TORSION UNITS IN THE INTEGRAL GROUP RING OF $PSL(3,4)$

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Abstract. We investigate the Zassenhaus Conjecture for the integral group ring of the simple group $PSL(3,4)$.

1. Introduction

Let $\mathbb{Z}G$ be the integral group ring of a finite group G and $U(\mathbb{Z}G)$ be the group of units of the integral group ring. The homomorphism $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ given by $\varepsilon\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g$ is called the augmentation mapping of $\mathbb{Z}G$. Let $u \in \sum_{g\in G} a_g g \in U(\mathbb{Z}G)$, it is well known that $U(\mathbb{Z}G) = \{\pm 1\} \times V(\mathbb{Z}G)$ where

$$V(\mathbb{Z}G) = \{u \mid \varepsilon(u) = 1\}.$$

G will represent a finite group throughout this article and and torsion units will always represent torsion units in $V(\mathbb{Z}G) \setminus \{1\}$. The next conjecture is one of the important in the theory of integral group rings.

Conjecture 1. *If G is a finite group, then for each torsion unit $u \in V(\mathbb{Z}G)$ there exists $g \in G$, such that $|u| = |g|$.*

Hans Zassenhaus formulated a stronger version of this conjecture in [34], which states that:

Conjecture 2. *A torsion unit in $V(\mathbb{Z}G)$ is said to be rationally conjugate to a group element if it is conjugate to an element of G by a unit of the rational group ring $\mathbb{Q}G$.*

This conjecture was confirmed some classes of solvable groups in [21] and for nilpotent groups in [31,33]. The main investigative tool for simple group in relation to the Zassenhaus conjecture is the Luthar-Passi Method, which was introduced in [27]. It was confirmed true for all groups up to order 71, A_5 and S_5, central extensions of S_5 and other simple finite groups in [3,4,23,27,28]. In [32], partial results were given for A_6 and the remaining cases were dealt with in [19]. It was also proved for $PSL(2,p)$ when $p \in \{7,11,13\}$ in [20] and $p \in \{8,17\}$ in [17]. See [22] for further results regarding $PSL(2,p)$.

Let H be a group with torsion part $t(H)$ of finite exponent and $\#H$ be the set of primes dividing the order of elements from $t(H)$. The prime graph of H (denoted by $\pi(H)$) is a graph with vertices labeled by primes from $\#H$, such that vertices p and q are adjacent if and only if there is an element of order pq in the group. The following, which was composed as a problem in [30] (Problem 37):

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Question 1. *(Prime Graph Question) If G is a finite group, then $\pi(G) = \pi(V(\mathbb{Z}G))$.*

This question was upheld for Solvable and Frobenius groups in [25]. Also it was confirmed for some Sporadic Simple groups in [2, 5–15]. We use the Luthar-Passi Method to obtain our result. Our main result is the following:

Theorem 1. *Let $G = PSL(3,4)$ and u be a torsion unit of $V(\mathbb{Z}G)$. Then,*

(i) If $|u| \in \{2,3\}$, then u is rationally conjugate to some $g \in G$.
(ii) There are no elements of order 10, 14, 15, 21 or 35 in $V(\mathbb{Z}G)$.
(iii) If $|u| = 4$, then $\nu_{rx} = 0\,\forall\, rx \notin \{\nu_{2a}, \nu_{4a}, \nu_{4b}, \nu_{4c}\}$ and

$$\begin{aligned}(\nu_{2a}, \nu_{4a}, \nu_{4b}, \nu_{4c}) \in \{&(2,-1,-1,1),(2,-1,0,0),(0,-1,0,2),(2,-1,1,-1),\\ &(0,-1,1,1),(0,-1,2,0),(-2,-1,2,2),(2,0,-1,0),\\ &(0,0,-1,2),(2,0,0,-1),(0,0,0,1),(0,0,1,0),\\ &(-2,0,1,2),(0,0,2,-1),(-2,0,2,1),(2,1,-1,-1),\\ &(0,1,-1,1),(0,1,0,0),(-2,1,0,2),(0,1,1,-1),\\ &(-2,1,1,1),(-2,1,2,0),(0,2,-1,0),(-2,2,-1,2),\\ &(0,2,0,-1),(-2,2,0,1),(-2,2,1,0),(-2,2,2,-1)\}.\end{aligned}$$

