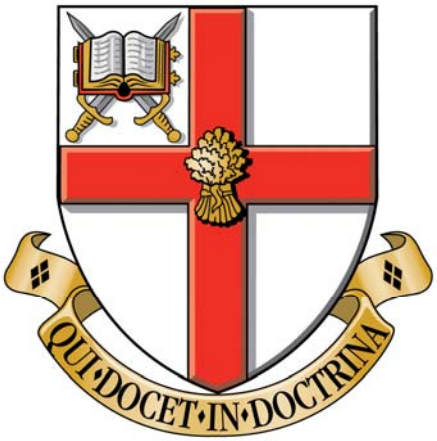

University of Chester

Author(s): Teresa Diogo; John T Edwards; Neville J Ford; Sophy M Thomas

Title: Numerical analysis of a singular integral equation

Date: 2005

Originally published in: Applied Mathematics and Computation

Example citation: Diogo, T., Edwards, J. T., Ford, N. J., & Thomas, S. M. (2005). Numerical analysis of a singular integral equation. *Applied Mathematics and Computation*, 167, 372-382

Version of item: Author's preprint

Available at: http://hdl.handle.net/10034/11851

Numerical analysis of a singular integral equation

Teresa Diogo, John T. Edwards, Neville J. Ford [1] and Sophy M. Thomas

Mathematics Department, University College Chester, Parkgate Road, Chester, CH1 4BJ, UK.
Neville Ford and John Edwards are members of the Manchester Centre for Computational Mathematics. Teresa Diogo is a member of the Centro de Matemática e Aplicações, IST, Lisbon

Abstract

In this paper we give details of the numerical analysis of an integral equation of the form

$$y(t) = f(t) + \int_0^t \frac{s^{\mu-1}}{t^{\mu}} y(s)ds \quad \mu > 0. \tag{1}$$

The distinctive feature of the equation is the presence of a singularity at $t = 0$ for all values of $\mu > 0$ and at $s = 0$ for all values of $t > 0$ for $0 < \mu < 1$. This means that conventional analytical and numerical theory does not apply. In fact, for $0 < \mu < 1$, the equation has an infinite family of solutions.

As background, we give details of the analytical results on existence and uniqueness of solutions to (1) and we give new results on the use of numerical schemes that will yield approximations to *any specific solution*. We conclude with some numerical results that show that our methods enable us to find approximations of any chosen order to any of the infinite class of exact solutions to (1).

1 Introduction

We consider in this paper the solution of equations of the form

$$y(t) = f(t) + \int_0^t \frac{s^{\mu-1}}{t^{\mu}} y(s)ds, \quad \mu > 0, \tag{2}$$

[1] corresponding author email: njford@chester.ac.uk

where $f(0)$ is finite. An initial inspection of (2) would lead the reader to expect that the analysis of the equation would be quite straightforward: the problem is a linear integral equation with separable kernel and might be expected to yield to quite a simple analysis. In fact this equation has been the subject of previous analysis in, for example, [6,7,9,10,14,4]. It is of interest because of the rather unusual singularity. For values of $\mu > 1$ the singularity at $t = 0$ does not *persist* for values of $t > 0$. Thus the solutions are quite well-behaved. However, for $0 < \mu < 1$ there is a singularity in the kernel for all values of $t > 0$. It turns out that, in this case, there are infinitely many solutions to (2) passing through the point $(0, f(0))$. Our main concern in this paper will be with the effective numerical approximation of *a chosen* solution to (2) for $0 < \mu < 1$.

Remark 1.1 *Note that* $\lim_{t\to 0} \int_0^t \frac{s^{\mu-1}}{t^\mu} y(s)ds = \frac{y(0)}{\mu}$ *and therefore, if* $y(0) \neq 0$ *we have* $y(0) \neq f(0)$*, more precisely* $y(0) = \frac{\mu}{\mu-1} f(0)$.

