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Stabilizing a Nonlinear System By Using Feedback Control

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M.Sc. Mathematics

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Abstract

In this work, we consider how to stabilize a nonlinear system by using feedback control. To stabilize a nonlinear system, we first need to find the unstable steady state. Then we consider the linearized problem at this steady state and solve the Riccati equation using the linear quadratic regulator (lqr). We then design the feedback controller on the linearized system. Finally, we apply the feedback controller on the original nonlinear system. We use the forward Euler method, backward Euler method and Trapezoidal method to consider the discretization of the nonlinear system. We design the algorithm and consider two numerical examples of ecological models and verify that the results obtained are in accordance with theoretical results.

Keywords

- Steady state
- Feedback control
- Riccati equation
- Linear quadratic regulator
- Forward Euler method
- Backward Euler method
- Trapezoidal method

This work is original and has not been previously submitted for any academic purpose.

Signed:

Date:

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1 Introduction

In this work, we will consider how to stabilize a nonlinear system by using feedback control. The feedback controller is a powerful tool in the stabilization process. This topic is very useful in Physics, Biology and Engineering. It also has applications in finance. It is very important to develop some numerical methods for stabilizing a nonlinear system. The feedback control idea generates from the theory of optimal control system.

In chapter two, we consider the basic concepts and results in optimal control theory. We look at the optimum of a function and a functional and some variational problems. Various steps used in finding the optimal solution to these variational problems are discussed. We also consider the extrema of functions and functionals with conditions. Also, the various steps in finding this extrema are highlighted. Examples on finding the extrema were given with their solutions.

In chapter three, we consider how to stabilize a nonlinear system. We first introduce the nonlinear system and the steady state of a nonlinear system, then we investigate which of the steady state(s) is/are stable or unstable. If it is stable, then we do not need any further stabilization. Otherwise, we stabilize it. We consider the linear quadratic optimal control and introduce the Riccati differential equation and Riccati algebraic equation. We first consider how to stabilize a linearized system by using the feedback control. Then we apply the feedback controller to the nonlinear system. To solve the nonlinear feedback control problem, we use finite difference method. We introduce the forward Euler, backward Euler and trapezoidal methods and discuss the stabilities of the different numerical methods.

Finally, we consider two examples of ecological models and designed an algorithm to find numerical solutions to the given problems. Numerical simulations shows that the results obtained are in accordance with theoretical results.

In the last chapter, conclusions and recommendations are given in terms of its usefulness and how it can be applied.

2 Background

Calculus of variation deals with finding the optimum (maximum or minimum) value of a functional.

2.1 Basic Concepts

2.1.1 Function and Functional

Function: A variable x is a function of a variable t written as $x(t) = f(t)$ e.g $x(t) = 2t^2 + 1$

Functional: A variable quantity J is a functional depending on a function $f(x)$ written as $J = J(f(x))$

Example 2.1:

Let $x(t) = 2t^2 + 1$, then

$$J(x(t)) = \int_0^1 x(t)dt = \int_0^1 (2t^2 + 1)dt = \frac{2}{3} + 1 = \frac{5}{3}$$

which is the area under the curve $x(t)$. If $v(t)$ is the velocity of a vehicle, then

$$J(v(t)) = \int_{t_0}^{t_1} v(t)dt$$

is the path traversed by the vehicle. Hence $x(t)$ and $v(t)$ are functions of t and J is a functional of $x(t)$ or $v(t)$.

2.1.2 Increment

(a.) **Increment of a function:** The increment of the function f denoted by Δf is defined as

$$\Delta f = f(t + \Delta t) - f(t)$$

clearly f depends on t and the increment Δt . We need to write the increment of function as $\Delta f(t, \Delta t)$. For example

$$\begin{aligned} \text{If } f(t) &= (t_1 + t_2)^2, \text{ then} \\ f(t) + \Delta t &= (t_1 + \Delta t_1 + t_2 + \Delta t_2)^2, \text{ and} \\ \Delta f &= (t_1 + \Delta t_1 + t_2 + \Delta t_2)^2 - f(t) \\ &= (t_1 + \Delta t_1 + t_2 + \Delta t_2)^2 - (t_1 + t_2)^2 \\ &= t_1^2 + t_1\Delta t_1 + t_1t_2 + t_1\Delta t_2 + t_1\Delta t_1 + (\Delta t_1)^2 + t_2\Delta t_1 + \Delta t_1\Delta t_2 \\ &\quad t_2\Delta t_1 + t_2^2 + t_2\Delta t_2 + t_1\Delta t_2 + \Delta t_1\Delta t_2 + (\Delta t_2)^2 - (t_1^2 + 2t_1t_2 + t_2^2) \\ \Delta f &= 2t_1\Delta t_1 + 2t_1\Delta t_2 + 2t_2\Delta t_1 + 2t_2\Delta t_2 + 2\Delta t_1\Delta t_2 + (\Delta t_1)^2 + (\Delta t_2)^2 \\ &= 2t_1\Delta t_1 + 2t_2\Delta t_1 + 2t_1\Delta t_2 + 2t_2\Delta t_2 + 2\Delta t_1\Delta t_2 + (\Delta t_1)^2 + (\Delta t_2)^2 \\ &= 2(t_1 + t_2)\Delta t_1 + 2(t_1 + t_2)\Delta t_2 + 2\Delta t_1\Delta t_2 + (\Delta t_1)^2 + (\Delta t_2)^2 \end{aligned}$$

Increment of a functional: This is denoted by ΔJ that is

$$\Delta J = J(x(t) + \delta x(t)) - J(x(t))$$

where $\delta x(t)$ is called the variation of the function $x(t)$. Since the increment of a functional is dependent upon the function $x(t)$ and its variation $\delta x(t)$, then we need to write the increment as $\Delta J(x(t))$. Let

$$J = \int_{t_0}^{t_f} [2x^2(t) + 1] dt$$

Then the increment of J is given by

$$\begin{aligned} \Delta J &= J(x(t) + \delta x(t)) - J(x(t)) \\ &= \int_{t_0}^{t_f} [2(x(t) + \delta x(t))^2 + 1] dt - \int_{t_0}^{t_f} [2x^2(t) + 1] dt \\ &= \int_{t_0}^{t_f} [2(x^2(t) + 2x(t)\delta x(t) + (\delta x(t))^2 + 1)] dt - \int_{t_0}^{t_f} [2x^2(t) + 1] dt \\ &= \int_{t_0}^{t_f} [2x^2(t) + 4x(t)\delta x(t) + 2(\delta x(t))^2 + 1] dt - \int_{t_0}^{t_f} [2x^2(t) + 1] dt \\ &= \int_{t_0}^{t_f} [4x(t)\delta x(t) + 2(\delta x(t))^2] dt \end{aligned}$$

2.1.3 Differential and Variation

a) **Differential of a function:** The increment of the function f at point t^* is defined by

$$\Delta f = f(t^* + \Delta t) - f(t^*) \quad (2.1)$$

By expanding $f(t^* + \Delta t)$ in a Taylor series about a point t^* that is

$$f(t^* + \Delta t) = f(t^*) + \Delta t f'(t^*) + \frac{(\Delta t)^2}{2!} f''(t^*) + \frac{(\Delta t)^3}{3!} f'''(t^*) + \dots$$

Neglecting the higher order terms in Δt , we get

$$f(t^* + \Delta t) \approx f(t^*) + \Delta t f'(t^*)$$

(2.1) now becomes

$$\begin{aligned} \Delta f &\approx f(t^*) + \Delta t f'(t^*) - f(t^*) \\ \Delta f &\approx \Delta t f'(t^*) \\ \Delta f &\approx \Delta t f'(t^*) = df \end{aligned} \quad (2.2)$$

Here df is called the differential of f at the point t^* . $f'(t^*)$ is the derivative or slope of f at t^* . In other words the differential df is the first order approximation to increment Δf .

Example 2.2

Let $f(t) = t^2 + 2t$. Find the increment and derivative of the function $f(t)$.

By definition, the increment Δf is

$$\begin{aligned}\Delta f &= f(t + \Delta t) - f(t) = (t + \Delta t)^2 + 2(t + \Delta t) - (t^2 + 2t) \\ &= t^2 + 2t\Delta t + (\Delta t)^2 + 2t + 2\Delta t - t^2 - 2t = 2t\Delta t + 2\Delta t + (\Delta t)^2\end{aligned}$$

Ignoring the higher order terms

$$\Delta f \approx 2t\Delta t + 2\Delta t = \dot{f}(t)\Delta t$$

That is

$$\dot{f}(t) = 2(t + 1)$$

b) Variation of a functional

Consider the increment of a functional

$$\Delta J = J(x(t) + \delta x(t)) - J(x(t)) \quad (2.3)$$

Using the Taylor series to expand $J(x(t) + \delta x(t))$, we have

$$\begin{aligned}J(x(t) + \delta x(t)) &= J(x(t) + \delta x(t))J'(x) + \frac{(\delta x(t))^2}{2!}J''(x) + \dots \\ &= Jx(t) + \delta J + \delta^2 J + \dots\end{aligned}$$

(2.3) becomes

$$\begin{aligned}\Delta J &= Jx(t) + \delta x(t)J'(x) + \frac{(\Delta x(t))^2}{2!}J''(x) + \dots - Jx(t) \\ &= \delta x(t)J'(x) + \frac{(\delta x(t))^2}{2!}J''(x) + \dots \\ \Delta J &= \delta J + \delta^2 J + \dots\end{aligned}$$

Where $\delta J = \delta x(t)J'(x)$ and $\delta^2 J = \frac{(\delta x(t))^2}{2!}J''(x)$ are called the first variation of the functional (or simply the variation) and the second variation of the functional J , respectively. The variation δJ of a functional J is the linear (or first order approximate) part (in $\delta x(t)$) of the increment ΔJ . [2]

Example 2.3: Evaluate the variation of the functional

$$J(x(t)) = \int_{t_0}^{t_f} [2x^2(t) + 3x(t) + 4]dt$$

We first form the increment and then extract the variation as the first order approximation.

$$\begin{aligned}\Delta J &= J(x(t) + \delta x(t)) - J(x(t)) \\ &= \int_{t_0}^{t_f} [2(x(t) + \delta x(t))^2 + 3(x(t) + \delta x(t)) + 4]dt - \int_{t_0}^{t_f} [2x^2(t) + 3x(t) + 4]dt \\ &= \int_{t_0}^{t_f} [4x(t)\delta x(t) + 2(\delta x(t))^2 + 3\delta x(t)]dt\end{aligned}$$

considering only the first order terms, we get the (first) variation as

$$\delta J(x(t), \delta x(t)) = \int_{t_0}^{t_f} (4x(t) + 3)\delta x(t)dt$$

2.2 Optimum of a Function and a Functional

Here, we give some definitions for optimum or extremum (max or min) of a function and a functional[2].

Definition: Optimum of a function

A function is said to have a relative optimum at the point t^* if there exist a positive parameter ϵ such that for all points t in a domain D that satisfy $|t - t^*| < \epsilon$, the increment of $f(t)$ has the same sign (positive or negative). In other words, if

$$\Delta f = f(t) - f(t^*) \geq 0 \quad (2.4)$$

then, $f(t^*)$ is a **relative local minimum**. On the other hand, if

$$\Delta f = f(t) - f(t^*) \leq 0 \quad (2.5)$$

then, $f(t^*)$ is a **relative local maximum**

If (2.4) and (2.5) are valid arbitrarily large ϵ , then $f(t^*)$ is said to have a **global absolute optimum**.

It is well known that the **necessary condition** for optimum of a function is that the (first) differential vanishes i.e $df = 0$. the **sufficient condition** for minimum is that the second differential is positive i.e $d^2f > 0$. For maximum is that the second differential is negative i.e $d^2f < 0$. if $d^2f = 0$, it corresponds to a **stationary or (inflection) point**.

Definition: Optimum of a functional

A functional J is said to have a relative optimum at x^* if there is a positive ϵ such that for all functions x in a domain Ω which satisfy $|x - x^*| < \epsilon$, the increment of J has the same sign. In other words, if

$$\Delta J = J(x) - J(x^*) \geq 0 \quad (2.6)$$

Then, $J(x^*)$ is a **relative minimum**. On the other hand, if

$$\Delta J = J(x) - J(x^*) \leq 0 \quad (2.7)$$

Then, $J(x^*)$ is a **relative maximum**. If (2.6) and (2.7) are satisfied for arbitrarily large ϵ , then $J(x^*)$ is a **global absolute optimum**.

Fundamental theorem of the calculus of Variations

For $x^*(t)$ to be a candidate for an optimum, the (first) variation of J must be zero on $x^*(t)$, i.e $\delta J(x^*(t), \delta x(t)) = 0$ for all admissible values of $\delta x(t)$. This is a necessary condition. As a sufficient condition for minimum, the second variation of $\delta^2 J > 0$, and for maximum $\delta^2 J < 0$ [2].

2.3 Variational Problem**2.3.1 Fixed-End Time and Fixed-End State System**

We address a fixed-end time and fixed-end state problem where both the initial time and final time and state are fixed or given a priori. Let $x(t)$ be a scalar function with continuous first derivatives and the vector case can be dealt with similarly. The problem is to find the optimal function $x^*(t)$ for which the functional

$$J(x(t)) = \int_{t_0}^{t_f} V(x(t), \dot{x}(t), t) dt \quad (2.8)$$

has a relative optimum.

It is assumed that the integrand V has continuous first and second partial derivatives with respect to all its arguments; t_0 and t_f are fixed (or given a priori) and the end points are fixed i.e

$$x(t = t_0) = x_0; x(t = t_f) = x_f \quad (2.9)$$

From the theorem above, we know that the necessary condition for an optimum is that the variation of a functional vanishes. Hence in our attempt to find the optimum of $x(t)$, we first define the increment for J , obtain its variation and finally apply the fundamental theorem of the calculus of variations (theorem).

The steps involved in finding the optimal solution to the fixed-end time and fixed-end state system are

- Assumption of an Optimum
- Variations and Increment
- First Variation
- Fundamental Theorem
- Fundamental Lemma
- Euler-Lagrange Equation

Assumption of an Optimum

Let $x^*(t)$ be the optimum attained for $x(t)$. Take some admissible function $x_a(t) = x^*(t) +$

$\delta x(t)$ close to $x^*(t)$ where $\delta x(t)$ is the variation of $x^*(t)$. The function $x_a(t)$ should also satisfy the boundary conditions (1.9) and hence it is necessary that

$$\delta x(t_0) = \delta x(t_f) = 0 \quad (2.10)$$

Variations and Increment

Let's define the increment as

$$\begin{aligned} \Delta J(x^*(t), \delta x(t)) &= J(x^*(t) + \delta x(t), \dot{x}^*(t) + \delta \dot{x}(t), t) - J(x^*(t), \dot{x}^*(t), t) \\ &= \int_{t_0}^{t_f} V(x^*(t) + \delta x(t), \dot{x}^*(t) + \delta \dot{x}(t)) - \int_{t_0}^{t_f} V(x^*(t), \dot{x}^*(t), t) dt \end{aligned}$$

which by combining the integrals can be written as

$$\Delta J(x^*(t), \delta x(t)) = \int_{t_0}^{t_f} [V(x^*(t) + \delta x(t), \dot{x}^*(t) + \delta \dot{x}(t), t) - V(x^*(t), \dot{x}^*(t), t)] dt \quad (2.11)$$

where $\dot{x}^*(t) = \frac{dx^*(t)}{dt}$ and $\delta \dot{x}(t) = \frac{d}{dt} \delta x(t)$

Expanding V in a Taylor series about the point $x^*(t)$ and $\dot{x}^*(t)$, the increment ΔJ becomes

$$\begin{aligned} \Delta J &= \Delta J(x^*(t), \delta x(t)) \\ &= \int_{t_0}^{t_f} \left[\frac{\delta V(x^*(t), \dot{x}^*(t), t)}{\delta x} \delta x(t) + \frac{\delta V(x^*(t), \dot{x}^*(t), t)}{\delta \dot{x}} \delta \dot{x}(t) \right. \\ &\quad \left. + \frac{1}{2!} \left\{ \frac{\delta^2 V(\dots)}{\delta x^2} (\delta x(t))^2 + \frac{\delta^2 V(\dots)}{\delta \dot{x}^2} (\delta \dot{x}(t))^2 + \frac{2\delta^2 V(\dots)}{\delta x \delta \dot{x}} \delta x(t) \delta \dot{x}(t) \right\} + \dots \right] dt \end{aligned} \quad (2.12)$$

the partial derivatives are with respect to $x(t)$ and $\dot{x}(t)$.

First Variation

We obtain the variation by retaining the terms that are linear in $\delta x(t)$ and $\delta \dot{x}(t)$ as

$$\delta J(x^*(t), \delta x(t)) = \int_{t_0}^{t_f} \left[\frac{\delta V(x^*(t), \dot{x}^*(t), t)}{\delta x} \delta x(t) + \frac{\delta V(x^*(t), \dot{x}^*(t), t)}{\delta \dot{x}} \delta \dot{x}(t) \right] dt \quad (2.13)$$

We can now express the relation for the first variation (2.13) entirely in terms containing $\delta x(t)$ (since $\delta \dot{x}(t)$ is dependent on committing the arguments in V for simplicity).

$$\begin{aligned} \int_{t_0}^{t_f} \left(\frac{\delta V}{\delta \dot{x}} \right)_* \delta \dot{x}(t) dt &= \int_{t_0}^{t_f} \left(\frac{\delta V}{\delta \dot{x}} \right)_* \frac{d}{dt} (\delta x(t)) dt \\ &= \int_{t_0}^{t_f} \left(\frac{\delta V}{\delta \dot{x}} \right)_* d(\delta x(t)) \\ &= \left[\left(\frac{\delta V}{\delta \dot{x}} \right)_* \delta x(t) \right]_{t_0}^{t_f} - \int_{t_0}^{t_f} \delta x(t) \frac{d}{dt} \left(\frac{\delta V}{\delta \dot{x}} \right)_* dt \end{aligned} \quad (2.14)$$

Using (2.14), the relation (2.13) for first variation becomes

$$\begin{aligned} \delta J(x^*(t), \delta x(t)) &= \int_{t_0}^{t_f} \left(\frac{\delta V}{\delta x} \right)_* \delta x(t) dt + \left[\left(\frac{\delta V}{\delta \dot{x}} \right)_* \delta x(t) \right]_{t_0}^{t_f} \\ &\quad - \int_{t_0}^{t_f} \frac{d}{dt} \left(\frac{\delta V}{\delta \dot{x}} \right)_* \delta x(t) dt \\ &= \int_{t_0}^{t_f} \left[\left(\frac{\delta V}{\delta x} \right)_* - \frac{d}{dt} \left(\frac{\delta V}{\delta \dot{x}} \right)_* \right] \delta x(t) dt + \left[\left(\frac{\delta V}{\delta \dot{x}} \right)_* \delta x(t) \right]_{t_0}^{t_f} \end{aligned} \quad (2.15)$$

Using (2.10) for boundary variations in (2.15) we have

$$\delta J(x^*(t), \delta x(t)) = \int_{t_f}^{t_f} \left[\left(\frac{\delta V}{\delta x} \right)_* - \frac{d}{dt} \left(\frac{\delta V}{\delta \dot{x}} \right)_* \right] \delta x(t) dt \quad (2.16)$$

Fundamental Theorem

We apply the fundamental theorem of the calculus variations i.e the variation of J must vanish for an optimum. That is, for the optimum $x^*(t)$ to exist, $\delta J(x^*(t), \delta x(t)) = 0$. Hence, (2.16) becomes

$$\int_{t_0}^{t_f} \left[\left(\frac{\delta V}{\delta x} \right)_* - \frac{d}{dt} \left(\frac{\delta V}{\delta \dot{x}} \right)_* \right] \delta x(t) dt = 0 \quad (2.17)$$

where $\delta x(t)$ must be zero at t_f and t_0 , but here it is completely arbitrary.