(iv) If $|u| = 5$, then then $\nu_{rx} = 0\,\forall\, rx \notin \{\nu_{5a}, \nu_{5b}\}$ and

$$(\nu_{5a}, \nu_{5b}) \in \{(2,-1),(1,0),(0,1),(-1,2)\}.$$

(v) If $|u| = 7$, then $\nu_{rx} = 0\,\forall\, rx \notin \{\nu_{7a}, \nu_{7b}\}$ and

$$(\nu_{7a}, \nu_{7b}) \in \{(2,-1),(1,0),(0,1),(-1,2)\}.$$

(vi) If $|u| = 6$, then $\nu_{rx} = 0\,\forall\, rx \notin \{\nu_{2a}, \nu_{3a}\}$ and

$$(\nu_{2a}, \nu_{3a}) \in \{(4,-3),(-2,3)\}.$$

Let $u = \sum a_g g$ be a torsion unit of $V(\mathbb{Z}G)$. Then, the sum $\sum_{g \in X^G} a_g \in \mathbb{Z}$ which is the partial augmentation (denoted by $\varepsilon_C(u)$) of u with respect to it's conjugacy classes X^G in G. Let $\nu_i = \varepsilon_{C_i}(u)$ be the i-th partial augmentation of u. It was proved that $\nu_1 = 0$ and $\nu_j = 0$ if the conjugacy class C_j consists of a central elements by G. Higman and S. D. Berman [1]. Therefore $\nu_2 + \nu_3 + \cdots + \nu_l = 1$ for any torsion u (since $u \in V(\mathbb{Z}G)$) where l denotes the class number of G.

Proposition 1. *([16]) Let u be torsion unit of $V(\mathbb{Z}G)$. The order of u divides the exponent of G.*

The following Propositions provide relationships between the partial augmentations and the order of a torsion unit.

Proposition 2. *(Proposition 3.1 in [18]) Let u be a torsion unit of $V(\mathbb{Z}G)$. Let C be a conjucay class of G. If p is a prime dividing the order of a representative of C but not the order of u then the partial augmentation $\varepsilon_C(u) = 0$.*

Proposition 3. *(Proposition 2.2 in [20]) Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$.*

(i) If u has order p^n, then $\varepsilon_x(u) = 0$ for every x of G whose p-part is of order strictly greater than p^n.
(ii) If x is an element of G whose p-part, for some prime, has order strictly greater than the order of the p-part of u, then $\varepsilon_x(u) = 0$.

Proposition 4. *([27] and Theorem 2.5 in [29]) Let u be a torsion unit of $V(\mathbb{Z}G)$ of order k. Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ iff for each d dividing k there is precisely one conjugacy class C_{i_d} with partial augmentation $\varepsilon_{C_{i_d}}(u^d) \neq 0$.*

For any character χ of G and any torsion unit of $V(\mathbb{Z}G)$, clearly $\chi(u) = \sum_{i=2}^{l} \nu_i \chi(h_i)$ where h_i is a representative of a conjugacy class C_i.

Proposition 5. *(Theorem 1 in [27] and [20]) Let p be equal to zero or a prime divisor of $|G|$. Suppose that u is an element of $V(\mathbb{Z}G)$ of order k. Let z be a primitive $k-th$ root of unity. Then for every integer l and any character χ of G, the number*

$$\mu_l(u,\chi,p) = \frac{1}{k}\sum_{d|k} Tr_{\mathbb{Q}(z^d)/\mathbb{Q}}\{\chi(u^d)z^{-dl}\}$$

is a non-negative integer.

We will use the notation $\mu_l(u,\chi,*)$ when $p=0$. The LAGUNA package [26] for the GAP system [24] is a very useful tool when calculating $\mu_l(u,\chi,p)$.

2. Proof of Theorem 1

Let $G = PSL(3,4)$. Clearly $|G| = 20160 = 2^6\cdot 3^2\cdot 5\cdot 7$ and $exp(G) = 420 = 2^2\cdot 3\cdot 5\cdot 7$. Initially for any torsion unit of $V(\mathbb{Z}G)$ of order k, we have that

$$\nu_{2a}+\nu_{3a}+\nu_{4a}+\nu_{4b}+\nu_{4c}+\nu_{5a}+\nu_{5b}+\nu_{7a}+\nu_{7b} = 1.$$

By Proposition 1, we need only to consider torsion units of $V(\mathbb{Z}G)$ of order 2, 3, 4, 5, 7, 6, 10, 14, 15, 21, 35.