There is a wide literature on certain classes of singular integral equations. Most notably previous authors have been concerned with convolution equations of the form

$$y(t) = f(t) + \int_0^t k(t-s)y(s)ds \tag{3}$$

and within the class of convolution equations, many authors concentrate on those where the kernel k has a singularity at the origin, such as the Abel integral equation

$$y(t) = f(t) + \int_0^t \frac{y(s)}{(t-s)^\alpha} ds \tag{4}$$

for some value of $\alpha > 0$ (see, for example, [1,11] and the references contained therein).

However, despite the fact that the problem (2) can be expressed in convolution form through the substitutions $\Sigma = \log s, \tau = \log t$ (see, for example [3]) it turns out that the nature of the singularity at the origin in (2) makes it more convenient to analyse the problem in its original form. We draw attention, in particular, to the fact that the singularity in the kernel of the Abel equation (4) persists for all values of $t > 0$ and arises always for the *current* value $s = t$. In our paper [3] we consider the equation:

$$y(e^\tau) = g(e^\tau) + \int_{-\infty}^{\tau} e^{\mu(\Sigma-\tau)} y(e^\Sigma) d\Sigma. \tag{5}$$

We remark that this is an equation with infinite memory and such equations are difficult to analyse). We refer the reader to [3] for further details.

2 The analytical background

Basic existence and uniqueness theory for linear Volterra integral equations of the form

$$y(t) = f(t) + \int_0^t k(t,s)y(s)ds \tag{6}$$

is well-established and can be found, for example, in the books [1,11]. The usual approach is to consider a suitable domain $\mathcal{D}$ and to require a Lipschitz condition on the kernel k with respect to its second argument with uniform Lipschitz constant $\mathcal{L}$ on $\mathcal{D}$. There is no restriction (as would be the case for Fredholm equations) on the value that $\mathcal{L}$ can take, save that it is finite, and one can then prove that the successive Picard iterates converge on $\mathcal{D}$ to a solution of (6) and that the solution is unique.

Now it is quite clear that in the case of (2) there is no such finite Lipschitz constant available for the kernel $k(t,s) = \frac{s^{\mu-1}}{t^{\mu}}$ on any domain $\mathcal{D}$ that contains the origin if $0 < \mu < 1$.

When $\mu > 1$ and $t > 0$ the integral $I_t := \int_0^t \frac{s^{\mu-1}}{t^{\mu}} y(s)ds$ is regular. Indeed, the limit $\lim_{t\to 0} I_t$ exists so that I_t has a removeable singularity at the origin. It follows that, for $\mu > 1$, (2) has a unique solution on $[0,T]$ for any finite $T > 0$.

We now turn our attention to the case $0 < \mu < 1$ since this is the case of greatest interest. As we remarked earlier the conventional existence/uniqueness theory cannot be applied in this case. We will show that there are infinitely many solutions to (2).

Define the linear operator:

$$L[y](t) = y(t) - \int_0^t \frac{s^{\mu-1}}{t^{\mu}} y(s)ds. \tag{7}$$

It is simple to confirm that $\alpha t^{1-\mu} \in Ker(L)$ for any $\alpha \in \mathbb{R}$ and that no other power of t is in the kernel.

Now it follows by linearity of L that, for any solution $\tilde{y}$ of

$$L[y](t) = f(t) \tag{8}$$

$$\tilde{y} + \alpha t^{1-\mu} \tag{9}$$

is also a solution.

We ask whether there exists a solution $\tilde{y}$ of (8). Obviously the question hinges on the precise nature of the function f. If we look for solutions $\tilde{y}$ in some function space $\mathcal{F}$ then a solution will exist precisely for functions $f \in L[\mathcal{F}]$, the image of $\mathcal{F}$ under the operator L. In the papers [9,10] the authors describe in detail suitable function spaces $V_{m,\beta}$ and we refer the interested reader to those papers. Here, for simplicity, we assume that the function f is smooth. It is then quite simple to show that there is a solution $\tilde{y}$ to $L[y](t) = f(t)$ (for example by looking for a smooth solution using a Taylor series method, and showing that the resulting formula solves the equation). Therefore we know that there is an infinite family of solutions of the form (9). Only one member of this family ($\alpha = 0$) is differentiable at the origin.

