integrals of vector fields and flux integrals we must comment on the way in which orientations need to be taken. The orientation of the surface S will induce the positive orientation of the boundary curve. To yield the positive orientation of the boundary curve, it is possible to imagine looking downwards on a surface and visualise its normal vectors protruding outwards. If those vectors are protruding in your direction, if they are coming towards you, then the positive direction on the boundary curve is in the anti-clockwise direction.

Let S be an oriented piecewise smooth surface that is enclosed by a simple, closed and piecewise smooth boundary curve ∂S with positive orientation and define the vector field $\vec{\Phi}\big(x_1, x_2, x_3\big)$ in three-dimensions, then Stokes' theorem is defined as the following, (1.38).

$$\oint_{\partial S} \vec{\Phi} \cdot \mathrm{d}\vec{\mathbf{r}} = \iint_S \big(\nabla \times \vec{\Phi}\big) \cdot \mathrm{d}\vec{S} \tag{1.38}$$

1.3.4 Gauss' divergence theorem

Gauss' divergence theorem, more commonly the divergence theorem relates the flux integral along a closed surface to a triple integral over the solid region, or volume enclosed by the surface [8]. Let D be a bounded solid region, or volume in $\mathbb{R}^3$ whose boundary surface ∂D consists of finitely many piecewise smooth, closed orientable surfaces, each of which is oriented by unit normals that point away from D and $\vec{\Phi}$ be a vector field whose domain includes D, then Gauss' divergence theorem is defined as the following, (1.39).

$$\oiint_{\partial D} \vec{\Phi} \cdot \mathrm{d}\vec{S} = \iiint_D \nabla \cdot \vec{\Phi}.\mathrm{d}V \tag{1.39}$$

Chapter 2

An introduction to differential forms

The calculus of differential forms gives a simpler and more flexible alternative to vector calculus but here, we first take a moment to state that differential forms and their 'meaning' are difficult to motivate initially.

2.1 Manifolds

Where previously we have restricted ourselves mostly to $\mathbb{R}^n$, we now branch out and take a more general viewpoint with no additional endeavour. We begin by discussing m–dimensional subspaces in $\mathbb{R}^n$; singularly, an *embedded manifold*. If we have such a case that $m = n - 1$, then the embedded manifold is called a *hypersurface*. A *manifold* itself, is a topological space that resembles the Euclidean space near each point. That is, each point on an n–dimensional manifold has a neighbourhood that is homeomorphic(i) to the n–dimensional Euclidian space $\mathbb{R}^n$ [3]. An m–dimensional smooth manifold has the following conditions [4]. Let D be bounded open region in $\mathbb{R}^m$ then consider the function $\vec{\chi} : D \to M \subset \mathbb{R}^n$ where $\vec{\chi}(\mathbf{u}) = \sum_{j=1}^{n} \chi_j(\mathbf{u})\hat{\mathbf{e}}_j$ with coordinate $\mathbf{u} = \big(u_1, u_2, \ldots, u_m\big)$ are defined on D.

(i) $\vec{\chi}$ is continuous and once differentiable, $\vec{\chi} \in C^1$(ii).

(ii) $\vec{\chi}$ is injective, $\forall\, \mathbf{u}, \mathbf{v} \in D$, if $\mathbf{u} \neq \mathbf{v}$ then $\vec{\chi}(\mathbf{u}) \neq \vec{\chi}(\mathbf{v})$.

(iii) The tangent vectors, $\dfrac{\partial \vec{\chi}}{\partial u_m} = \displaystyle\sum_{j=1}^{n} \frac{\partial \chi_j}{\partial u_m}\hat{\mathbf{e}}_j$ are linearly independent on D.

Although not directly related to manifolds, we must give a definition of a *chart* as we will be needing this in the succeeding chapter. We define M to be a locally Euclidean n–space,

(i) Recall the conditions of a homeomorphism; a one-to-one onto map from one set to another which is continuous and has continuous inverse.

(ii) Here $C^k(D)$ denotes the set of continuous k^{th} differentiable functions on the domain D. If $k = 0$ then the set is written $C(D)$. If D is absolutely everywhere then we can denote the set as C^k.

that is, every point in M has an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$. Then a chart of M is the ordered pair $\big(S, \psi\big)$ where $S \subseteq M$ and $\psi : S \to D$ is a homeomorphism of S onto $D \subseteq_\circ \mathbb{R}^n$[iii].

2.2 Differential forms

2.2.1 The directional derivative in the sense of Barrett O'Neill

For any tangent vector $\mathbf{v_p} \in \mathbb{R}^3$ there is an associated straight line $\mathbf{u} = \mathbf{p} + \lambda\mathbf{v}$, $\lambda \in \mathbb{R}$. Given a differentiable scalar field $f \in \mathbb{R}^3$ then $f(\mathbf{u}) \in \mathbb{R}$ is an ordinary differentiable function and for $\lambda = 0$ the derivative of this function gives the initial rate of change of f as $\mathbf{p}$ moves in the direction $\mathbf{v}$. We define the new notation (2.1) to be the derivative of f with respect to $\mathbf{v_p}$, and so the directional derivative in the sense of Barrett O'Neill [26].

$$\mathbf{v_p}[f] = \left.\frac{\mathrm{d}f(\mathbf{u})}{\mathrm{d}\lambda}\right|_{\lambda=0} \tag{2.1}$$

We then extend O'Neill's derivative of f with respect to $\mathbf{v_p}$ to $\mathbb{R}^n$ with tangent vector $\mathbf{v_p} = \displaystyle\sum_{i=1}^{n} v_i\hat{\mathbf{e}}_i$ and variables u_i, $i = 1, 2, \ldots, n$ further defining a new notation with proof (2.2).

$$\mathbf{v_p}[f] = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial u_i}(\mathbf{p}) \tag{2.2}$$

Proof. Let $\mathbf{p} = \big(\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}\big)$ then $\mathbf{u} = \mathbf{p} + \lambda\mathbf{v} = \big(p_1 + \lambda v_1, p_2 + \lambda v_2, \ldots, p_n + \lambda v_n\big)$ and we have,

$$\begin{aligned}
\mathbf{v_p}[f] &= \left.\frac{\mathrm{d}f(\mathbf{u})}{\mathrm{d}\lambda}\right|_{\lambda=0} \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial u_i}(\mathbf{p}) \cdot \frac{\mathrm{d}\big(p_i + \lambda v_i\big)}{\mathrm{d}\lambda} \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial u_i}(\mathbf{p}) \cdot v_i \\
&= \sum_{i=1}^{n} v_i \frac{\partial f}{\partial u_i}(\mathbf{p})
\end{aligned}$$

□

As with the directional derivative in the vector calculus sense (see page 2) the directional derivative in the sense of Barrett O'Neill has the following properties, where f, g are scalar

[iii] The notation here $\subseteq_\circ$ is used to represent an *open* subset.

fields, $\mathbf{v_p}, \mathbf{w_p}$ are tangent vectors and $\alpha, \beta \in \mathbb{R}$ [26].

$$\big(\alpha\mathbf{v_p} + \beta\mathbf{w_p}\big)[f] = \alpha\mathbf{v_p}[f] + \beta\mathbf{w_p}[f] \tag{2.3a}$$

$$\mathbf{v_p}[\alpha f + \beta g] = \alpha\mathbf{v_p}[f] + \beta\mathbf{w_p}[g] \tag{2.3b}$$

$$\mathbf{v_p}[fg] = \mathbf{v_p}[f]g(\mathbf{p}) + f(\mathbf{p}) \cdot \mathbf{v_p}[g] \tag{2.3c}$$

Example 2.2.1. Compute $\mathbf{v_p}[f]$ for $f = x^2\cos(y)$ at the point $\mathbf{p} = (2, \pi)$.

Solution 2.2.1. We have,

$$\frac{\partial f}{\partial x} = 2x\cos(y) \qquad \frac{\partial f}{\partial y} = -x^2\sin(y)$$

which at the point $\mathbf{p} = (2, \pi)$ yield,

$$\frac{\partial f}{\partial x}\mathbf{p} = -4 \qquad \frac{\partial f}{\partial y}\mathbf{p} = 0.$$

Hence $\mathbf{v_p}[f] = -4 + 0 = -4$ and the function f is decreasing initially as $\mathbf{p}$ moves in the direction of $\mathbf{v}$.

Example 2.2.2. Compute $\mathbf{v_p}[f]$ for $f = x^2y + e^z$ at the point $\mathbf{p} = (2, -1, 0)$.

Solution 2.2.2. We have,

$$\frac{\partial f}{\partial x} = 2xy \qquad \frac{\partial f}{\partial y} = x^2 \qquad \frac{\partial f}{\partial z} = e^z$$

which at the point $\mathbf{p} = (2, -1, 0)$ yield,

$$\frac{\partial f}{\partial x}\mathbf{p} = -4 \qquad \frac{\partial f}{\partial y}\mathbf{p} = 4 \qquad \frac{\partial f}{\partial y}\mathbf{p} = 1.$$

Hence $\mathbf{v_p}[f] = -4 + 4 + 1 = 1$ and the function f is increasing initially as $\mathbf{p}$ moves in the direction of $\mathbf{v}$.

2.2.2 Differential k-form

Quite simply, a differential form is an integrand(iv), often noted by a lowercase letter from the greek alphabet. Differential forms are an approach to multivariable calculus that is independent of a coordinate system. If we begin our discussion in $\mathbb{R}^n$ with variables $u_1, u_2, \ldots, u_n$ and write $\mathbf{u}$ to be the coordinate $\mathbf{u} = \big(u_1, u_2, \ldots, u_n\big)$ then we can first define a differential 0-form ω to be simply a scalar field on $\mathbb{R}^n$,

$$\omega = \phi\big(u_1, u_2, \ldots, u_n\big).$$

(iv) An integrand is a quantity being integrated.

A differential 1-form, or simply a 1-form, ω in $\mathbb{R}^n$ is an expression of the form,

$$\omega = \phi_1(\mathbf{u})\mathrm{d}u_1 + \phi_2(\mathbf{u})\mathrm{d}u_2 + \cdots + \phi_n(\mathbf{u})\mathrm{d}u_n = \sum_{j=1}^{n} \phi_j(\mathbf{u})\mathrm{d}u_j$$

where $\phi_j(\mathbf{u})$ are functions of $\mathbf{u}$. Recall Green's theorem (1.22),

$$\oint_{\partial D} M.\mathrm{d}x_1 + N.\mathrm{d}x_2 = \iint_D \left(\frac{\partial N}{\partial x_1} - \frac{\partial M}{\partial x_2}\right).\mathrm{d}x_2\mathrm{d}x_1,$$

although here we use the notation for the boundary of D, $\gamma = \partial D$. The integrand on the left-hand-side of the above is an example of a 1-form, so in fact we have already met 1-forms without realising. A differential 1-form can be thought of, although loosely, as a function. The *basic* or *elementary* 1-form $\mathrm{d}u_j$ acts on a vector $\mathbf{v} = \langle v_1, v_2, \ldots, v_n\rangle$ and outputs the j^{th} element of that vector,

$$\mathrm{d}u_j(\mathbf{v}) = v_j.$$

Using this we construct a general 1-form $\omega = \sum_{j=1}^{n} \phi_j(\mathbf{u})\mathrm{d}u_j$ acting on $\mathbf{v} = \langle v_1, v_2, \ldots, v_n\rangle$ as (2.4).

$$\omega(\mathbf{v}) = \phi_1(\mathbf{u})v_1 + \phi_2(\mathbf{u})v_2 + \cdots + \phi_n(\mathbf{u})v_n = \sum_{j=1}^{n} \phi_j(\mathbf{u})v_j \tag{2.4}$$

Similarly to a differential 1-form, the basic or elementary 2-form written as $\omega = \mathrm{d}u_i \wedge \mathrm{d}u_j$, where $1 \leq i, j \leq n$ and the symbol $\wedge$ denotes the *wedge* or *exterior* product, can be thought of, although loosely, as a function. The elementary differential 2-form ω acts on 2 vectors $\mathbf{v} = \langle v_1, v_2, \ldots, v_n\rangle$ and $\mathbf{x} = \langle x_1, x_2, \ldots, x_n\rangle$ and has the following definition, (2.5).

$$\omega(\mathbf{v}, \mathbf{x}) = \mathrm{d}u_i \wedge \mathrm{d}u_j(\mathbf{v}, \mathbf{x}) = \begin{vmatrix} \mathrm{d}u_i(\mathbf{v}) & \mathrm{d}u_i(\mathbf{x}) \\ \mathrm{d}u_j(\mathbf{v}) & \mathrm{d}u_j(\mathbf{x}) \end{vmatrix} \tag{2.5}$$

Based on the properties of the determinant from linear algebra, we have the following properties for differential 2-forms, simply a 2-form.

(i) $\mathrm{d}u_i \wedge \mathrm{d}u_j = 0, \forall\, i = j$.

(ii) $\mathrm{d}u_i \wedge \mathrm{d}u_j = -\mathrm{d}u_j \wedge \mathrm{d}u_i$, the *anti-commutativity* property.

(iii) There are $\frac{1}{2}(n^2 - n)$ independent differential 2-forms.

A general differential 2-form ω is an expression of the form (2.6),

$$\omega = \sum_{1 \leq i < j \leq n} \phi_{ij}(\mathbf{u})\mathrm{d}u_i \wedge \mathrm{d}u_j \tag{2.6}$$

which when acting upon the two vectors $\mathbf{v}$ and $\mathbf{x}$ defined above yields (2.7),

$$\omega(\mathbf{v},\mathbf{x}) = \sum_{1\le i<j\le n} \phi_{ij}(\mathbf{u})\mathrm{d}u_i \wedge \mathrm{d}u_j(\mathbf{v},\mathbf{x}) = \sum_{i,j>i} \phi_{ij}(\mathbf{u}) \begin{vmatrix} \mathrm{d}u_i(\mathbf{v}) & \mathrm{d}u_i(\mathbf{x}) \\ \mathrm{d}u_j(\mathbf{v}) & \mathrm{d}u_j(\mathbf{x}) \end{vmatrix}. \tag{2.7}$$

In general then we define the basic or elementary differential k-form ω in n–dimensions as an expression of the form (2.8),

$$\omega = \mathrm{d}u_{i_1} \wedge \mathrm{d}u_{i_2} \wedge \cdots \wedge \mathrm{d}u_{i_k} \tag{2.8}$$

where $1 \le i_j \le n\ \forall\, j$, which acts upon k vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ and yields (2.9),

$$\omega\big(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\big) = \begin{vmatrix} \mathrm{d}u_{i_1}\big(\mathbf{x}_1\big) & \mathrm{d}u_{i_1}\big(\mathbf{x}_2\big) & \cdots & \mathrm{d}u_{i_1}\big(\mathbf{x}_k\big) \\ \mathrm{d}u_{i_2}\big(\mathbf{x}_1\big) & \mathrm{d}u_{i_2}\big(\mathbf{x}_2\big) & \cdots & \mathrm{d}u_{i_2}\big(\mathbf{x}_k\big) \\ \vdots & \vdots & \ddots & \cdots \\ \mathrm{d}u_{i_k}\big(\mathbf{x}_1\big) & \mathrm{d}u_{i_k}\big(\mathbf{x}_2\big) & \cdots & \mathrm{d}u_{i_k}\big(\mathbf{x}_k\big) \end{vmatrix}. \tag{2.9}$$

Based on the properties of the determinant from linear algebra, we have the following properties for differential k-forms, simply a k-form.

(i) $\mathrm{d}u_{i_1} \wedge \cdots \wedge \mathrm{d}u_{i_j} \wedge \cdots \wedge \mathrm{d}u_{i_j} \wedge \cdots \wedge \mathrm{d}u_{i_k} = 0$.