Case (i). Let $u\in V(\mathbb{Z}G)$ be of order 2. By Proposition 2, $\nu_{rx} = 0\ \forall\ rx \in \{3a,4a,4b,4c,5a,5b,7a,7b\}$. Therefore u is rationally conjugated to some element $g\in G$ by Proposition 4.

Case (ii). Let $u\in V(\mathbb{Z}G)$ be of order 3. By Proposition 2, $\nu_{rx} = 0\ \forall\ rx \in \{2a,4a,4b,4c,5a,5b,7a,7b\}$. Therefore u is rationally conjugated to some element $g\in G$ by Proposition 4.

Case (iii). Let $u\in V(\mathbb{Z}G)$ be of order 4. Using Propositions 2 & 3, $\nu_{2a}+\nu_{4a}+\nu_{4b}+\nu_{4c} = 1$. Now $\chi(u^2) = \chi(2a)$. Applying Proposition 5 to the character tables (Tables 1 & 3), we obtain the following system of inequalities:

$$\begin{gathered}
\mu_0(u,\chi_2,*) = \tfrac{1}{4}(8\nu_{2a}+24)\ge 0;\quad \mu_2(u,\chi_2,*) = \tfrac{1}{4}(-8\nu_{2a}+24)\ge 0;\\
\mu_0(u,\chi_3,*) = \tfrac{1}{4}(2\gamma_1+38)\ge 0;\quad \mu_2(u,\chi_3,*) = \tfrac{1}{4}(-2\gamma_1+38)\ge 0;\\
\mu_0(u,\chi_4,*) = \tfrac{1}{4}(2\gamma_2+38)\ge 0;\quad \mu_2(u,\chi_4,*) = \tfrac{1}{4}(-2\gamma_2+38)\ge 0;\\
\mu_0(u,\chi_2,3) = \tfrac{1}{4}(-\gamma_3+14)\ge 0;\quad \mu_2(u,\chi_2,3) = \tfrac{1}{4}(\gamma_3+14)\ge 0;\\
\mu_0(u,\chi_3,3) = \tfrac{1}{4}(-\gamma_4+14)\ge 0;\quad \mu_2(u,\chi_3,3) = \tfrac{1}{4}(\gamma_4+14)\ge 0;\\
\mu_0(u,\chi_4,3) = \tfrac{1}{4}(-\gamma_5+14)\ge 0;\quad \mu_2(u,\chi_4,3) = \tfrac{1}{4}(\gamma_5+14)\ge 0;\\
\mu_0(u,\chi_5,3) = \tfrac{1}{4}(\gamma_6+22)\ge 0
\end{gathered}$$

where $\gamma_1 = 3\nu_{2a}+3\nu_{4a}-\nu_{4b}-\nu_{4c}$, $\gamma_2 = 3\nu_{2a}-\nu_{4a}+3\nu_{4b}-\nu_{4c}$, $\gamma_3 = 2\nu_{2a}-6\nu_{4a}+2\nu_{4b}+2\nu_{4c}$, $\gamma_4 = 2\nu_{2a}+2\nu_{4a}-6\nu_{4b}+2\nu_{4c}$, $\gamma_5 = 2\nu_{2a}+2\nu_{4a}+2\nu_{4b}-6\nu_{4c}$ and $\gamma_6 = 6\nu_{2a}-2\nu_{4a}-2\nu_{4b}-2\nu_{4c}$. Solving the system of inequalities, $\nu_{2a}\in\{k \mid -3\le k\le 3\}$, $\gamma_1 \in \{1+2k \mid -10\le k\le 9\}$ and $\gamma_2\in\{1+2k\mid -10\le k\le 9\}$. It follows that the only possible integer solutions for $(\nu_{2a},\nu_{4a},\nu_{4b},\nu_{4c})$ are listed in part (iii) of Theorem 1.