In previous papers on the numerical solution of (2) the authors (see [9,10,14]) have adopted the following approach: approximate the smooth solution as accurately as possible and then add the appropriate multiple of $t^{1-\mu}$ to provide the solution sought. That approach

has attractions from a theoretical viewpoint in that the analysis provides every possible trajectory. However from a practical viewpoint (as we will discuss later) it turns out that the approximate trajectory chosen in any particular case may exhibit the wrong qualitative behaviour. It is the preservation of correct qualitative behaviour that the methods presented here achieve more effectively.

3 Uniqueness of a specific trajectory

In this section we will present an analysis for a *particular solution trajectory* of equation (2). From now on we shall always assume that $0 < \mu < 1$.

Our approach is based on the premise that in a practical application a particular solution will be required. The fact that the underlying equation posseses infinitely many such solution trajectories implies that some further data is required in order to specify the particular path to be approximated.

From previous experience with differential and integral equations we might anticipate that this extra data should be given either in terms of an additional point on the chosen trajectory (for example the value of $y(r)$ for some $r > 0$) or as some initial data about the values of derivatives of y at the initial time. As we have seen, all the non-smooth solutions to (2) have infinite derivative at $t = 0$ and so it will not be appropriate to expect data to be given in terms of derivatives of the solution path at the origin. Accordingly we shall assume that the value of $y(r)$ is provided (at least approximately) for some fixed $r > 0$.

Now consider the expression

$$y(r) = f(r) + \int_0^r \frac{s^{\mu-1}}{r^\mu} y(s)ds. \tag{10}$$

For fixed $r > 0$, this expression provides us with a method of finding the initial integral

$$\int_0^r s^{\mu-1} y(s)ds = r^\mu (y(r) - f(r)). \tag{11}$$

We shall use this expression later.

We return to the analysis of (6): we note that the non-uniqueness problem arose because of the singularity of (2) when $s = 0$. If $\mathcal{D}$ does not contain a neighbourhood of the line $s = 0$, then $k(t, s) = \frac{s^{\mu-1}}{t^\mu}$ satisfies a suitable Lipschitz condition on $\mathcal{D}$. This property provides us with the key to analysing the exact solutions to (2).

We reformulate equation (2) in the following way. First we choose some fixed value of $\epsilon > 0$ and we subdivide the integral (for $t > \epsilon$):

$$y(t) = f(t) + \int_0^\epsilon \frac{s^{\mu-1}}{t^\mu} y(s)ds + \int_\epsilon^t \frac{s^{\mu-1}}{t^\mu} y(s)ds. \tag{12}$$

This is equivalent to the equation

$$y(t) = \tilde{f}(t) + \int_{\epsilon}^{t} \frac{s^{\mu-1}}{t^{\mu}} y(s)ds, \quad t > \epsilon. \tag{13}$$

Now, with a simple change of origin, the standard existence and uniqueness theory applies to (13) and we can deduce that there exists a unique solution for y on the interval $[\epsilon, T]$ for any fixed $T > \epsilon$.

We turn our attention next to the function $\tilde{f} = f(t) + \int_0^{\epsilon} \frac{s^{\mu-1}}{t^{\mu}} y(s)ds$. For a fixed forcing term f the values of $\tilde{f}$ vary according to the integral term. Therefore, suppose that we know a particular solution $\tilde{y}$ on $[0, \epsilon]$, for which the integral $\int_0^{\epsilon} s^{\mu-1}\tilde{y}(s)ds = c$. It then follows that there is, for each possible value of c obtained in this way, a unique solution to (2) that continues $\tilde{y}$ to $[0, T]$ for any fixed value of T. We conclude that, for each possible value that c can take, defined in this way, there is precisely one solution y to (12).

Finally we can use the formula for c given in (11) above. We choose $\epsilon = r$ and conclude that, given the additional data point $(r, y(r))$ there is now a unique solution $y(t)$ to (2) for any interval $[r, T]$ and any finite $T > r$. In other words there is precisely one solution to (2) on $[r, T]$ crossing each point on the line $t = r$.