(ii) If any two rows or columns of the determinant are switched, the sign of the determinant is negated.

(iii) There are ${}^{n}\mathrm{C}_k = \dfrac{n!}{k!(n-k)!}$ independent differential k-forms.

A general k-form ω is an expression of the form (2.10),

$$\omega = \sum_{1\le i_1<\cdots<i_k\le n} \phi_{i_1,\ldots,i_k}(\mathbf{u})\mathrm{d}u_{i_1} \wedge \mathrm{d}u_{i_2} \wedge \cdots \wedge \mathrm{d}u_{i_k} \tag{2.10}$$

which when acting upon the k vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ yields (2.11) [2, 4],

$$\begin{aligned} \omega\big(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\big) &= \sum_{1\le i_1<\cdots<i_k\le n} \phi_{i_1,\ldots,i_k}(\mathbf{u})\mathrm{d}u_{i_1} \wedge \mathrm{d}u_{i_2} \wedge \cdots \wedge \mathrm{d}u_{i_k} \\ &= \sum_{i_1, i_2>i_1,\ldots,i_k>i_{k-1}} \phi_{i_1,\ldots,i_k}(\mathbf{u}) \begin{vmatrix} \mathrm{d}u_{i_1}\big(\mathbf{x}_1\big) & \mathrm{d}u_{i_1}\big(\mathbf{x}_2\big) & \cdots & \mathrm{d}u_{i_1}\big(\mathbf{x}_k\big) \\ \mathrm{d}u_{i_2}\big(\mathbf{x}_1\big) & \mathrm{d}u_{i_2}\big(\mathbf{x}_2\big) & \cdots & \mathrm{d}u_{i_2}\big(\mathbf{x}_k\big) \\ \vdots & \vdots & \ddots & \cdots \\ \mathrm{d}u_{i_k}\big(\mathbf{x}_1\big) & \mathrm{d}u_{i_k}\big(\mathbf{x}_2\big) & \cdots & \mathrm{d}u_{i_k}\big(\mathbf{x}_k\big) \end{vmatrix}. \end{aligned} \tag{2.11}$$

We end here describing the notation of an m-form ω in the dimension space $\mathbb{R}^n$ as $\omega \in \Lambda^m\big(\mathbb{R}^n\big)$.

Example 2.2.3. Given $\omega = (u_1+u_2)\mathrm{d}u_2\wedge\mathrm{d}u_3+\tan(u_3)\mathrm{d}u_3\wedge\mathrm{d}u_1 \in \Lambda^2(\mathbb{R}^3)$. Find $\omega(\mathbf{x}_1,\mathbf{x}_2)$ when $\mathbf{x}_1 = \langle 1,2,3\rangle$ and $\mathbf{x}_2 = \langle 2,1,1\rangle$.

Solution 2.2.3.

$$\begin{aligned}\omega(\mathbf{x}_1,\mathbf{x}_2) &= (u_1+u_2)\begin{vmatrix}\mathrm{d}u_2(\mathbf{x}_1) & \mathrm{d}u_2(\mathbf{x}_2)\\ \mathrm{d}u_3(\mathbf{x}_1) & \mathrm{d}u_3(\mathbf{x}_2)\end{vmatrix} + \tan(u_3)\begin{vmatrix}\mathrm{d}u_3(\mathbf{x}_1) & \mathrm{d}u_3(\mathbf{x}_2)\\ \mathrm{d}u_1(\mathbf{x}_1) & \mathrm{d}u_1(\mathbf{x}_2)\end{vmatrix}\\ &= (u_1+u_2)\begin{vmatrix}2 & 1\\ 3 & 1\end{vmatrix} + \tan(u_3)\begin{vmatrix}3 & 1\\ 1 & 2\end{vmatrix}\\ &= -(u_1+u_2) + 5\tan(u_3)\end{aligned}$$

Example 2.2.4. Given $\omega = (3u_1+u_2)\mathrm{d}u_3\wedge\mathrm{d}u_1+4u_2\mathrm{d}u_1\wedge\mathrm{d}u_2 \in \Lambda^2(\mathbb{R}^3)$. Find $\omega(\mathbf{x}_1,\mathbf{x}_2)$ when $\mathbf{x}_1 = \langle 3,3,0\rangle$ and $\mathbf{x}_2 = \langle -4,0,11\rangle$.

Solution 2.2.4.

$$\begin{aligned}\omega(\mathbf{x}_1,\mathbf{x}_2) &= (3u_1+u_2)\begin{vmatrix}\mathrm{d}u_3(\mathbf{x}_1) & \mathrm{d}u_3(\mathbf{x}_2)\\ \mathrm{d}u_1(\mathbf{x}_1) & \mathrm{d}u_1(\mathbf{x}_2)\end{vmatrix} + 4u_2\begin{vmatrix}\mathrm{d}u_1(\mathbf{x}_1) & \mathrm{d}u_1(\mathbf{x}_2)\\ \mathrm{d}u_2(\mathbf{x}_1) & \mathrm{d}u_2(\mathbf{x}_2)\end{vmatrix}\\ &= (3u_1+u_2)\begin{vmatrix}0 & 11\\ 3 & -4\end{vmatrix} + 4u_2\begin{vmatrix}3 & -4\\ 3 & 0\end{vmatrix}\\ &= -33(3u_1+u_2) + 12\cdot 4u_2\\ &= 15u_2 - 99u_1\end{aligned}$$

2.3 Calculus of differential forms

2.3.1 Wedge product

We begin our discussion of the calculus of differential forms[v] by considering *exterior multiplication*, also called the *wedge product*. In fact, we have already visited the notation of the wedge product of two vectors $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$, expressed as $\mathbf{u}\wedge\mathbf{v}$ although this is often omitted when using differential forms, i.e. $\mathrm{d}x\wedge\mathrm{d}y$ can be denoted $\mathrm{d}x\mathrm{d}y$.

Suppose we have the scalar field (0-form) f and the k-form ω described by (2.10) then the wedge product $f\wedge\omega$, commonly $f\omega$ with the wedge omitted, is given as (2.12) where $\mathbf{u}$ denotes the coordinate $\mathbf{u} = (u_1,u_2,\ldots,u_n)$,

$$f\wedge\omega = f\omega = \sum_{1\le i_1<\cdots<i_k\le n} f\phi_{i_1,\ldots,i_k}(\mathbf{u})\mathrm{d}u_{i_1}\wedge\mathrm{d}u_{i_2}\wedge\cdots\wedge\mathrm{d}u_{i_k}. \tag{2.12}$$

Suppose that we have an l-form,

$$\gamma = \sum_{1\le j_1<\cdots<j_l\le n} \psi_{j_1,\ldots,j_k}(\mathbf{u})\mathrm{d}u_{j_1}\wedge\mathrm{d}u_{j_2}\wedge\cdots\wedge\mathrm{d}u_{j_l}$$

[v] Often called exterior differential calculus.

then the wedge product $\gamma \wedge \omega$ yields (2.13)

$$\gamma \wedge \omega = \sum \phi_{i_1,\ldots,i_k}(\mathbf{u})\psi_{j_1,\ldots,j_k}(\mathbf{u})\mathrm{d}u_{i_1} \wedge \mathrm{d}u_{i_2} \wedge \cdots \wedge \mathrm{d}u_{i_k} \wedge \mathrm{d}u_{j_1} \wedge \mathrm{d}u_{j_2} \wedge \cdots \wedge \mathrm{d}u_{j_l}. \quad (2.13)$$

Given (2.12) and (2.13) we can define the following properties for the wedge product with the following definitions.

$$F, f_{1,i_{1,1},\ldots,i_{1,k}}, f_{2,i_{2,1},\ldots,i_{2,k}}, f_{3,i_{3,1},\ldots,i_{3,l}}, f_{4,i_{4,1},\ldots,i_{4,p}} \in \Lambda^0(\mathbb{R}^n) \quad (2.14)$$

$$\alpha = \sum_{1\le i_{1,1}<\cdots<i_{1,k}\le n} f_{1,i_{1,1},\ldots,i_{1,k}}(\mathbf{u})\mathrm{d}u_{i_{1,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{1,k}} \in \Lambda^k(\mathbb{R}^n) \quad (2.15)$$

$$\beta = \sum_{1\le i_{2,1}<\cdots<i_{2,k}\le n} f_{2,i_{2,1},\ldots,i_{2,k}}(\mathbf{u})\mathrm{d}u_{i_{2,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{2,k}} \in \Lambda^k(\mathbb{R}^n) \quad (2.16)$$

$$\gamma = \sum_{1\le i_{3,1}<\cdots<i_{3,l}\le n} f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u})\mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}} \in \Lambda^l(\mathbb{R}^n) \quad (2.17)$$

$$\delta = \sum_{1\le i_{4,1}<\cdots<i_{4,p}\le n} f_{4,i_{4,1},\ldots,i_{4,p}}(\mathbf{u})\mathrm{d}u_{i_{4,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{4,p}} \in \Lambda^p(\mathbb{R}^n). \quad (2.18)$$

(i) $\big(\alpha+\beta\big)\wedge\gamma=\alpha\wedge\gamma+\beta\wedge\gamma$.

(ii) $\big(\alpha\wedge\gamma\big)\wedge\delta=\alpha\wedge\big(\gamma\wedge\delta\big)$.

Proof. Computing the left-hand-side of the above expanding and factorising we have the following proof.

$$
\begin{aligned}
\big(\alpha\wedge\gamma\big)\wedge\delta =&\Bigg(\sum_{1\le i_{1,1}<\cdots<i_{1,k}\le n} f_{1,i_{1,1},\dots,i_{1,k}}(\mathbf{u})\mathrm{d}u_{i_{1,1}}\wedge\cdots\wedge\mathrm{d}u_{i_{1,k}}\\
&\sum_{1\le i_{3,1}<\cdots<i_{3,l}\le n} f_{3,i_{3,1},\dots,i_{3,l}}(\mathbf{u})\mathrm{d}u_{i_{3,1}}\wedge\cdots\wedge\mathrm{d}u_{i_{3,l}}\Bigg)\wedge\\
&\sum_{1\le i_{4,1}<\cdots<i_{4,p}\le n} f_{4,i_{4,1},\dots,i_{4,p}}(\mathbf{u})\mathrm{d}u_{i_{4,1}}\wedge\cdots\wedge\mathrm{d}u_{i_{4,p}}\\
=&\Bigg(\sum f_{1,i_{1,1},\dots,i_{1,k}}(\mathbf{u})f_{3,i_{3,1},\dots,i_{3,l}}(\mathbf{u})\mathrm{d}u_{i_{1,1}}\wedge\cdots\wedge\mathrm{d}u_{i_{1,k}}\wedge\mathrm{d}u_{i_{3,1}}\wedge\cdots\wedge\mathrm{d}u_{i_{3,l}}\Bigg)\wedge\\
&\sum_{1\le i_{4,1}<\cdots<i_{4,p}\le n} f_{4,i_{4,1},\dots,i_{4,p}}(\mathbf{u})\mathrm{d}u_{i_{4,1}}\wedge\cdots\wedge\mathrm{d}u_{i_{4,p}}\\
=&\sum f_{1,i_{1,1},\dots,i_{1,k}}(\mathbf{u})f_{3,i_{3,1},\dots,i_{3,l}}(\mathbf{u})f_{4,i_{4,1},\dots,i_{4,p}}(\mathbf{u})\mathrm{d}u_{i_{1,1}}\wedge\cdots\wedge\mathrm{d}u_{i_{1,k}}\wedge\\
&\quad\mathrm{d}u_{i_{3,1}}\wedge\cdots\wedge\mathrm{d}u_{i_{3,l}}\wedge\mathrm{d}u_{i_{4,1}}\wedge\cdots\wedge\mathrm{d}u_{i_{4,p}}\\
=&\sum_{1\le i_{1,1}<\cdots<i_{1,k}\le n} f_{1,i_{1,1},\dots,i_{1,k}}(\mathbf{u})\mathrm{d}u_{i_{1,1}}\wedge\cdots\wedge\mathrm{d}u_{i_{1,k}}\wedge\\
&\Bigg(\sum f_{3,i_{3,1},\dots,i_{3,k}}(\mathbf{u})f_{4,i_{4,1},\dots,i_{4,l}}(\mathbf{u})\mathrm{d}u_{i_{3,1}}\wedge\cdots\wedge\mathrm{d}u_{i_{3,k}}\wedge\mathrm{d}u_{i_{4,1}}\wedge\cdots\wedge\mathrm{d}u_{i_{4,k}}\Bigg)\\
=&\sum_{1\le i_{1,1}<\cdots<i_{1,k}\le n} f_{1,i_{1,1},\dots,i_{1,k}}(\mathbf{u})\mathrm{d}u_{i_{1,1}}\wedge\cdots\wedge\mathrm{d}u_{i_{1,k}}\wedge\\
&\Bigg(\sum_{1\le i_{3,1}<\cdots<i_{3,l}\le n} f_{3,i_{3,1},\dots,i_{3,l}}(\mathbf{u})\mathrm{d}u_{i_{3,1}}\wedge\cdots\wedge\mathrm{d}u_{i_{3,l}}\wedge\\
&\sum_{1\le i_{4,1}<\cdots<i_{4,p}\le n} f_{4,i_{4,1},\dots,i_{4,p}}(\mathbf{u})\mathrm{d}u_{i_{4,1}}\wedge\cdots\wedge\mathrm{d}u_{i_{4,p}}\Bigg)=\alpha\wedge\big(\gamma\wedge\delta\big)
\end{aligned}
$$

□

(iii) $\beta \wedge \big(F\gamma\big) = F\big(\beta \wedge \gamma\big) = \big(F\beta\big) \wedge \gamma$

Proof. Computing the left-hand-side of the above, we are able to expand and rearrange to evaluate the centre and right-hand-side of the property above.

$$\begin{aligned}
\beta \wedge \big(F\gamma\big) &= \sum_{1\leq i_{2,1}<\cdots<i_{2,k}\leq n} f_{2,i_{2,1},\ldots,i_{2,k}}(\mathbf{u})\mathrm{d}u_{i_{2,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{2,k}} \wedge \\
&\quad \left(F \sum_{1\leq i_{3,1}<\cdots<i_{3,l}\leq n} f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u})\mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}}\right) \\
&= \sum_{1\leq i_{2,1}<\cdots<i_{2,k}\leq n} f_{2,i_{2,1},\ldots,i_{2,k}}(\mathbf{u})\mathrm{d}u_{i_{2,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{2,k}} \wedge \\
&\quad \sum_{1\leq i_{3,1}<\cdots<i_{3,l}\leq n} F f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u})\mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}} \\
&= \sum f_{2,i_{2,1},\ldots,i_{2,k}}(\mathbf{u}) F f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u})\mathrm{d}u_{i_{2,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{2,k}} \wedge \mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}} \\
&= F \sum f_{2,i_{2,1},\ldots,i_{2,k}}(\mathbf{u}) f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u})\mathrm{d}u_{i_{2,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{2,k}} \wedge \mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}} \\
&= F\big(\beta \wedge \gamma\big) \\
&= F \sum f_{2,i_{2,1},\ldots,i_{2,k}}(\mathbf{u}) f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u})\mathrm{d}u_{i_{2,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{2,k}} \wedge \mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}} \\
&= \sum F f_{2,i_{2,1},\ldots,i_{2,k}}(\mathbf{u}) f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u})\mathrm{d}u_{i_{2,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{2,k}} \wedge \mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}} \\
&= \sum_{1\leq i_{2,1}<\cdots<i_{2,k}\leq n} F f_{2,i_{2,1},\ldots,i_{2,k}}(\mathbf{u})\mathrm{d}u_{i_{2,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{2,k}} \wedge \\
&\quad \sum_{1\leq i_{3,1}<\cdots<i_{3,l}\leq n} f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u})\mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}} \\
&= \left(F \sum_{1\leq i_{2,1}<\cdots<i_{2,k}\leq n} f_{2,i_{2,1},\ldots,i_{2,k}}(\mathbf{u})\mathrm{d}u_{i_{2,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{2,k}}\right) \wedge \\
&\quad \sum_{1\leq i_{3,1}<\cdots<i_{3,l}\leq n} f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u})\mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}} \\
&= \big(F\beta\big) \wedge \gamma
\end{aligned}$$

□

(iv) $\alpha \wedge \gamma = \big(-1\big)^{kl} \gamma \wedge \alpha$

Proof. Since $\alpha \in \Lambda^k$ and $\gamma \in \Lambda^l$, to move each element γ to the left-hand-side of α individually, we would need to move them k positions to the left. Due to the anti-commutativity property of differential forms this would mean a multiplication of $\big(-1\big)^k$ for each of the l elements, that is multiplying by $\big(-1\big)^k$, l times $\Rightarrow$ $\big((-1)^k\big)^l = \big(-1\big)^{kl}$ to move the entirety of γ to the left-hand-side of α. Thus we have the relation $\alpha \wedge \gamma = \big(-1\big)^{kl}\gamma \wedge \alpha$ and the proof is complete. □

Example 2.3.1. Given $\omega = \cos\big(u_1\big)\mathrm{d}u_1 \wedge \mathrm{d}u_2 + u_2\mathrm{d}u_2 \wedge \mathrm{d}u_3 \in \Lambda^2\big(\mathbb{R}^3\big)$ and $\gamma = 2\sin\big(u_1\big)\mathrm{d}u_3 \in \Lambda^1\big(\mathbb{R}^3\big)$ calculate $\omega \wedge \gamma$ and $\gamma \wedge \omega$.