Case (iv). Let $u\in V(\mathbb{Z}G)$ be of order 5. Using Propositions 2 & 3, $\nu_{5a}+\nu_{5b} = 1$. Applying Proposition 5 to the character tables (Table 2), we obtain the following system of inequalities:

$$\mu_1(u,\chi_2,2) = \tfrac{1}{5}(\gamma_1+8)\ge 0;\quad \mu_2(u,\chi_2,2) = \tfrac{1}{5}(\gamma_2+8)\ge 0$$

where $\gamma_1 = -3\nu_{5a}+2\nu_{5b}$ and $\gamma_2 = 2\nu_{5a}-3\nu_{5b}$. It follows that the only possible integer solutions for (ν_{5a},ν_{5b}) are listed in part (iv) of Theorem 1.

Case (v). Let $u \in V(\mathbb{Z}G)$ be of order 7. Using Propositions 2 & 3, $\nu_{7a} + \nu_{7b} = 1$. Applying Proposition 5 to the character tables (Tables 1 & 2), we obtain the following system of inequalities:

$$\mu_1(u,\chi_6,*) = \tfrac{1}{7}(\gamma_1 + 45) \geq 0; \quad \mu_3(u,\chi_6,*) = \tfrac{1}{7}(\gamma_2 + 45) \geq 0;$$
$$\mu_1(u,\chi_4,2) = \tfrac{1}{7}(\gamma_3 + 9) \geq 0; \quad \mu_3(u,\chi_4,2) = \tfrac{1}{7}(\gamma_4 + 9) \geq 0;$$

where $\gamma_1 = 4\nu_{7a} - 3\nu_{7b}$, $\gamma_2 = -3\nu_{7a} + 4\nu_{7b}$, $\gamma_3 = 5\nu_{7a} - 2\nu_{7b}$ and $\gamma_4 = -2\nu_{7a} + 5\nu_{7b}$. It follows that the only possible integer solutions for (ν_{7a}, ν_{7b}) are listed in part (v) of Theorem 1.

Case (vi). Let $u \in V(\mathbb{Z}G)$ be of order 6. Using Propositions 2 & 3, $\nu_{2a} + \nu_{3a} = 1$. Applying Proposition 5 to the character tables (Table 1), we obtain the following system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{6}(2\gamma_1 + 28) \geq 0; \quad \mu_3(u,\chi_2,*) = \tfrac{1}{6}(-2\gamma_1 + 20) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \tfrac{1}{6}(\gamma_2 + 36) \geq 0; \quad \mu_1(u,\chi_6,*) = \tfrac{1}{6}(-3\nu_{2a} + 48) \geq 0$$

where $\gamma_1 = 2\nu_{2a} + \nu_{3a}$ and $\gamma_2 = 6\nu_{2a} - 2\nu_{3a}$. Clearly $\gamma_1 \in \{-7, -4, -1, 2, 5\}$. It follows that the only possible integer solutions for (ν_{2a}, ν_{3a}) are listed in part (vi) of Theorem 1.

Case (vii). Let $u \in V(\mathbb{Z}G)$ be of order 10. Using Propositions 2 & 3, $\nu_{2a} + \nu_{5a} + \nu_{5b} = 1$. Consider the cases $\chi(u^5) = \chi(2a)$ and $\chi(u^2) = m_1\chi(5a) + m_2\chi(5b)$ where $(m_1, m_2) \in \{(1,0), (0,1), (2,-1), (-1,2)\}$. Applying Proposition 5 to the character table (Table 1), we obtain the following system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{10}(16\nu_{2a} + 24) \geq 0; \quad \mu_5(u,\chi_2,*) = \tfrac{1}{10}(-16\nu_{2a} + 16) \geq 0;$$
$$\mu_1(u,\chi_3,*) = \tfrac{1}{10}(3\nu_{2a} + 32) \geq 0.$$

Clearly there are no possible solutions for ν_{2a} or $(\nu_{2a}, \nu_{5a}, \nu_{5b})$.