Fundamental Lemma

Let us take advantage of the following lemma without proof called the fundamental lemma of the calculus of variations to simplify the condition obtained in (2.17)

Lemma: If for fixed function $g(t)$ which is continuous,

$$\int_{t_0}^{t_f} g(t) \delta x(t) dt = 0 \quad (2.18)$$

where $\delta x(t)$ is any continuous function in $[t_0, t_f]$, then the function $g(t)$ must be zero everywhere throughout the interval $[t_0, t_f]$. [2]

Euler-Lagrange Equation:

Applying the previous lemma to (2.17) a necessary condition for $x^*(t)$ to be an optimal of the functional J given by (2.8) is

$$\left(\frac{\delta V(x^*(t), \dot{x}^*(t), t)}{\delta x} \right)_* - \frac{d}{dt} \left(\frac{\delta V(x^*(t), \dot{x}^*(t), t)}{\delta \dot{x}} \right)_* = 0 \quad (2.19)$$

which is written as

$$\left(\frac{\delta V}{\delta x} \right)_* - \frac{d}{dt} \left(\frac{\delta V}{\delta \dot{x}} \right)_* = 0 \quad (2.20)$$

for all $t \in [t_0, t_f]$. This is called Euler equation. [2]

2.3.2 Euler-Lagrange Equation

The following are to be noted for the following Euler-Lagrange(E-L) equation

1. The Euler-Lagrange equation (2.19) can be written in many different forms. Thus (2.19) becomes

$$V_x - \frac{d}{dt}(V_{\dot{x}}) = 0 \quad (1.21)$$

where,

$$V_x = \frac{\delta V}{\delta x} = V_x(x^*(t), \dot{x}^*(t), t) \quad (2.22)$$

Since V is a function of three arguments $x^*(t)$, $\dot{x}^*(t)$, and t , and that $\dot{x}^*(t)$ and $x^*(t)$ are in turn functions of t , we get

$$\begin{aligned} \frac{d}{dt} \left(\frac{\delta V}{\delta \dot{x}} \right) &= \frac{d}{dt} \left(\frac{\delta V(x^*(t), \dot{x}^*(t), t)}{\delta \dot{x}} \right)_* \\ &= \left(\frac{\delta^2 V}{\delta x \delta \dot{x}} \right)_* \left(\frac{dx}{dt} \right)_* + \left(\frac{\delta^2 V}{\delta \dot{x} \delta \dot{x}} \right)_* \left(\frac{d^2 x}{dt^2} \right)_* + \left(\frac{\delta^2 V}{\delta t \delta \dot{x}} \right)_* \\ &= V_{x\dot{x}} \dot{x}(t) + V_{\dot{x}\dot{x}} \ddot{x}^*(t) + V_t \dot{x} \end{aligned} \quad (2.23)$$

Combining (2.21) and (2.23), we get an alternate form for the E-L equation as

$$V_x - V_{t\dot{x}} - V_{x\dot{x}} \dot{x}^*(t) - V_{\dot{x}\dot{x}} \ddot{x}^*(t) = 0 \quad (2.24)$$

2. The presence of $\frac{d}{dt}$ and/or $\dot{x}^*(t)$ in the E-L equation (2.19) means that it is a differential equation.
3. In the E-L equation (2.19), the term $\frac{\delta V(x^*(t), \dot{x}^*(t), t)}{\delta \dot{x}}$ is in general a function of $x^*(t)$, $\dot{x}^*(t)$, and t . Thus when this function is differentiated with respect to t , $\ddot{x}^*(t)$ may be present. This means that the differential equation (2.19) is in general of second order. This is also evident from the alternate form (2.24) for the E-L equation.
4. There may also be terms involving products or powers of $\ddot{x}^*(t)$, $\dot{x}^*(t)$, and $x^*(t)$, in which case, the differential equation becomes nonlinear.
5. The explicit presence of t in the arguments indicates that the coefficients may be time-varying.
6. The conditions at initial point $t = t_0$ and final point $t = t_f$ leads to a boundary value problem.

7. Thus, the Euler-Lagrange equation (2.19) is, in general, a nonlinear, time-varying, two-point boundary value, second order, ordinary differential equation. Thus, we often have a nonlinear two-point boundary value problem (TPBVP). The solution of the nonlinear TPBVP is quite a formidable task and often done using numerical techniques. This is the price we pay for demanding optimal performance!
8. Compliance with the Euler-Lagrange equation is only a necessary condition for the optimum. Optimal may sometimes not yield either a maximum or a minimum; just as inflection points where the derivative vanishes in differential calculus. However, if the Euler-Lagrange equation is not satisfied for any function, this indicates that the optimum does not exist for that functional.

2.3.3 Cases for Euler-Lagrange Equation

Case 1: V is dependent of $\dot{x}(t)$ and t . That is $V = V(\dot{x}(t), t)$. Then the Euler-Lagrange equation (2.21) becomes

$$\frac{d}{dt}(V_{\dot{x}}) = 0 \quad (2.25)$$

This leads to

$$V_{\dot{x}} = \frac{\delta V(\dot{x}^*(t), t)}{\delta \dot{x}} = C$$

Where C is a constant of integration.

Case 2: V is dependent of $\dot{x}(t)$ only. That is $V = V(\dot{x}(t))$. Then $V_x = 0$. The Euler-Lagrange equation (2.21) becomes

$$\frac{d}{dt}(V_{\dot{x}}) = 0 \quad (2.26)$$

which implies that the solution of (2.26) becomes

$$V_{\dot{x}^*} = C_1 \text{ and} \quad (2.27)$$

$$V(\dot{x}^*(t)) = C_1 \dot{x}^*(t) + C_2 \quad (2.28)$$

Case 3: V is dependent of $x(t)$ and $\dot{x}(t)$.

That is, $V = V(x(t), \dot{x}(t))$. Then $v_{t\dot{x}} = 0$. Using the other form of the Euler-Lagrange equation (2.24), we get

$$V_x - V_{x\dot{x}}\dot{x}^*(t) - V_{x\ddot{x}}\ddot{x}^*(t) - V_{\dot{x}\ddot{x}}\ddot{x}^*(t) = 0 \quad (2.29)$$

Multiplying the previous equation by $x^*(t)$, we have

$$\dot{x}(t)[V_x - V_{x\dot{x}}\dot{x}^*(t) - V_{x\ddot{x}}\ddot{x}^*(t) - V_{\dot{x}\ddot{x}}\ddot{x}^*(t)] = 0 \quad (2.30)$$

This can be rewritten as

$$\frac{d}{dt}(V - \dot{x}^*(t)V_{\dot{x}}) = 0 \longrightarrow V - \dot{x}^*(t)V_{\dot{x}} = C \quad (2.31)$$

The previous equation can be solved using any of the techniques such as, separation of variables.

Case 4: V is dependent of $x(t)$, and t , i.e., $V = V(x(t), t)$. Then, $V_{\dot{x}} = 0$ and the Euler-Lagrange equation (2.21) becomes

$$\frac{\delta V(x^*(t), t)}{\delta x} = 0 \quad (2.32)$$

The solution of this equation does not contain any arbitrary constants and therefore generally speaking does not satisfy the boundary conditions $x(t_0)$ and $x(t_f)$. Hence, in general, no solution exists for this variational problem. Only in rare cases, when the function $x(t)$ satisfies the given boundary conditions $x(t_0)$ and $x(t_f)$, it becomes an optimal function.[2]

2.4 Extrema of Functions with Conditions

Here we discuss two methods: the direct method using simple calculus and the Lagrange multiplier method using the Lagrange multiplier method.

2.4.1 Direct Method

Consider the extrema of a function $f(x_1, x_2)$ with two independent variables x_1 and x_2 subject to the condition.

$$g(x_1, x_2) = 0 \quad (2.33)$$

A necessary condition for extrema is

$$df = \frac{\delta f}{\delta x_1} dx_1 + \frac{\delta f}{\delta x_2} dx_2 = 0 \quad (2.34)$$

dx_1 and dx_2 are not arbitrary but are related by the condition

$$dg = \frac{\delta g}{\delta x_1} dx_1 + \frac{\delta g}{\delta x_2} dx_2 = 0 \quad (2.35)$$

It is impossible to conclude as in the case of extremization of functions without conditions that

$$\frac{\delta f}{\delta x_1} = 0 \text{ and } \frac{\delta f}{\delta x_2} = 0 \quad (2.36)$$

Assuming that $\frac{\delta g}{\delta x_2} \neq 0$, then (2.35) becomes

$$dx_2 = \left(\frac{\frac{\delta g}{\delta x_1}}{\frac{\delta g}{\delta x_2}} \right) dx_1 \quad (2.37)$$

substituting (2.37) in (2.34)

$$df = \left[\frac{\delta f}{\delta x_1} - \frac{\delta f}{\delta x_2} \left(\frac{\delta g}{\delta x_1} \right) \right] dx_1 = 0 \quad (2.38)$$

where dx_1 is independent, we now consider it to be arbitrary and conclude that in order to satisfy (2.38) we have the coefficient of dx_1 to be zero. i.e

$$\left(\frac{\delta f}{\delta x_1} \right) \left(\frac{\delta g}{\delta x_2} \right) - \left(\frac{\delta f}{\delta x_2} \right) \left(\frac{\delta f}{\delta x_1} \right) = 0 \quad (2.39)$$

We now solve (2.39) and (2.33) simultaneously for the optimal solutions x_1^* and x_2^* . We can write (2.39) as

$$\begin{vmatrix} \frac{\delta f}{\delta x_1} & \frac{\delta f}{\delta x_2} \\ \frac{\delta g}{\delta x_1} & \frac{\delta g}{\delta x_2} \end{vmatrix} = 0 \quad (2.40)$$

which is known as the jacobian of f and with respect to x_1 and x_2 . This method of eliminating the dependent variables is tedious for higher order problems.

2.4.2 Lagrange Multiplier Method

This method will be illustrated with example.

Example 2.4: Find the extrema of the function $f(x_1, x_2)$ subject to $g(x_1, x_2) = 0$

Step 1: Introduce the Lagrangian function

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2) \quad (2.41)$$

We take x_1, x_2 and λ to be independent variable.

Step 2: We need to find the first variation. i.e

$$\delta L = \frac{\delta L}{\delta x_1} dx_1 + \frac{\delta L}{\delta x_2} dx_2 + \frac{\delta L}{\delta \lambda} d\lambda \quad (2.42)$$

The critical points satisfy

$$\frac{\delta L}{\delta x_1} = 0, \frac{\delta L}{\delta x_2} = 0, \frac{\delta L}{\delta \lambda} = 0$$

$$\frac{\delta L}{\delta x_1} = \frac{\delta f(x_1, x_2)}{\delta x_1} + \lambda \frac{\delta g(x_1, x_2)}{\delta x_1} = 0 \quad (2.43)$$

$$\frac{\delta L}{\delta x_2} = \frac{\delta f(x_1, x_2)}{\delta x_2} + \lambda \frac{\delta g(x_1, x_2)}{\delta x_2} = 0 \quad (2.44)$$

$$\frac{\delta L}{\delta \lambda} = g(x_1, x_2) = 0 \quad (2.45)$$

The preceding three equations are to be solved simultaneously to obtain x_1^*, x_2^* and λ^* . By eliminating λ^* between (2.43) and (2.44) we obtain (x_1^*, x_2^*) as the required extreme point and $f(x_1^*, x_2^*)$ is the required extreme value.[2]

2.5 Extrema of Functionals with Condition

We extend our ideas to functionals based on those developed in the last section for functions.

Langrange Multiplier method

We illustrate this method with example

Example 1.5: Find the extrema of

$$J(x_1(t), x_2(t), t) = \int_{t_0}^{t_f} V(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), t) dt$$

(Here V is a functional of $x_1, x_2, \dot{x}_1, \dot{x}_2$) subject to $g(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t)) = 0$ with fixed-end point condition.

$$\begin{aligned} x_1(t_0) &= x_{10}, x_1(t_f) = x_{1f} \\ x_2(t_0) &= x_{20}, x_2(t_f) = x_{2f} \end{aligned}$$

Step 1: Introduce the Lagrangian function

$$\begin{aligned} L &= L(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), \lambda(t), t) \\ &= V(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), t) + \lambda(t)g(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t)) \end{aligned} \quad (2.46)$$

We take $x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), \lambda(t)$ to be independent variables.

Consider $J = \int_{t_0}^{t_f} L(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), \lambda(t), t) dt$

Step 2: Find the first variation of J

Given the increment $\Delta x_1, \Delta x_2$, we have

$$\begin{aligned} &J(x_1 + \Delta x_1, x_2 + \Delta x_2, \dot{x}_1 + \Delta \dot{x}_1, \dot{x}_2 + \Delta \dot{x}_2, \lambda, t) - J(x_1, x_2, \dot{x}_1, \dot{x}_2, \lambda, t) \\ &= \int_{t_0}^{t_f} [L(x_1 + \Delta x_1, x_2 + \Delta x_2, \dot{x}_1 + \Delta \dot{x}_1, \dot{x}_2 + \Delta \dot{x}_2, \lambda, t) - L(x_1, x_2, \dot{x}_1, \dot{x}_2, \lambda, t)] dt \end{aligned} \quad (2.47)$$

Step 3: First variation: using the Taylor series expansion, equation (2.47) becomes

$$\begin{aligned} &= \int_{t_0}^{t_f} \left(\frac{\delta L}{\delta x_1} \Delta x_1 + \frac{\delta L}{\delta x_2} \Delta x_2 + \frac{\delta L}{\delta \dot{x}_1} \Delta \dot{x}_1 + \frac{\delta L}{\delta \dot{x}_2} \Delta \dot{x}_2 + \text{higher order} \right) dt \\ &\approx \int_{t_0}^{t_f} \left(\frac{\delta L}{\delta x_1} \Delta x_1 + \frac{\delta L}{\delta x_2} \Delta x_2 + \frac{\delta L}{\delta \dot{x}_1} \delta \dot{x}_1 + \frac{\delta L}{\delta \dot{x}_2} \Delta \dot{x}_2 \right) dt \end{aligned} \quad (2.48)$$

Using integration by parts

$$\begin{aligned}
 \int_{t_0}^{t_f} \frac{\delta L}{\delta \dot{x}_1} \Delta \dot{x}_1 dt &= \int_{t_0}^{t_f} \left(\frac{\delta L}{\delta \dot{x}_1} \right) \frac{d}{dt} (\Delta x_1) dt \\
 &= \int_{t_0}^{t_f} \left(\frac{\delta L}{\delta \dot{x}_1} \right) d(\Delta x_1) \\
 &= \left[\frac{\delta L}{\delta \dot{x}_1} \Delta x_1 \right]_{t_0}^{t_f} - \int_{t_0}^{t_f} \Delta x_1 \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{x}_1} \right) dt \quad (2.49)
 \end{aligned}$$

Similarly

$$\int_{t_0}^{t_f} \frac{\delta L}{\delta \dot{x}_2} \Delta \dot{x}_2 dt = \left[\frac{\delta L}{\delta \dot{x}_2} \Delta x_2 \right]_{t_0}^{t_f} - \int_{t_0}^{t_f} \Delta x_2 \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{x}_2} \right) dt \quad (2.50)$$

Substituting (2.49) and (2.50) in (2.48) gives

$$\begin{aligned}
 &= \int_{t_0}^{t_f} \left[\frac{\delta L}{\delta x_1} \Delta x_1 + \frac{\delta L}{\delta x_2} \Delta x_2 - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{x}_1} \right) \Delta x_1 - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{x}_2} \right) \Delta x_2 \right] dt \\
 &\quad + \left[\frac{\delta L}{\delta \dot{x}_1} \Delta x_1 + \frac{\delta L}{\delta \dot{x}_2} \Delta x_2 \right]_{t=t_0}^{t=t_f} \quad (2.51)
 \end{aligned}$$

Since the problem is a fixed-final time and fixed-final state, no variations are allowed at the final point. This means

$$\Delta x_1(t_0) = \Delta x_2(t_0) = \Delta x_1(t_f) = \Delta x_2(t_f) = 0$$

$$= \int_{t_0}^{t_f} \left[\frac{\delta L}{\delta x_1} - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{x}_1} \right) \right] \Delta x_1 dt + \int_{t_0}^{t_f} \left[\frac{\delta L}{\delta x_2} - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{x}_2} \right) \right] \Delta x_2 dt \quad (2.52)$$

Therefore, the necessary conditions for the extreme point are

$$\begin{aligned}
 \frac{\delta L}{\delta x_1} - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{x}_1} \right) &= 0 \\
 \frac{\delta L}{\delta x_2} - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{x}_2} \right) &= 0 \\
 \text{Also } \frac{\delta L}{\delta \lambda} &= 0 \text{ (from(2.46))} \quad (2.53)
 \end{aligned}$$

Example 2.6: Minimize the performannce Index

$$J = \int_0^1 [x^2(t) + u^2(t)] dt \quad (2.54)$$

with boundary conditions

$x(0) = 1; x(1) = 0$ subject to the condition

$$\dot{x}(t) = -x(t) + u(t) \quad (2.55)$$

Solution: (Lagrange Method)

The condition is $g(x, \dot{x}, u) = \dot{x}(t) + x(t) - u(t) = 0$ introduce the Lagrangian function as $L(x, \dot{x}, u, \lambda) = x^2(t) + u^2(t) + \lambda(t)(\dot{x}(t) + x(t) - u(t))$

The necessary conditions are

$$\begin{aligned}\frac{\delta L}{\delta x} - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{x}} \right) &= 0 \\ \frac{\delta L}{\delta u} - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{u}} \right) &= 0 \\ \frac{\delta L}{\delta \lambda} &= 0\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\delta L}{\delta x} &= 2x + \lambda; \quad \frac{\delta L}{\delta \dot{x}} = \lambda \\ \frac{\delta L}{\delta u} &= 2u - \lambda; \quad \frac{\delta L}{\delta \dot{u}} = 0 \\ \frac{\delta L}{\delta \lambda} &= \dot{x} + x - u\end{aligned}$$

Thus, we have

$$\begin{aligned}2x + \lambda - \lambda' &= 0 \\ 2u - \lambda &= 0 \\ \dot{x} - x - u &= 0\end{aligned}\tag{2.56}$$

We solve the above simultaneously to get

$$\begin{aligned}\lambda &= 2(\dot{x} + x) \\ 2x + 2(\dot{x} + x) - 2(\ddot{x} + \dot{x}) &= 0 \\ \ddot{x} - 2x &= 0\end{aligned}\tag{2.57}$$

Solving (2.57) gives

$$x(t) = C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t}\tag{2.58}$$

Using the boundary conditions gives the constants C_1 and C_2

$$C_1 = \frac{1}{1 - e^{-2\sqrt{2}}}; C_2 = \frac{1}{1 - e^{2\sqrt{2}}}$$

Finally, we find the optimal control $u(t)$ knowing $x(t)$ from (2.55)

$$\begin{aligned}u(t) &= \dot{x}(t) + x(t) \\ &= C_1(1 - \sqrt{2})e^{-\sqrt{2}t} + C_2(1 + \sqrt{2})e^{\sqrt{2}t}\end{aligned}\tag{2.59}$$

Solution:(Direct method)

Here, we eliminate $u(t)$ in the performance index and the condition to get

$$\begin{aligned} J &= \int_0^1 (x^2(t) + (\dot{x}(t) + x(t))^2) dt \\ &= \int_0^1 (2x^2(t) + \dot{x}^2(t) + 2x(t)\dot{x}(t)) dt \end{aligned}$$

Applying the Euler-Lagrange equation

$$\frac{\delta V}{\delta x} - \frac{d}{dt} \left(\frac{\delta V}{\delta \dot{x}} \right) = 0$$

Where

$$V = 2x^2(t) + \dot{x}^2(t) + 2x(t)\dot{x}(t)$$

We have

$$4x(t) + 2\dot{x}(t) - \frac{d}{dt}(2\dot{x}(t) + 2x(t)) = 0$$

Simplifying the above

$$\ddot{x}(t) = C_1 e^{1\sqrt{2}t} + C_2 e^{\sqrt{2}t}$$

Applying the boundary conditions gives

$$C_1 = \frac{1}{1 - e^{-2\sqrt{2}}}; C_2 = \frac{1}{1 + e^{2\sqrt{2}}}$$

Therefore, since we know $x(t)$, $u(t)$ is given as

$$\begin{aligned} u(t) &= \dot{x}(t) + x(t) \\ &= C_1(1 - \sqrt{2})e^{-\sqrt{2}t} + C_2(1 + \sqrt{2})e^{\sqrt{2}t} \end{aligned}$$

2.6 Variational Approach to Optimal Control System

Here, we use variational techniques to approach the optimal control system, and in the process introduce the Hamiltonian function, which was used by Pontryagin and his associates to develop the famous minimum principle.