Solution 2.3.1.

$$\begin{aligned}
\omega \wedge \gamma &= \Big(\cos\big(u_1\big)\mathrm{d}u_1 \wedge \mathrm{d}u_2 + u_2\mathrm{d}u_2 \wedge \mathrm{d}u_3\Big) \wedge \Big(2\sin\big(u_1\big)\mathrm{d}u_3\Big) \\
&= 2\sin\big(u_1\big)\cos\big(u_1\big)\mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \mathrm{d}u_3 + 2u_2\sin\big(u_1\big)\mathrm{d}u_2 \wedge \mathrm{d}u_3 \wedge \mathrm{d}u_3 \\
&= \sin\big(2u_1\big)\mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \mathrm{d}u_3
\end{aligned}$$

$$\begin{aligned}
\gamma \wedge \omega &= \Big(2\sin\big(u_1\big)\mathrm{d}u_3\Big) \wedge \Big(\cos\big(u_1\big)\mathrm{d}u_1 \wedge \mathrm{d}u_2 + u_2\mathrm{d}u_2 \wedge \mathrm{d}u_3\Big) \\
&= 2\sin\big(u_1\big)\cos\big(u_1\big)\mathrm{d}u_3 \wedge \mathrm{d}u_1 \wedge \mathrm{d}u_2 + 2u_2\sin\big(u_1\big)\mathrm{d}u_3 \wedge \mathrm{d}u_2 \wedge \mathrm{d}u_3 \\
&= -\sin\big(2u_1\big)\mathrm{d}u_1 \wedge \mathrm{d}u_3 \wedge \mathrm{d}u_2 \\
&= \sin\big(2u_1\big)\mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \mathrm{d}u_3
\end{aligned}$$

2.3.2 Exterior differentiation

Exterior differentiation is defined by the linear operator $\mathrm{d} : \Lambda^k\big(\mathbb{R}^n\big) \to \Lambda^{k+1}\big(\mathbb{R}^n\big)$. That is, the operator d maps a differential k-form into a differential $(k+1)$-form in an n–dimensional space $\mathbb{R}^n$. d obeys the chain and product rules yielding (2.19) and (2.20) respectively where f, g are scalar fields.

$$\mathrm{d}\big(f(g)\big) = f'(g)\mathrm{d}g \tag{2.19}$$

$$\mathrm{d}\big(f \wedge g\big) = f\mathrm{d}g + g\mathrm{d}f \tag{2.20}$$

Applied to a scalar field $f \in \Lambda^0\big(\mathbb{R}^n\big)$ the operator yields the *exterior derivative* (2.21),

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial u_i}\mathrm{d}u_i \tag{2.21}$$

which then allows us to define the following properties of the exterior derivative on differential forms where $f, g \in \Lambda^0$ and α and γ are described as (2.15) and (2.17).

(i) Given $\alpha \in \Lambda^k\big(\mathbb{R}^n\big)$, $\mathrm{d}\alpha \in \Lambda^{k+1}\big(\mathbb{R}^n\big)$ is described as,

$$\mathrm{d}\alpha = \sum_{1 \le i_{1,1} < \cdots < i_{1,k} \le n} \mathrm{d}f_{1,i_{1,1},\ldots,i_{1,k}}(\mathbf{u}) \wedge \mathrm{d}u_{i_{1,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{1,k}}.$$

(ii) *Leibiz rule*: $\mathrm{d}(\alpha \wedge \gamma) = \mathrm{d}\alpha \wedge \gamma + (-1)^k \alpha \wedge \mathrm{d}\gamma$

Proof.

$$\begin{aligned}
\mathrm{d}(\alpha \wedge \gamma) &= \mathrm{d}\Bigg(\sum_{1 \le i_{1,1} < \cdots < i_{1,k} \le n} f_{1,i_{1,1},\ldots,i_{1,k}}(\mathbf{u}) \mathrm{d}u_{i_{1,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{1,k}} \wedge \\
&\qquad \sum_{1 \le i_{3,1} < \cdots < i_{3,l} \le n} f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u}) \mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}} \Bigg) \\
&= \mathrm{d} \sum f_{1,i_{1,1},\ldots,i_{1,k}}(\mathbf{u}) f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u}) \mathrm{d}u_{i_{1,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{1,k}} \wedge \mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}} \\
&= \sum \mathrm{d}\big(f_{1,i_{1,1},\ldots,i_{1,k}}(\mathbf{u}) f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u})\big) \wedge \mathrm{d}u_{i_{1,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{1,k}} \wedge \mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}} \\
&= \sum f_{1,i_{1,1},\ldots,i_{1,k}}(\mathbf{u}) \mathrm{d}\big(f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u})\big) \wedge \mathrm{d}u_{i_{1,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{1,k}} \wedge \mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}} + \\
&\quad \sum f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u}) \mathrm{d}\big(f_{1,i_{1,1},\ldots,i_{1,k}}(\mathbf{u})\big) \wedge \mathrm{d}u_{i_{1,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{1,k}} \wedge \mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}} \\
&= (-1)^k \sum f_{1,i_{1,1},\ldots,i_{1,k}}(\mathbf{u}) \wedge \mathrm{d}u_{i_{1,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{1,k}} \wedge \mathrm{d}\big(f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u})\big) \wedge \\
&\qquad \mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}} + \\
&\quad \sum f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u}) \mathrm{d}\big(f_{1,i_{1,1},\ldots,i_{1,k}}(\mathbf{u})\big) \wedge \mathrm{d}u_{i_{1,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{1,k}} \wedge \mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}} \\
&= (-1)^k \Bigg(\sum_{1 \le i_{1,1} < \cdots < i_{1,k} \le n} f_{1,i_{1,1},\ldots,i_{1,k}}(\mathbf{u}) \mathrm{d}u_{i_{1,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{1,k}} \wedge \\
&\qquad \mathrm{d}\Bigg(\sum_{1 \le i_{3,1} < \cdots < i_{3,l} \le n} f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u}) \mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}} \Bigg) \Bigg) + \\
&\quad \mathrm{d}\Bigg(\sum_{1 \le i_{1,1} < \cdots < i_{1,k} \le n} f_{1,i_{1,1},\ldots,i_{1,k}}(\mathbf{u}) \mathrm{d}u_{i_{1,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{1,k}} \Bigg) \wedge \\
&\qquad \sum_{1 \le i_{3,1} < \cdots < i_{3,l} \le n} f_{3,i_{3,1},\ldots,i_{3,l}}(\mathbf{u}) \mathrm{d}u_{i_{3,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{3,l}} \\
&= (-1)^k \alpha \wedge \mathrm{d}\gamma + \mathrm{d}\alpha \wedge \gamma \\
&= \mathrm{d}\alpha \wedge \gamma + (-1)^k \alpha \wedge \mathrm{d}\gamma
\end{aligned}$$

□

(iii) $\mathrm{d}(\mathrm{d}\alpha) = 0$

Proof. Let $k = 0$ then α is some function in $\mathbb{R}^n$.

$$\begin{aligned}
\mathrm{d}(\mathrm{d}\alpha) &= \mathrm{d}\left(\sum_{i=1}^{n} \frac{\partial \alpha}{\partial u_i}\mathrm{d}u_i\right) \\
&= \sum_{i,j} \frac{\partial^2 \alpha}{\partial u_i \partial u_j}\mathrm{d}u_j \wedge \mathrm{d}u_i \\
&= \sum_{i<j} \left(\frac{\partial^2 \alpha}{\partial u_j \partial u_i} - \frac{\partial^2 \alpha}{\partial u_i \partial u_j}\right)\mathrm{d}u_i \wedge \mathrm{d}u_j \\
&= \sum_{i<j} 0 \cdot \mathrm{d}u_i \wedge \mathrm{d}u_j \\
&= 0
\end{aligned}$$

As u_i represents the i^{th} coordinate in $\mathbb{R}^n$ it is logical to conclude that $\mathrm{d}\big(\mathrm{d}u_i\big) = 0$. Now let us consider the case where $k > 0$, then we have the following results considering Leibniz rule.

$$\begin{aligned}
\mathrm{d}(\mathrm{d}\alpha) = \mathrm{d}&\left(\sum_{1 \le i_{1,1} < \cdots < i_{1,k} \le n} f_{1,i_{1,1},\ldots,i_{1,k}}(\mathbf{u})\mathrm{d}u_{i_{1,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{1,k}}\right) \\
= &\sum_{1 \le i_{1,1} < \cdots < i_{1,k} \le n} \Big(\mathrm{d}f_{1,i_{1,1},\ldots,i_{1,k}}(\mathbf{u}) \wedge \mathrm{d}u_{i_{1,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{1,k}} \\
&\qquad - f_{1,i_{1,1},\ldots,i_{1,k}}(\mathbf{u}) \wedge \mathrm{d}\big(\mathrm{d}u_{i_{1,1}}\big) \wedge \cdots \wedge \mathrm{d}u_{i_{1,k}} + \cdots - \cdots\Big) \\
= &\sum_{1 \le i_{1,1} < \cdots < i_{1,k} \le n} \Big(0 \wedge \mathrm{d}u_{i_{1,1}} \wedge \cdots \wedge \mathrm{d}u_{i_{1,k}} \\
&\qquad - f_{1,i_{1,1},\ldots,i_{1,k}}(\mathbf{u}) \wedge 0 \wedge \cdots \wedge \mathrm{d}u_{i_{1,k}} + \cdots - \cdots\Big) \\
= &\ 0
\end{aligned}$$

□

Example 2.3.2. Calculate $\mathrm{d}\omega$, where $\omega = u_1^2 u_3^3 \mathrm{d}u_1 \wedge \mathrm{d}u_2 \in \Lambda^2\big(\mathbb{R}^3\big)$.

Solution 2.3.2.

$$\begin{aligned}
\mathrm{d}\omega &= \left(\frac{\partial}{\partial u_1}\Big(u_1^2 u_3^3\Big)\mathrm{d}u_1 + \frac{\partial}{\partial u_2}\Big(u_1^2 u_3^3\Big)\mathrm{d}u_2 + \frac{\partial}{\partial u_3}\Big(u_1^2 u_3^3\Big)\mathrm{d}u_3\right)\mathrm{d}u_1 \wedge \mathrm{d}u_2 \\
&= 3u_1^2 u_3^2 \mathrm{d}u_3 \wedge \mathrm{d}u_1 \wedge \mathrm{d}u_2 \\
&= -3u_1^2 u_3^2 \mathrm{d}u_1 \wedge \mathrm{d}u_3 \wedge \mathrm{d}u_2 \\
&= 3u_1^2 u_3^2 \mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \mathrm{d}u_3
\end{aligned}$$

Example 2.3.3. Calculate $\mathrm{d}\gamma$, where $\gamma = u_1 \cos\big(u_2\big)\mathrm{d}u_1 \in \Lambda^1\big(\mathbb{R}^3\big)$.

Solution 2.3.3.

$$\begin{aligned}
\mathrm{d}\gamma &= \left(\frac{\partial}{\partial u_1}\Big(u_1 \cos\big(u_2\big)\Big)\mathrm{d}u_1 + \frac{\partial}{\partial u_2}\Big(u_1 \cos\big(u_2\big)\Big)\mathrm{d}u_2 + \frac{\partial}{\partial u_3}\Big(u_1 \cos\big(u_2\big)\Big)\mathrm{d}u_3\right)\mathrm{d}u_1 \\
&= -u_1 \sin\big(u_2\big)\mathrm{d}u_2 \wedge \mathrm{d}u_1 + 0\mathrm{d}u_3 \wedge \mathrm{d}u_1 \\
&= u_1 \sin\big(u_2\big)\mathrm{d}u_1 \wedge \mathrm{d}u_2
\end{aligned}$$

2.4 Orientation

The *orientation* is the sense of twist in space. An example of orientation would be the anti-clockwise twist in $\mathbb{R}^2$–space [31] used to measure the size of an angle from the positive x–axis in the direction of the positive y–axis. If this angle was measured from the positive x–axis in the direction of the negative y–axis, then the angle would be measured clockwise and would have an opposite orientation.

The right-hand screw rule for $\mathbb{R}^3$

The *right-hand screw rule* or sometimes, the *right-hand thumb rule* or simply the *right-hand rule* is a movement of the right hand that yields the direction of the vector perpendicular to two other vectors; commonly used with the cross product (see subsection A.2.2, p.47). To apply the right hand rule, point your index and middle fingers in the directions of the first and second vectors respectively; your thumb will then point in the direction of the cross product, the vector perpendicular to both the first and second vectors. The rule is also known as curling the fingers on your right hand from the first vector in the direction of the second vector; the thumb will again point in the direction of the vector perpendicular to the first and second.

The idea of orientation using the right-hand rule is strongly connected to and is the reason for the anti-commutativity of differential forms but is also a well known source of mistakes. The variables used must retain a 'correct' order to express orientation properly. In applications, the foremost place in which orientation occurs in $\mathbb{R}^n$ is in n- and $(n-1)$-forms [31]. If we define the n variables used in $\mathbb{R}^n$ to be $u_1, u_2, \ldots, u_n$(vi) then the *correct* order for the variables is the way in which they are numbered; (2.22a) and (2.22b) are in correct and incorrect order respectively and are related by (2.22c).

$$\mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_{n-1} \wedge \mathrm{d}u_n \tag{2.22a}$$

$$\mathrm{d}u_2 \wedge \mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_{n-1} \wedge \mathrm{d}u_n \tag{2.22b}$$

$$\mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_{n-1} \wedge \mathrm{d}u_n = -\,\mathrm{d}u_2 \wedge \mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_{n-1} \wedge \mathrm{d}u_n \tag{2.22c}$$

(vi) Using $u_1, u_2, \ldots, u_n$ we draw attention to the fact that the variables are general and are not necessarily rectangular, cylindrical, spherical or even orthogonal.