Remark 3.1 Uniqueness of a trajectory on $[0, r]$**.** *It is natural to ask whether there may be several solutions to (2) on the interval* $[0, r]$ *passing through a single point* (r, β)*. The answer to this is "*No*" for the following reason: if we choose some value* $\delta_1 : 0 < \delta_1 < r$ *and consider equation (2)* but with time running backwards *it is easy to show that there is a unique solution on* $[\delta_1, r]$ *that passes through* (r, β)*. Now choose a sequence* $\delta_1 > \delta_2 > \ldots > \delta_n > \ldots > 0$ *such that* $\delta_n \to 0$*. It follows that there is a unique solution through* (r, β) *on each interval* $[\delta_n, r]$ *therefore each solution trajectory on* $[0, r]$ *is unique.*

4 Numerical schemes

In previous papers ([9,10]) the singularity at the origin has played a big part in the choice of suitable numerical schemes. The need to integrate accurately over the initial interval has prompted the development of product integration methods to ensure a consistent approach. However the analysis of the previous section shows that, in fact, in our quest for a particular (possibly non-smooth) solution we have reformulated our problem into one where there is no singularity. To be precise: on the domain $\{(s, t) : r \leq s \leq t \leq T\}$ the kernel of (2) satisfies a uniform Lipschitz condition. It follows that the standard theory for numerical approximations now applies to the equation over $[r, T]$ (see, for example [1] chapters 3-5, [11] chapter 7) and therefore as long as the function f is sufficiently smooth we can apply (for example) either a linear $\theta-$method or a reducible quadrature scheme (subject to a suitable starting procedure) and obtain the same order of convergence on $[r, T]$ as we would over $[0, T]$ for an integral equation whose kernel was smooth.

To make our approach clear, we fix $h > 0$ and approximate the solution to (2) at gridpoints $t_n = r + nh, y_n \approx y(t_n)$ using the formula

$$y_n = f(t_n) + \int_0^r \frac{s^{\mu-1}}{t_n^\mu} y(s)ds + I_{n,h} \tag{14}$$

where

$$I_{n,h} = \sum_{j=0}^{n} \omega_{n,j} \frac{(r+jh)^{\mu-1}}{(r+nh)^\mu} y_j \tag{15}$$

for quadrature weights $\omega_{n,j}, j = 0, \ldots, n; n \in \mathbb{N}$.

The analysis of the numerical schemes follows in a quite straightforward way. For $t \geq r$ we put $\tau = t - r$ and define the function u for $\tau > 0$ by

$$u(\tau) = g(\tau) + \int_0^\tau \frac{(r+\sigma)^{\mu-1}}{(r+\tau)^\mu} u(\sigma)d\sigma \tag{16}$$

where

$$g(\tau) = f(\tau + r) + \frac{A}{(\tau + r)^\mu} \tag{17}$$

and the constant A is chosen to represent $\int_0^r s^{\mu-1}y(s)ds$ for the particular solution y that is sought.

With u defined in this way it follows that the functions u and y satisfy $u(\tau) = y(\tau + r)$. Further (16) satisfies the standard hypotheses and so the well-known theory on convergence of numerical schemes applies to (16) (see, for example, [1,11]). We can deduce, for example, that the trapezium rule is convergent of order 2 and the Euler rules are convergent of order 1. One could also apply higher order methods quite simply and the established analytical theory would apply here too. We give the results of some simple numerical experiments in the next section.

5 Numerical results

For the numerical experiments we have chosen to set

$$f(t) = 1 + t \tag{18}$$

since in this case we can find an exact solution to (2) for *any* choice of $\mu \in (0, 1)$.

The exact solution is

$$y(t) = \frac{\mu}{\mu - 1} + \frac{\mu + 1}{\mu} t + \alpha t^{1-\mu} \tag{19}$$

and we can use this formula to check the errors in our numerical approximations.