We illustrate this approach with examples.

Example 2.7: Find the extrema of

$$J(u) = S(x(t_f), t_f) + \int_{t_0}^{t_f} V(x(t), u(t), t) dt$$

subject to the condition

$$\begin{aligned}\dot{x}(t) - f(x(t), u(t), t) &= 0 \\ x(t_0) &= x_0\end{aligned}$$

$x(t_f)$ is free, t_f is free. Here S is a functional of $x(t)$. V is a functional of $x(t), u(t)$. This question is equivalent to the following

Find the extrema of

$$J(u) = \int_{t_0}^{t_f} \left[V(x(t), u(t), t) + \frac{dS}{dt} \right] dt$$

subject to

$$\begin{aligned}\dot{x}(t) - f(x(t), u(t), t) &= 0 \\ x(t_0) &= x_0\end{aligned}$$

$x(t_f)$ is free, t_f is free

Solution:

Step 1: Introduce the Lagrangian function L i.e

$$L(x, \dot{x}, u, \lambda, t) = \left[V(x(t), u(t), t) + \frac{dS}{dt} \right] + \lambda(t) \{ f(x(t), u(t), t) - \dot{x}(t) \}$$

Also introduce

$$J_a = J_a(x, \dot{x}, u, \lambda, t_f) = \int_{t_0}^{t_f} L(x, \dot{x}, u, \lambda, t) dt$$

Step 2: Find the first variation given increments $\Delta x, \Delta \dot{x}, \Delta u, \Delta t_f$

$$\frac{\delta J_a}{\delta x} = 0; \frac{\delta J_a}{\delta \dot{x}} = 0; \frac{\delta J_a}{\delta u} = 0; \frac{\delta J_a}{\delta \lambda} = 0$$

$$\begin{aligned} & J_a(x + \Delta x, \dot{x} + \Delta \dot{x}, u + \Delta u, t_f + \Delta t_f) - J_a(x, \dot{x}, u, t_f) \\ &= \int_{t_0}^{t_f + \Delta t_f} L(x + \Delta x, \dot{x} + \Delta \dot{x}, u + \Delta u, \lambda) dt - \int_{t_0}^{t_f} L(x, \dot{x}, u, \lambda) dt \\ &= \int_{t_0}^{t_f} [L(x + \Delta x, \dot{x} + \Delta \dot{x}, u + \Delta u, \lambda) - L(x, \dot{x}, u, \lambda)] dt \\ &\quad + \int_{t_f}^{t_f + \Delta t_f} L(x + \Delta x; \dot{x} + \Delta \dot{x}, u + \Delta u, \lambda) dt \end{aligned}$$

By Taylor series expansion

$$= \int_{t_0}^{t_f} \left(\frac{\delta L}{\delta x} \Delta x + \frac{\delta L}{\delta \dot{x}} \Delta \dot{x} + \frac{\delta L}{\delta u} \Delta u + \text{higher order} \right) dt + L(x, \dot{x}, u, \lambda) \Big|_{t=t_f} \Delta t_f + \text{higher order} \quad (2.60)$$

Using integration by parts

$$\int_{t_0}^{t_f} \frac{\delta L}{\delta \dot{x}} \Delta \dot{x} dt = \int_{t_0}^{t_f} \frac{\delta L}{\delta \dot{x}} d(\Delta x) = \frac{\delta L}{\delta \dot{x}} \Delta x \Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} (\Delta x) \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{x}} \right) dt$$

Equation (2.60) becomes

$$= \int_{t_0}^{t_f} \left(\frac{\delta L}{\delta x} - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{x}} \right) \right) \Delta x dt + \int_{t_0}^{t_f} \frac{\delta L}{\delta u} \Delta u dt + L(x, \dot{x}, u, \lambda) \Big|_{t_f} \Delta t_f + \frac{\delta L}{\delta \dot{x}} \Delta x \Big|_{t_0}^{t_f} + \text{higher order}$$

The necessary conditions for the extreme point are

$$\begin{aligned} \frac{\delta L}{\delta x} - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{x}} \right) &= 0 \\ \frac{\delta L}{\delta u} &= 0 \\ L(x, \dot{x}, u, \lambda) \Big|_{t=t_f} \Delta t_f + \frac{\delta L}{\delta \dot{x}} \Delta x \Big|_{t_0}^{t_f} &= 0 \\ \frac{\delta L}{\delta \lambda} &= 0 \end{aligned}$$

Step 3: Introduce Hamiltonian function $H(x, u, \lambda, t)$

$$H(x, u, \lambda, t) = V(x, u, t) + \lambda(t)f(x, u, t)$$

$$\text{But } L = V(x, u, t) + \frac{dS}{dt} + \lambda(t)(f(x, u, t) - \dot{x})$$

$$L = H(x, u, \lambda, t) + \frac{dS}{dt} - \lambda(t)\dot{x}(t) \quad (2.61)$$

$$\frac{\delta L}{\delta x} - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{x}} \right) = \frac{\delta H}{\delta x} + \frac{\delta}{\delta x} \left(\frac{dS}{dt} \right) - 0 - \frac{d}{dt} \left(\frac{\delta}{\delta \dot{x}} \left(\frac{dS}{dt} \right) - \lambda(t) \right)$$

Note that

$$\frac{dS}{dt} = \frac{dS(x(t), t)}{dt} = \frac{\delta S(x, t)}{\delta x} \dot{x} + \frac{\delta S(x, t)}{\delta t}$$

We have

$$\begin{aligned} \frac{\delta L}{\delta x} - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{x}} \right) &= \frac{\delta H}{\delta x} + \frac{\delta^2 S(x, t)}{\delta x^2} \dot{x} + \frac{\delta^2 S(x, t)}{\delta x \delta t} - \frac{d}{dt} \left(\frac{\delta S(x, t)}{\delta x} - \lambda(t) \right) \\ &= \frac{\delta H}{\delta x} + \frac{\delta^2 S(x, t)}{\delta x^2} \dot{x} + \frac{\delta^2 S(x, t)}{\delta x \delta t} - \left[\frac{\delta^2 S(x, t)}{\delta x^2} \dot{x} + \frac{\delta^2 S(x, t)}{\delta x \delta t} - \frac{d\lambda}{dt} \right] \\ &= \frac{\delta H}{\delta x} + \frac{d\lambda}{dt} = 0 \end{aligned}$$

from (2.61)

$$\begin{aligned}\frac{\delta L}{\delta u} &= \frac{\delta H}{\delta u} = 0 \\ \frac{\delta L}{\delta \lambda} &= \frac{\delta H}{\delta \lambda} - \dot{x}(t) = 0\end{aligned}$$

The necessary conditions are

$$\begin{aligned}\frac{\delta L}{\delta x} - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{x}} \right) &= 0 \implies \frac{\delta H}{\delta x} = -\dot{\lambda}(t) \\ \frac{\delta L}{\delta u} = 0 &\implies \frac{\delta H}{\delta u} = 0 \\ L(x, \dot{x}, u, \lambda) \Big|_{t=t_f} \Delta t_f + \frac{\delta L}{\delta \dot{x}} \Delta x \Big|_{t_0}^{t_f} &= 0 \\ \implies \left[H + \frac{\delta S}{\delta t} \right]_{t_f} \Delta t_f + \left[\frac{\delta S}{\delta x} - \lambda(t) \right]_{t_f} \Delta x_f &= 0 \\ \frac{\delta L}{\delta \lambda} = 0 &\implies \frac{\delta H}{\delta \lambda} = \dot{x}(t)\end{aligned}$$

2.6.1 Types of Systems

We obtain different cases depending on the statement of the problem regarding the final time t_f and the final state $x(t_f)$. These are summarised below.

Type A: Fixed final time and fixed final state ($\Delta t_f = 0, \Delta x_f = 0$).

The necessary conditions are

$$\begin{aligned}\frac{\delta H}{\delta x} &= -\frac{d\lambda}{dt} \\ \frac{\delta H}{\delta u} &= 0\end{aligned}$$

All equal to zero (not considered)

$$\frac{\delta H}{\delta \lambda} - \dot{x}(t) = 0.$$

Type B: Fixed final state and free final time ($\Delta t_f \neq 0, \Delta x_f = 0$).

The necessary conditions are

$$\begin{aligned}\frac{\delta H}{\delta x} &= -\frac{d\lambda}{dt} \\ \frac{\delta H}{\delta u} &= 0 \\ \left(H + \frac{\delta S}{\delta t} \right)_{t_f} &= 0 \\ \frac{\delta H}{\delta \lambda} - \dot{x}(t) &= 0\end{aligned}$$

Type C: Fixed final time and free final state ($\Delta t_f = 0, \Delta x_f \neq 0$)

The necessary conditions are

$$\begin{aligned}\frac{\delta H}{\delta x} &= -\frac{\delta \lambda}{dt} \\ \frac{\delta H}{\delta x} &= 0 \\ \left(H + \frac{\delta S}{\delta t}\right)_{t_f} &= 0 \\ \frac{\delta H}{\delta \lambda} - \dot{x}(t) &= 0\end{aligned}$$

Type D: Free final time and free final state ($\Delta t_f \neq 0, \Delta x_f \neq 0$)

The necessary conditions are

$$\begin{aligned}\frac{\delta H}{\delta \lambda} &= -\frac{d\lambda}{dt} \\ \frac{\delta H}{\delta u} &= 0 \\ \left(H + \frac{\delta S}{\delta t}\right)_{t_f} \Delta t_f + \left(\frac{\delta S}{\delta x} - \lambda\right)_{t_f} \Delta x_f &= 0 \\ \frac{\delta H}{\delta \lambda} - \dot{x}(t) &= 0\end{aligned}$$

Table 1 summarises the different types of systems.

Type	Substitutions	Boundary Conditions
(a)	$\Delta t_f = 0, \Delta x_f = 0$	$x(t_0) = x_0, x(t_f) = x_f$
(b)	$\Delta t_f \neq 0, \Delta x_f = 0$	$x_f, \left[H + \frac{\delta S}{\delta t}\right]_{t_f} = 0$
(c)	$\Delta t_f = 0, \Delta x_f \neq 0$	$x(t_0) = x_0, \lambda^*(t_f) = \left(\frac{\delta S}{\delta x}\right)_{*t_f}$
(d)	$\Delta t_f \neq 0, \Delta x_f \neq 0$	$\Delta x(t_0) = x_0, \left[H + \frac{\delta S}{\delta t}\right]_{t_f} = 0$ $\left[\left(\frac{\delta S}{\delta x}\right) - \lambda^*(t)\right]_{t_f} = 0$

Table 2.1: Types of System

Example 2.8 (Type A)

Find the extrema (i.e the optimal control and optimal state)[2]

$$J(u) = \frac{1}{2} \int_{t_0}^{t_f} u^2(t) dt \tag{2.62}$$

Under the conditions

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u(t) \\ t_0 &= 0, t_f = 2\end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (t_0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (t_f) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution

From the problem, we identify the following

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad V(x, u, t) = V(u(t)) = \frac{1}{2}u^2(t)$$

$$f(x, u, t) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \text{ where } f_1 = x_2, f_2 = u$$

Step 1: We formulate the Hamiltonian function as

$$H = H(x_1(t), x_2(t), u(t), \lambda_1(t), \lambda_2(t)) = V + \lambda f$$

$$= \frac{1}{2}u^2 + (\lambda_1, \lambda_2) \begin{pmatrix} x_2 \\ u \end{pmatrix}$$

$$H = \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u$$

Step 2: find u from

$$\frac{\delta H}{\delta u} = 0 \implies u + \lambda_2 = 0$$

$$u = -\lambda_2$$

Step 3: Using the results of step 2 in step 1, find the optimal H

$$H = \frac{1}{2}\lambda_2^2 + \lambda_1 x_2 - \lambda_2^2$$

$$= \lambda_1 x_2 - \frac{1}{2}\lambda_2^2$$

Step 4: Obtain the state and costate equations from

$$H(x_1, x_2, \lambda_1, \lambda_2) = \lambda_1 x_2 - \frac{1}{2}\lambda_2^2$$

$$\frac{\delta H}{\delta x_1} = 0; \quad \frac{\delta H}{\delta x_2} = \lambda_1;$$

$$\frac{\delta H}{\delta \lambda_1} = x_2; \quad \frac{\delta H}{\delta \lambda_2} = -\lambda_2$$

$$\begin{bmatrix} \frac{\delta H}{\delta x_1} \\ \frac{\delta H}{\delta x_2} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 \\ \lambda_1 \end{bmatrix} + \begin{bmatrix} \frac{d\lambda_1}{dt} \\ \frac{d\lambda_2}{dt} \end{bmatrix} = 0$$

$$\frac{d\lambda_1}{dt} = 0 \implies \lambda_1 = C_1 \quad (2.63)$$

$$\lambda_1 + \frac{d\lambda_2}{dt} = 0 \implies C_1 + \frac{d\lambda_2}{dt} = 0. \implies \lambda_2 = -C_1 t = C_2 \quad (2.64)$$

$$\begin{aligned} \frac{\delta H}{\delta \lambda_2} &= -\lambda_2 = u \\ \dot{x}_2 &= \lambda_2 \implies \dot{x}_2 = -(-C_1 t + C_2) \\ \dot{x}_2 &= C_1 t - C_2 \\ \frac{dx_2}{dt} &= C_1 t - C_2 \\ x_2 &= \frac{C_1 t^2}{2} - C_2 t + C_3 \end{aligned} \quad (2.65)$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_1 &= \frac{C_1 t^2}{2} - C_2 t + C_3 \\ x_1 &= \frac{C_1 t^3}{6} - \frac{C_2 t^2}{2} + C_3 t + C_4 \end{aligned} \quad (2.66)$$

Step 5: Obtain the optimal control from

$$\begin{aligned} u &= -\lambda_2 \\ u &= C_1 t - C_2 \end{aligned} \quad (2.67)$$

where C_1, C_2, C_3 and C_4 are constants evaluated using the boundary conditions given that is

$$\begin{aligned} \begin{pmatrix} \frac{C_1 t^3}{6} - \frac{C_2 t^2}{2} + C_3 t + C_4 \\ \frac{C_1 t^2}{2} - C_2 t + C_3 \end{pmatrix} (t_0) &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ \begin{pmatrix} C_4 \\ C_3 \end{pmatrix} &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ \implies C_4 &= 1, C_3 = 2 \end{aligned}$$

Also

$$\begin{aligned} \begin{pmatrix} \frac{C_1 t^3}{6} - \frac{C_2 t^2}{2} + C_3 t + C_4 \\ \frac{C_1 t^2}{2} - C_2 t + C_3 \end{pmatrix} (t_f) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \frac{8C_1 t^3}{6} - \frac{4C_2 t^2}{2} + 2C_3 t + C_4 \\ \frac{4C_1}{2} - 2C_2 + C_3 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

Solving these equations simultaneously gives

$$C_2 = 4, C_1 = 3$$

$$\therefore C_1 = 3, C_2 = 4, C_3 = 2, C_4 = 1$$

Finally, we have the optimal states, costates and the control as

$$\begin{aligned}x_1 t &= 0.5t^3 - 2t^2 + 2t + 1 \\x_2(t) &= 1.5t^2 - 4t + 2 \\ \lambda_1(t) &= 3 \\ \lambda_2(t) &= -3t + 4 \\ u(t) &= 3t - 4\end{aligned}$$

Example 2.9: (Type C)

Consider the same example 2.8 with changed boundary conditions as

$$\begin{aligned}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (x_0) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ x_1(2) &= 0 \\ x_2(2) \text{ is free.} &\implies t_f \text{ is fixed.} \\ \Delta t_f &= 0, \Delta x_f \neq 0\end{aligned}$$

Find the optimal control and optimal states.[2]

Solution

we follow the same procedure as in Example 2.8 and in table 2.1, Type(c) to get the same optimal states, costates and control as given in (2.63) and (2.67) above. i.e

$$\begin{aligned}\lambda_1(t) &= C_1 \\ \lambda_2(t) &= -C_1 t + C_2 \\ x_2(t) &= \frac{C_1 t^2}{2} - C_2 t + C_3 \\ x_1(t) &= \frac{C_1 t^3}{6} - \frac{C_2 t^2}{2} + C_3 t + C_4 \\ u(t) &= C_1 t - C_2\end{aligned}$$

The only difference is in solving for the constants C_1 to C_4 . Note that the performer index (1.62) does not contain the terminal cost function S . So, $S = 0$ and $t_f = 2$.

Since $x_2(2)$ is free, Δx_{2f} is arbitrary and corresponding final condition on the costate becomes

$$\lambda_2(t_f) = \left(\frac{\delta S}{\delta x_2} \right)_{t_f} = 0$$

Hence we have four boundary conditions given as

$$x_1(0) = 1, x_2(0) = 2, x_1(2) = 0, \lambda_2(2) = 0$$

With these boundary conditions substituted in we have

$$\begin{aligned} C_3 &= 2, C_4 = 1 \\ -2C_1 + C_2 &= 0 \\ 4C_1 - 6C_2 &= -15 \end{aligned}$$

Solving simultaneously gives

$$C_1 = \frac{15}{8}, C_2 = \frac{15}{4}$$

$$\therefore C_1 = \frac{15}{8}, C_2 = \frac{15}{4}, C_3 = 2 \text{ and } C_4 = 1$$

Finally, we have the optimal states, costates and the control given as

$$\begin{aligned} x_1(t) &= \frac{5}{16}t^3 - \frac{15}{8}t^2 + 2t + 1 \\ x_2(t) &= \frac{15}{16}t^2 - \frac{15}{4}t + 2 \\ \lambda_1(t) &= \frac{15}{8} \\ \lambda_2(t) &= -\frac{15}{8}t + \frac{15}{4} \\ u(t) &= \frac{15}{8}t - \frac{15}{4} \end{aligned}$$

Example 2.10: (Type D)

Consider the same Example 2.8 with the changed boundary conditions [2] as

$$x(0) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$x_1(t_f) = 3, x_2(t_f)$ is free.

Find the optimal control and optimal state.