Due to the anti-commutativity property of differential forms, interchanging any two of the $\mathrm{d}u_j$ variables reverses the sign and changes correct to incorrect order or vice versa. Further, an odd or even number of permutations of $\mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_{n-1} \wedge \mathrm{d}u_n$ yield incorrect and correct order respectively. The same cannot be said for $(n-1)$-forms; with $\mathrm{d}u_j$ removed from the listing, the correct order of an $(n-1)$-form is given by (2.23).

$$(-1)^{j-1}\mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_{j-1} \wedge \mathrm{d}u_{j+1} \wedge \cdots \wedge \mathrm{d}u_{n-1} \wedge \mathrm{d}u_n \tag{2.23}$$

Example 2.4.1. Given the 5-form $\omega = \mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \mathrm{d}u_3 \wedge \mathrm{d}u_4 \wedge \mathrm{d}u_5 \in \Lambda^5\left(\mathbb{R}^5\right)$ find the correctly ordered 4-forms with $\mathrm{d}u_2$ and $\mathrm{d}u_3$ removed.

Solution 2.4.1. The correctly ordered 4-form of ω with $\mathrm{d}u_2$ removed is,

$$\begin{aligned}(-1)^{2-1}\mathrm{d}u_1 \wedge \mathrm{d}u_3 \wedge \mathrm{d}u_4 \wedge \mathrm{d}u_5 &= (-1)^1\mathrm{d}u_1 \wedge \mathrm{d}u_3 \wedge \mathrm{d}u_4 \wedge \mathrm{d}u_5 \\ &= -\mathrm{d}u_1 \wedge \mathrm{d}u_3 \wedge \mathrm{d}u_4 \wedge \mathrm{d}u_5.\end{aligned}$$

The correctly ordered 4-form of ω with $\mathrm{d}u_3$ removed is,

$$\begin{aligned}(-1)^{3-1}\mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \mathrm{d}u_4 \wedge \mathrm{d}u_5 &= (-1)^2\mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \mathrm{d}u_4 \wedge \mathrm{d}u_5 \\ &= \mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \mathrm{d}u_4 \wedge \mathrm{d}u_5.\end{aligned}$$

2.5 Hodge star operator

Where ${}^n\mathrm{C}_k$ represents the dimension of the differential k-forms $\Lambda^k\left(\mathbb{R}^n\right)$; by the fact that ${}^n\mathrm{C}_k = {}^n\mathrm{C}_{n-k}$, there are just as many $(n-k)$-forms as there are k-forms [7]. We define a specific linear transformation, the *Hodge star* operator, for converting from k-forms to $(n-k)$-forms by the mapping $\star : \Lambda^k\left(\mathbb{R}^n\right) \to \Lambda^{n-k}\left(\mathbb{R}^n\right)$. When applying the Hodge star to a k-form $\omega \in \Lambda^k\left(\mathbb{R}^n\right)$ the resulting $(n-k)$-form, $\star\omega \in \Lambda^{n-k}\left(\mathbb{R}^n\right)$ is called the *dual* of ω. This leads to the Hodge star operator sometimes being called the *Hodge dual*; further $\star\omega$ is seen as the complement to ω by the following.

Given a differential form $\omega = \varsigma_1 \wedge \varsigma_2 \wedge \cdots \wedge \varsigma_k \in \Lambda^k\left(\mathbb{R}^n\right)$ then the Hodge star operator yields (2.24) where $\varsigma \in \mathcal{S}_n$. Here $\mathcal{S}_n$ denotes the symmetric group of permutations $(1, 2, \ldots, n)$ [27], $\varsigma_1 < \varsigma_2 < \cdots < \varsigma_k$ and $\varsigma_{k+1} < \varsigma_{k+2} < \cdots < \varsigma_n$.

$$\star\left(\varsigma_1 < \varsigma_2 < \cdots < \varsigma_k\right) = \mathrm{sgn}(\varsigma)\varsigma_{\mathrm{k+1}} < \varsigma_{\mathrm{k+2}} < \cdots < \varsigma_{\mathrm{n}} \tag{2.24}$$

Example 2.5.1. Calculate in $\mathbb{R}^3$, $\star \mathrm{d}x$, $\star \mathrm{d}y$, $\star \mathrm{d}z$ and $\star 1$.

Solution 2.5.1.

$$\begin{aligned}
\star \mathrm{d}x &= \mathrm{d}y \wedge \mathrm{d}z \\
\star \mathrm{d}y &= \mathrm{d}z \wedge \mathrm{d}x \\
\star \mathrm{d}z &= \mathrm{d}x \wedge \mathrm{d}y \\
\star 1 &= \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z
\end{aligned}$$

Chapter 3

Integration of differential forms

As previously stated, a differential form is an integrand and hence a k-form must be integrated over a k dimensional region. When integrating a differential form, the form must be placed in correct order and wedges are to be removed to yield an ordinary multiple integral. The differential form can then be integrated in the normal way. As an example can let $\omega = f\mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}y \in \Lambda^3(\mathbb{R}^3)$ where $f \in \Lambda^0(\mathbb{R}^3)$, then the integral of this form over the region $V \subseteq \mathbb{R}^3$ can be found as follows.

$$\begin{aligned}\int_V \omega &= \int_V f\mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}y \\ &= -\int_V f\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\ &= -\iiint_V f\mathrm{d}x\mathrm{d}y\mathrm{d}z\end{aligned}$$

3.1 Div, grad and curl

Where we have previously discussed div, grad and curl in the sense of vector calculus we now discuss them in terms of differential forms and in $\mathbb{R}^3$. The fact that these three vector operators naturally occur with differential forms is a testament to the elegance of forms and their uses in mathematics. As we restrict ourselves to $\mathbb{R}^3$ we will only consider 0-forms, 1-forms and 2-forms.

The gradient

Given any scalar field $f(x, y, z) \in \Lambda^0(\mathbb{R}^3)$ then by the exterior derivative (2.21), we can compute $\mathrm{d}f$ (3.1).

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z \tag{3.1}$$

Notice then that we see the components of the gradient in (3.1),

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

and so the gradient has fallen out naturally when computing the exterior derivative of $f(x,y,z) \in \Lambda^0(\mathbb{R}^3)$.

Curl

Let us compute the exterior derivative of $\omega = f\mathrm{d}x + g\mathrm{d}y + h\mathrm{d}z \in \Lambda^1(\mathbb{R}^3)$ where $f,g,h \in \Lambda^0(\mathbb{R}^3)$.

$$
\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}\big(f\mathrm{d}x + g\mathrm{d}y + h\mathrm{d}z\big) \\
&= \mathrm{d}f \wedge \mathrm{d}x + \mathrm{d}g \wedge \mathrm{d}y + \mathrm{d}h \wedge \mathrm{d}z \\
&= \left(\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}x + \left(\frac{\partial g}{\partial x}\mathrm{d}x + \frac{\partial g}{\partial y}\mathrm{d}y + \frac{\partial g}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}y \\
&\quad + \left(\frac{\partial h}{\partial x}\mathrm{d}x + \frac{\partial h}{\partial y}\mathrm{d}y + \frac{\partial h}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}z \\
&= \frac{\partial f}{\partial y}\mathrm{d}y \wedge \mathrm{d}x + \frac{\partial f}{\partial z}\mathrm{d}z \wedge \mathrm{d}x + \frac{\partial g}{\partial x}\mathrm{d}x \wedge \mathrm{d}y + \frac{\partial g}{\partial z}\mathrm{d}z \wedge \mathrm{d}y + \frac{\partial h}{\partial x}\mathrm{d}x \wedge \mathrm{d}z + \frac{\partial h}{\partial y}\mathrm{d}y \wedge \mathrm{d}z \\
&= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y
\end{aligned}
$$

Here we notice that the components of the curl (1.8) have fallen out naturally for $\vec{F} = f\hat{\mathbf{i}} + g\hat{\mathbf{j}} + h\hat{\mathbf{k}}$,

$$
\nabla \times \vec{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{\mathbf{k}}
$$

by finding the exterior derivative of $\omega \in \Lambda^1(\mathbb{R}^3)$.

Divergence

Let $\gamma = f\mathrm{d}y \wedge \mathrm{d}z + g\mathrm{d}z \wedge \mathrm{d}x + h\mathrm{d}x \wedge \mathrm{d}y \in \Lambda^2(\mathbb{R}^3)$ then computing the exterior derivative $\mathrm{d}\gamma$ we have the following.

$$
\begin{aligned}
\mathrm{d}\gamma &= \mathrm{d}\big(f\mathrm{d}y \wedge \mathrm{d}z + g\mathrm{d}z \wedge \mathrm{d}x + h\mathrm{d}x \wedge \mathrm{d}y\big) \\
&= \mathrm{d}f \wedge \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}g \wedge \mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}h \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial g}{\partial x}\mathrm{d}x + \frac{\partial g}{\partial y}\mathrm{d}y + \frac{\partial g}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}z \wedge \mathrm{d}x \\
&\quad + \left(\frac{\partial h}{\partial x}\mathrm{d}x + \frac{\partial h}{\partial y}\mathrm{d}y + \frac{\partial h}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \frac{\partial f}{\partial x}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \frac{\partial g}{\partial y}\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x + \frac{\partial h}{\partial z}\mathrm{d}x \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z
\end{aligned}
$$

Here then notice that the coefficient of $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ is simply the divergence of $\vec{F}$, $\nabla \cdot \vec{F}$

$$\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

and we have seen that each of the three vector operators occur naturally and elegantly through differential forms.

3.2 Links to vector calculus

It is possible to establish a correspondence between differential forms and vector calculus by giving a basic overview of the discussions in this text[(i)].

	Differential form	Type of field
0-forms	ϕ_1	Scalar: ϕ_1
1-forms	$\phi_1 \mathrm{d}x + \phi_2 \mathrm{d}y + \phi_3 \mathrm{d}z$	Vector: $\phi_1 \hat{\mathbf{i}} + \phi_2 \hat{\mathbf{j}} + \phi_3 \hat{\mathbf{k}}$
2-forms	$\phi_1 \mathrm{d}y \wedge \mathrm{d}z + \phi_2 \mathrm{d}z \wedge \mathrm{d}x + \phi_3 \mathrm{d}x \wedge \mathrm{d}y$	Vector: $\phi_1 \hat{\mathbf{i}} + \phi_2 \hat{\mathbf{j}} + \phi_3 \hat{\mathbf{k}}$
3-forms	$\phi_1 \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$	Scalar: ϕ_1

Table 3.1: A correspondence between differential forms and vector calculus in $\mathbb{R}^3$.

Using the definitions defined now with exterior calculus and the Divergence, Gradient and Curl we are able to define the following.

(i) 0-form: $\mathrm{d}\phi_1 \to \nabla \phi_1$

(ii) 1-form: $\mathrm{d}\alpha \to \nabla \times \vec{\Phi}$

(iii) 2-form: $\mathrm{d}\beta \to \nabla \cdot \vec{\Phi}$

Figure 3.1 uses the above correspondence to produce the following commutative diagram.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \Lambda^0 & \xrightarrow{\mathrm{d}} & \Lambda^1 & \xrightarrow{\mathrm{d}} & \Lambda^2 & \xrightarrow{\mathrm{d}} & \Lambda^3 & \longrightarrow & 0 \\
 & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\
0 & \longrightarrow & \text{Scalar field} & \xrightarrow{\text{grad}} & \text{Vector field} & \xrightarrow{\text{curl}} & \text{Vector field} & \xrightarrow{\text{div}} & \text{Scalar field} & \longrightarrow & 0
\end{array}$$

Figure 3.1: The exterior derivative alongside the Divergence, Gradient and Curl.

(i) For this section we have the following definitions $\phi_1, \phi_2, \phi_3 \in \Lambda^0(\mathbb{R}^3)$, $\vec{\Phi} = \phi_1 \hat{\mathbf{i}} + \phi_2 \hat{\mathbf{j}} + \phi_3 \hat{\mathbf{k}}$, $\alpha \in \Lambda^1(\mathbb{R}^3)$ and $\beta \in \Lambda^2(\mathbb{R}^3)$

From figure 3.1 we are able to identify that $\mathrm{d}(\mathrm{d}\omega) = 0$ for any differential form ω, and the following vector identities shown on page 3, where here, f and $\vec{F}$ denote a scalar and vector field respectively.

$$\operatorname{curl}\big(\operatorname{grad} f\big) = \nabla \times \big(\nabla f\big) = 0 \qquad \operatorname{div}\big(\operatorname{curl} \vec{F}\big) = \nabla \cdot \big(\nabla \times \vec{F}\big) = 0$$

3.3 Integral theorems and their proofs

Where we have previously seen the integral theorems described in vector calculus we now take a look at proving them using differential forms and exterior calculus.

3.3.1 Generalised Stokes' theorem

The Generalised Stokes' theorem is one of the most powerful theorems in mathematics, containing within it the fundamental theorem of Calculus, Green's theorem, Gauss' divergence theorem and the ordinary Stokes' theorem. It allows us to take the integral of a form over a closed boundary and convert it into the integral of the exterior derivative of the form over the region inside the boundary.

[19] Let Ω be a compact region of a k–dimensional oriented manifold $M \subseteq \mathbb{R}^n$. Then the boundary of Ω is given as $\partial\Omega$ and let $\omega \in \Lambda^{k-1}\big(\mathbb{R}^n\big)$ defined on an open set S containing Ω then,

$$\oint_{\partial\Omega} \omega = \int_{\Omega} \mathrm{d}\omega. \tag{3.2}$$

Proof. [29] Let $\mathbf{u} = \big(u_1, u_2, \ldots, u_n\big) : S \subseteq \Omega \to \mathbb{R}^n$ and $\sup \omega \subseteq S$. Applying a translation to $\mathbf{u}$ we can suppose that $u(S) \subseteq \mathbb{H}^n = \Big\{\mathbf{u} \in \mathbb{R}^n : u_1 < 0\Big\}$. Given the basic n-form

$$\omega = \sum_{i=1}^{n} f_i(\mathbf{u})\mathrm{d}u_1 \wedge \cdots \wedge \widehat{\mathrm{d}u_i} \wedge \cdots \wedge \mathrm{d}u_n$$

where $\widehat{\mathrm{d}u_i}$ represents that the $\mathrm{d}u_i$ term is missing, then the forms $\widehat{\mathrm{d}u_i}$ disappear on the tangent space to $\mathbb{H}^n$ $\forall\, i > 1$. Thus,

$$\oint_{\partial\mathbb{H}^n} \omega = \oint_{\partial\mathbb{H}^n} f_1(\mathbf{u})\widehat{\mathrm{d}u_1}$$

and by the exterior derivative

$$\begin{aligned}
\mathrm{d}\omega &= \sum_{i=1}^{n} \mathrm{d}f_i \wedge \widehat{\mathrm{d}u_i} \\
&= \sum_{i=1}^{n} \frac{\partial f_i}{\partial u_i} \mathrm{d}u_i \wedge \widehat{\mathrm{d}u_i} \\
&= \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial u_i} \right) \mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n.
\end{aligned}$$

Using the above we have that,

$$\int_{\mathbb{H}^n} \mathrm{d}\omega = \sum_{i=1}^{n} \int_{\mathbb{H}^n} \frac{\partial f_i}{\partial u_i} \mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n$$

and we consider the cases where $i > 1$ and $i = 1$. First we consider $i > 1$:

$$\int_{\mathbb{H}^n} \frac{\partial f_i}{\partial u_i} \mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n = \iint \cdots \int \left(\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial u_i} \right) \widehat{\mathrm{d}u_i} = 0$$

by Fubini's theorem [18]. Now considering the case where $i = 1$ we have,

$$\int_{\mathbb{H}^n} \frac{\partial f_i}{\partial u_i} \mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n = \iint \cdots \int \left(\int_{-\infty}^{0} \frac{\partial f_i}{\partial u_i} \right) \widehat{\mathrm{d}u_i} = \oint_{\partial \mathbb{H}^n} f_1(\mathbf{u}) \widehat{\mathrm{d}u_1}$$

and hence,

$$\oint_{\partial \Omega} \omega = \int_{\Omega} \mathrm{d}\omega.$$

□

3.3.2 Green's theorem

As stated on page 7, Green's theorem says that if we let ∂D be a positively oriented, piecewise smooth, simple, closed curve parameterised by $\vec{\mathbf{r}}(t)$ and let D be the region enclosed by ∂D. Then with the differentiable vector field $\vec{\Phi}(x, y) = M(x, y)\hat{\mathbf{i}} + N(x, y)\hat{\mathbf{j}}$ we have the following relation.