For a simple analysis of errors we have considered $\theta-$methods with $\theta = 0, \theta = 1$ and $\theta = \frac{1}{2}$ (the explicit and implicit Euler rules and the trapezoidal rule). We put $r = 0.2$ and

$\mu = 0.4$ and tabulated the errors in the approximation of the smooth solution ($\alpha = 0$ in (19)) at $T = 10$. Of course we put in the *exact* value of the integral over the initial interval $[0, r]$.

h	Explicit Euler	Trapezoidal rule	Implicit Euler
0.1	1.6181	0.0728	1.8885
0.05	0.8431	0.0187	0.9118
0.025	0.4302	0.00476	0.4475
0.0125	0.2173	0.00119	0.2216
0.00625	0.1092	0.000299	0.1103
0.003125	0.0547	0.0000747	0.0550
0.0015625	0.0274	0.0000187	0.0275

Table 1 Errors at $T = 10$ for different step sizes, $\alpha = 0, r = 0.2, \mu = 0.4$

The expected convergence rates are seen in the figures with both Euler rules displaying $\mathcal{O}(h)$ convergence and the trapezoidal rule exhibiting $\mathcal{O}(h^2)$ convergence.

In fact the same pattern of errors is found with any other member of the family of solutions: with r and μ unchanged but this time with $\alpha = 1$ the corresponding errors for the trapezoidal rule look like this:

h	Trapezoidal rule
0.1	0.1065
0.05	0.02762
0.025	0.006976
0.0125	0.001749
0.00625	0.0004375
0.003125	0.0001094
0.0015625	0.00002735

Table 2 Errors at $T = 10$ for different step sizes, $\alpha = 1, r = 0.2, \mu = 0.4$

6 Propagation of errors in the given data

The example calculations given in the previous section are based on the values chosen for r, μ and the value of the data point (r, β) that determines the solution path that we are

approximating. It is natural to ask how the erors in the numerical solution depend on the choices of r, μ and the accuracy of the data.

First we consider very briefly the effect of changing the value of μ. This situation is covered by the existing theory and we can predict that the convergence rates for the methods will continue to be preserved. Of course as μ increases the derivative of the solution y decreases and therefore the absolute errors in the numerical approximations are likely to reduce. Whether this reduction is significant depends on the value of the constant α in (19). We see this phenomenon in Table 3.

μ	Trapezoidal rule
0.1	0.001866
0.2	0.001749
0.4	0.001193
0.5	0.0009193
0.6	0.0006859
0.8	0.00003526

Table 3 Errors at $T = 10$ for $h = 0.125, \alpha = 0, r = 0.2$

Next we consider the effect of varying r. Here we need to take note that an increase in the value of r means that we are specifying the exact solution at some later instant in time (and in fact we are giving the exact integral over the interval $[0, r]$). This is important because it explains the reduction in error that we can see when r increases.

In fact one would expect to be able to predict this change in error using conventional analytical methods. We recall the discrete Gronwall lemmas that can be used to predict the growth in error in the numerical solution of an integral equation (see, for example, [1] p40-44 or [11] p101-102). In each case, we obtain a bound on the error at a fixed time point in terms of an exponential function of the time elapsed since $t = 0$ Now the effect of increasing r is that we decrease the time between our new origin at r and T. We would expect this to show up as a reduction in the size of the error bound E which should take the form

$$E \leq Be^{k(T-r)}; \tag{20}$$

however this simple analysis does not take into account the singularity in the kernel at the origin. In fact this singularity implies that the smaller the value of r, the larger is the Lipschitz constant associated with the Gronwall lemma. Thus the uniform Lipschitz constant on which the estimate (20) is based under-estimates the improvement in error as r increases and we see this in fig.1. In fact we have numerical evidence that the actual error in our approximation varies as $\exp(T-r)^2$, when using the trapezoidal rule.

Finally, and most importantly, we consider how the data reading $y(r)$ influences the approximate solution we calculate. Here we recall that there exists precisely one solution

to (2) passing through each point (r, β). Therefore if an incorrect data value is given it will *correspond to the correct data value for the wrong solution!* It is easy to see that the exact solutions passing through the points (r, β_1) and (r, β_2) separate at a rate proportional to $t^{1-\mu}$ but it is less clear what will happen to the approximate solutions. However it can be shown (see [3]) that numerical methods for solving (2) exhibit the correct exponential order of growth over the long term. Thus the choice of the *wrong* solution trajectory to follow will result in an error at time t that grows with exponential order $1 - \mu$. Note that this is in agreement with the results about propagation of errors in initial data which were presented in [4].