Solution: The same procedure illustrated above and in table 1 (type (d)) will be required to get the same optimal states, costates and control as given in (2.63) - (2.67) above i.e

$$\begin{aligned} \lambda_1(t) &= C_1 \\ \lambda_2(t) &= -C_1t + C_2 \\ x_2(t) &= \frac{C_1t^2}{2} - C_2t + C_3 \\ x_1(t) &= \frac{C_1}{6}t^3 - \frac{C_2}{2}t^2 + C_3t + C_4 \\ u(t) &= C_1t - C_2 \end{aligned}$$

We solve for the constants C_1 to C_4 and the unknown t_f . t_f is unspecified in the given boundary conditions, hence Δt_f is free in the general boundary conditions (see Type (d) above). This leads to the specific final condition.

$$\begin{aligned} \left(H + \frac{\delta S}{\delta t} \right)_{t_f} &= 0 \\ \lambda_1(t_f)x_2(t_f) - \frac{1}{2}\lambda_2^2(t_f) &= 0 \end{aligned} \quad (2.68)$$

Since $x_2(t_f)$ is free, Δx_{2f} is arbitrary and the general boundary conditions becomes

$$\lambda_2(t_f) = \frac{\delta S}{\delta x_2} = 0 \quad (2.69)$$

Where H is given as in Step 3 above. Combining the boundary conditions in this example with (2.68) and (2.69) leads us to the following 5 boundary conditions.

$$\begin{aligned} x_1(0) &= 1; x_2(0) = 2; x_1(t_f) = 3; \\ \lambda_2(t_f) &= 0; \lambda_1(t_f)x_2(t_f) - \frac{1}{2}\lambda_2^2(t_f) = 0 \end{aligned}$$

With these boundary conditions substituted in, we have

$$\begin{aligned} C_3 &= 2, C_4 = 1 \\ -C_1 t_f + C - 2 &= 0 \\ C_1 t_f^2 - 2C_2 t_f &= -4 \\ \frac{C_1}{6} t_f^3 - \frac{C_2}{2} t_f^2 &= 2 - 2t_f \end{aligned}$$

Solving these equations for C_1, C_2 and t_f gives

$$C_1 = \frac{4}{9}, C_2 = \frac{4}{3}, C_3 = 2, C_4 = 1, t_f = 3$$

Finally, we have the optimal states, costates and the control given as

$$\begin{aligned} x_1(t) &= \frac{4}{54}t^3 - \frac{2}{3}t^2 + 2t + 1 \\ x_2(t) &= \frac{4}{18}t^2 - \frac{4}{3}t + 2 \\ \lambda_1(t) &= \frac{4}{9} \\ \lambda_2(t) &= -\frac{4}{9}t + \frac{4}{3} \\ u(t) &= \frac{4}{9}t - \frac{4}{3} \end{aligned}$$

Example 2.11: (Type B)

Consider the same Example 1.8 with changed performance index [2]

$$J = \frac{1}{2} [x_1(2) - 4]^2 + \frac{1}{2} [x_2(2) - 2]^2 + \frac{1}{2} \int_0^2 u^2 dt \quad (2.70)$$

and boundary conditions as
 $x(0) = [1 \ 2] : x(2)$ is free

With the same procedure illustrated above and as on table 1 (Type (b)), we get the same optimal control, states and costates as given in (2.63)- (2.67) i.e

$$\begin{aligned}\lambda_1(t) &= C_1 \\ \lambda_2(t) &= -C_1t + C_2 \\ x_2(t) &= \frac{C_1}{2}t^2 - C_2t + C_3 \\ x_1(t) &= \frac{C_1}{6}t^3 - \frac{C_2}{2}t^2 + C_3t + C_4 \\ u(t) &= C_1t - C_2\end{aligned}$$

We solve the constants C_1 to C_4 using the given and obtained boundary conditions. t_f is specified as 2, hence Δt_f is zero and since $x(2)$ is unspecified, Δx_f is free in the boundary condition which reduces to

$$\lambda(t_f) = \left(\frac{\delta S}{\delta x} \right)_{t_f} \quad (2.71)$$

where

$$S(x(t_f)) = \frac{1}{2}[x_1(2) - 4]^2 + \frac{1}{2}[x_2(2) - 2]^2$$

Thus, (2.71) becomes

$$\begin{aligned}\lambda_1(t_f) &= \left(\frac{\delta S}{\delta x_1} \right)_{t_f} \\ \lambda_1(2) &= x_1(2) - 4\end{aligned} \quad (2.72)$$

$$\begin{aligned}\lambda_2(t_f) &= \left(\frac{\delta S}{\delta x_2} \right)_{t_f} \\ \lambda_2(2) &= x_2(2) - 2\end{aligned} \quad (2.73)$$

We now have two given initial conditions from the question and two final conditions from (2.72) and (2.73) to solve the constants C_1 to C_4 i.e

$$\begin{aligned}C_4 &= 1, C_3 = 2 \\ C_1 - 6C_2 &= -3 \\ 4C_1 - 3C_2 &= 0\end{aligned}$$

Solving simultaneously gives

$$C_1 = \frac{3}{7}, C_2 = \frac{4}{7}, C_3 = 2, C_4 = 1$$

Finally, we have the optimal states, costates and control given as

$$\begin{aligned}x_1(t) &= \frac{1}{14}t^3 - \frac{2}{7}t^2 + 2t + 1 \\x_2(t) &= \frac{3}{14}t^2 - \frac{4}{7}t + 2 \\ \lambda_1(t) &= \frac{3}{7} \\ \lambda_2(t) &= -\frac{3}{7}t + \frac{4}{7} \\ u(t) &= \frac{3}{7}t - \frac{4}{7}\end{aligned}$$

3 Stabilizing a Nonlinear System

A linear first-order ordinary differential equation (ODE) has the standard form

$$\frac{dx}{dt} + p(t)x = q(t) \quad (3.1)$$

where the function $p(t)$ and $q(t)$ are piecewise continuous on some interval I . A nonlinear first ODE is one that cannot be manipulated into the form of equation (3.1). To verify whether an ODE is linear, we see if the equation can be re-arranged into the form of (3.1). For example, the system $x^2\dot{x} + 3t = e^{-x}$ is nonlinear.

Nonlinear ODEs frequently occur as mathematical models for problem in engineering, physics and biology too. Therefore, we cannot ignore such equations. Now we will look at the stability for some numerical methods. We say that a numerical method is stable if small changes or perturbation in the initial conditions produce corresponding small changes in the subsequent approximations. That is, a stable method is one whose results depend continuously on the initial data. We shall consider the stability for some numerical methods with a test problem later in this chapter.

3.1 Steady State for a Nonlinear System

A system in a **steady state** has numerous properties that are not changing in time. This means that for any dependable variable $x(t)$ of the system, the partial derivative with respect to time t is zero. That is

$$\frac{dx}{dt} = 0 \quad (3.2)$$

The concept of steady state has relevance in many fields and is a more general situation than dynamic equilibrium. If a system is in steady state, then the observed behaviour of the system will continue into the future.

A system that is in steady state, may not necessary be in a state of dynamic equilibrium, because some of the processes involved are not reversible. For instance, the flow of electricity through a network or the flow of fluid through a tube could be in a steady state because there is a constant flow of electricity or fluid. Conversely, a tank which is being drained or filled with fluid would be an example of a system in transient state, because the volume of fluid contained in it changes with time.

Example 3.1

Consider an autonomous finite-dimensional system

$$\frac{dx}{dt} + F(x) = 0, x \in R \quad (3.3)$$

where F is a nonlinear function. The steady state x^* is given as

$$\frac{dx}{dt} = 0 \implies F(x^*) = 0 \quad (3.4)$$

If x^* is unstable i.e

$$\lim_{t \rightarrow \infty} x(t) \neq x^*$$

We now need to stabilize the system at x^* by linearization which shall be discussed later in this section.

Example 3.2 Consider the nonlinear systems

$$\begin{aligned} \frac{dx}{dt} &= F(x, y), x \in R \\ \frac{dy}{dt} &= G(x, y), y \in R \end{aligned} \quad (3.5)$$

The steady states are

$$\begin{aligned} F(x^*, y^*) &= 0 \\ G(x^*, y^*) &= 0 \end{aligned} \quad (3.6)$$

if (x^*, y^*) is unstable i.e

$$\lim_{t \rightarrow \infty} (x(t), y(t)) \not\rightarrow (x^*, y^*)$$

We need to stabilize the system at (x^*, y^*) by linearization.

3.2 Linearization of Nonlinear System at Steady state

Here, we intend to perform linearization of systems described by nonlinear differential equations. The Taylor series expansion is employed for the procedure.

In **Example 3.1**, we linearize the system at x^* using Taylor series expansion

$$\begin{aligned} F(x) &= F(x^*) + F'(x^*)(x - x^*) + \text{Higher order terms} \\ F(x) &\approx F(x^*) + F'(x^*)(x - x^*) \end{aligned} \quad (3.7)$$

We remark that $F'(x^*)$ is a scalar number.

Substitute (3.7) in (3.3) to get

$$\frac{dx}{dt} + F(x^*) + F'(x^*)(x - x^*) = 0$$

Using (3.4), we have

$$\frac{dx}{dt} + F'(x^*)(x - x^*) = 0 \quad (3.8)$$

Equation (3.8) is the linearised system of (3.3)

Now, let $z = x - x^* \implies x = z + x^*$

$x \rightarrow x^* \implies y \rightarrow 0$

Then (3.8) becomes

$$\frac{dz}{dt} + F'(x^*)z = 0$$

Let $A = F'(x^*)$

$$\frac{dz}{dt} + Az = 0 \quad (3.9)$$

Also, in **Example 3.2**, we linearize the systems of two autonomous finite-dimensional systems at (x^*, y^*) using Taylor series expansion.

$$F(x, y) = F(x^*, y^*) + \frac{\delta F}{\delta x}(x^*, y^*)(x - x^*) + \frac{\delta F}{\delta y}(x^*, y^*)(y - y^*)$$

$$G(x, y) = G(x^*, y^*) + \frac{\delta G}{\delta x}(x^*, y^*)(x - x^*) + \frac{\delta G}{\delta y}(x^*, y^*)(y - y^*)$$

The problem are now linearised about (x^*, y^*)

$$F(x, y) = F(x^*, y^*) + \frac{\delta F}{\delta x}(x^*, y^*)(x - x^*) + \frac{\delta F}{\delta y}(x^*, y^*)(y - y^*)$$

$$G(x, y) = G(x^*, y^*) + \frac{\delta G}{\delta x}(x^*, y^*)(x - x^*) + \frac{\delta G}{\delta y}(x^*, y^*)(y - y^*)$$

From (3.6), $F(x^*, y^*) = G(x^*, y^*) = 0$

The linearized systems are

$$\frac{dy}{dt} = \frac{\delta F}{\delta x}(x^*, y^*)(x - x^*) + \frac{\delta F}{\delta y}(x^*, y^*)(y - y^*) \quad (3.10)$$

$$\frac{dy}{dt} = \frac{\delta G}{\delta x}(x^*, y^*)(x - x^*) + \frac{\delta G}{\delta y}(x^*, y^*)(y - y^*) \quad (3.11)$$

Let $x - x^* = X$

$y - y^* = Y$

Then (3.10) and (3.11) becomes

$$\frac{dX}{dt} = \frac{\delta F}{\delta x}(x^*, y^*)X + \frac{\delta F}{\delta y}Y \quad (3.12)$$

$$\frac{dY}{dt} = \frac{\delta G}{\delta x}(x^*, y^*)X + \frac{\delta G}{\delta y}(x^*, y^*)Y \quad (3.13)$$

So, let

$$A = \begin{bmatrix} \frac{\delta F}{\delta x}(x^*, y^*) & \frac{\delta F}{\delta y}(x^*, y^*) \\ \frac{\delta G}{\delta x}(x^*, y^*) & \frac{\delta G}{\delta y}(x^*, y^*) \end{bmatrix}$$

Note that A is a 2x2 matrix

So

$$Z = \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\frac{dZ}{dt} = AZ$$

Example 3.3: Consider the nonlinear system

$$\begin{aligned} \frac{dx}{dt} &= x^3 - xy \\ \frac{dy}{dt} &= y^3 - xy \end{aligned} \quad (3.14)$$

The steady states of (3.14) are

$$\begin{aligned} x^3 - xy &= 0 \\ y^3 - xy &= 0 \end{aligned} \quad (3.15)$$

Obviously, one solution of (3.15) is $z = 0, y = 0$ and the other is $x = 1, y = 1$

$$F(x, y) = \begin{bmatrix} x^3 - xy \\ y^3 - xy \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$F'(x^*, y^*) = \begin{bmatrix} \frac{\delta F_1}{\delta x}(x^*, y^*) & \frac{\delta F_1}{\delta y}(x^*, y^*) \\ \frac{\delta F_2}{\delta x}(x^*, y^*) & \frac{\delta F_2}{\delta y}(x^*, y^*) \end{bmatrix} = A$$

$$A = \begin{bmatrix} 3x^* - y^* & -x^* \\ -y^* & 3y^* - x^* \end{bmatrix}$$

3.3 Riccati Differential Equation

In order to stabilize our linearized system in section 3.2, we need to introduce the Riccati equation. Let us consider the linear quadratic optimal control system.

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (3.16)$$

$$y(t) = C(t)x(t) \quad (3.17)$$

with a cost functional (CF) or performance index (PI)

$$\begin{aligned} J(u(t)) &= J(x(t_0), u(t), t_0) \\ &= \frac{1}{2}[z(t_f) - y(t_f)]'F(t_f)[z(t_f) - y(t_f)] \\ &\quad + \frac{1}{2} \int_{t_0}^{t_f} [z(t) - y(t)]'Q(t)[z(t) - y(t)] + [u'(t)R(t)u(t)] \end{aligned} \quad (3.18)$$

where $x(t)$ is n th state vector, $y(t)$ is m th output vector, $z(t)$ is method reference or designed output vector (or n th desired state vector if the state $x(t)$ is available), $u(t)$ is r th control vector and $e(t) = z(t) - y(t)$ (or $e(t) = z(t) - x(t)$, if the state $x(t)$ is directly available) is the method error vector. $A(t)$ is $n \times n$ state matrix, $B(t)$ is $n \times r$ control matrix and $C(t)$ is $m \times n$ output matrix. We assume that the control $u(t)$ is unconstrained. $0 < m \leq r \leq n$, and all the states and/or outputs are completely measurable.

The cost functional (3.18) contains quadratic terms in error $e(t)$ and control $u(t)$. Hence, it is called the quadratic cost functional. We consider some assumptions on the various matrices in the (3.18). Under these assumptions, we will find that the optimal control $u(t)$ is a function of the state $x(t)$ or the output $y(t)$.

We also consider the following categories of systems.[2]

1. If our objective is to keep the state $x(t)$ near zero (i.e $z(t) = 0$ and $C = 1$), then we call it state regulator system. In other words, the objective is to obtain a control $u(t)$ which takes the systems (3.16) and (3.17) from a non zero state to zero state.
2. If our interest is to keep the output $y(t)$ near zero (i.e $z(t) = 0$), then it is called the output regulator system.
3. If we try to keep the output or state near a desired state or output, then we are dealing with a tracking system.

Finite-Time Linear Quadratic Regulator

We proceed with the linear quadratic regulator (LQR) system, that is to keep the state near zero during the interval of interest. For the sake of completeness, we shall repeat the plant and performance index equations described earlier. Consider a linear time-varying system described by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{3.19}$$

with a cost functional

$$\begin{aligned} J(u) &= J(x, (t_0), u(t), t_0) \\ &= \frac{1}{2}x'(t_f)F(t_f)x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt \\ &= \frac{1}{2}x'(t_f)F(t_f)x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [x'(t)u'(t)] \begin{bmatrix} Q(t) & 0 \\ 0 & R(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \end{aligned}$$

Note that the reference or desired state $z(t) = 0$ and the error $e(t) = 0 - x(t)$ itself is the state, thereby implying a state regulator system.

We consider the following assumptions

1. The control $u(t)$ is unconstrained
2. The initial condition $x(t = t_0) = x_0$ is given. The terminal time t_f is specified and the final state $x(t_f)$ is not specified.
3. The terminal cost matrix $F(t_f)$ and the error weighted matrix $Q(t)$ are $n \times n$ positive semidefinite matrices respectively, and the control weighted matrix $R(t)$ is an $r \times r$ positive definite matrix
4. Finally, the function $\frac{1}{2}$ in the cost functional (3.20) is associated mainly to cancel a 2 that would have otherwise been carried on throughout the result.

We follow the Pontryagin procedure discussed in chapter one (Table 1) to obtain optimal solution and then proposed the closed-loop configuration. The following steps are required.[2]

- Step 1: Formulate the Hamiltonian
 Step 2: Obtain the optimal control
 Step 3: Obtain the state and costate equations
 Step 4: Formulate a closed-loop optimal control
 Step 5: Matrix Differential Riccatic equation

Example 3.4

Find the extrema of

$$J(u) = \frac{1}{2} \int_{t_0}^{t_f} (x'(t)Q(t)x(t) + u'(t)R(t)u(t))dt$$

subject to the condition

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ x(t_0) &= x_0, x(t_f) \text{ is fixed} \end{aligned}$$

Solution:

Step 1: Formulate the Hamiltonian

$$H(x(t), u(t), \lambda(t)) = \frac{1}{2}[x'(t)Q(t)x(t) + u'(t)R(t)u(t)] + \lambda(t)[A(t)x(t) + B(t)u(t)]$$

Step 2: Obtain the optimal control $u(t)$

$$\frac{\delta H}{\delta u} = 0 \implies R(t)u(t) + B'(t)\lambda(t) = 0$$

This gives

$$u(t) = -R^{-1}(t)B'(t)\lambda(t) \tag{3.21}$$

Step 3: Obtain the state and costate equations

$$\begin{aligned} \frac{\delta H}{\delta \lambda} &= \dot{x}(t) \\ \implies A(t)x(t) + B(t)u(t) &= \dot{x}(t) \end{aligned}$$

$$\implies A(t)x(t) - B(t)R^{-1}(t)B'(t)\lambda(t) = \dot{x}(t) \quad (3.22)$$

$$\frac{\delta H}{\delta x} = -\dot{\lambda}(t)$$

$$\implies -Q(t)x(t) - A'(t)\lambda(t) = \dot{\lambda}(t) \quad (3.23)$$

$$\left[H + \frac{\delta S}{\delta t} \right]_{t_f=0} \Delta t_f + \left[\frac{\delta S}{\delta x} - \lambda(t) \right]_{t=t_f} \Delta x_f = 0 \quad (3.24)$$

Since $x(t_f)$ is fixed, $\Delta x_f = 0$

Let

$$S = \frac{1}{2}x'(t)F(t)x(t)$$

$$\frac{\delta S}{\delta x} = F(t)x(t) \implies F(t_f)x(t_f) = \lambda(t_f) \quad (3.25)$$

Step 4: Closed-Loop Optimal Control

$$\text{Let } \lambda(t) = P(t)x(t) \quad (3.26)$$

$$\dot{\lambda}(t) = \dot{P}(t)x(t) + P(t)\dot{x}(t) \quad (3.27)$$

substitute (3.26) in (3.21), (3.22) and (3.23) to have

$$u(t) = -R^{-1}(t)B'(t)P(t)x(t) \quad (3.28)$$

$$\dot{x}(t) = A(t)x(t) - B(t)R^{-1}(t)B'(t)P(t)x(t) \quad (3.29)$$

$$\dot{\lambda}(t) = -Q(t)x(t) - A'(t)P(t)x(t) \quad (3.30)$$

substitute (3.27) in (3.30) to get

$$\dot{P}(t)x(t) + P(t)\dot{x}(t) = -Q(t)x(t) - A'(t)P(t)x(t) \quad (3.31)$$

Also, (3.29) in (3.31) gives

$$\begin{aligned} \dot{P}(t)x(t) + P(t)[A(t)x(t) - B(t)R^{-1}(t)B'(t)P(t)x(t)] &= -Q(t)x(t) - A'(t)P(t)x(t) \\ \dot{P}(t) + P(t)A(t) + A'(t)P(t) + Q(t) - P(t)B(t)R^{-1}(t)B'(t)P(t) &= 0 \end{aligned} \quad (3.32)$$

Also, from (3.25) and (3.26)

$$P(t_f) = F(t_f) \text{ (Final Condition)} \quad (3.33)$$

This is called the Riccati Differential Equation.[2]

Finite-Time LQR System: Time Varying Case

In this case, the following steps are required.[2]

Step 1: Solve the matrix differential Riccati equation from (3.32) with final condition $P(t = t_f) = F(t_f)$

Step 2: solve the optimal state $x(t)$ from (3.29) with initial condition $x(t_0) = x_0$.

Step 3: Obtain the optimal control $u(t)$ as in (3.28)

Step 4: Obtain the optimal performance index from $J = \frac{1}{2}x'(t)P(t)x(t)$

Example 3.5:

Find the extrema of

$$J = \frac{1}{2}x'(t_f)F(t_f)x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt$$

subject to

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \text{ (The performance index)}$$

$x(t_0) = x_0, t_f$ is fixed and $x(t_f)$ is free.