$$\oint_{\partial D} M \mathrm{d}x + N \mathrm{d}y = \iint_{D} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) .\mathrm{d}y\mathrm{d}x \tag{3.3}$$

Proof. Let $\omega = M\mathrm{d}x + N\mathrm{d}y \in \Lambda^1\big(\mathbb{R}^2\big)$ and $D \subseteq \mathbb{R}^2$ be some region whose boundary is ∂D then if we apply the Generalised Stokes' theorem proven in subsection 3.3.1, we have,

$$\begin{aligned}
\oint_{\partial D} M\mathrm{d}x + N\mathrm{d}y &= \oint_{\partial D} \omega \\
&= \int_D \mathrm{d}\omega \\
&= \int_D \mathrm{d}\big(M\mathrm{d}x + N\mathrm{d}y\big) \\
&= \int_D \mathrm{d}M \wedge \mathrm{d}x + \mathrm{d}N \wedge \mathrm{d}y \\
&= \int_D \left(\frac{\partial M}{\partial x}\mathrm{d}x + \frac{\partial M}{\partial y}\mathrm{d}y\right) \wedge \mathrm{d}x + \left(\frac{\partial N}{\partial x}\mathrm{d}x + \frac{\partial N}{\partial y}\mathrm{d}y\right) \wedge \mathrm{d}y \\
&= \int_D \frac{\partial M}{\partial y}\mathrm{d}y \wedge \mathrm{d}x + \frac{\partial N}{\partial x}\mathrm{d}x \wedge \mathrm{d}y \\
&= \int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y \\
&= \int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x\mathrm{d}y
\end{aligned}$$

and so the proof is complete. □

3.3.3 Stokes' theorem

As stated on page 10, Stokes' theorem says that if we let S be an oriented piecewise smooth surface that is enclosed by a simple, closed and piecewise smooth boundary curve ∂S with positive orientation and define the vector field $\vec{\Phi}(x,y,z) = F(x,y,z)\hat{\mathbf{i}} + G(x,y,z)\hat{\mathbf{j}} + H(x,y,z)\hat{\mathbf{k}}$, then Stokes' theorem is defined as the following relation.

$$\oint_{\partial S} \vec{\Phi} \cdot \mathrm{d}\vec{\mathbf{r}} = \iint_S \big(\nabla \times \vec{\Phi}\big) \cdot \mathrm{d}\vec{S} \tag{3.4}$$

Proof. Given $\vec{\Phi}(x,y,z)$, we define $\omega = \vec{\Phi} \cdot \mathrm{d}\vec{\mathbf{r}} = F\mathrm{d}x + G\mathrm{d}y + H\mathrm{d}z \in \Lambda^1\big(\mathbb{R}^3\big)$ and $S \subseteq \mathbb{R}^3$ be some region whose boundary is ∂S, then if we apply the Generalised Stokes' theorem proven in subsection 3.3.1, we have,

$$\begin{aligned}
\oint_{\partial S} \vec{\Phi} \cdot \mathrm{d}\vec{\mathbf{r}} &= \oint_{\partial S} F\mathrm{d}x + G\mathrm{d}y + H\mathrm{d}z \\
&= \oint_{\partial S} \omega \\
&= \int_S \mathrm{d}\omega \\
&= \int_S \mathrm{d}\big(F\mathrm{d}x + G\mathrm{d}y + H\mathrm{d}z\big) \\
&= \int_S \mathrm{d}F \wedge \mathrm{d}x + \mathrm{d}G \wedge \mathrm{d}y + \mathrm{d}H \wedge \mathrm{d}z \\
&= \int_S \left(\frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y + \frac{\partial F}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}x + \left(\frac{\partial G}{\partial x}\mathrm{d}x + \frac{\partial G}{\partial y}\mathrm{d}y + \frac{\partial G}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}y \\
&\qquad\qquad + \left(\frac{\partial H}{\partial x}\mathrm{d}x + \frac{\partial H}{\partial y}\mathrm{d}y + \frac{\partial H}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}z \\
&= \int_S \frac{\partial F}{\partial y}\mathrm{d}y \wedge \mathrm{d}x + \frac{\partial F}{\partial z}\mathrm{d}z \wedge \mathrm{d}x + \frac{\partial G}{\partial x}\mathrm{d}x \wedge \mathrm{d}y + \frac{\partial G}{\partial z}\mathrm{d}z \wedge \mathrm{d}y \\
&\qquad\qquad + \frac{\partial H}{\partial x}\mathrm{d}x \wedge \mathrm{d}z + \frac{\partial H}{\partial y}\mathrm{d}y \wedge \mathrm{d}z \\
&= \int_S \left(\frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}\right)\mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}\right)\mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y \\
&= \int_S \left(\frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}\right)\mathrm{d}y\mathrm{d}z + \left(\frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}\right)\mathrm{d}z\mathrm{d}x + \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right)\mathrm{d}x\mathrm{d}y \\
&= \int_S \big(\nabla \times \vec{\Phi}\big) \cdot \mathrm{d}\vec{S}
\end{aligned}$$

where $\mathrm{d}\vec{S} = \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}x \wedge \mathrm{d}y \in \Lambda^2\big(\mathbb{R}^3\big)$ and so the proof is complete. □

3.3.4 Gauss' divergence theorem

As stated on page 10, Stokes' theorem says that if we let $D \subseteq \mathbb{R}^3$ be a bounded solid region, whose boundary surface ∂D consists of finitely many piecewise smooth, closed orientable surfaces, each of which is oriented by unit normals that point away from D and $\vec{\Phi}(x, y, z) = F(x, y, z)\hat{\mathbf{i}} + G(x, y, z)\hat{\mathbf{j}} + H(x, y, z)\hat{\mathbf{k}}$ be a vector field whose domain includes D, then Gauss' divergence theorem is defined as the following relation.

$$\oiint_{\partial D} \vec{\Phi} \cdot \mathrm{d}\vec{S} = \iiint_D \nabla \cdot \vec{\Phi}.\mathrm{d}V \tag{3.5}$$

Proof. Given $\vec{\Phi}(x, y, z)$ and $\mathrm{d}\vec{S}$ defined above as $\mathrm{d}\vec{S} = \mathrm{d}y\wedge\mathrm{d}z + \mathrm{d}z\wedge\mathrm{d}x + \mathrm{d}x\wedge\mathrm{d}y \in \Lambda^2\big(\mathbb{R}^3\big)$ we can define $\omega = \vec{\Phi} \cdot \mathrm{d}\vec{S} = F\mathrm{d}y \wedge \mathrm{d}z + G\mathrm{d}z \wedge \mathrm{d}x + H\mathrm{d}x \wedge \mathrm{d}y \in \Lambda^2\big(\mathbb{R}^3\big)$ and $D \subseteq \mathbb{R}^3$, some bounded solid region enclosed by ∂D, then if we apply the Generalised Stokes' theorem proven in subsection 3.3.1, we have,

$$\begin{aligned}
\oint_{\partial D} \vec{\Phi} \cdot \mathrm{d}\vec{S} &= \oint_{\partial D} F\mathrm{d}y \wedge \mathrm{d}z + G\mathrm{d}z \wedge \mathrm{d}x + H\mathrm{d}x \wedge \mathrm{d}y \\
&= \oint_{\partial D} \omega \\
&= \int_D \mathrm{d}\omega \\
&= \int_D \mathrm{d}\big(F\mathrm{d}y \wedge \mathrm{d}z + G\mathrm{d}z \wedge \mathrm{d}x + H\mathrm{d}x \wedge \mathrm{d}y\big) \\
&= \int_D \mathrm{d}F \wedge \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}G \wedge \mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}H \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \int_D \left(\frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y + \frac{\partial F}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&\qquad + \left(\frac{\partial G}{\partial x}\mathrm{d}x + \frac{\partial G}{\partial y}\mathrm{d}y + \frac{\partial G}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}z \wedge \mathrm{d}x \\
&\qquad\qquad + \left(\frac{\partial H}{\partial x}\mathrm{d}x + \frac{\partial H}{\partial y}\mathrm{d}y + \frac{\partial H}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \int_D \frac{\partial F}{\partial x}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \frac{\partial G}{\partial y} \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x + \frac{\partial H}{\partial z}\mathrm{d}z \wedge x \wedge \mathrm{d}y \\
&= \int_D \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= \int_D \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_D \nabla \cdot \vec{\Phi}\mathrm{d}V
\end{aligned}$$

where $\mathrm{d}V = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ and so the proof is complete. □

Chapter 4

Applications of differential forms

4.1 Maxwell's equations

Maxwell's equations[(i)] are four of the most well known equations in science and are most commonly known in the order; Gauss' law for electric fields, Gauss' law for magnetic fields, Faraday's law and Ampere's law[(ii)] [14] . They are the most general equations describing the *electromagnetic field* and in fact are used to solve any problem involving the electromagnetic field, even if it is not too obvious at first [28]. In this paper we will not go into too great a detail for the equations defining all of their properties but we will consider the deduction of the differential versions of Maxwell's equations using differential forms and give definitions of the notation used[(iii)] (see subsection B).

Maxwell's equations are given as the following; (4.1a), Gauss' law for electric fields; (4.1b), Gauss' law for magnetic fields; (4.1c), Faraday's law; and (4.1d), Ampere-Maxwell's law.

$$\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\varepsilon_0} \tag{4.1a}$$

$$\nabla \cdot \vec{\mathbf{B}} = 0 \tag{4.1b}$$

$$\nabla \times \vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{B}}}{\partial t} = 0 \tag{4.1c}$$

$$\nabla \times \vec{\mathbf{B}} = \mu_0 \left(\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \right) \tag{4.1d}$$

We give the definitions of Maxwell's equations; Gauss' law for electric fields, (4.1a) says that the electric field produced by electric charge diverges and converges from and upon positive and negative charge respectively; Gauss' law for magnetic fields, (4.1b) says that the divergence of the magnetic field at any point is zero; Faraday's law, (4.1c) says that a circulating electric field is produced by a magnetic field that changes with time; and

(i) The main resource used for this section will be 'A student's guide to Maxwell's equations' by Daniel Fleisch, [14].
(ii) The notation used to describe Maxwell's equations will differ among texts.
(iii) For implementations of Maxwell's equations, one should consider other texts [14]

Ampere-Maxwell's law, (4.1d) says that a circulating magnetic field is produced by an electric current and by an electric field that changes with time [14].

4.1.1 Maxwell's equations in terms of differential forms

Choosing units such that $\mu_0 = \varepsilon_0 = 1$ we are able to rewrite (4.1a), (4.1b), (4.1c) and (4.1d) as the following set of equations.

$$\nabla \cdot \vec{\mathbf{E}} = \rho \tag{4.2a}$$

$$\nabla \cdot \vec{\mathbf{B}} = 0 \tag{4.2b}$$

$$\nabla \times \vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{B}}}{\partial t} = 0 \tag{4.2c}$$

$$\nabla \times \vec{\mathbf{B}} - \frac{\partial \vec{\mathbf{E}}}{\partial t} = \vec{\mathbf{J}} \tag{4.2d}$$

Where previously we defined the electric and magnetic fields separately, we will now describe them as one single field, the electromagnetic field $\eta \in \Lambda^2(\mathbb{R}^4)$ in space-time with variables (x, y, z, t). The electric and magnetic fields are then described as the 1- and 2-forms, (4.3) and (4.4) respectively[(iv)].

$$\mathcal{E} = \mathcal{E}_1 \mathrm{d}x + \mathcal{E}_2 \mathrm{d}y + \mathcal{E}_3 \mathrm{d}z \tag{4.3}$$

$$\mathcal{B} = \mathcal{B}_1 \mathrm{d}y \wedge \mathrm{d}z + \mathcal{B}_2 \mathrm{d}z \wedge \mathrm{d}x + \mathcal{B}_3 \mathrm{d}x \wedge \mathrm{d}y \tag{4.4}$$

Define the electromagnetic field η,

$$\begin{aligned}\eta &= \mathcal{E} \wedge \mathrm{d}t + \mathcal{B} \\ &= \mathcal{E}_1 \mathrm{d}x \wedge \mathrm{d}t + \mathcal{E}_2 \mathrm{d}y \wedge \mathrm{d}t + \mathcal{E}_3 \mathrm{d}z \wedge \mathrm{d}t + \mathcal{B}_1 \mathrm{d}y \wedge \mathrm{d}z + \mathcal{B}_2 \mathrm{d}z \wedge \mathrm{d}x + \mathcal{B}_3 \mathrm{d}x \wedge \mathrm{d}y\end{aligned}$$

(iv) Note the difference in notation where the vectors $\vec{E}$ and $\vec{B}$ have been replaced with the differential forms $\mathcal{E}$ and $\mathcal{B}$ respectively.

then we fine the exterior derivative of η as the following.