We include a computed example from the paper [3] that shows that this is indeed the case. For the equation

$$y(t) = 1 + t + \int_0^t \frac{s^{-\frac{1}{2}}}{t^{\frac{1}{2}}} y(s) ds. \tag{21}$$

whose exact solutions take the form $y(t) = -1 + 3t + \alpha t^{\frac{1}{2}}$. We consider the numerical approximations to two trajectories of the solution (respectively for $\alpha = 10, 20$) using the trapezium rule. We set $r = 1$ and $h = 0.1$. We give, in Table 1, very clear numerical evidence that the two approximate trajectories separate at a rate with dominant term proportional to $t^{\frac{1}{2}}$ as $t \to \infty$. This example is discussed more fully in [3].

t	Separation
10	10.00000000000000
20	14.14213562373095
30	17.32050807568878
40	20.00000000000000
50	22.36067977499789
60	24.49489742783177
70	26.45751311064588
80	28.28427124746188
90	29.99999999999998
100	31.62277660168375

Table 4 Separation between solution trajectories for $\alpha = 10, 20$ varying t.

7 Conclusions

The approach we have described in this paper shows that the singular integral equation (2) can be solved using conventional quadrature schemes and that these schemes can provide

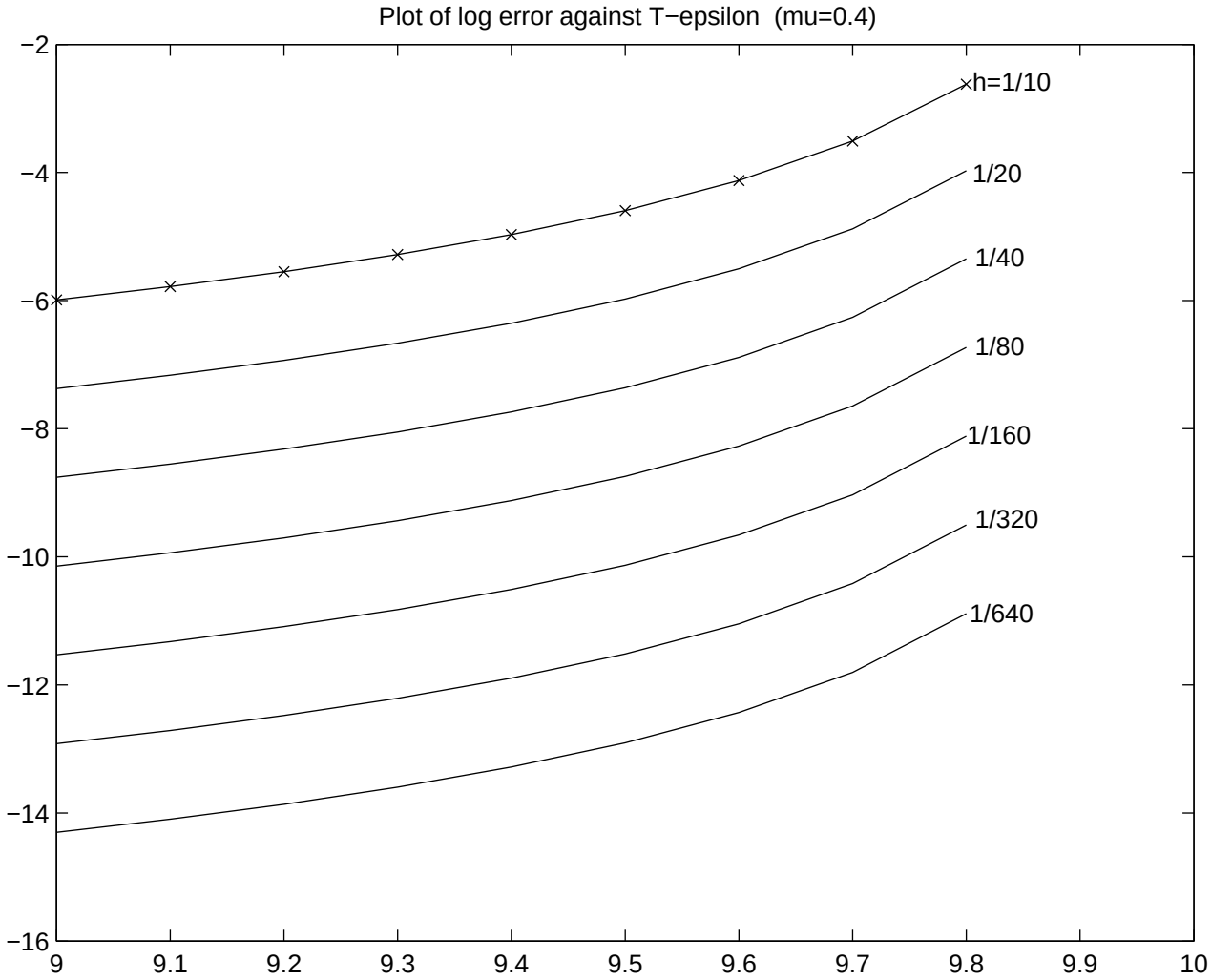

Fig. 1. Plots of $\log(E)$ against $T - r$ for different step sizes h

a good approximation to *any chosen* solution path over any fixed time interval as long as we are given accurate data to enable the correct path to be chosen. We have shown that the numerical schemes are well-behaved and therefore an extrapolation procedure of the type used in previous work ([9]) for product quadratures approximating the smooth solution is likely to be successful in this case too. We leave this investigation for a sequel.

In our sequel we will also discuss the solution of (2) viewed as a boundary value problem, where the boundary value is used to specify the particular path to be followed. The discussion will extend our observations made above about the use of reversed time in predicting solutions over earlier time intervals.