Solution: Using the steps stated above

Step 1: Obtain the solution of the matrix differential Riccati equation.

$$\dot{P} = -P(t)A(t) - A'(t)P(t) - Q(t) + P(t)B(t)R^{-1}(t)B'(t)P(t)$$

Satisfying the final condition

$$P(t_f) = F(t_f)$$

Step 2: Find $x(t)$ the optimal state of

$$\dot{x}(t) = [A(t) - B(t)R^{-1}(t)B'(t)P(t)]x(t)$$

$$x(0) = x_0$$

Step 3: The optimal control $u(t)$ is given as

$$u(t) = -R^{-1}(t)B'(t)\lambda(t)$$

$$u(t) = -R^{-1}(t)B'(t)P(t)x(t)$$

where $\lambda(t) = P(t)x(t)$

Analytical Solution to Matrix Differential Riccati Equation

Example 3.6: Given a double integral system

$$\dot{x}_1(t) = x_2(t), x_1(0) = 2$$

$$\dot{x}_2(t) = -2x_1(t) + x_2(t) + u(t), x_2(0) = -3 \tag{3.34}$$

and the performance index (PI)

$$J = \frac{1}{2}[x_1^2(5) + x_1(5)x_2(5) + 2x_2^2(5)] + \frac{1}{2} \int_0^5 [2x_1^2(t) + 6x_1(t)x_2(t) + 5x_2^2(t) + 0.25u^2(t)]dt \tag{3.35}$$

Obtain the feedback control law.

Solution: We compare (3.34) and (3.35) with the general formulations of the system (3.19) and (3.20) respectively. We identify the various quantities as

$$\begin{aligned} A(t) &= \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}; B(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ F(t_f) &= \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix}; Q(t) = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \\ R(t) &= r(t) = \frac{1}{4}, t_0 = 0, t_f = 5 \end{aligned}$$

We check that the system (3.34) is unstable. Let $P(t)$ be the 2x2 symmetric matrix i.e

$$P(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix}; \quad (3.35)$$

Then, the optimal control (3.28) is given by

$$\begin{aligned} u(t) &= -4 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &= -4 [P_{12}(t)x_1(t) + P_{22}(t)x_2(t)] \end{aligned} \quad (3.36)$$

where $P(t)$, positive definite matrix, is the solution of the matrix (3.32)

$$\begin{aligned} \dot{P}(t) &= -P(t)A(t) - A'(t)P(t) - Q(t) + P(t)B(t)R^{-1}(t)B'(t)P(t) \\ \begin{bmatrix} \dot{P}_{11}(t) & \dot{P}_{12}(t) \\ \dot{P}_{12}(t) & \dot{P}_{22}(t) \end{bmatrix} &= - \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 4 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \\ &= - \begin{bmatrix} -2P_{12}(t) & P_{11}(t) + P_{12}(t) \\ -2P_{22}(t) & P_{12}(t) + P_{22}(t) \end{bmatrix} - \begin{bmatrix} -2P_{12}(t) & -2P_{22}(t) \\ P_{11}(t) + P_{12}(t) & P_{12}(t) + P_{22}(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} P_{12}(t) \\ P_{22}(t) \end{bmatrix} \begin{bmatrix} 4P_{12}(t) & 4P_{22}(t) \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2P_{12}(t) & -P_{11}(t) - P_{12}(t) \\ 2P_{22}(t) & -P_{12}(t) - P_{22}(t) \end{bmatrix} - \begin{bmatrix} -2P_{12}(t) & -2P_{22}(t) \\ P_{11}(t) + P_{12}(t) & P_{12}(t) + P_{22}(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} 4P_{12}^2(t) & 4P_{12}(t)P_{22}(t) \\ 4P_{12}(t)P_{22}(t) & 4P_{22}^2(t) \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4P_{12}(t) & -P_{11}(t) - P_{12}(t) + 2P_{22}(t) \\ 2P_{22}(t) - P_{11}(t) - P_{12}(t) & -2P_{12}(t) - 2P_{22}(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} 4P_{12}^2(t) - 2 & 4P_{12}(t)P_{22}(t) - 3 \\ 4P_{12}(t)P_{22}(t) - 3 & 4P_{22}^2(t) - 5 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 4P_{12}(t) + 4P_{12}^2(t) - 2 & -P_{11}(t) - P_{12}(t) + 2P_{22}(t) + 4P_{12}(t)P_{22}(t) - 3 \\ 2P_{22}(t) - P_{11}(t) - P_{12}(t) + 4P_{12}(t)P_{22}(t) - 3 & -2P_{12}(t) - 2P_{22}(t) + 4P_{22}^2(t) - 5 \end{bmatrix} \quad (3.37)$$

satisfying the final condition (3.33)

$$\begin{bmatrix} P_{11}(5) & P_{12}(5) \\ P_{12}(5) & P_{22}(5) \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix} \quad (3.38)$$

Simplifying the matrix Differential Riccati Equation in (2.23), we get

$$\begin{aligned} \dot{P}_{11}(t) &= 4P_{12}^2(t) + 4P_{12}(t) - 2 & P_{11}(5) &= 1 \\ \dot{P}_{12}(t) &= -P_{11}(t) - P_{12}(t) + 2P_{22}(t) + 4P_{12}(t)P_{22}(t) - 3 & P_{12}(5) &= 0.5 \\ \dot{P}_{22}(t) &= -2P_{12}(t) - 2P_{22}(t) + 4P_{22}^2(t) - 5 & P_{22}(5) &= 2 \end{aligned} \quad (3.39)$$

Infinite-Time LQR System

Here, we make the terminal (final) time t_f to be infinite in the previous linear, time-varying, quadratic regulator system. And this is called the Infinite-time linear quadratic regulator system.

Example 3.7:

Find the extrema of

$$J = \frac{1}{2} \int_0^{\infty} [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt \quad (3.40)$$

subject to

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ x(t_0) &= x_0 \end{aligned} \quad (3.41)$$

Solution:

Note: It makes no sense to have a terminal cost term with terminal time being infinite. Some special conditions will be required before solving this problem.

if any one of the states is uncontrollable and/or unstable, the corresponding performance measure J will be infinite which makes no sense. Conversely, with the finite-time system, the performance measure is always finite. Hence, we need to impose the condition that the system (3.41) is completely controllable.

With the results in the finite final time t_f case (See Table 1), the optimal control for the infinite-time linear regulator system is obtained as

$$u(t) = -R^{-1}(t)B'(t)\tilde{P}(t)x(t) \quad (3.42)$$

where

$$\tilde{P}(t) = \lim_{t_f \rightarrow \infty} \{P(t)\}$$

is the solution of the matrix differential Riccati equation.

$$\dot{\tilde{P}} = -\tilde{P}(t)A(t) - A'(t)\tilde{P}(t) - Q(t) + \tilde{P}(t)B(t)R^{-1}(t)\tilde{P}(t) \quad (3.43)$$

satisfying the final condition[2]

$$\lim_{t_f \rightarrow \infty} \tilde{P}(t_f) = 0$$

Infinite-Time LQR system Time Varying case

The following steps are employed.[2]

Step 1: Solve the matrix differential Riccati equation in (3.43) with final condition $\tilde{P}(t = t_f) = 0$

Step 2: Solve the optimal state $x(t)$ from (3.29) with initial condition $x(t_0) = x_0$

Step 3: Obtain the optimal control $u(t)$ from (3.42)

Step 4: Obtain the optimal performance index from

$$J = \frac{1}{2}x(t)\tilde{P}(t)x(t)$$

Infinite-Time LQR system: Time Invariant case

The infinite time interval case is considered if

1. We want to measure that the state-regulator stays near zero state after the initial transient.
2. We want to include any special case of large final time

Example 3.8

Find the extrema

$$J = \frac{1}{2} \int_0^{\infty} [x^1(t)Qx(t) + u'(t)Ru(t)]dt$$

subject to

$$\dot{x} = Ax(t) + Bu(t)$$

Solution

Step 1: Solve the matrix differential Riccate Algebraic equation with boundary condition $P(t_f) = 0$ i.e

$$\frac{dP}{dt} = 0 = -PA - A'P + PBR^{-1}B'P - Q \quad (3.44)$$

Step 2: Find $x(t)$ from

$$\begin{aligned}\dot{x}(t) &= [A - BR^{-1}B'P]x(t) \\ x(0) &= x_0\end{aligned}$$

Step 3: Find the optimal control $u(t)$ from

$$u(t) = -R^{-1}B'Px(t) \quad (3.45)$$

Step 4: The Optimal cost is given by

$$J = \frac{1}{2}x'(t)Px(t) \quad (3.46)$$

Example 3.9 Obtain the feedback optimal control of

$$J = \frac{1}{2} \int_0^{\infty} [2x_1^2(t) + 6x_1(t)x_2(t) + 5x_2^2(t) + 0.25u^2(t)]dt$$

subject to the conditions

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -2x_1(t) + x_2(t) + u(t) \\ x_1(0) &= 2; x_2(0) = -3\end{aligned}$$

Solution

From the problem, we identify the various matrices as

$$\begin{aligned}A &= \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ Q &= \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}; R = r = \frac{1}{4} \\ t_0, t_f &= \infty\end{aligned}$$

Let P be the 2×2 symmetric matrix i.e

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$$

Then, the optimal control (3.28) is given by

$$\begin{aligned}u(t) &= -4[0 \quad 1] \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &= -4[P_{12}x_1(t) + P_{22}x_2(t)]\end{aligned}$$

Where P the positive definite matrix is the solution of the matrix algebraic Riccati equation.

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = & - \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \\ & + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 4^{[01]} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \end{aligned}$$

Simplifying this equation as in example 3.6 gives

$$\begin{aligned} 4P_{12}^2 + 4P_{12} - 2 &= 0 \\ -P_{11} - P_{12} + 2P_{22} + 4P_{12}P_{22} - 3 &= 0 \\ -2P_{12} - 2P_{22} + 4P_{22}^2 - 5 &= 0 \end{aligned} \tag{3.47}$$

We now solve (3.47) for positive definiteness of P . that is the first and third equations can be solved quadratically for P_{12} and P_{22} to have $P_{12} = 0.3660$ and $P_{22} = 1.4729$. These values can now be substituted in the second equation to get P_{11} . That is

$$4P_{12}^2 + 4P_{12} - 2 = 0$$

Using the General formula
 $a = 4, b = 4, c = -2$

$$\begin{aligned} P_{12} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-4 \pm \sqrt{4^2 - 4(4)(-2)}}{2 \times 4} \\ &= \frac{2.9282}{8} \text{ or } \frac{-10.9282}{8} \\ P_{12} &= 0.3660 \text{ or } -1.3660 \end{aligned}$$

We choose $P_{12} = 0.3660$ (for positive definiteness)

Also,

$$\begin{aligned} 4P_{22}^2 - 2P_{22} - 2P_{12} - 5 &= 0 \\ 4P_{22}^2 - 2P_{22} - 2(0.3660) - 5 &= 0 \\ 4P_{22}^2 - 2P_{22} - 0.732 - 5 &= 0 \\ 4P_{22}^2 - 2P_{22} - 5.732 &= 0 \end{aligned}$$

$$a = 4, b = -2, c = -5.732$$

$$\begin{aligned} P_{22} &= \frac{2 \pm \sqrt{(-2)^2 - 4(4)(-5.732)}}{2 \times 4} \\ &= \frac{2 + 9.78325}{8} \text{ or } \frac{2 - 9.78325}{8} \\ P_{22} &= 1.4729 \text{ or } -0.9729 \end{aligned}$$

We choose $P_{22} = 1.4729$ (for positive definiteness).

We now substitute the values of P_{12} and P_{22} in $-P_{11} - P_{12} + 2P_{22} + 4P_{12}P_{22} - 3 = 0$ to find P_{11} .

$$\begin{aligned} P_{11} &= P_{12} - 2P_{22} - 4P_{12}P_{22} + 3 \\ P_{11} &= 0.3660 - 2(1.4729) - 4(0.3660)(1.4729) + 3 \\ &= 1.7363 \end{aligned}$$

Hence

$$P = \begin{bmatrix} 1.7363 & 0.3660 \\ 0.3660 & 1.4729 \end{bmatrix}$$

With these Riccati coefficients, the closed-loop optimal control is given by

$$\begin{aligned} u(t) &= -4[0.366x_1(t) + 1.4729x_2(t)] \\ &= -[1.464x_1(t) + 5.8916x_2(t)] \end{aligned}$$

Using the closed-loop optimal control above in the original open-loop system in this system in this problem, the closed-loop optimal system becomes

$$\begin{aligned} \dot{x}_1 &= x_2(t) \\ \dot{x}_2 &= -2x_1(t) + x_2(t) - 4[0.366x_1(t) + 1.4729x_2(t)] \end{aligned}$$

Finally, using the initial conditions and the Riccati coefficient matrix calculated above, we obtain the optimal cost as

$$\begin{aligned} J &= \frac{1}{2}x'(0)Px(0) \\ &= \frac{1}{2}[2 \quad -3] \begin{bmatrix} 1.7363 & 0.3660 \\ 0.3660 & 1.4729 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} \\ &= 7.9047 \end{aligned}$$

3.4 Feedback Control

Now, we will use the feedback control to stabilize the linearized systems in Examples 3.1 and 3.2.

In **Example 3.1**, we design the feedback control for equation (3.9) as follows

$$\frac{dz}{dt} + Az = Bu(t)$$

$$\text{where } B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$u(t) = -B'Pz(t)$$

and P satisfies Riccati equation

$$AP + A'P + Q - B'PBPR^{-1} = 0 \quad (3.48)$$

where,

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = 1$$

P is a 2x2 symmetric matrix and can be calculated from (3.48) using Matlab Linear quadratic regulator (LQR).

In **Example 3.2**, we design the feedback control as in the previous example for the linearized equation.

$$\frac{dZ}{dt} = AZ$$
$$\frac{dZ}{dt} = AZ + Bu(t)$$

Here also $u(t) = -B'PZ(t)$ and P satisfies Riccati equation

$$AP + A'P + Q - PB'PBR^{-1} = 0$$

where B and Q are given above and

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \text{ is a 2x2 symmetric matrix}$$

3.5 Finite Difference Methods

We now look at stability for some numerical methods for the test problem[17]

$$\begin{aligned}\frac{dy}{dx} &= \lambda y, \\ y(0) &= 1, x > 0,\end{aligned}$$

where λ is a constant. That is $\lambda \in \mathbf{C}$.

The exact solution is

$$y = e^{\lambda x}$$

In particular, when $\lambda < 0$, we see that the solution tends to 0 when $x \rightarrow \infty$.

The Forward Euler Method

Let $0 = x_0 < x_1 < x_2 < \dots$ be the partition of the interval $[0, +\infty)$.

Let $x_n - x_{n-1} = h$. We then have

$$\left. \frac{dy}{dx} \right|_{x=x_n} = \lambda y(x_n).$$

Note that

$$\left. \frac{dy}{dx} \right|_{x=x_n} = \lim_{h \rightarrow 0} \frac{y(x_{n+1}) - y(x_n)}{h} \approx \frac{y(x_{n+1}) - y(x_n)}{h}$$

We then have

$$\frac{y(x_{n+1}) - y(x_n)}{h} \approx \lambda y(x_n).$$

let $y_n \approx y(x_n), y_{n+1} \approx y(x_{n+1})$. We get the forward Euler method,

$$\frac{y_{n+1} - y_n}{h} \approx \lambda y_n,$$

or

$$y_{n+1} = (1 + h\lambda)y_n,$$

or

$$y_n = (1 + h\lambda)^n y_0$$

Denote $Q(z) = 1 + z$. The forward Euler method is

$$y_n = Q(h\lambda)^n y_0$$

Definition: The region of the absolute stability for one step method is

$$R = \{h\lambda \in \mathbf{C}, |Q(h\lambda)| < 1\}.$$

Here $\lambda \in \mathbf{C}$.

For the forward Euler method, we have

$$|Q(h\lambda)| = |1 + h\lambda| < 1.$$

The absolute stability region is

$$|h\lambda - (-1)| < 1.$$

In particular, when $\lambda < 0$, we have

$$|h\lambda + 1| < 1,$$

which implies that $-2 < h\lambda < 0$, or $0 < h < \frac{2}{-\lambda} = \frac{2}{|\lambda|}$. Thus, when $\lambda < 0$, the forward Euler method is stable only when $h < \frac{2}{|\lambda|}$. In other words, the stability of the Euler method depends on the step size h . The h should be small enough to make the method stable.

The Backward Euler Method

Let $0 = x_0 < x_1 < x_2 < \dots$ be the partition of the interval $[0, +\infty)$.

Let $x_n - x_{n-1} = h$. We then have

$$\left. \frac{dy}{dx} \right|_{x=x_n} = \lambda y(x_n)$$

Note that

$$\left. \frac{dy}{dx} \right|_{x=x_n} = \lim_{h \rightarrow 0} \frac{y(x_n) - y(x_{n-1})}{h} \approx \frac{y(x_n) - y(x_{n-1})}{h}$$

We then have

$$\frac{y(x_n) - y(x_{n-1})}{h} \approx \lambda y(x_n)$$

Let $y_n \approx y(x_n)$, $y_{n-1} \approx y(x_{n-1})$. We get the forward Euler method,

$$\frac{y_n - y_{n-1}}{h} \approx \lambda y_n,$$

or

$$y_n = \frac{1}{1 - h\lambda} y_{n-1}$$

or

$$y_n = \frac{1}{(1 - h\lambda)^n} y_0.$$

Denote $Q(z) = \frac{1}{1-z}$. The forward Euler method is

$$y_n = Q(h\lambda)^n y_0$$

For the backward Euler method, we have

$$|Q(h\lambda)| = \left| \frac{1}{1 - h\lambda} \right| < 1$$

The absolute stability region is

$$\left| \frac{1}{1 - h\lambda} \right| < 1$$

Let $h\lambda = (x + iy)$. we have

$$\left| \frac{1}{1 - h\lambda} \right| = \left| \frac{1}{1 - (x + iy)} \right| = \frac{1}{\sqrt{(1-x)^2 + y^2}} < 1, \text{ for any } x < 0.$$

That is the absolute stability region for backward Euler method is the left half plane for $h\lambda$. We call such method is A-stable method.

In particular, when $\lambda < 0$, we have

$$\left| \frac{1}{1 - h\lambda} \right| < 1, \text{ for all } h > 0.$$

which implies that the backward Euler method is unconditional stable.

The stability is independent of h , it is very useful in practice.

The Trapezoidal Method

Let $0 = x_0 < x_1 < x_2 < \dots$ be the partition of the interval $[0, +\infty)$.

Let $x_n - x_{n-1} = h$. Consider

$$\frac{dy}{dx} = \lambda y,$$

or

$$dy = \lambda y dx.$$

We have

$$\int_{x_n}^{x_{n+1}} dy = \int_{x_n}^{x_{n+1}} \lambda y dx \quad (3.49)$$

We use the trapezoidal method to approximate the right integral in (3.49). We get

$$\int_{x_n}^{x_{n+1}} dy \approx \lambda \frac{y(x_{n+1}) + y(x_n)}{2} h$$

That is

$$y(x_{n+1}) - y(x_n) \approx \lambda \frac{y(x_{n+1}) + y(x_n)}{2} h$$

Let $y_n \approx y(x_n)$, $y_{n+1} \approx y(x_{n+1})$. We get the trapezoidal method

$$y_{n+1} - y_n = \lambda h \frac{y_{n+1} + y_n}{2}$$

or

$$y_{n+1} = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} y_n$$

or

$$y_n = \left(\frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} \right)^n y_0$$

Denote $Q(z) = \frac{1+\frac{z}{2}}{1-\frac{z}{2}}$. The backward Euler method is

$$y_n Q(h\lambda)^n y_0.$$

For the trapezoidal method, we have

$$|Q(h\lambda)| = \left| \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} \right| < 1.$$

The absolute stability region is

$$\left| \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} \right| < 1.$$

let $h\lambda = x + iy$. We have

$$\left| \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} \right| = \left| \frac{2 + x + iy}{2 - x - iy} \right| = \frac{\sqrt{(2+x)^2 + y^2}}{\sqrt{(2-x)^2 + y^2}} < 1, \text{ for any } x < 0.$$

That is the absolute stability region for the trapezoidal method is the left half plane for $h\lambda$. Thus this method is also a A -stable method.