$$\begin{aligned}
\mathrm{d}\eta &= \mathrm{d}\big(\mathcal{E} \wedge \mathrm{d}t + \mathcal{B}\big) \\
&= \mathrm{d}\mathcal{E} \wedge \mathrm{d}t + \mathrm{d}\mathcal{B} \\
&= \left(\frac{\partial \mathcal{E}_1}{\partial x}\mathrm{d}x + \frac{\partial \mathcal{E}_1}{\partial y}\mathrm{d}y + \frac{\partial \mathcal{E}_1}{\partial z}\mathrm{d}z + \frac{\partial \mathcal{E}_1}{\partial t}\mathrm{d}t\right)\mathrm{d}x \wedge \mathrm{d}t \\
&\quad + \left(\frac{\partial \mathcal{E}_2}{\partial x}\mathrm{d}x + \frac{\partial \mathcal{E}_2}{\partial y}\mathrm{d}y + \frac{\partial \mathcal{E}_2}{\partial z}\mathrm{d}z + \frac{\partial \mathcal{E}_2}{\partial t}\mathrm{d}t\right)\mathrm{d}y \wedge \mathrm{d}t \\
&\quad + \left(\frac{\partial \mathcal{E}_3}{\partial x}\mathrm{d}x + \frac{\partial \mathcal{E}_3}{\partial y}\mathrm{d}y + \frac{\partial \mathcal{E}_3}{\partial z}\mathrm{d}z + \frac{\partial \mathcal{E}_3}{\partial t}\mathrm{d}t\right)\mathrm{d}z \wedge \mathrm{d}t \\
&\quad + \left(\frac{\partial \mathcal{B}_1}{\partial x}\mathrm{d}x + \frac{\partial \mathcal{B}_1}{\partial y}\mathrm{d}y + \frac{\partial \mathcal{B}_1}{\partial z}\mathrm{d}z + \frac{\partial \mathcal{B}_1}{\partial t}\mathrm{d}t\right)\mathrm{d}y \wedge \mathrm{d}z \\
&\quad + \left(\frac{\partial \mathcal{B}_2}{\partial x}\mathrm{d}x + \frac{\partial \mathcal{B}_2}{\partial y}\mathrm{d}y + \frac{\partial \mathcal{B}_2}{\partial z}\mathrm{d}z + \frac{\partial \mathcal{B}_2}{\partial t}\mathrm{d}t\right)\mathrm{d}z \wedge \mathrm{d}x \\
&\quad + \left(\frac{\partial \mathcal{B}_3}{\partial x}\mathrm{d}x + \frac{\partial \mathcal{B}_3}{\partial y}\mathrm{d}y + \frac{\partial \mathcal{B}_3}{\partial z}\mathrm{d}z + \frac{\partial \mathcal{B}_3}{\partial t}\mathrm{d}t\right)\mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{\partial \mathcal{B}_1}{\partial x} + \frac{\partial \mathcal{B}_2}{\partial y} + \frac{\partial \mathcal{B}_3}{\partial z}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \frac{\partial \mathcal{B}_1}{\partial t}\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t + \frac{\partial \mathcal{B}_2}{\partial t}\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}t \\
&\quad + \frac{\partial \mathcal{B}_3}{\partial t}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t + \frac{\partial \mathcal{E}_3}{\partial y}\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t + \frac{\partial \mathcal{E}_2}{\partial z}\mathrm{d}z \wedge \mathrm{d}y \wedge \mathrm{d}t + \frac{\partial \mathcal{E}_1}{\partial z}\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}t \\
&\quad + \frac{\partial \mathcal{E}_3}{\partial x}\mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}t + \frac{\partial \mathcal{E}_2}{\partial x}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t + \frac{\partial \mathcal{E}_1}{\partial y}\mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}t \\
&= \left(\frac{\partial \mathcal{B}_1}{\partial x} + \frac{\partial \mathcal{B}_2}{\partial y} + \frac{\partial \mathcal{B}_3}{\partial z}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \left[\left(\left[\frac{\partial \mathcal{E}_3}{\partial y} - \frac{\partial \mathcal{E}_2}{\partial z}\right] + \frac{\partial \mathcal{B}_1}{\partial t}\right)\mathrm{d}y \wedge \mathrm{d}z\right. \\
&\quad \left. + \left(\left[\frac{\partial \mathcal{E}_1}{\partial z} - \frac{\partial \mathcal{E}_3}{\partial x}\right] + \frac{\partial \mathcal{B}_2}{\partial t}\right)\mathrm{d}z \wedge \mathrm{d}x + \left(\left[\frac{\partial \mathcal{E}_2}{\partial x} - \frac{\partial \mathcal{E}_1}{\partial y}\right] + \frac{\partial \mathcal{B}_3}{\partial t}\right)\mathrm{d}x \wedge \mathrm{d}y\right] \wedge \mathrm{d}t
\end{aligned}$$

Now if we set $\mathrm{d}\eta = 0$ and equate terms we obtain (4.5) and (4.6), Gauss' law for magnetic fields (4.2b) and Faraday's law (4.2c) respectively.

$$\begin{aligned}
\frac{\partial \mathcal{B}_1}{\partial x} + \frac{\partial \mathcal{B}_2}{\partial y} + \frac{\partial \mathcal{B}_3}{\partial z} &= 0 \\
\Rightarrow \nabla \cdot \vec{\mathbf{B}} &= 0
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&\left(\left[\frac{\partial \mathcal{E}_3}{\partial y} - \frac{\partial \mathcal{E}_2}{\partial z}\right] + \frac{\partial \mathcal{B}_1}{\partial t}\right)\mathrm{d}y \wedge \mathrm{d}z \\
&\qquad + \left(\left[\frac{\partial \mathcal{E}_1}{\partial z} - \frac{\partial \mathcal{E}_3}{\partial x}\right] + \frac{\partial \mathcal{B}_2}{\partial t}\right)\mathrm{d}z \wedge \mathrm{d}x \\
&\qquad\qquad + \left(\left[\frac{\partial \mathcal{E}_2}{\partial x} - \frac{\partial \mathcal{E}_1}{\partial y}\right] + \frac{\partial \mathcal{B}_3}{\partial t}\right)\mathrm{d}x \wedge \mathrm{d}y = 0 \\
&\qquad\qquad\qquad \Rightarrow \nabla \times \vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{B}}}{\partial t} = 0
\end{aligned} \tag{4.6}$$

If we now calculate the Hodge dual of η and calculate the exterior derivative we are able to yield the following results.

$$\star\eta = \mathcal{E}_1 \mathrm{d}y \wedge \mathrm{d}z + \mathcal{E}_2 \mathrm{d}z \wedge \mathrm{d}x + \mathcal{E}_3 \mathrm{d}x \wedge \mathrm{d}y + \mathcal{B}_1 \mathrm{d}x \wedge \mathrm{d}t + \mathcal{B}_2 \mathrm{d}y \wedge \mathrm{d}t + \mathcal{B}_3 \mathrm{d}z \wedge \mathrm{d}t$$

$$\begin{aligned}
\mathrm{d}(\star\eta) &= \mathrm{d}\big(\mathcal{E}_1 \mathrm{d}y \wedge \mathrm{d}z + \mathcal{E}_2 \mathrm{d}z \wedge \mathrm{d}x + \mathcal{E}_3 \mathrm{d}x \wedge \mathrm{d}y + \mathcal{B}_1 \mathrm{d}x \wedge \mathrm{d}t + \mathcal{B}_2 \mathrm{d}y \wedge \mathrm{d}t + \mathcal{B}_3 \mathrm{d}z \wedge \mathrm{d}t\big) \\
&= \left(\frac{\partial \mathcal{E}_1}{\partial x}\mathrm{d}x + \frac{\partial \mathcal{E}_1}{\partial y}\mathrm{d}y + \frac{\partial \mathcal{E}_1}{\partial z}\mathrm{d}z + \frac{\partial \mathcal{E}_1}{\partial t}\mathrm{d}t\right)\mathrm{d}y \wedge \mathrm{d}z \\
&\quad + \left(\frac{\partial \mathcal{E}_2}{\partial x}\mathrm{d}x + \frac{\partial \mathcal{E}_2}{\partial y}\mathrm{d}y + \frac{\partial \mathcal{E}_2}{\partial z}\mathrm{d}z + \frac{\partial \mathcal{E}_2}{\partial t}\mathrm{d}t\right)\mathrm{d}z \wedge \mathrm{d}x \\
&\quad + \left(\frac{\partial \mathcal{E}_3}{\partial x}\mathrm{d}x + \frac{\partial \mathcal{E}_3}{\partial y}\mathrm{d}y + \frac{\partial \mathcal{E}_3}{\partial z}\mathrm{d}z + \frac{\partial \mathcal{E}_3}{\partial t}\mathrm{d}t\right)\mathrm{d}x \wedge \mathrm{d}y \\
&\quad + \left(\frac{\partial \mathcal{B}_1}{\partial x}\mathrm{d}x + \frac{\partial \mathcal{B}_1}{\partial y}\mathrm{d}y + \frac{\partial \mathcal{B}_1}{\partial z}\mathrm{d}z + \frac{\partial \mathcal{B}_1}{\partial t}\mathrm{d}t\right)\mathrm{d}x \wedge \mathrm{d}t \\
&\quad + \left(\frac{\partial \mathcal{B}_2}{\partial x}\mathrm{d}x + \frac{\partial \mathcal{B}_2}{\partial y}\mathrm{d}y + \frac{\partial \mathcal{B}_2}{\partial z}\mathrm{d}z + \frac{\partial \mathcal{B}_2}{\partial t}\mathrm{d}t\right)\mathrm{d}y \wedge \mathrm{d}t \\
&\quad + \left(\frac{\partial \mathcal{B}_3}{\partial x}\mathrm{d}x + \frac{\partial \mathcal{B}_3}{\partial y}\mathrm{d}y + \frac{\partial \mathcal{B}_3}{\partial z}\mathrm{d}z + \frac{\partial \mathcal{B}_3}{\partial t}\mathrm{d}t\right)\mathrm{d}z \wedge \mathrm{d}t \\
&= \left(\frac{\partial \mathcal{E}_1}{\partial x} + \frac{\partial \mathcal{E}_2}{\partial y} + \frac{\partial \mathcal{E}_3}{\partial z}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \frac{\partial \mathcal{E}_1}{\partial t}\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t + \frac{\partial \mathcal{E}_2}{\partial t}\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}t \\
&\quad + \frac{\partial \mathcal{E}_3}{\partial t}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t + \frac{\partial \mathcal{B}_3}{\partial y}\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t + \frac{\partial \mathcal{B}_2}{\partial z}\mathrm{d}z \wedge \mathrm{d}y \wedge \mathrm{d}t + \frac{\partial \mathcal{B}_1}{\partial z}\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}t \\
&\quad + \frac{\partial \mathcal{B}_3}{\partial x}\mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}t + \frac{\partial \mathcal{B}_2}{\partial x}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t + \frac{\partial \mathcal{B}_1}{\partial y}\mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}t \\
&= \left(\frac{\partial \mathcal{E}_1}{\partial x} + \frac{\partial \mathcal{E}_2}{\partial y} + \frac{\partial \mathcal{E}_3}{\partial z}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \left[\left(\left[\frac{\partial \mathcal{B}_3}{\partial y} - \frac{\partial \mathcal{B}_2}{\partial z}\right] - \frac{\partial \mathcal{E}_1}{\partial t}\right)\mathrm{d}y \wedge \mathrm{d}z\right. \\
&\quad \left. + \left(\left[\frac{\partial \mathcal{B}_1}{\partial z} - \frac{\partial \mathcal{B}_3}{\partial x}\right] - \frac{\partial \mathcal{E}_2}{\partial t}\right)\mathrm{d}z \wedge \mathrm{d}x + \left(\left[\frac{\partial \mathcal{B}_2}{\partial x} - \frac{\partial \mathcal{B}_1}{\partial y}\right] - \frac{\partial \mathcal{E}_3}{\partial t}\right)\mathrm{d}x \wedge \mathrm{d}y\right] \wedge \mathrm{d}t
\end{aligned}$$

Where $\vec{\mathbf{J}}$ defines the current vector we define the 4-current $\vec{\mathbf{I}} = \big(\rho, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\big)$ and the corresponding differential form $\omega_{\mathcal{J}} \in \Lambda^1\big(\mathbb{R}^4\big)$.

$$\omega_{\mathcal{J}} = \rho \mathrm{d}t + \mathcal{J}_1 \mathrm{d}x + \mathcal{J}_2 \mathrm{d}y + \mathcal{J}_3 \mathrm{d}z$$

Applying the Hodge star operator to $\omega_{\mathcal{J}}$ we have the following.

$$\star\omega_{\mathcal{J}} = \rho \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \mathcal{J}_1 \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t + \mathcal{J}_2 \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}t + \mathcal{J}_3 \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t$$

If we consider the form equation,

$$\mathrm{d}(\star\eta) = \star\omega_{\mathcal{J}}$$

and equate coefficients we are able to yield (4.7) and (4.8), Gauss' law for electric fields

(4.1a) and Ampere-Maxwell's law (4.1d) respectively.

$$\begin{aligned} \frac{\partial \mathcal{E}_1}{\partial x} + \frac{\partial \mathcal{E}_2}{\partial y} + \frac{\partial \mathcal{E}_3}{\partial z} &= \rho \\ \Rightarrow \nabla \cdot \vec{\mathbf{E}} &= \rho \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\left(\left[\frac{\partial \mathcal{B}_3}{\partial y} - \frac{\partial \mathcal{B}_2}{\partial z}\right] - \frac{\partial \mathcal{E}_1}{\partial t}\right)\mathrm{d}y \wedge \mathrm{d}z \\ &\quad + \left(\left[\frac{\partial \mathcal{B}_1}{\partial z} - \frac{\partial \mathcal{B}_3}{\partial x}\right] - \frac{\partial \mathcal{E}_2}{\partial t}\right)\mathrm{d}z \wedge \mathrm{d}x \\ &\qquad + \left(\left[\frac{\partial \mathcal{B}_2}{\partial x} - \frac{\partial \mathcal{B}_1}{\partial y}\right] - \frac{\partial \mathcal{E}_3}{\partial t}\right)\mathrm{d}x \wedge \mathrm{d}y = \mathcal{J}_1 \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t \\ &\qquad\qquad + \mathcal{J}_2 \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}t \\ &\qquad\qquad + \mathcal{J}_3 \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t \\ &\Rightarrow \nabla \times \vec{\mathbf{B}} - \frac{\partial \vec{\mathbf{E}}}{\partial t} = \vec{\mathbf{J}} \end{aligned} \tag{4.8}$$

In fact, by forming the electromagnetic field and current density in terms of differential forms we have been able to reduce Maxwell's four equations (4.1a), (4.1b), (4.1c) and (4.1d) to the two equations (4.9) and (4.10).

$$\mathrm{d}\eta = 0 \tag{4.9}$$

$$\mathrm{d}(\star\eta) = \star\omega_{\mathcal{J}} \tag{4.10}$$

This is a second example shown in this text of the elegance of differential forms alongside the Generalised Stokes' theorem (3.2) (see page 29).

4.2 Current research

4.2.1 Differential forms and its applications

'Differential forms and its applications' [17], uses differential forms in Pullback calculations and proves the main theorems of vector calculus, (1.22), (1.38) and (1.39). Examples of Pullback calculations can be found in texts such as [11, 21, 26]. In particular the paper gives a convenient test for a transformation from a set of independent coordinates to another to be canonical is introduced with examples shown based on differential forms. This new test was an advancement and is seen to be more suitable than the tests provided by Goldstein et. al [15].

Described in terms of differential forms, a canonical transformation of a set of coordinates, $\left(q^i, p_i\right)$ to another, $\left(Q^i, P^i\right)$ is seen as one which leaves ω in the same form[(v)]. That is, a transformation is canonical if and only if the following condition holds.

$$\mathrm{d}q^i \wedge \mathrm{d}p_i = \mathrm{d}Q^i \wedge \mathrm{d}P^i$$

The following example is taken from Gupta and Sharma [17] but here we show extra steps.

Example 4.2.1. Show that the transformation from coordinates (x, y) to (s, t) is canonical with s and t defined by the following equations.

$$s = \ln\left(\frac{\sin(y)}{x}\right) \qquad\qquad t = x\cot(y)$$

Solution 4.2.1.

$$\begin{aligned}
\mathrm{d}s \wedge \mathrm{d}t &= \left(\frac{\partial s}{\partial x}\mathrm{d}x + \frac{\partial s}{\partial y}\mathrm{d}y\right) \wedge \left(\frac{\partial t}{\partial x}\mathrm{d}x + \frac{\partial t}{\partial y}\mathrm{d}y\right) \\
&= \mathrm{d}\left(\ln\left(\frac{\sin(y)}{x}\right)\right) \wedge \mathrm{d}\big(x\cot(y)\big) \\
&= \left(-\frac{1}{x}\mathrm{d}x + \cot(y)\mathrm{d}y\right) \wedge \left(\cot(y)\mathrm{d}x - x\csc^2(y)\right) \\
&= \csc^2(y)\mathrm{d}x \wedge \mathrm{d}y + \cot^2(y)\mathrm{d}y \wedge \mathrm{d}x \\
&= \csc^2(y)\mathrm{d}x \wedge \mathrm{d}y - \cot^2(y)\mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\csc^2(y) - \cot^2(y)\right)\mathrm{d}x \wedge \mathrm{d}y \\
&= \mathrm{d}x \wedge \mathrm{d}y
\end{aligned}$$

Hence the transformation is canonical since $\mathrm{d}s \wedge \mathrm{d}t = \mathrm{d}x \wedge \mathrm{d}y$.

[(v)] This text describes this concept in a basic way and suggests the reading of [17] for further information into canonical transformations.