Finally we are pleased to acknowledge the support and advice we have received from a number of quarters. Sophy Thomas acknowledges a University College Chester student bursary. Neville Ford and Teresa Diogo acknowledge support from the British Council. Teresa Diogo acknowledges support from the FCT, project POCTI/MAT/45700/2002. We are also pleased to acknowledge the useful discussions held with Pedro Lima, Chris

Poole and Judy Ford.

References

[1] H. Brunner & P.J. van der Houwen, *The numerical solution of Volterra equations*, North Holland, Amsterdam, 1986.

[2] H. Brunner & J.D. Lambert, Stability of numerical methods for Volterra integro-differential equations, Computing 12, 75–89, 1974.

[3] T. Diogo, J.M. Ford, N.J. Ford, & P. Lima, Non-integrable resolvent kernels for some convolution integral equations and applications to an equation with singular kernel, submitted.

[4] T. Diogo, N.J. Ford, P.Lima & S.Valtchev, Numerical methods for a Volterra integral equation with nonsmooth solutions, in preparation.

[5] E. Hairer, S.P. Nörsett & G. Wanner, *Solving Ordinary Differential Equations I*, Springer-Verlag, Berlin, 1991.

[6] W. Han, Existence, uniqueness and smoothness results for second-kind Volterra equations with weakly singular kernels, J. Int. Eqns. Appl., 9, 365–384, 1994.

[7] W. Lamb, A spectral approach to an integral equation, Glasgow Math. J. 26, 83–89, 1985.

[8] J.D. Lambert, *Numerical methods for ordinary differential equations,*Wiley, Chichester, 1991.

[9] P. Lima & T. Diogo, An extrapolation method for a Volterra integral equation with weakly singular kernel, Applied Numerical Mathematics 24, 131–149, 1997.

[10] P. Lima & T. Diogo, Numerical solution of a non-uniquely solvable Volterra integral equation using extrapolation methods, JCAM, 140, 537–557, 2002.

[11] P. Linz, *Analytical and numerical methods for Volterra equations*, SIAM Publications, Philadelphia, 1974.

[12] Ch. Lubich, On the stability of linear multistep methods for Volterra convolution equations, IMA J. Numer. Anal. 3, 439–465, 1983.

[13] J. Matthys, A-stable linear multistep methods for Volterra integro-differential equations, Numer. Math., 27, 85–96, 1976.

[14] T. Tang, S. McKee & T. Diogo, Product integration methods for an integral equation with logarithmic singular kernel, Applied Numerical Mathematics 9, 259–266, 1992.