In particular, when $\lambda < 0$, we have

$$\left| \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} \right| < 1, \text{ for all } h > 0.$$

which implies that the trapezoidal method is unconditional stable. The stability is independent of h , it is very useful in practice.

Stabilization of a Linearized System

Consider the system

$$\frac{dx}{dt} = Ax + Bu \tag{3.50}$$

$$x(0) = x_0$$

$$\frac{dx}{dt} = Ax + B(-B'Px)$$

$$\frac{dx}{dt} = (A - BB'P)x \tag{3.51}$$

where $u = -B'Px$

Forward Euler Method

$$\begin{aligned}\frac{dx}{dt}\Big|_{t=t_n} &\approx \frac{x(t_{n+1}) - x(t_n)}{h} \\ \frac{x(t_{n+1}) - x(t_n)}{h} &= [A - BB'P]x(t_n) \\ x^{n+1} &= x^n + h[A - BB'P]x^n \\ x^0 &\rightarrow x^1 \rightarrow x^2 \rightarrow x^3 \rightarrow \dots\end{aligned}$$

Backward Euler Method

$$\begin{aligned}\frac{dx}{dt}\Big|_{t=t_n} &\approx \frac{x(t_n) - x(t_{n-1})}{h} \\ \frac{x(t_n) - x(t_{n-1})}{h} &= [A - BB'P]x(t_n) \\ x^n - x^{n-1} &= h[A - BB'P]x^n \\ x^{n-1} &= x^n - h[A - BB'P]x^n \\ x^{n-1} &= [I - h(A - BB'P)]x^n\end{aligned}$$

Where $[I - h(A - BB'P)]$ is a matrix.

$$\begin{aligned}x^n &= [I - h(A - BB'P)]^{-1}x^{n-1} \\ x^0 &\rightarrow x^1 \rightarrow x^2 \rightarrow x^3 \rightarrow \dots\end{aligned}$$

Trapezoidal Method

The trapezoidal rule is a numerical method that approximates the value of a definite integral. Consider the integral

$$\int_a^b f(x)dx$$

Let $f(x)$ be continuous on $[a, b]$ and we divide $[a, b]$ into n subintervals of equal length called h i.e

$$h = \frac{b - a}{n}$$

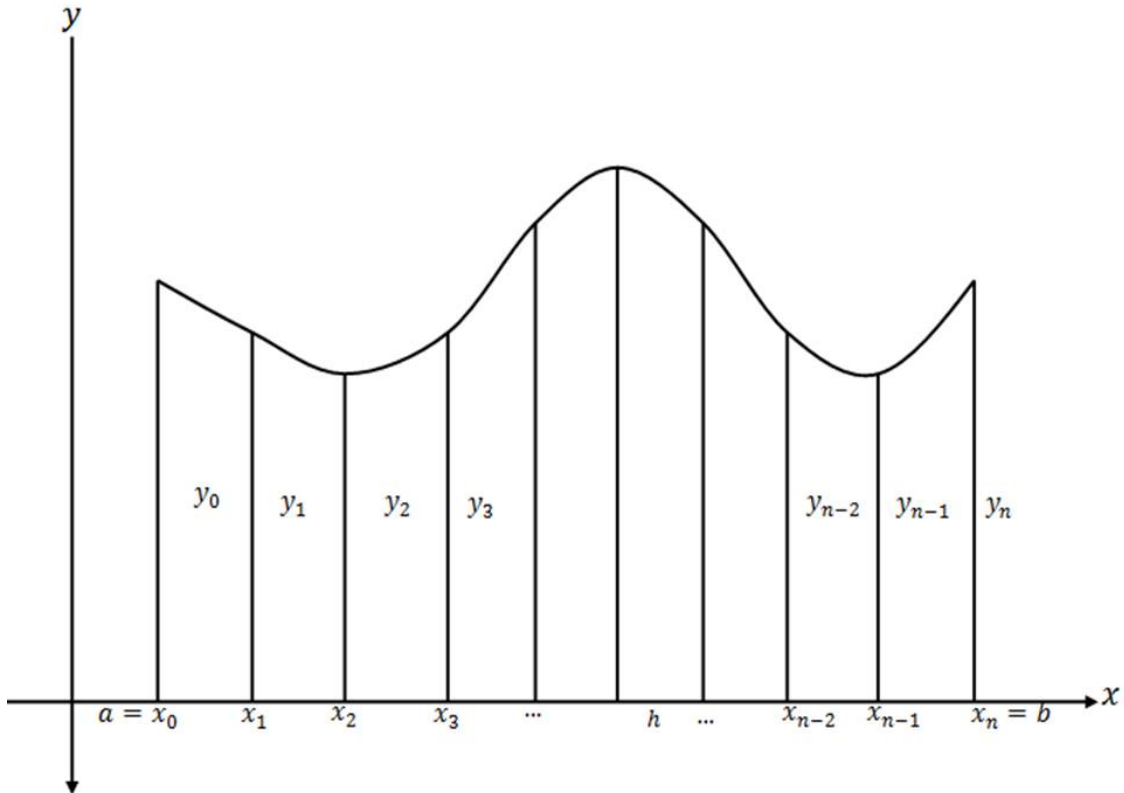
Using the $n + 1$ points

$$x_0 = a, x_1 = a + h, x_2 = a + 2h, \dots, x_n = a + nh = b$$

The value of $f(x)$ can be calculated from these points

$$y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2) \dots y_n = f(x_n)$$

We approximate the integral by using n trapezoids formed by using straight line segments between the points (x_{i-1}, y_{i-1}) and (x_i, y_i) for $1 \leq i \leq n$ as shown in the figure below.



The area of a trapezoid is obtained by adding the area of a triangle

$$A = y_0h + \frac{1}{2}(y_1 - y_0)h = \frac{(y_0 + y_1)h}{2}$$

By adding the area of the n trapezoids, we obtain the approximation

$$\int_a^b f(x)dx \approx \frac{(y_0 + y_1)h}{2} + \frac{(y_1 + y_2)h}{2} + \frac{(y_2 + y_3)h}{2} + \dots + \frac{(y_{n-1} + y_n)h}{2}$$

which now gives the trapezoidal rule formula

$$\int_a^b f(x)dx \approx \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n)$$

Stabilization of a Nonlinear System

The following theorem [18] is required without proof.

Theorem 3.1 Assume that $\begin{bmatrix} y^* \\ z^* \end{bmatrix}$ is unstable. Then

$$V = -R^{-1}B'P \begin{bmatrix} y - y^* \\ z - z^* \end{bmatrix}$$

will stabilize exponentially the nonlinear system

$$\frac{d}{dt} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} F(y, z) \\ G(y, z) \end{bmatrix} + BV(t).$$

More precisely, there exists $\rho > 0$ such that for all

$$\begin{bmatrix} y_0 \\ z_0 \end{bmatrix} : \left\| \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} - \begin{bmatrix} y^* \\ z^* \end{bmatrix} \right\| < \rho$$

there exists a unique solution $\begin{bmatrix} y \\ z \end{bmatrix} \in C^1(0, \infty, R^2)$, such that, with some constant C and $\gamma > 0$,

$$\left\| \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} - \begin{bmatrix} y^* \\ z^* \end{bmatrix} \right\| < Ce^{-\gamma t} \left\| \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \right\|$$

Now, let $0 = t_0 < t_1 < t_2 < \dots$ be time points, and $k = t_j - t_{j-1}$ be time step. We use y^n to denote the approximation of $y(t_n)$. Using the backward Euler method, we define the following difference scheme

$$\begin{bmatrix} \frac{y^n - y^{n-1}}{k} \\ \frac{z^n - z^{n-1}}{k} \end{bmatrix} = \begin{bmatrix} F(y^n, z^n) \\ G(y^n, z^n) \end{bmatrix} - R^{-1}BB^*P \begin{bmatrix} y^{n-1} - y^* \\ z^{n-1} - z^* \end{bmatrix} \quad (3.52)$$

with initial value (y^0, z^0) . Then we get the sequences $(y^n, z^n), n = 1, 2, \dots$. From **Theorem 3.1**, we have $(y^n, z^n) \rightarrow (y^*, z^*)$ as $n \rightarrow \infty$.

Let us consider the error estimate of (3.52). Recall that our nonlinear system has the form

$$\frac{d}{dt} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} F(y, z) \\ G(y, z) \end{bmatrix} + BV(t) \quad (3.53)$$

substituting $y - y^*$ and $z - z^*$ by Y and Z , we get

$$\frac{d}{dt} \begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} F(Y + y^*, Z + z^*) \\ G(Y + y^*, Z + z^*) \end{bmatrix} + BV(t) \quad (3.54)$$

Here $V(t) = -RB^*P \begin{bmatrix} Y \\ Z \end{bmatrix}$. Denote $u = \begin{bmatrix} Y \\ Z \end{bmatrix}$ and $F(u) = \begin{bmatrix} F(Y + y^*) \\ G(Z + z^*) \end{bmatrix}$

Then we write (3.53) into

$$\frac{du}{dt} = F(u) - RBB^*Pu \quad (3.55)$$

or

$$\frac{du}{dt} + A_1u = F(u) \quad (3.56)$$

Here $A_1 = RBB'P$.

Denote U^n as the approximation of $u(t_n)$. We define the following backward Euler method for the abstract form (3.56)

$$\frac{U^n - U^{n-1}}{k} + A_1 U^n = F(U^{n-1}) \quad (3.57)$$

or, with $r(\lambda) = (1 + \lambda)^{-1}$,

$$U^n = r(kA_1)U^{n-1} + kr(kA_1)U^{n-1} \quad (3.58)$$

4 Numerical Simulation

4.1 Numerical Methods

Recall that we defined what a steady state is in the previous chapter. Now, let us consider the steady states of the following general nonlinear system [3].

$$\frac{dy}{dt} = y(t)(a_1 - b_1y(t) - c_1z(t)) \quad (4.1)$$

$$\frac{dz}{dt} = z(t)(a_2 - b_2y(t) - c_2z(t)) \quad (4.2)$$

with initial conditions $y(0) = y_0 > 0, z(0) = z_0 > 0$.

where

a_1 = Intrinsic growth rate for species y

b_1 = Intraspecific coefficient of species y

c_1 = Interspecific coefficient of species y

a_2 = Intrinsic growth rate for species z

b_2 = Intraspecific coefficient of species z

c_2 = Interspecific coefficient of species z

The task is to find the steady states for the systems (4.1) and (4.2) at (y^*, z^*) . We set

$$\frac{dy}{dt} = \frac{dz}{dt} = 0$$

Then we have

$$y^*(a_1 - b_1y^* - c_1z^*) = 0 \quad (4.3)$$

$$z^*(a_2 - b_2y^* - c_2z^*) = 0 \quad (4.4)$$

Assumption: At steady state, let

$$F_1(y^*, z^*) = a_1 - b_1y^* - c_1z^*$$

$$F_2(y^*, z^*) = a_2 - b_2y^* - c_2z^*$$

Since (y^*, z^*) is a steady state solution

$$y^*F_1(y^*, z^*) = 0$$

$$z^*F_2(y^*, z^*) = 0$$

Note that F_1 and F_2 are two continuous differentiable functions at steady state (y^*, z^*) .

$$\text{If } F_1(y^*, z^*) \neq 0 \implies y^* = 0$$

$$F_2(y^*, z^*) \neq 0 \implies z^* = 0$$

Here, the steady state $(y^*, z^*) = (0, 0)$.

It means that the two species are driven to extinction which is not physically meaningful.

$$\begin{aligned}\text{If } y^* = 0, z^* &\neq 0 \\ z^*(a_2 - c_2 z^*) &= 0 \\ a_2 - c_2 z^* &= 0 \\ z^* &= \frac{a_2}{c_2}\end{aligned}$$

Here the steady state $(y^*, z^*) = (0, \frac{a_2}{c_2})$

$$\begin{aligned}\text{If } y^* &\neq 0, z^* = 0 \\ y^*(a_1 - b_1 y^*) &= 0 \\ a_1 - b_1 y^* &= 0 \\ y^* &= \frac{a_1}{b_1}\end{aligned}$$

Here, the steady state $(y^*, z^*) = (\frac{a_1}{b_1}, 0)$

Remark: The carrying capacity is the maximum size that plant species population can support.

Here, $k_1 = \frac{a_1}{b_1}$ is the carrying capacity of species y . At steady state $(\frac{a_1}{b_1}, 0)$, the species y will survive at its carrying capacity and species z will go to extinction. Then $k_2 = \frac{a_2}{c_2}$ is the carrying capacity of species z . At steady state $(0, \frac{a_2}{c_2})$, the species z will survive at its carrying capacity and species y will go to extinction.

$$\begin{aligned}\text{If } y^* &\neq 0, z^* \neq 0 \\ F_1(y^*, z^*) &= 0 \\ F_2(y^*, z^*) &= 0\end{aligned}$$

$$a_1 - b_1 y^* - c_1 z^* = 0 \tag{4.5}$$

$$a_2 - b_2 y^* - c_2 z^* = 0 \tag{4.6}$$

Solving (4.5) and (4.6) simultaneously to have

$$b_1 y^* + c_1 z^* = a_1 \tag{4.7}$$

$$b_2 y^* + c_2 z^* = a_2 \tag{4.8}$$

Multiply (4.7) by b_2 and (4.8) by b_1

$$b_1 b_2 y^* + c_1 b_2 z^* = a_1 b_2 \tag{4.9}$$

$$b_1 b_2 y^* + b_1 c_2 z^* = a_2 b_1 \tag{4.10}$$

Subtract (4.10) from (4.19) to get

$$\begin{aligned}c_1 b_2 z^* - b_1 c_2 z^* &= a_1 b_2 - a_1 b_2 - a_2 b_1 \\(c_1 b_2 - b_1 c_2) z^* &= a_1 b_2 - a_2 b_1 \\z^* &= \frac{a_1 b_2 - a_2 b_1}{c_1 b_2 - b_1 c_2}\end{aligned}$$

Substitute z^* in (4.7) to find y^*

$$\begin{aligned}y^* &= \frac{a_1 - c_1 z^*}{b_1} \\&= \frac{a_1 - c_1 \left(\frac{a_1 b_2 - a_2 b_1}{c_1 b_2 - b_1 c_2} \right)}{b_1} \\&= \frac{a_1}{b_1} - \frac{c_1 (a_1 b_2 - a_2 b_1)}{b_1 (c_1 b_2 - b_1 c_2)} \\&= \frac{a_1 (c_1 b_2 - b_1 c_2) - c_1 (a_1 b_2 - a_2 b_1)}{b_1 (c_1 b_2 - b_1 c_2)} \\&= \frac{-a_1 b_1 c_2 + a_2 b_1 c_1}{b_1 (c_1 b_2 - b_1 c_2)} \\&= \frac{b_1 (a_2 c_1 - a_1 c_2)}{b_1 (c_1 b_2 - b_1 c_2)} = \frac{a_2 c_1 - a_1 c_2}{c_1 b_2 - b_1 c_2}\end{aligned}$$

Here, the steady state

$$(y^*, x^*) = \left(\frac{a_2 c_1 - a_1 c_2}{c_1 b_2 - b_1 c_2}, \frac{a_1 b_2 - a_2 b_1}{c_1 b_2 - b_1 c_2} \right)$$

In summary, we have the following steady states for the systems (4.1) to (4.2)

$$\begin{aligned}y^* &= 0, z^* = 0 \\y^* &= 0, z^* = \frac{a_2}{c_2} \\y^* &= \frac{a_1}{b_1}, z^* = 0 \\y^* &= \frac{a_2 c_1 - a_1 c_2}{c_1 b_2 - b_1 c_2}, z^* = \frac{a_1 b_2 - a_2 b_1}{c_1 b_2 - b_1 c_2}\end{aligned}$$

We now determine the stability of the steady state by considering the linearised systems of (4.1) to (4.2) about (y^*, z^*) using the method described in chapter 3.

By this, we present some numerical examples of model equations of competition and other types of interactions.

4.2 Numerical Examples

In this section, I will consider two examples to demonstrate the numerical method described in chapter 3 and in section 4.1 above. The Matlab program used in the simulation of these problems are in the Appendix .

Example 4.1: Consider a system of nonlinear first order ordinary differential equations from [3].

$$\frac{dN_1}{dt} = N_1(t)(0.168 - 0.0020339N_1(t) - 0.0005N_2(t)) \quad (4.11)$$

$$\frac{dN_2}{dt} = N_2(t)(0.002 - 0.00002N_1(t) - 0.000015N_2(t)) \quad (4.12)$$

with initial starting values $N_1 = 4$ grams per area of plant species cover, $N_2 = 10$ grams per area of plant species cover.

Steady State

Solution: My task is to investigate the steady states for the above equations.

Therefore, using the method described in section 4.1, the steady states are $(0,0)$, $(0, 133.3333)$, $(82.59993, 0)$ and $(74.1156, 34.5125)$.

Stability: Here, we find out which of the steady state(s) is/are stable or unstable.

From theory

$$\begin{aligned} F_1(N_1, N_2) &= 0.168N_1 - 0.0020339N_1^2 - 0.0005N_1N_2 \\ F_2(N_1, N_2) &= 0.002N_2 - 0.00002N_1N_2 - 0.000015N_2^2 \\ \implies \lim_{N_1 \rightarrow N_1^*} F_1(N_1, N_2) &= (N_1^*, N_2^*); \quad \lim_{N_2 \rightarrow N_2^*} F_1(N_1, N_2) = (N_1^*, N_2^*) \\ \implies \lim_{N_1 \rightarrow N_1^*} F_1(N_1, N_2) &= (N_1^*, N_2^*); \quad \lim_{N_2 \rightarrow N_2^*} F_2(N_1, N_2) = (N_1^*, N_2^*) \end{aligned}$$

So,

$$\begin{aligned} a_{11} &= \left. \frac{\delta F_1}{\delta N_1^*} \right|_{N_2^*} = 0.168 - 0.0040678N_1^* - 0.0005N_2^* \\ a_{12} &= \left. \frac{\delta F_1}{\delta N_2^*} \right|_{N_1^*} = -0.0005N_1^* \\ a_{21} &= \left. \frac{\delta F_2}{\delta N_1^*} \right|_{N_2^*} = -0.00002N_2^* \\ a_{22} &= \left. \frac{\delta F_2}{\delta N_2^*} \right|_{N_1^*} = 0.002 - 0.00002N_1^* - 0.00003N_2^* \end{aligned}$$

At steady state $(0, 0)$

$$a_{11} = 0.168$$

$$a_{12} = 0$$

$$a_{21} = 0$$

$$a_{22} = 0.002$$

Comparing this problem with that in (4.1) to (4.2) we have that

$$a_1 = 0.168, \quad b_1 = 0.0020339, \quad c_1 = 0.0005$$

$$a_2 = 0.002, \quad b_2 = 0.00002, \quad c_2 = 0.000015$$

The steady states are

$$N_1^* = 0, N_2^* = 0$$

$$\implies (N_1^*, N_2^*) = (0, 0)$$

$$N_1^* = 0, N_2^* = \frac{a_2}{c_2} = \frac{0.002}{0.000015} = 133.3333$$

$$\implies (N_1^*, N_2^*) = (0, 133.3333)$$

$$N_1^* = \frac{a_1}{b_1} = \frac{0.168}{0.0020339} = 82.59993, N_2^* = 0$$

$$\implies (N_1^*, N_2^*) = (82.59993, 0)$$

$$N_1^* = \frac{a_2c_1 - a_1c_2}{c_1b_2 - b_1c_2} = \frac{(0.002)(0.0005) - (0.168)(0.000015)}{(0.0005)(0.00002) - (0.0020339)(0.000015)} \\ = 74.1156$$

$$N_2^* = \frac{a_1b_2 - a_2b_1}{c_1b_2 - b_1c_2} = \frac{(0.168)(0.00002) - (0.002)(0.0020339)}{(0.0005)(0.00002) - (0.0020339)(0.000015)} \\ = 34.5125$$

$$\implies (N_1^*, N_2^*) = (74.1156, 34.5125)$$

We now have the Jacobian Matrix A

$$A = \begin{pmatrix} 0.168 & 0 \\ 0 & 0.002 \end{pmatrix}$$

whose eigenvalues are

$$\lambda_1 = 0.168, \lambda_2 = 0.002$$

Since the two eigenvalues have positive sign, it follows that $(0,0)$ is unstable.