4.2.2 Fractional differential forms

'Fractional differential forms' [10], describes an approach to a generalisation of exterior calculus by allowing the partial derivatives in the exterior differential to assume fractional orders; a fractional exterior derivative is defined as (4.11), where ν is the order of the fractional coordinate differential and a_i is the initial point of the derivative.

$$\mathrm{d}^{\nu} f = \sum_{i=1}^{n} \frac{\partial^{\nu} f}{\partial \big(u_i - a_i\big)^{\nu}} \mathrm{d}u_i \tag{4.11}$$

The paper gives a brief overview of fractional calculus defining the Riemann-Louiville fractional integral and differential and their properties. It defines the fractional order derivative of x^p the same way as many texts do with,

$$\frac{\partial^{\nu} x^p}{\partial x^{\nu}} = \frac{\Gamma(p+1)}{\Gamma(p-\nu+1)} x^{p-\nu} \qquad \forall\, p < -1,\ q \geq 0$$

where $\Gamma(z)$ is the Gamma function. With the above, the paper defines, in two dimensions (x, y), the fractional exterior derivative order ν of x^p, with the initial point taken to be the origin as (4.12).

$$\mathrm{d}^{\nu} x^p = \frac{\Gamma(p+1)}{\Gamma(p-\nu+1)} x^{p-\nu} \mathrm{d}x^{\nu} + \frac{x^p}{y^{\nu}\Gamma(1-\nu)} \mathrm{d}y^{\nu} \tag{4.12}$$

As an extension to this paper, we can define in n–dimensions $\big(u_1, \ldots, u_n\big)$, the fractional exterior derivative order ν of u_1^p, with the initial point taken to be $(0, \ldots, 0)$ as (4.13).

$$\mathrm{d}^{\nu} u_1^p = \frac{\Gamma(p+1)}{\Gamma(p-\nu+1)} u_1^{p-\nu} \mathrm{d}u_1^{\nu} + \sum_{i=2}^{n} \frac{u_1^p}{u_i^{\nu}\Gamma(1-\nu)} \mathrm{d}u_i^{\nu} \tag{4.13}$$

The paper defines a fractional form space $F(\nu, m, n)$ where ν is the fractional order of the differential m-forms in n variables. The paper defines, as an example, the basis set for $F(\nu, 1, n)$ as $\big\{\mathrm{d}x_1^{\nu}, \mathrm{d}x_2^{\nu}, \ldots, \mathrm{d}x_n^{\nu}\big\}$ and gives an arbitrary element of $F(\nu, 1, n)$ as (4.14).

$$\alpha = \sum_{i=1}^{n} \alpha_i \mathrm{d}x_i^{\nu} \in F(\nu, 1, n) \tag{4.14}$$

For a fixed ν the above defines an n–dimensional vector space and for $\nu = 1$ the 1-forms from standard exterior calculus are recovered. The paper defines the basis set and an arbitrary element for $F(\nu, 2, n)$ as (4.15) and (4.16) respectively and an arbitrary element of $F(\nu, 3, n)$ as (4.17) with $\beta_{ii}\big(\mu, \mu\big) = 0$ and $\beta_{iii}\big(\mu, \mu, \mu\big) = 0$.

$$F(\nu, 2, n) = \big\{\mathrm{d}x_1^{\mu_{11}} \wedge \mathrm{d}x_1^{\mu_{21}}, \mathrm{d}x_1^{\mu_{11}} \wedge \mathrm{d}x_2^{\mu_{31}}, \ldots, \mathrm{d}x_n^{\mu_{n-1m}} \wedge \mathrm{d}x_n^{\mu_{nm}} \mid \mu_{ij} + \mu_{kj} = \nu\big\} \tag{4.15}$$

$$\beta = \sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^{\nu}\Big(\beta_{ij}\big(\nu_i, \nu-\nu_i\big)\mathrm{d}x_i^{\nu_i}\wedge \mathrm{d}x_j^{\nu-\nu_i}\Big)\mathrm{d}\nu_i \tag{4.16}$$

$$\beta = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\int_0^{\nu}\int_0^{\nu-\nu_i}\Big(\beta_{ijk}\big(\nu_i, \nu-\nu_j, \nu-\nu_i-\nu_j\big)\mathrm{d}x_i^{\nu_i}\wedge \mathrm{d}x_j^{\nu_j}\wedge \mathrm{d}x_k^{\nu-\nu_i-\nu_j}\Big)\mathrm{d}\nu_i \tag{4.17}$$

As the fractional differential forms increase to become m-forms the basis set of $F(\nu, m, n)$ is defined in the paper to be (4.18).

$$F(\nu, m, n) = \left\{\mathrm{d}x_{i_1}^{\mu_{i_1 1}}\wedge \mathrm{d}x_{i_2}^{\mu_{i_2 1}}\wedge\cdots\wedge \mathrm{d}x_{i_m}^{\mu_{i_m 1}}, \ldots \;\middle|\; \sum_{k=1}^{m}\mu_{i_k j} = \nu\right\} \tag{4.18}$$

The paper then goes on to define properties for fractional exterior calculus including closed and exact fractional forms and a coordinate transformations for the fractional differentials to curvilinear coordinates.

4.2.3 Applications of fractional exterior differential in three–dimensional space

'Applications of fractional exterior differential in three–dimensional space' [35] is an article that is, somewhat of an extension to *'Fractional differential forms'* [10]. The fractional exterior transition to curvilinear coordinates at the origin are discussed and the two coordinate transformations for fractional differentials for three–dimensional Cartesian coordinates to spherical and cylindrical coordinates are obtained, respectively. In particular for $\nu = m = 1$ in the form space $F(\nu, m, n)$, the usual exterior transformations, between spherical and cylindrical coordinates and the Cartesian coordinates are found respectively, from fractional exterior transformation[(vi)].

Where $\{x_i\}$ and $\{y_i\}$ are taken to be the Cartesian and curvilinear coordinate systems respectively, the assumption is made that $\{x_i\}$ can be written smoothly in terms of $\{y_i\}$; that is, $x_i = x_i(y)$. Applying the ν^{th} order fractional exterior derivative to both sides of this relation we yield (4.19).

$$\sum_{i=1}^{n}\frac{\partial^{\nu}}{\partial x_i^{\nu}}\mathrm{d}x_i^{\nu} = \sum_{i=1}^{n}\frac{\partial^{\nu}}{\partial y_i^{\nu}}\mathrm{d}y_i^{\nu} \tag{4.19}$$

The following example is taken from Yong et. al [35]. The example considers the coordinate transformation for three–dimensional Cartesian to cylindrical coordinates respectively in terms of the fractional exterior differential.

Example 4.2.2. Find the fractional coordinate transformation for the cylindrical coordinate

[(vi)] For some information on transition to curvilinear coordinates in the standard exterior calculus, refer to texts such as [7, 31].

system.

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$$

Solution 4.2.2. Applying the fractional exterior differential we have,

$$\mathrm{d}x = \frac{\Gamma(2\nu - m + 1)}{\Gamma(\nu+1)\Gamma(\nu - 2m + 1)} \frac{z^{\nu-m}\cos^{\nu}(\theta)}{\sin^{m-\nu}(\theta)} r^{\nu-m}\mathrm{d}r^{\nu} \tag{4.20}$$

$$+ \frac{r^{2\nu-m}z^{\nu-m}}{\Gamma(\nu+1)} \frac{\partial^{\nu}}{\partial\theta^{\nu}} \left(\frac{\cos^{\nu}(\theta)}{\sin^{m-\nu}(\theta)}\right) \mathrm{d}\theta^{\nu} \tag{4.21}$$

$$+ \frac{r^{2\nu-m}\Gamma(\nu - m + 1)}{z^m\Gamma(1-m)\Gamma(\nu+1)} \frac{\cos^{\nu}(\theta)}{\sin^{m-\nu}(\theta)} \mathrm{d}z^{\nu} \tag{4.22}$$

$$\mathrm{d}y = \frac{\Gamma(2\nu - m + 1)}{\Gamma(\nu+1)\Gamma(\nu - 2m + 1)} \frac{z^{\nu-m}\sin^{\nu}(\theta)}{\cos^{m-\nu}(\theta)} r^{\nu-m}\mathrm{d}r^{\nu} \tag{4.23}$$

$$+ \frac{r^{2\nu-m}z^{\nu-m}}{\Gamma(\nu+1)} \frac{\partial^{\nu}}{\partial\theta^{\nu}} \left(\frac{\sin^{\nu}(\theta)}{\cos^{m-\nu}(\theta)}\right) \mathrm{d}\theta^{\nu} \tag{4.24}$$

$$+ \frac{r^{2\nu-m}\Gamma(\nu - m + 1)}{z^m\Gamma(1-m)\Gamma(\nu+1)} \frac{\sin^{\nu}(\theta)}{\cos^{m-\nu}(\theta)} \mathrm{d}z^{\nu} \tag{4.25}$$

$$\mathrm{d}z = \frac{\Gamma(2\nu - m + 1)}{\Gamma(\nu+1)\Gamma(\nu - 2m + 1)} \frac{z^{\nu-m}\cos^{\nu}(\theta)}{\sin^{m-\nu}(\theta)} r^{\nu-2m}\mathrm{d}r^{\nu} \tag{4.26}$$

$$+ \frac{r^{2\nu-2m}z^{\nu}}{\Gamma(\nu+1)} \frac{\partial^{\nu}}{\partial\theta^{\nu}} \left(\frac{\cos^{\nu-m}(\theta)}{\sin^{m-\nu}(\theta)}\right) \mathrm{d}\theta^{\nu} \tag{4.27}$$

$$+ \frac{r^{2\nu-2m}\cos^{\nu-m}(\theta)}{\sin^{m-\nu}(\theta)} \mathrm{d}z^{\nu}. \tag{4.28}$$

For $\nu = m = 1$ the above simply reduces to,

$$\begin{aligned} \mathrm{d}x &= \cos(\theta)\mathrm{d}r - r\sin(\theta)\mathrm{d}\theta \\ \mathrm{d}y &= \sin(\theta)\mathrm{d}r + r\cos(\theta)\mathrm{d}\theta \\ \mathrm{d}z &= \mathrm{d}z, \end{aligned}$$

which is the same as the usual exterior transformation between the Cartesian and cylindrical coordinate systems in three–dimensional space.

In summary, the paper finds the two coordinate transformations for the fractional differentials for three–dimensional Cartesian coordinates to spherical and cylindrical coordinates. In particular, for $\nu = m = 1$, the transformations obtained are the same as the standard results obtained from the exterior calculus.

Appendices

Appendix A

A brief in vectors

Note that the main resources used for this chapter are [8, 12, 25, 24, 32, 33].

A.1 Scalars and vectors

The most common quantity used within mathematics and physics has magnitude but no direction within any space [24]; the *scalar*. The scalar is made up using an appropriate choice of *unit* and real number called the *measure*; common examples being, temperature, speed, volume, time, distance and mass. The second most common quantity then has both magnitude and direction; the *vector*. The vector is made up using an appropriate choice of unit and measure, combined with the specification of direction in a given space; common examples being, velocity, force and displacement. Usually thought of to be an arrow connecting two points in space, the vector is commonly denoted using a boldfaced letter or $\overrightarrow{OP}$ where O and P represent the *tail* and *tip* of the arrow, or the *initial* and *terminal* points respectively as in figure A.1, [33].

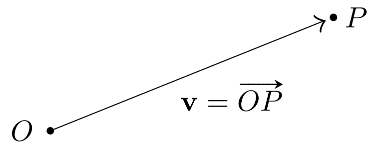

Figure A.1: A vector from an initial point O to a terminal point P

The vector $\mathbf{v} = \overrightarrow{OP}$ has a magnitude denoted by v, $|\mathbf{v}|$, $\left|\overrightarrow{OP}\right|$ or OP depending on the notation used. The magnitude of a vector is simply length from its initial to terminal point. In this text, we will denote a vector $\mathbf{v}$ or $\overrightarrow{OP}$ as having magnitudes, v and OP respectively. A vector is made up of n *components* that provide its direction in an n–dimensional space, denoted $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \; \forall \, \alpha_i \in \mathbb{R}$, where the notation $\langle \cdot \rangle$ of a vector is not to be confused with the notation of a coordinate $(\cdot)$. In this text we will define $\mathbb{R}^n$ to be the n–dimensional real space.

Figure A.2, as the caption describes, shows the direction of the vector $\mathbf{v} = \langle 5, 4 \rangle$ in $\mathbb{R}^2$,

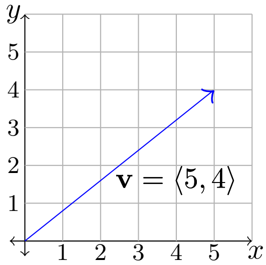

Figure A.2: The vector $\mathbf{v} = \langle 5, 4\rangle$ in $\mathbb{R}^2$, the xy–plane.

the xy–plane, where 5 and 4 are the x and y components of the vector $\mathbf{v}$ respectively. By the pythagorean theorem, the magnitude of the vector $\mathbf{v}$ is given as $v = \sqrt{5^2 + 4^2} = \sqrt{34}$. When extended to a vector of n–components, the vector $\mathbf{v} = \langle \alpha_1, \alpha_2, \ldots, \alpha_n\rangle$ is described to have magnitude, $v = \sqrt{(\alpha_1)^2 + (\alpha_2)^2 + \cdots + (\alpha_n)^2}$.

A.2 Vector algebra

Vector algebra can take place in n dimensions, commonly known as a Euclidean n-space[(i)] and with slight variations, the algebra's defined on numbers, or scalars, can be extended to the algebra of vectors including the following fundamental definitions.

Definition A.2.1. [33] Two vectors $\mathbf{u}$ and $\mathbf{v}$ are defined to be *equal* iff they have the same direction and equal magnitude.

Definition A.2.2. [33] A vector $\mathbf{v}$ when negated, $-\mathbf{v}$, has same magnitude but opposite direction.

Then the operations of addition and multiplication can be extended from $\mathbb{R}$ to $\mathbb{R}^n$ with the following two definitions.

Definition A.2.3 (Vector addition & subtraction)**.** [12] Given two vectors $\mathbf{u} = \langle \alpha_1, \alpha_2, \ldots, \alpha_n\rangle$, $\mathbf{v} = \langle \beta_1, \beta_2, \ldots, \beta_n\rangle \in \mathbb{R}^n$ we define their sum as the following.

$$\mathbf{u} \pm \mathbf{v} = \langle \alpha_1, \alpha_2, \ldots, \alpha_n\rangle \pm \langle \beta_1, \beta_2, \ldots, \beta_n\rangle = \langle \alpha_1 \pm \beta_1, \alpha_2 \pm \beta_2, \ldots, \alpha_n \pm \beta_n\rangle$$

Definition A.2.4 (Scalar Multiplication)**.** [12] Given the vector $\mathbf{u} = \langle \alpha_1, \alpha_2, \ldots, \alpha_n\rangle \in \mathbb{R}^n$ and the scalar $\lambda \in \mathbb{R}$ then we define scalar multiplication as the following.

$$\lambda\mathbf{u} = \lambda\langle \alpha_1, \alpha_2, \ldots, \alpha_n\rangle = \langle \lambda\alpha_1, \lambda\alpha_2, \ldots, \lambda\alpha_n\rangle$$

By definition A.2.3, a vector $\overrightarrow{AB}$ who's initial and terminal points are $\langle \alpha_1, \alpha_2, \ldots, \alpha_n\rangle$ and $\langle \beta_1, \beta_2, \ldots, \beta_n\rangle$ respectively with neither at the origin, has components $\langle \beta_1 - \alpha_1, \beta_2 - \alpha_2, \ldots, \beta_n - \alpha_n\rangle$. This type of vector is called a *displacement vector*, whereas a vector with

(i) In this paper we define the Euclidean n-space with the notation $\mathbb{R}^n$ opposed to the notation $\mathbb{E}^n$

its tail at the origin is called a *position vector*. Vectors, when written out, can be noted in different ways depending on the area of maths they are used in[(ii)]. A 'column' vector is commonly used within linear algebra, angled brackets, $\langle\cdot\rangle$, are used in vector calculus and the expanded basis form of a vector, $\alpha_1\hat{\mathbf{e}}_1 + \alpha_2\hat{\mathbf{e}}_n + \cdots + \alpha_n\hat{\mathbf{e}}_n$ is commonly used for mechanics, where $\hat{\mathbf{e}}_i$, $\forall\, 1 \leq i \leq n$ represent n basis vectors for $\mathbb{R}^n$[(iii)(iv)]. Given any non zero vector $\mathbf{v} \in \mathbb{R}^n$, we can define the unit vector of $\mathbf{v}$, $\hat{\mathbf{v}}$ by the following relation.

$$\hat{\mathbf{v}} = \frac{1}{v}\mathbf{v} \tag{A.1}$$

Given definitions A.2.3 and A.2.4 we can summarise the laws of vector algebra [32] as follows,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \tag{A.2a}$$
$$(\mathbf{u} + \mathbf{v}) + \mathbf{x} = \mathbf{u} + (\mathbf{v} + \mathbf{x}) \tag{A.2b}$$
$$\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v} \tag{A.2c}$$
$$(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v} \tag{A.2d}$$
$$\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v} \tag{A.2e}$$

where $\mathbf{u}, \mathbf{v}$ and $\mathbf{x}$ are any three vectors and $\alpha, \beta \in \mathbb{R}$. Here (A.2a) is the commutative law of addition, (A.2b) and (A.2c) are the associative laws of addition and scalar multiplication and (A.2d) and (A.2e) are the distributive laws of addition and scalar multiplication [24].