Case $(viii)$. Let $u \in V(\mathbb{Z}G)$ be of order 14. Using Propositions 2 & 3, $\nu_{2a} + \nu_{7a} + \nu_{7b} = 1$. Consider the cases $\chi(u^7) = \chi(2a)$ and $\chi(u^2) = m_1\chi(7a) + m_2\chi(7b)$ where $(m_1, m_2) \in \{(1,0), (0,1), (2,-1), (-1,2)\}$. Applying Proposition 5 to the character table (Table 1), we obtain the following system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{14}(6\gamma_1 + 18) \geq 0; \quad \mu_7(u,\chi_2,*) = \tfrac{1}{14}(-6\gamma_1 + 10) \geq 0;$$
$$\mu_1(u,\chi_6,*) = \tfrac{1}{14}(-\gamma_2 + t_1) \geq 0; \quad \mu_2(u,\chi_6,*) = \tfrac{1}{14}(\gamma_2 + t_2) \geq 0$$

where $\gamma_1 = 4\nu_{2a} - \nu_{7a} - \nu_{7b}$, $\gamma_2 = 3\nu_{2a} + 4\nu_{7a} - 3\nu_{7b}$, $(t_1, t_2) = (52, 46)$ when $(m_1, m_2) = (1,0)$, $(t_1, t_2) = (45, 39)$ when $(m_1, m_2) = (0,1)$, $(t_1, t_2) = (59, 53)$ when $(m_1, m_2) = (2,-1)$ and $(t_1, t_2) = (38, 32)$ when $(m_1, m_2) = (-1, 2)$. Clearly there are no possible solutions for $(\nu_{2a}, \nu_{7a}, \nu_{7b})$.

Case (ix). Let $u \in V(\mathbb{Z}G)$ be of order 15. Using Propositions 2 & 3, $\nu_{3a} + \nu_{5a} + \nu_{5b} = 1$. Consider the cases $\chi(u^5) = \chi(3a)$ and $\chi(u^3) = m_1\chi(5a) + m_2\chi(5b)$ where $(m_1, m_2) \in \{(1,0), (0,1), (2,-1), (-1,2)\}$. Applying Proposition 5 to the character table (Table 1), we obtain the following system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{15}(16\nu_{3a} + 24) \geq 0; \quad \mu_5(u,\chi_2,*) = \tfrac{1}{15}(-8\nu_{3a} + 18) \geq 0.$$

Clearly there are no possible solutions for $(\nu_{2a}, \nu_{5a}, \nu_{5b})$.

Case (x). Let $u \in V(\mathbb{Z}G)$ be of order 21. Using Propositions 2 & 3, $\nu_{3a} + \nu_{7a} + \nu_{7b} = 1$. Consider the cases $\chi(u^7) = \chi(3a)$ and $\chi(u^3) = m_1\chi(7a) + m_2\chi(7b)$ where $(m_1, m_2) \in \{(1,0), (0,1), (2,-1), (-1,2)\}$. Applying Proposition 5 to the character table (Table 1), we obtain the following system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{21}(12\gamma_1 + 18) \geq 0; \quad \mu_7(u,\chi_2,*) = \tfrac{1}{21}(-6\gamma_1 + 12) \geq 0;$$
$$\mu_1(u,\chi_3,*) = \tfrac{1}{21}(-\nu_{3a} + 36) \geq 0; \quad \mu_1(u,\chi_6,*) = \tfrac{1}{21}(\gamma_2 + t) \geq 0;$$
$$\mu_9(u,\chi_6,*) = \tfrac{1}{21}(-2\gamma_2 + t) \geq 0$$

where $\gamma_1 = 2\nu_{3a} - \nu_{7a} - \nu_{7b}$, $\gamma_2 = 3\nu_{7a} - 4\nu_{7b}$, $t = 49$ when $(m_1, m_2) = (1, 0)$, $t = 42$ when $(m_1, m_2) = (0, 1)$, $t = 56$ when $(m_1, m_2) = (2, -1)$ and $t = 35$ when $(m_1, m_2) = (-1, 2)$. Clearly there are no possible solutions for $(\nu_{3a}, \nu_{7a}, \nu_{7b})$.

Case $(x1)$. Let $u \in V(\mathbb{Z}G)$ be of order 35. Using Propositions 2 & 3, $\nu_{5a} + \nu_{5b} + \nu_{7a} + \nu_{7b} = 1$. Consider the cases $\chi(u^7) = m_1\chi(5a) + m_2\chi(5b)$ and $\chi(u^5) = m_3\chi(7a) + m_4\chi(7b)$ where

$$\begin{aligned}(m_1, m_2, m_3, m_4) \in \{&(1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), (1,0,2,-1),\\ &(0,1,2,-1), (1,0,-1,2), (0,1,-1,2), (2,-1,1,0),\\ &(2,-1,0,1), (2,-1,2,-1), (2,-1,-1,2), (-1,2,1,0),\\ &(-1,2,0,1), (-1,2,2,-1), (-1,2,-1,2)\}.\end{aligned}$$