At steady state $(0, 133.3333)$

$$\begin{aligned}a_{11} &= 0.1013 \\a_{12} &= 0 \\a_{21} &= -0.0027 \\a_{22} &= -0.002\end{aligned}$$

The Jacobian Matrix A is

$$A = \begin{pmatrix} 0.1013 & 0 \\ -0.0027 & -0.002 \end{pmatrix}$$

The eigenvalues are

$$\lambda_1 = -0.002, \lambda_2 = 0.1013$$

Since these eigenvalues are of opposite sign, it follows that $(0,133.3333)$ is unstable.

At steady state $(82.59993, 0)$

$$\begin{aligned}a_{11} &= -0.168 \\a_{12} &= 0.0413 \\a_{21} &= 0 \\a_{22} &= 0.0003\end{aligned}$$

The Matrix A is

$$A = \begin{pmatrix} -0.168 & 0.0413 \\ 0 & 0.0003 \end{pmatrix}$$

whose eigenvalues are

$$\lambda_1 = -0.168, \lambda_2 = 0.003$$

Since these eigenvalues are of opposite sign, it follows that $(82.59993,0)$ is unstable.

At steady state $(74.1156, 34.5125)$

$$\begin{aligned}a_{11} &= -0.1507 \\a_{12} &= -0.0371 \\a_{21} &= -0.0007 \\a_{22} &= -0.0005\end{aligned}$$

The Matrix A is

$$A = \begin{pmatrix} -0.1507 & -0.0371 \\ -0.0007 & -0.0005 \end{pmatrix}$$

And the eigenvalues are

$$\lambda_1 = -0.1509, \lambda_2 = -0.0003$$

Since the two eigenvalues have negative sign, it follows that $(74.1156, 34.5125)$ is stable. We have found the above results analytically which are consistent with [3], and [13].

Remark: It is observed that the first three steady states are not stable and would need to be stabilised whereas the last one is stable and may not require any further stabilization.

Simulation: In the simulation, I choose the initial values to be $(4, 10)$ and final time (T_f) to be 100. The program used for the simulation can be found in Appendix 1 for the stabilization process of the trivial steady state with step size $h = 0.01$

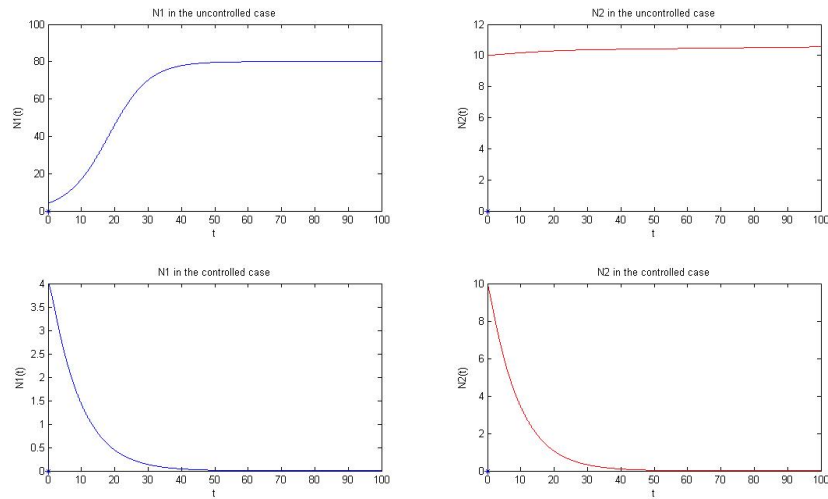


Figure 4.1: Uncontrolled and Controlled solution Trajectories of Trivial Steady state $(0, 0)$

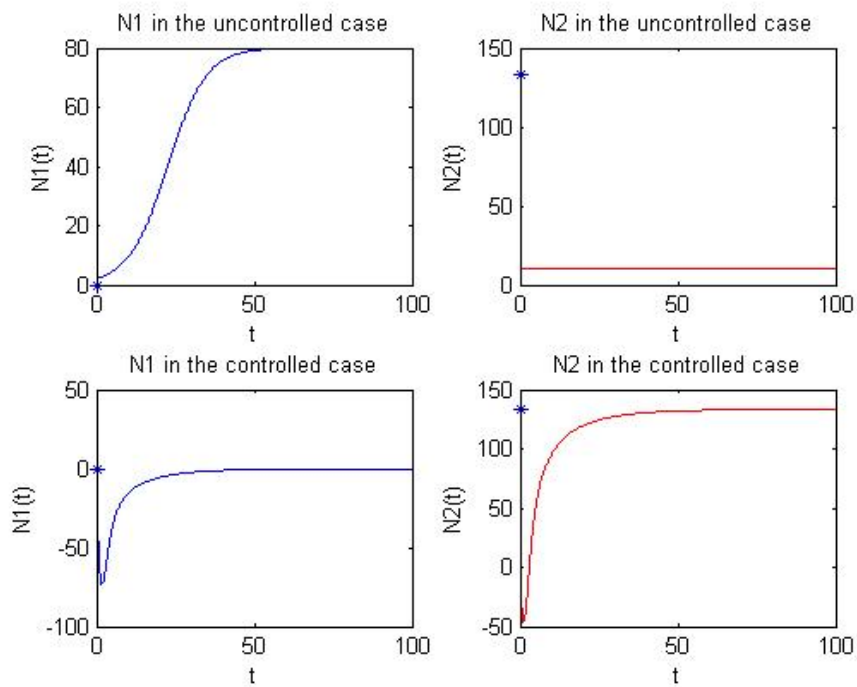


Figure 4.2: Uncontrolled and Controlled solution Trajectories of Steady state $(0, 133.3333)$

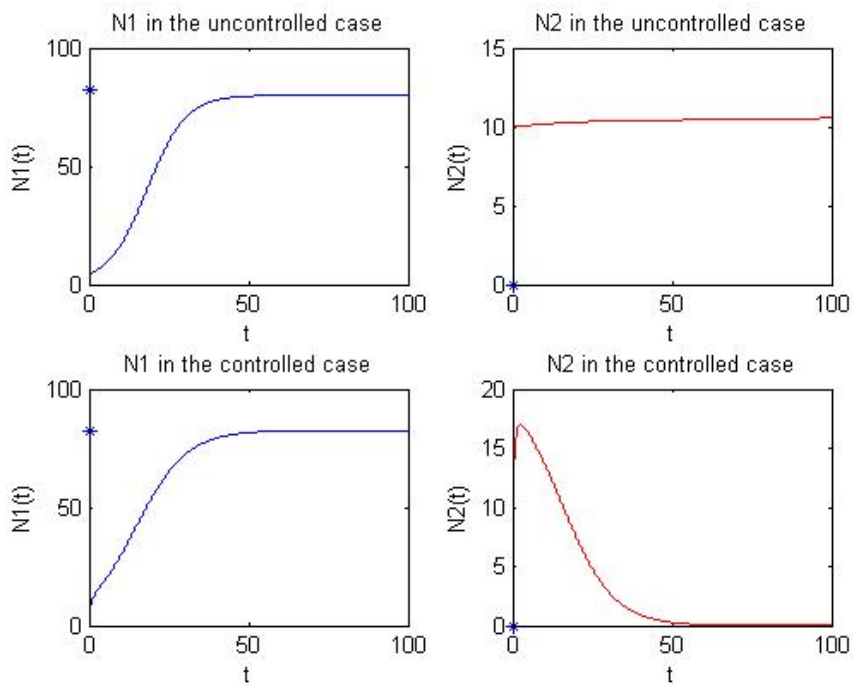


Figure 4.3: Uncontrolled and Controlled solution Trajectories of Steady state $(82.59993, 0)$

In the controlled case, we observe that N_1 converges to 80.0095 and N_2 converges to 10.5526 which is expected when $t_f = 100$. In comparison in the controlled case $N_1 \rightarrow 0.0000$ and $N_2 \rightarrow 0.0001$ which shows that the trivial steady state $(0, 0)$ converges to the same point. Hence the unstable steady state $(0, 0)$ is stabilized.

In the uncontrolled case, we cannot guarantee where the unstable steady state will converge. However, in the controlled stabilization of unstable steady state, our optimal control methods show that we can stabilize unstable steady state to converge.

Note: Further investigation can be made for changing patterns of final time.

Other Extensions of these important model Parameter

For $a = 0.084$, The eigenvalues are

$$\begin{aligned} \text{At } (0, 0), \quad \lambda_1 &= 0.002, \lambda_2 = 0.184 \\ \text{At } (0, 133.3333) \quad \lambda_1 &= -0.002, \lambda_2 = 0.0173 \\ \text{At } (41.3, 0) \quad \lambda_1 &= -0.084, \lambda = 0.0012 \\ \text{At } (12.6777, 116.4298) \quad \lambda_1 &= -0.0264, \lambda_2 = -0.0011 \end{aligned}$$

For $a = 0.0336$

$$\begin{aligned} \text{At } (0, 0) \quad \lambda_1 &= 0.002, \lambda_2 = 0.0336 \\ \text{At } (0, 133.333) \quad \lambda_1 &= -0.002, \lambda_2 = -0.331 \\ \text{At } (16.52, 0) \quad \lambda_1 &= -0.0336, \lambda = 0.0017 \\ \text{At } (-24.1851, 165.5801) \quad \lambda_1 &= 0.0484, \lambda_2 = -0.0017 \end{aligned}$$

Steady state is invalid or degenerate. It shows no physical ecological meaning.

For $a = 0.0168$

$$\begin{aligned} \text{At } (0, 0) \quad \lambda_1 &= 0.002, \lambda_2 = 0.0168 \\ \text{At } (0, 133.3333) \quad \lambda &= -0.002, \lambda = -0.05 \\ \text{At } (8.26, 0) \quad \lambda_1 &= -0.0168, \lambda = 0.0018 \\ \text{At } (-36.4727, 181.9636) &\quad \text{Invalid} \end{aligned}$$

For $a = 0.1697$

$$\begin{aligned} \text{At } (0, 0) \quad \lambda_1 &= 0.002, \lambda_2 = 0.7697 \\ \text{At } (0, 133.3333) \quad \lambda_1 &= -0.002, \lambda_2 = 0.103 \\ \text{At } (83.4239, 0) \quad \lambda_1 &= -0.1697, \lambda_2 = 0.0003 \\ \text{At } (75.3444, 32.8742) \quad \lambda_1 &= -0.1534, \lambda_2 \end{aligned}$$

For $a = 0.1714$

$$\begin{aligned} \text{At } (0, 0) \quad \lambda_1 = 0.002, \lambda_2 = 0.1714 \\ \text{At } (0, 133.3333) \quad \lambda_1 = -0.002, \lambda_2 = 0.1047 \\ \text{At } (84.2519, 0) \quad \lambda_1 = -0.1714, \lambda_2 = 0.0003 \\ \text{At } (76.5731, 31.2358) \quad \lambda_1 = -0.1559, \lambda_2 = -0.0003 \end{aligned}$$

For $a = 0.1764$

$$\begin{aligned} \text{At}(0, 0) \quad \lambda_1 = 0.002, \lambda_2 = 0.1764 \\ \text{At}(0, 133.3333) \quad \lambda_1 = -0.002, \lambda_2 = 0.1097 \\ \text{At}(86.7299, 0) \quad \lambda_1 = -0.1764, \lambda_2 = 0.0003 \\ \text{At}(80.2594, 26.3208) \quad \lambda_1 = -0.1634, \lambda_2 = -0.0003 \end{aligned}$$

For $a = 0.1848$

$$\begin{aligned} \text{At } (0, 0) \quad \lambda_1 = 0.002, \lambda_2 = 0.1848 \\ \text{At } (0, 133, 3333) \quad \lambda_1 = -0.002, \lambda_2 = 0.1181 \\ \text{At } (90.8599, 0) \quad \lambda_1 = -0.1848, \lambda_2 = 0.0002 \\ \text{At } (86.4032, 18.1291) \quad \lambda_1 = -0.1758, \lambda_2 = -0.0002 \end{aligned}$$

For $a = 0.168 \times 1.5 = 0.252$

$$\begin{aligned} \text{At } (0, 0) \quad \lambda_1 = 0.002, \lambda_2 = 0.252 \\ \text{At } (0, 133.3333) \quad \lambda_1 = -0.002, \lambda_2 = 0.1853 \\ \text{At } (123.8999, 0) \quad \lambda_1 = -0.252, \lambda_2 = -0.0005 \\ \text{At } (135.5536, -47.4047) \quad \lambda_1 = -0.02755, \lambda_2 = 0.0005 \end{aligned}$$

Summary

These eight further extensions of model parameters found in the works of [3] and [18] show interesting further stability features of steady states subject to a variation of intrinsic growth rate of species y .

In particular, we observe an invalid or degenerate steady state when the value Of $a = 0.168$ is reduced below 50% and also when the value of $a = 0.168$ is increased above 50%.

Example 4.2:

This example is a competitive system governed by

$$\frac{dN_1}{dt} = \frac{r_1}{K_1} N_1(t)(K_1 - N_1(t) - \alpha N_2(t)) \quad (4.13)$$

$$\frac{dN_2}{dt} = \frac{r_2}{K_2} N_2(t)(K_2 - N_2(t) - \beta N_1(t)) \quad (4.14)$$

Where $r_1, r_2, K_1, K_2, \alpha, \beta$ are positive numbers. [3].

Let the initial value be $(1.5, 0.5)$ and the final time $t_f = 20$.

Solution:

We use the same procedure in Example 4.1 to find the steady states to first order nonlinear differential equations.

At steady state

$$\frac{dN_1^*}{dt} = \frac{dN_2^*}{dt} = 0$$

This implies that

$$\begin{aligned}\frac{r_1}{K_1}N_1^*(K_1 - N_1^* - \alpha N_2^*) &= 0 \\ \frac{r_2}{K_2}N_2^*(K_2 - N_2^* - \beta N_1^*) &= 0\end{aligned}$$

These equations can be written as

$$\begin{aligned}N_1^*(r_1 - \frac{r_1}{K_1}N_1^* - \frac{\alpha r_1}{K_1}N_2^*) &= 0 \\ N_2^*(r_2 - \frac{r_2}{K_2}N_2^* - \frac{\beta r_2}{K_2}N_1^*) &= 0\end{aligned}$$

We are interested to investigate the steady state of these equations.

If both brackets are assumed to be nonzero, then

$$N_1^* = 0, N_2^* = 0$$

Next, assume $N_2^* \neq 0$ and $N_1^* = 0$

$$\begin{aligned}N_1^* = 0, N_2^* = \frac{r_2}{K_2} &= K_2 \\ \implies (N_1^*, N_2^*) &= (0, K_2)\end{aligned}$$

Also

$$\begin{aligned}N_1^* = \frac{r_1}{K_1} = K_1, N_2^* = 0 \\ \implies (N_1^*, N_2^*) &= (K_1, 0)\end{aligned}$$

Following our previous procedure in this chapter when $N_1^* \neq 0$ and $N_2^* \neq 0$

$$\begin{aligned}r_1 - \frac{r_1}{K_1}N_1^* - \frac{\alpha r_1}{K_1}N_2^* &= 0 \\ \frac{N_1^*}{K_1} + \frac{\alpha}{K_1}N_2^* &= 1\end{aligned}\tag{4.15}$$

similarly,

$$\frac{N_2^*}{K_2} + \frac{\beta}{K_2}N_1^* = 1\tag{4.16}$$

Express (4.15) and (4.16) as

$$N_1^* - \alpha N_2^* = K_1 \quad (4.17)$$

$$N_2^* + \beta N_1^* = K_2 \quad (4.18)$$

Solving simultaneously from (4.17) we have

$$N_1^* = K_1 - \alpha N_2^*$$

Substitute this in (4.18)

$$N_2^* + \beta(K_1 - \alpha N_2^*) = K_2$$

$$N_2^* + \beta K_1 - \alpha \beta N_2^* = K_2$$

$$(1 - \alpha \beta) N_2^* = K_2 - \beta K_1$$

$$N_2^* = \frac{K_2 - \beta K_1}{1 - \alpha \beta}$$

Then

$$\begin{aligned} N_1^* &= K_1 - \alpha \left(\frac{K_2 - \beta K_1}{1 - \alpha \beta} \right) \\ &= \frac{K_1(1 - \alpha \beta) - \alpha(K_2 - \beta K_1)}{1 - \alpha \beta} \\ &= \frac{K_1 - \alpha \beta K_1 - \alpha K_2 + \alpha \beta K_1}{1 - \alpha \beta} \\ &= \frac{K_1 - \alpha K_2}{1 - \alpha \beta} \\ \implies (N_1^*, N_2^*) &= \left(\frac{K_1 - \alpha K_2}{1 - \alpha \beta}, \frac{K_2 - \beta K_1}{1 - \alpha \beta} \right) \end{aligned}$$

This point (N_1^*, N_2^*) is a positive steady state[12] provided

1. $\frac{K_1}{K_2} > \alpha$
2. $\alpha \beta < 1$
3. $\frac{K_2}{K_1} > \beta$

Therefore, on the basis of our calculations which are also consistent with the work of [12], species N_1 and N_2 will coexist and survive in an ecological environment.

In summary we have obtained four steady states for the systems of equations [12]

$$(N_1^*, N_2^*) = (0, 0)$$

$$(N_1^*, N_2^*) = (0, K_2)$$

$$(N_1^*, N_2^*) = (K_1, 0)$$

$$(N_1^*, N_2^*) = \left(\frac{K_1 - \alpha K_2}{1 - \alpha \beta}, \frac{K_2 - \beta K_1}{1 - \alpha \beta} \right)$$

We now investigate which of these four steady states is stable and unstable. Following the theory developed by [12] and [18]

$$\begin{aligned} F(N_1^*, N_2^*) &= r_1 N_1^{*2} - \frac{r_1 N_1^*}{K_1} - \frac{r_1 \alpha}{K_1} N_1^* N_2^* \\ G(N_1^*, N_2^*) &= r_2 N_2^{*2} - \frac{r_2 N_2^*}{K_2} - \frac{r_2 \beta}{K_2} N_1^* N_2^* \end{aligned}$$

Now we linearized [18] at arbitrary (N_1^*, N_2^*)

$$\begin{aligned} a_{11} &= \frac{\partial F}{\partial N_1^*} = r_1 - \frac{2r_1 N_1^*}{K_1} - \frac{r_1 \alpha}{K_1} N_2^* \\ a_{12} &= \frac{\partial F}{\partial N_2^*} = -\frac{r_1}{K_1} \alpha N_1^* \\ a_{21} &= \frac{\partial G}{\partial N_1^*} = -\frac{r_2 \beta}{K_2} N_2^* \end{aligned}$$

Since N_2^* is treated as a constant

$$a_{22} = \frac{\partial G}{\partial N_2^*} = r_2 - \frac{2r_2 N_2^*}{K_2} - \frac{r_2 \beta}{K_2} N_1^*$$

We want to check the stability property of each steady state.