A.2.1 Scalar product

The *scalar product*, also called the *dot* or *inner product*, is defined as the following equation,

$$\mathbf{u} \cdot \mathbf{v} = uv\cos(\theta) \tag{A.3}$$

where θ defines the angle between the two vectors $\mathbf{u}$ and $\mathbf{v}$. Texts commonly define only the left hand side of (A.3) as the dot or inner product, whilst the whole equation is seen to be the scalar product. Given two vectors $\mathbf{u} = \langle u_1, u_2, \ldots, u_n\rangle$, $\mathbf{v} = \langle v_1, v_2, \ldots, v_n\rangle \in \mathbb{R}^n$ the dot product can be defined as the following, (A.4).

$$\mathbf{u} \cdot \mathbf{v} = \langle \mathbf{u}, \mathbf{v}\rangle = \langle u_1, u_2, \ldots, u_n\rangle \cdot \langle v_1, v_2, \ldots, v_n\rangle = \langle u_1v_1, u_2v_2, \ldots, u_nv_n\rangle = \sum_{j=1}^{n} u_jv_j\hat{\mathbf{e}}_j \tag{A.4}$$

A number of properties for the scalar product follow from (A.3) [25].

(i) The dot product is commutative, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

(ii) All of these notations will be used within this text.

(iii) $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are commonly used to represent the x, y and z axes respectively in the 3–dimensional Cartesian space.

(iv) This paper will employ the use of all vector notation.

(ii) The dot product is distributive over addition, $\mathbf{u} \cdot (\mathbf{v} + \mathbf{x}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{x}$.

(iii) The dot product of a vector with itself is the square of its magnitude, $\mathbf{u} \cdot \mathbf{u} = u^2$.

(iv) If two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular (orthogonal) then $\mathbf{u} \cdot \mathbf{v} = 0$.

(v) Conversely, if $\mathbf{u} \cdot \mathbf{v} = 0$ then either the two vectors are perpendicular or one of the vectors is the zero vector.

A.2.2 Vector product

The *vector product*, also called the *cross product* is a vector quantity, written $\mathbf{u} \times \mathbf{v}$, where $\mathbf{u} = \langle u_1, u_2, u_3 \rangle, \mathbf{v} = \langle v_1, v_2, v_3 \rangle \in \mathbb{R}^3$ and is defined by (A.5).

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \big(u_2v_3 - u_3v_2\big)\hat{\mathbf{i}} + \big(u_3v_1 - u_1v_3\big)\hat{\mathbf{j}} + \big(u_1v_2 - u_2v_1\big)\hat{\mathbf{k}} \tag{A.5}$$

By definition as the vector product produces a vector, it must have magnitude and direction [25]. The magnitude of $\mathbf{u} \times \mathbf{v} = uv \sin(\theta)$, where θ is the angle between the two vectors $\mathbf{u}$ and $\mathbf{v}$. The direction of $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ in the right-handed sense[(v)]. Given this we can write another definition of the vector product as,

$$\mathbf{u} \times \mathbf{v} = uv \sin(\theta)\mathbf{x} \tag{A.6}$$

where $\hat{\mathbf{x}}$ is a unit vector perpendicular to $\mathbf{u}$ and $\mathbf{v}$ in the right-handed sense.

A number of properties can be defined then for the vector product [25].

(i) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

(ii) The vector product is distributive over addition, $\mathbf{u} \times (\mathbf{v} + \mathbf{x}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{x}$.

(iii) If two vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel then $\mathbf{u} \times \mathbf{v} = \vec{0}$. Hence, $\mathbf{u} \times \mathbf{u} = \vec{0}$.

A.2.3 The triple products

The *scalar triple product*, known seldomly as the *mixed* or *box product*. of three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{x}$ is defined to be (A.7) [25, 34].

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{x}) \tag{A.7}$$

Combining (A.4) and (A.5) then we are able to write the scalar triple product in terms of the components of the vectors themselves (A.8), where the vectors are $\mathbf{u} = \langle u_1, u_2, u_3 \rangle, \mathbf{v} =$

[(v)] Imagine a right-handed screw rotated from $\mathbf{u}$ towards $\mathbf{v}$, then this screw moves in the direction of $\mathbf{u} \times \mathbf{v}$; the right-handed sense.

$\langle v_1, v_2, v_3 \rangle, \mathbf{x} = \langle x_1, x_2, x_3 \rangle \in \mathbb{R}^3$.

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{x}) = u_1\big(v_2x_3 - v_3x_2\big)\hat{\mathbf{i}} + u_2\big(v_3x_1 - v_1x_3\big)\hat{\mathbf{j}} + u_3\big(v_1x_2 - v_2x_1\big)\hat{\mathbf{k}} \tag{A.8}$$

A number of properties can be defined then for the scalar triple product [25].

(i) The dot and cross products can be interchanged, i.e. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{x}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{x}$.

(ii) The scalar triple product can be written in the form of a determinant of a matrix.

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{x}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ x_1 & x_2 & x_3 \end{vmatrix}$$

(iii) The vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{x}$ can be permuted cyclically, i.e. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{x}) = \mathbf{x} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{x} \times \mathbf{u})$.

(iv) If two vectors are equal then the scalar triple product is equal to the zero vector, $\vec{0}$.

The *vector triple product* of three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{x}$ yields a vector quantity and is defined to be (A.9) [25, 34].

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{x}) \tag{A.9}$$

A couple of properties can be defined for the vector triple product [25].

(i) $\mathbf{u} \times (\mathbf{v} \times \mathbf{x}) = (\mathbf{u} \cdot \mathbf{x})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{x}$.

(ii) $\mathbf{x} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \mathbf{x})\mathbf{u} - (\mathbf{u} \cdot \mathbf{x})\mathbf{v}$.

Above (i) represents what is known as the *triple product expansion* or *Lagrange's formula*(vi) [20].

A.3 Vector-valued functions

A vector whose components are mathematical functions of one or more variables, and has an image comprised of multidimensional vectors is called a *vector-valued function* or simply a *vector function*. That is for an n-dimensional vector and n functions $\phi_1, \phi_2, \ldots, \phi_n$ of m variables $x_1, x_2, \ldots, x_m$, a vector-valued function $\vec{\Phi}$ can be defined as,

$$\vec{\Phi}\big(x_1, x_2, \ldots, x_m\big) = \sum_{j=1}^{n} \phi_j\big(x_1, x_2, \ldots, x_m\big)\hat{\mathbf{e}}_j.$$

A number of properties in calculus can be defined for vector-valued functions.

(vi) Lagrange's formula is a name used for several formulae.

(i) The partial derivative of an n-dimensional vector function $\vec{\Phi}$ with respect to a variable ξ is defined as,

$$\frac{\partial \vec{\Phi}}{\partial \xi} = \sum_{j=1}^{n} \frac{\partial \phi_j}{\partial \xi} \hat{\mathbf{e}}_j$$

where ϕ_j are the component functions of $\vec{\Phi}$.

(ii) By the above, the derivative of an n-dimensional vector function $\vec{\Phi}$ is defined as,

$$\vec{\Phi}' = \sum_{j=1}^{n} \phi_j' \hat{\mathbf{e}}_j$$

where ϕ_j' are the derivatives of the component functions of $\vec{\Phi}$.

(iii) For an n-dimensional vector function $\vec{\Phi}$ the limit of $\vec{\Phi}$ as a variable $\xi \to \alpha$ is defined as,

$$\lim_{\xi \to \alpha} \vec{\Phi} = \sum_{j=1}^{n} \lim_{\xi \to \alpha} \phi_j \hat{\mathbf{e}}_j$$

where ϕ_j are the component functions of $\vec{\Phi}$.

(iv) The indefinite integral of an n-dimensional vector function $\vec{\Phi}$ with respect to a variable ξ is defined as,

$$\int \vec{\Phi}.\mathrm{d}\xi = \sum_{j=1}^{n} \left(\int \phi_j.\mathrm{d}\xi + c_j \right) \hat{\mathbf{e}}_j$$

where ϕ_j are the component functions of $\vec{\Phi}$ and c_j are the constants of integration.

(v) The definite integral of an n-dimensional vector function $\vec{\Phi}$ with respect to a variable ξ is defined as,

$$\int_{\alpha}^{\beta} \vec{\Phi}.\mathrm{d}\xi = \sum_{j=1}^{n} \int_{\alpha}^{\beta} \phi_j.\mathrm{d}\xi \hat{\mathbf{e}}_j$$

where ϕ_j are the component functions of $\vec{\Phi}$.

A number of further properties can be evaluated from the definitions above with respect to a variable ξ where $\mathbf{u}$ and $\mathbf{v}$ are vectors, $\lambda \in \mathbb{R}$ and $\phi(t)$ is a function.

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\big(\mathbf{u}+\mathbf{v}\big) = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\xi} + \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\xi} \tag{A.10a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\big(\lambda\mathbf{u}\big) = \lambda\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\xi} \tag{A.10b}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\big(\phi(\xi)\mathbf{u}(\xi)\big) = \frac{\mathrm{d}\phi}{\mathrm{d}\xi}\mathbf{u}(\xi) + \phi(\xi)\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\xi} \tag{A.10c}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\big(\mathbf{u}\cdot\mathbf{v}\big) = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\xi}\mathbf{v} + \mathbf{u}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\xi} \tag{A.10d}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\big(\mathbf{u}\times\mathbf{v}\big) = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\xi}\times\mathbf{v} + \mathbf{u}\times\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\xi} \tag{A.10e}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\Big(\mathbf{u}\big(\phi(\xi)\big)\Big) = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\xi} \tag{A.10f}$$

A.4 Fields

If there corresponds a scalar $\phi(x, y, z)$ to each point (x, y, z) of a region R in space then ϕ is called a *scalar point function* or *scalar function of position* and ϕ is a *scalar field* in R [32]. The most common example of a scalar field would be the temperature inside of a room; such temperature has differing values at differing points in space, hence the temperature T is a function of position and is a scalar field. Other examples of scalar fields are gravitational and electrostatic potential. A scalar field which is independent of time is called a *stationary* or *steady-state scalar field*.

Similar to the above, if there corresponds a vector $\vec{\Phi}(x, y, z)$ to each point (x, y, z) of a region R in space then $\vec{\Phi}$ is called a *vector point function* or *vector function of position* and $\vec{\Phi}$ is a *vector field* in R [32]. A vector field can be obtained by applying the gradient operator (see chapter 1.1 section 1.1.1) to a scalar field. The most common examples of vector fields are the gravitational and electromagnetic fields. A vector field which is independent of time is called a *stationary* or *steady-state vector field*.

Appendix B

Definitions used in electromagnetic theory

The electric field

The electric field whose notation is $\vec{\mathbf{E}}$ is not usually directly defined and is commonly stated to just simply exist. In fact, it was Maxwell that identified the most practical explanation of the electric field as the space in which electric forces act around an electrified object [14]. Further the electric field is defined as the electric force in Newton's[(i)], per unit charge in Coulomb's[(ii)]. With this relation the electric field may be defined by (B.1),

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}_{\mathrm{e}}}{q_0} \tag{B.1}$$

where $\vec{\mathbf{F}}_{\mathrm{e}}$ is the electrical force on the electric charge q_0 [14] and so it is obvious that $\vec{\mathbf{E}}$ has unit Newton's per Coulomb, NC^{-1} [23] and is parallel to $\vec{\mathbf{F}}_{\mathrm{e}}$.

The magnetic field

The magnetic field whose notation is $\vec{\mathbf{B}}$ is, as with the electric field, not usually directly defined and is known sometimes to simply exists. This stated, it is often observed to be the magnetic force experienced by a moving charged particle. If a particle is stationary then it will experience no magnetic force this is shown by the Lorentz equation of magnetic force (B.2),

$$\vec{\mathbf{F}}_b = q\mathbf{v} \times \vec{\mathbf{B}} \tag{B.2}$$

where $\vec{\mathbf{F}}_b$ is the magnetic force, q is the particles charge and $\mathbf{v}$ represents the particles velocity with respect to the magnetic field $\vec{\mathbf{B}}$. If we apply then the vector cross product relation (A.6), page 47, then (B.2) becomes (B.3) where θ defines the angle between $\mathbf{v}$ and

(i) The Newton, denoted N, is the unit of force [23].

(ii) The Coulomb, denoted C, is the unit of electrical charge [23].

$\vec{\mathbf{B}}$.

$$\left|\vec{\mathbf{B}}\right| = \frac{\left|\vec{\mathbf{F}}_b\right|}{qv\sin(\theta)} \tag{B.3}$$

Current density

The current density whose notation is $\vec{\mathbf{J}}$ is the electric current per unit area, or cross section of a surface and has units Amperes(iii) per metres squared, Am^{-2}. The cross-sectional area is always taken perpendicular to the electric current. Assuming that the cross-sectional area has some electric conductivity given as σ then the current density can be related to the Electric field by (B.4), Ohm's law [16].

$$\vec{\mathbf{J}} = \sigma\vec{\mathbf{E}} \tag{B.4}$$

Charge density

The charge density whose notation is ρ is the amount of electric current per unit volume and has units Coulombs per metres cubed and Cm^{-3}.

Vacuum permittivity

The vacuum permittivity whose notation is ε_0 [23], is one of the most well known physical constants and is commonly known as the permittivity of free space or the electric constant. It has units farads(iv) per metre, Fm^{-1} and value shown by (B.5) measuring the resistance encountered forming an electric field in a vacuum.

$$\varepsilon_0 = 8.854187\cdots \times 10^{-12}Fm^{-1} \tag{B.5}$$

Vacuum permeability

The vacuum permeability whose notation is μ_0 [23], is a well known physical constant and is known commonly as the permeability of free space or the magnetic constant. It has units henrys(v) per metre, Hm^{-1} and value shown by (B.6) measuring the ability a vacuum has to support the formation of a magnetic field.

$$\mu_0 = 1.256637\cdots \times 10^{-6}Hm^{-1} \tag{B.6}$$

(iii) The Ampere, denoted A, is the unit of electric current [23].
(iv) The farad, denoted F, is the unit of electrical capacitance [23].
(v) The Henry, denoted H, is the unit of inductance [23].

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