Applying Proposition 5 to the character table (Tables 1 & 3, we obtain the following system of inequalities:

$$\mu_0(u,\chi_2,3) = \tfrac{1}{35}(24\gamma + 21) \geq 0; \quad \mu_0(u,\chi_2,*) = \tfrac{1}{35}(-24\gamma + 14) \geq 0$$

where $\gamma = \nu_{7a} + \nu_{7b}$. Clearly there are no possible solutions for $(\nu_{5a}, \nu_{5b}, \nu_{7a}, \nu_{7b})$.

Table 1. Character Table of $PSL(3,4)$ ($p = 0$, see [24])

	$1a$	$2a$	$3a$	$4a$	$4b$	$4c$	$5a$	$5b$	$7a$	$7b$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	20	4	2	0	0	0	0	0	-1	-1
χ_3	35	3	-1	3	-1	-1	0	0	0	0
χ_4	35	3	-1	-1	3	-1	0	0	0	0
χ_5	35	3	-1	-1	-1	3	0	0	0	0
χ_6	45	-3	0	1	1	1	0	0	$\mathcal{B}$	$\overline{\mathcal{B}}$
χ_7	45	-3	0	1	1	1	0	0	$\overline{\mathcal{B}}$	$\mathcal{B}$
χ_8	63	-1	0	-1	-1	-1	$\mathcal{A}$	$\mathcal{A}^*$	0	0
χ_9	63	-1	0	-1	-1	-1	$\mathcal{A}^*$	$\mathcal{A}$	0	0
χ_{10}	64	0	1	0	0	0	-1	-1	1	1

where $\mathcal{A} = -\alpha - \alpha^4$ and $\mathcal{B} = \zeta + \zeta^2 + \zeta^4$, $\alpha = e^{\frac{2\pi i}{5}}$ and $\zeta = e^{\frac{2\pi i}{7}}$.

TABLE 2. Character Table of $PSL(3,4)$ ($p = 2$, see [24])

	$1a$	$3a$	$5a$	$5b$	$7a$	$7b$
χ_1	1	1	1	1	1	1
χ_2	8	-1	$\mathcal{A}$	$\mathcal{A}^*$	1	1
χ_3	8	-1	$\mathcal{A}^*$	$\mathcal{A}$	1	1
χ_4	9	0	-1	-1	$\mathcal{B}$	$\overline{\mathcal{B}}$
χ_5	9	0	-1	-1	$\overline{\mathcal{B}}$	B
χ_6	64	1	-1	-1	1	1

where $\mathcal{A} = -\alpha - \alpha^4$ and $\mathcal{B} = 2\zeta + 2\zeta^2 + \zeta^3 + 2\zeta^4 + \zeta^5 + \zeta^6$, $\alpha = e^{\frac{2\pi i}{5}}$ and $\zeta = e^{\frac{2\pi i}{7}}$.

TABLE 3. Character Table of $PSL(3,4)$ ($p = 3$, see [24])

	$1a$	$2a$	$4a$	$4b$	$4c$	$5a$	$5b$	$7a$	$7b$
χ_1	1	1	1	1	1	1	1	1	1
χ_2	15	-1	3	-1	-1	0	0	1	1
χ_3	15	-1	-1	3	-1	0	0	1	1
χ_4	15	-1	-1	-1	3	0	0	1	1
χ_5	19	3	-1	-1	-1	-1	-1	-2	-2
χ_6	45	-3	1	1	1	0	0	$\mathcal{B}$	$\overline{\mathcal{B}}$
χ_7	45	-3	1	1	1	0	0	$\overline{\mathcal{B}}$	$\mathcal{B}$
χ_8	63	-1	-1	-1	-1	$\mathcal{A}$	$\mathcal{A}^*$	0	0
χ_9	63	-1	-1	-1	-1	$\mathcal{A}^*$	$\mathcal{A}$	0	0

where $\mathcal{A} = -\alpha - \alpha^4$ and $\mathcal{B} = \zeta + \zeta^2 + \zeta^4$, $\alpha = e^{\frac{2\pi i}{5}}$ and $\zeta = e^{\frac{2\pi i}{7}}$.

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