For steady state $(0, 0)$, we substitute $N_1^* = 0$ and $N_2^* = 0$ in the four equations above to obtain

$$\begin{aligned} a_{11} &= r_1 & a_{12} &= 0 \\ a_{21} &= 0 & a_{22} &= r_2 \end{aligned}$$

For this scenario, the Jacobian matrix at steady state $(0,0)$ is

$$A = \begin{pmatrix} r_1 & 0 \\ 0 & r_1 \end{pmatrix}$$

For this 2 X 2 matrix, the eigenvalues of A from linear algebra theory are $\lambda = r_1$ and $\lambda_2 = r_2$. Hence the trivial steady state $(0, 0)$ is unstable because the eigenvalues calculated are of the same positive sign.

Next, substitute $N_1^* = K_1$ and $N_2^* = 0$ to obtain

$$\begin{aligned} a_{11} &= r_1 - 2r_1 = -r_1, a_{12} = -r_1 \alpha \\ a_{21} &= 0, a_{22} = r_2 - \frac{r_2 \beta K_1}{K_2} \end{aligned}$$

The Jacobian matrix is

$$A = \begin{pmatrix} -r_1 & -r_1\alpha \\ 0 & r_2 - \frac{r_2\beta K_1}{K_2} \end{pmatrix}$$

Similarly, using the same theory of triangular matrix the eigenvalues of this matrix are

$$\begin{aligned} \lambda_1 = -r_1 \text{ and } \lambda_2 &= \left(r_2 - \frac{r_2\beta K_1}{K_2} \right) \\ \text{If } \lambda_1 < 0, &\implies r_1 > 0 \\ \text{If } \lambda_2 > 0, &\implies \left(r_2 - \frac{r_2\beta K_1}{K_2} \right) > 0 \\ &r_2 > \frac{r_2\beta K_1}{K_2} \\ &1 > \frac{\beta K_1}{K_2} \\ &K_2 > \beta K_1 \\ &\frac{K_2}{K_1} > \beta \end{aligned}$$

In summary steady state $(K_1, 0)$ is unstable provided

$$r_1 > 0 \text{ and } \frac{K_2}{K_1} > \beta.$$

But steady state $(K_1, 0)$ can also be stable if

$$\begin{aligned} r_1 > 0, \left(r_2 - \frac{r_2\beta K_1}{K_2} \right) &< 0 \\ r_2 &< \frac{r_2\beta K_1}{K_2} \\ K_2 &< \beta K_1 \\ \frac{K_2}{K_1} &< \beta \end{aligned}$$

$(K_1, 0)$ will have two negative eigenvalues.

In this work, we are interested to stabilize only unstable steady states.

When $\alpha = 1, \beta = 3, r_1 = 20, r_2 = 1, K_1 = 2, K_2 = 4$.

It has already been analysed by [18] and shown that unsteady state $(1, 1)$ can be stabilized using optimal control numerical techniques of constructing a controller.

In this work, we slightly modify the model parameters in [18] such that

$\alpha = 1.2, \beta = 3, r_1 = 20, r_2 = 1, K_1 = 2, K_2 = 4$.

In this scenario, the steady state $(1.0769, 0.7692)$ is unstable and we want to find out using the techniques in [18] to investigate the stabilization.

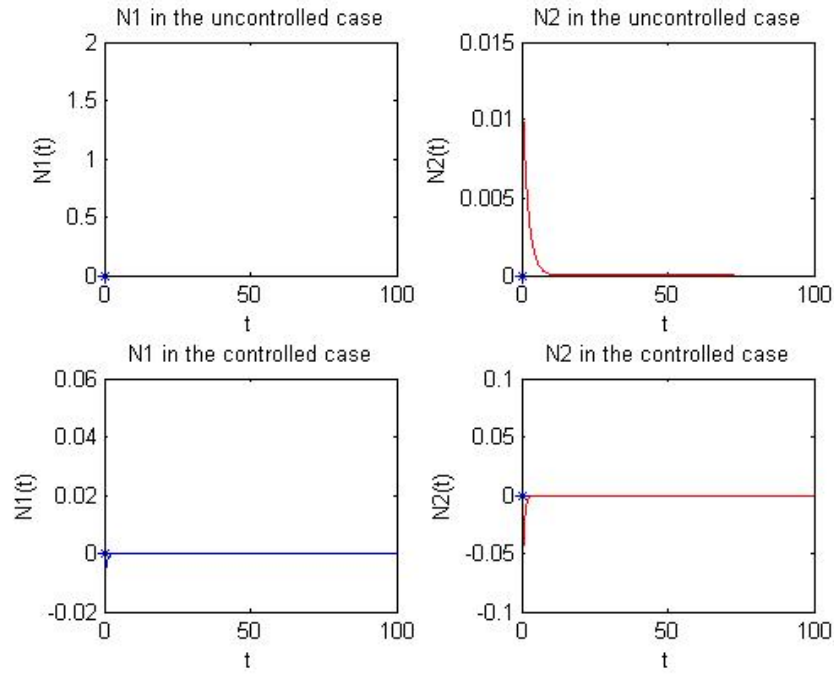


Figure 4.4: Uncontrolled and Controlled solution Trajectories of Steady state $(0, 0)$

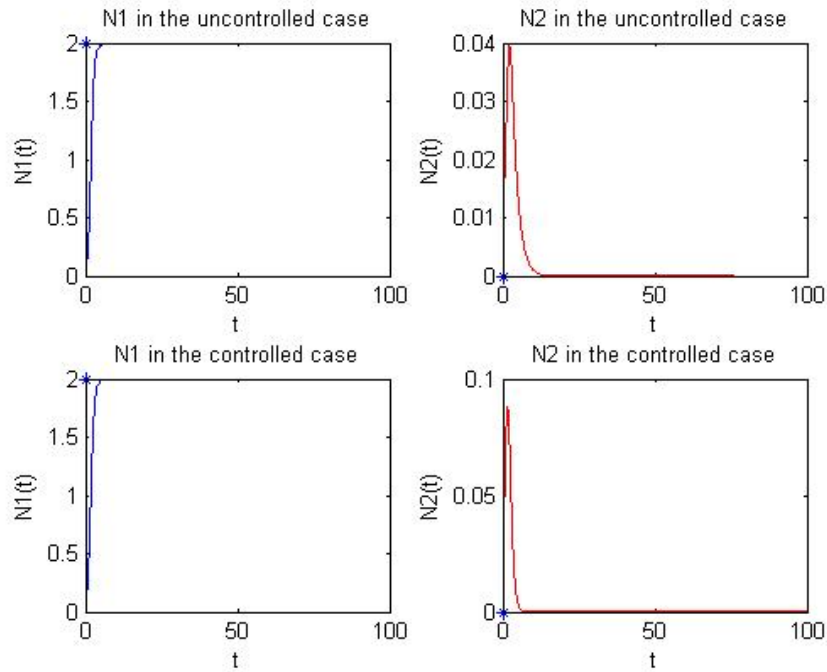


Figure 4.5: Uncontrolled and Controlled solution Trajectories of Steady state $(2, 0)$

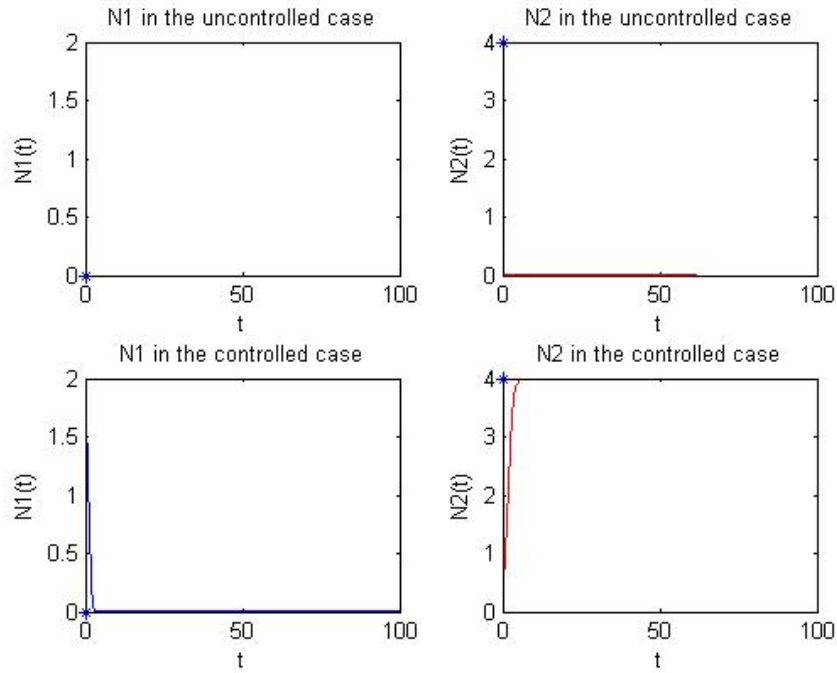


Figure 4.6: Uncontrolled and Controlled solution Trajectories of Steady state $(0, 4)$

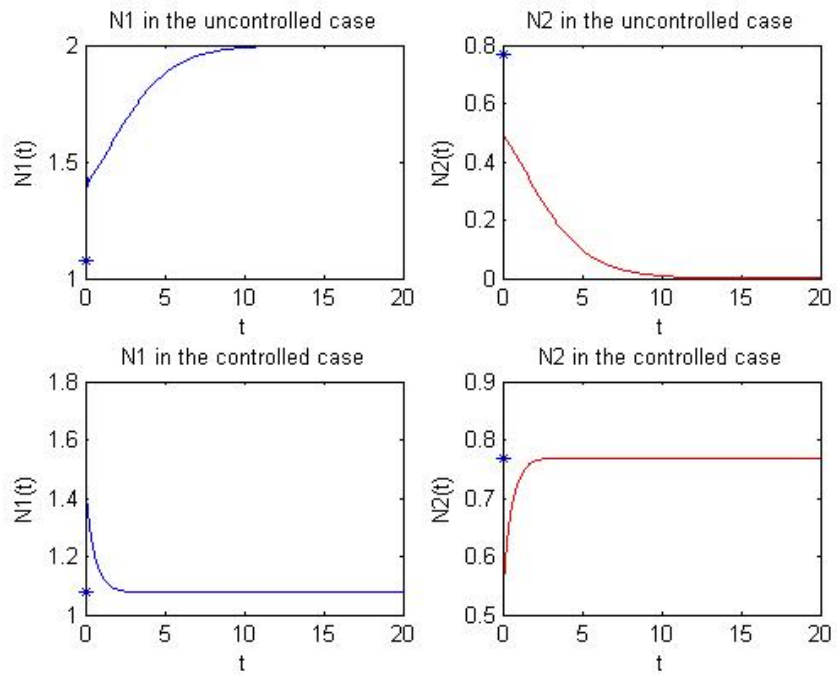


Figure 4.7: Uncontrolled and Controlled solution Trajectories of Steady state $(1.0769, 0.7692)$

In the figures above, we observe that the controlled system is stable at the steady state.

5 Conclusion and Recommendation

We discussed the theory of optimal control systems with examples. We looked at the procedures on how to stabilize a nonlinear system. We first found the steady states and found out which of the steady state(s) is/are stable or unstable. We know that if a steady state is stable, then we do not need any further stabilization process but if it is unstable, we need to stabilize it by using the feedback controller (which is a powerful tool in the stabilization process) on the linearized system. We used the linear quadratic regulator (lqr) to solve the Riccati equation and substituted the resulted 2x2 symmetric matrix in the feedback controller.

The finite difference method was used to discretize the nonlinear system and we designed the algorithm to solve some test problems of nonlinear ecological models. The numerical results obtained are in accordance with theoretical results.

In the future, we will consider how to stabilize a partial differential equation using the feedback control. Also, the feedback control may be extended to modelling of complex ecological interactions and dynamics of tank reactor phenomena.

6 Appendices

Note: The use of these programs are subject to the problem at hand. In other words, the program can be modified with the parameters given in a particular problem.

Appendix 1: Matlab code for Example 4.1

```
% Three unstable steady states (0,0), (0,133.3333), (82.59993,0)
Tfinal =default('Final time, (default is Tfinal=20)', 20);
Y0_1 =default('first element of initial value Y0=(Y0_1, Y0_2), (default is Y0_1=1.5)',1.5);
Y0_2 =default('second element of initial value Y0=(Y0_1, Y0_2), (default is Y0_2=0.5)',0.5);

alpha=1.2;
%beta=1/2; % stable
beta=3; %unstable
r1=20;
r2=1;
K1=2;
K2=4;

%coefficients
a= r1;
b= r1/K1;
c=alpha*r1/K1;
d=r2;
e=r2*beta/K2;
f=r2/K2;
%steady states
N1e = 1/(1-alpha*beta) *(K1-alpha*K2);
N2e = 1/(1-alpha*beta) * (K2-beta*K1);

%coefficients of the linearized system
a11 = a -2*b*N1e - c*N2e;
a12 = -c*N1e;
a21 =-e*N2e;
a22 = d - e*N1e - 2*f*N2e;

A=[a11 a12;a21 a22];
B=[1 1]';
Q=[1 0;0 1];

%construct the feedback control
R=1;
N1=0;
[K, P, E]=lqr(A,B,Q,R,N1);
Pi=P;
```

```
k=0.1; % time step
M=Tfinal/k; % number of loops
Y0=[Y0_1 Y0_2]; %initial value
tt=[0:k:Tfinal];
N=zeros(2*(M+1),2);
for j=1:2
    %m=0 means the uncontrolled case, m=1 means the controlled the case
    m=j-1;

    %[T,Y]=ode23(@myfunction,[0:Tfinal],Y0, 'RelTol=0.001');
    N1=Y0(1); N2=Y0(2);
    Y=[N1 N2];
    mBVO=[];
    for i=1:M
        mBV=m*(-B*B'*Pi*[N1-N1e;N2-N2e])
        N1=N1+k*(N1*(a-b*N1-c*N2)) +k*mBV(1);
        N2=N2+k*(N2*(d-e*N1-f*N2)) +k*mBV(2);
        Z=[N1 N2];
        Y=[Y;Z];
        mBVO=[mBVO mBV];
    end

    N1=Y(:,1);
    N2=Y(:,2);

    N(:,j)=[N1;N2];
end

figure(1)
subplot(2,2,1);
N1_uncontrolled=N(1:(M+1),1);
plot(tt, N1_uncontrolled,'b')
hold on
plot(0, N1e,'*')
title('N1 in the uncontrolled case')
xlabel('t')
%gtext('N_1(t) uncontrolled')
ylabel('N1(t)')

subplot(2,2,2);
N2_uncontrolled=N((M+2):(2*M+2),1);
plot(tt, N2_uncontrolled,'r')
hold on
plot(0, N2e,'*')
title('N2 in the uncontrolled case')
```

```
xlabel('t')
ylabel('N2(t)')
%gtext('N_2 uncontrolled')

subplot(2,2,3);
N1_controlled=N(1:(M+1),2);
plot(tt, N1_controlled,'b')
title('N1 in the controlled case')
hold on
plot(0, N1e,'*')
xlabel('t')
ylabel('N1(t)')
%gtext('N_1 controlled')

subplot(2,2,4);
N2_controlled=N((M+2):(2*M+2),2);

plot(tt,N2_controlled,'r')
title('N2 in the controlled case')
hold on
plot(0, N2e,'*')
xlabel('t')
ylabel('N2(t)')
%gtext('N_2 controlled')

% figure(2)
% subplot(2,1,1);
% plot(tt(2:end), mBV0(1,:))
% title('Feedback control BV1 on the first equation')
% xlabel('t')
% ylabel('BV1(t)')
% gtext('Feedback control on the first equation')
% subplot(2,1,2);
% plot(tt(2:end), mBV0(2,:))
% title('Feedback control BV2 on the second equation')
% xlabel('t')
% ylabel('BV2(t)')
% gtext('Feedback control on the second equation')

N1=N1_controlled;
N2=N2_controlled;

N1uc=N1_uncontrolled;
N2uc=N2_uncontrolled;

ANS=[ N1uc N2uc N1 N2]
```

```
C=[B A*B];
```

```
d1=det(C);  
d2=rank(C);
```

```
RS1=[d1 d2]'
```

Appendix 2: Matlab code for Example 4.2

```
% Four unstable steady states (0,0), (2,0), (0,4) and (1.0769, 0.7692)
N1e =default('N1e is the first element of the steady state (N1e, N2e), (default is N1e=0)',0);
N2e =default('N1e is the second element of the steady state (N1e, N2e), (default is N2e=0)',0);
Tfinal =default('Final time, (default is Tfinal=100)',100);
Y0_1 =default('first element of initial value Y0=(Y0_1, Y0_2), (default is Y0_1=2)',2);
Y0_2 =default('second element of initial value Y0=(Y0_1, Y0_2), (default is Y0_2=10)',10);

%model parameters
r1=20
k1=2
k2=4
r2=1
alpha=1
beta=3

%coefficients of the linearied system
a11 = r1-(2*r1*N1e/k1)-(r1*alpha*N2e/k1);
a12 = -(r1*alpha*N1e)/k1
a21 = -(r2*beta*N2e)/k2
a22 = r2-(2*r2*N2e)/k2-(r2*beta*N1e)/k2;

A=[a11 a12;a21 a22];
B=[1 1]';
Q=[1 0;0 1];

%construct the feedback control
R=1;
N1=0;
[K, P, E]=lqr(A,B,Q,R,N1);
Pi=P;

%solve the nonlinear system
k=0.01; % time step
M=Tfinal/k; % number of loops
Y0=[Y0_1 Y0_2]; %initial value
tt=[0:k:Tfinal];
N=zeros(2*(M+1),2);
for j=1:2
% m=0 means the uncontrolled case, m=1 means the controlled the case
m=j-1;
%[T,Y]=ode23(@myfunction,[0:Tfinal],Y0, 'RelTol=0.001');
N1=Y0(1); N2=Y0(2);
Y=[N1 N2];
mBVO=[];
```

```
for i=1:M
    mBV=m*(-B*B'*Pi*[N1-N1e;N2-N2e]);
    N1=N1+k*(N1*(r1-r1*N1/k1-r1*alpha*N2/k1)) +k*mBV(1);
    N2=N2+k*(N2*(r2-r2*N2/k2-r2*beta*N1/k2)) +k*mBV(2);
    Z=[N1 N2];
    Y=[Y;Z];
    mBVO=[mBVO mBV];
end

N1=Y(:,1);
N2=Y(:,2);

N(:,j)=[N1;N2];
end

figure(1)
subplot(2,2,1);
N1_uncontrolled=N(1:(M+1),1);
plot(tt, N1_uncontrolled,'b')
hold on
plot(0, N1e,'*')
title('N1 in the uncontrolled case')
xlabel('t')
%gtext('N_1(t) uncontrolled')
ylabel('N1(t)')

subplot(2,2,2);
N2_uncontrolled=N((M+2):(2*M+2),1);
plot(tt, N2_uncontrolled,'r')
hold on
plot(0, N2e,'*')
title('N2 in the uncontrolled case')
xlabel('t')
ylabel('N2(t)')
%gtext('N_2 uncontrolled')

subplot(2,2,3);
N1_controlled=N(1:(M+1),2);
plot(tt, N1_controlled,'b')
title('N1 in the controlled case')
hold on
plot(0, N1e,'*')
xlabel('t')
ylabel('N1(t)')
%gtext('N_1 controlled')

subplot(2,2,4);
```

```
N2_controlled=N((M+2):(2*M+2),2);

plot(tt, N2_controlled,'r')
title('N2 in the controlled case')
hold on
plot(0, N2e,'*')
xlabel('t')
ylabel('N2(t)')
%gtext('N_2 controlled')

% figure(2)
% subplot(2,1,1);
% plot(tt(2:end), mBV0(1,:))
% title('Feedback control BV1 on the first equation')
% xlabel('t')
% ylabel('BV1(t)')
% %gtext('Feedback control on the first equation')
% subplot(2,1,2);
% plot(tt(2:end), mBV0(2,:))
% title('Feedback control BV2 on the second equation')
% xlabel('t')
% ylabel('BV2(t)')
%gtext('Feedback control on the second equation')

ANS=[N1_uncontrolled N2_uncontrolled N1_controlled N2_controlled]

C=[B A*B];
d1=det(C);
d2=rank(C);

D=[d1 d2]'
```

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